THIRD EDITION

THE STRUCTURE Of ECONOMICS

A MATHEMATICAL ANALYSIS

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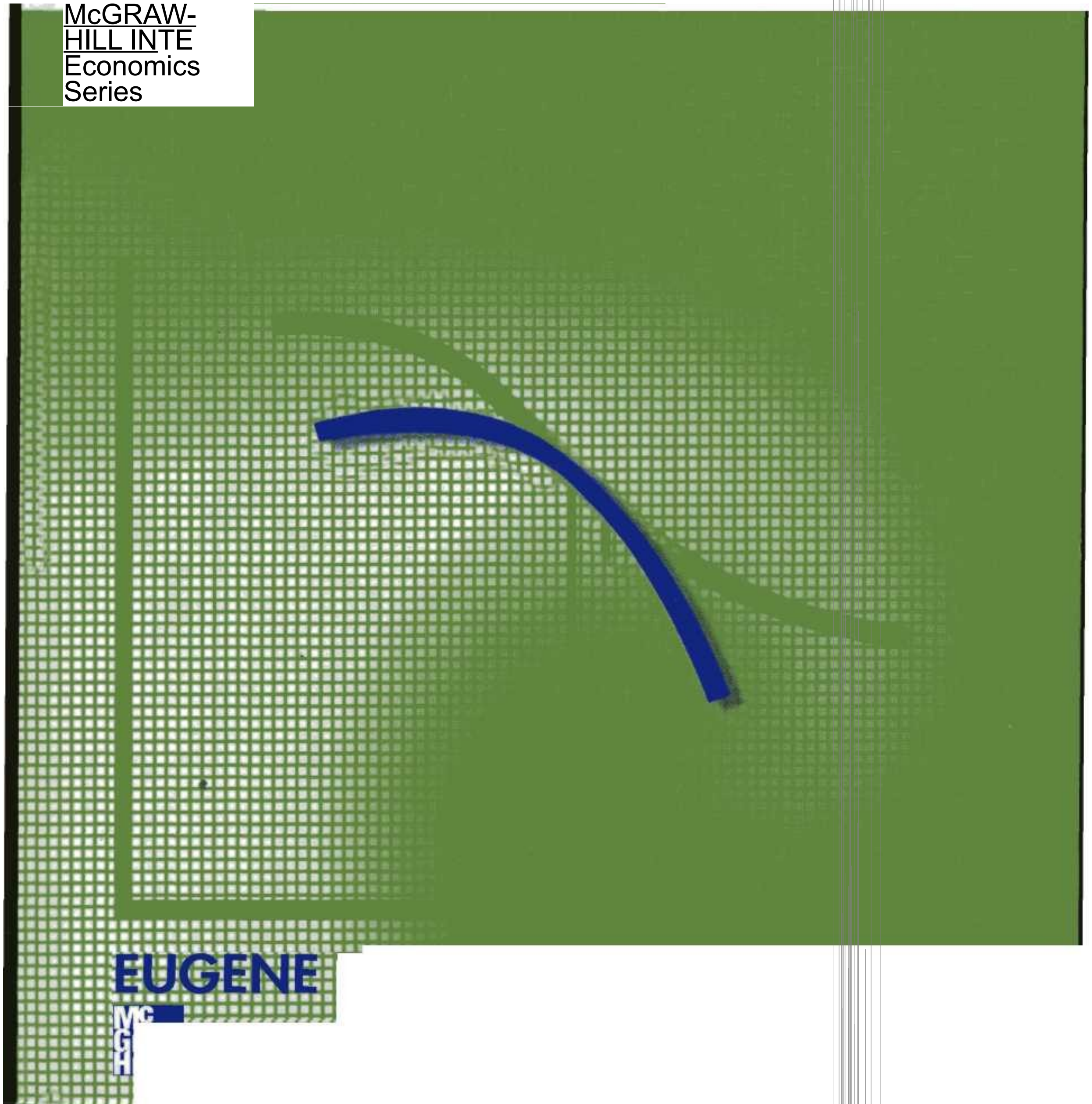

EUGENE

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A MATHEMATICAL ANALYSIS

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PREFACE

It's safe to say that the most interesting and important developments in microeco-nomic theory since the publication of the second edition of this work in 1990 are in the area of choice under imperfect information. With uncertainty, the choices individuals make may reflect the problems of moral hazard and adverse selection, and the operation of the market changes as well to reflect these actions. In the third edition, therefore, we expand the scope of the text to include these new developments in economic theory. In particular, the new Chapter 15, "Contracts and Incentives," covers the recent developments in contract theory, and the new Chapter 16, "Markets with Imperfect Information," covers recent developments in information economics. Wing Suen, of the University of Hong Kong, penned these chapters. Wing was also the secret author in the second edition of Chapter 13, "Behavior Under Uncertainty," to which we have added a few examples.

To accommodate this new material, we discarded the old Chapter 19 on stability of equilibrium. We feel that this material is now less relevant to today's economics courses, both absolutely and relative to the new material. Also, since today's students are much better prepared mathematically than students were when the first edition was first published, we discarded most of the material in Chapter 2, "Review of Calculus (One Variable)," assuming that students have rudimentary knowledge of the calculus of one variable. We maintained the discussion of calculus of several variables but deleted some of the formalisms, in order to make the material accessible to students whose knowledge of that material is less than in working order. Various other changes in the traditional parts of the book include a discussion of discriminating monopoly in Chapter 4, "Profit Maximization"; a theorem and application related to complementary factors of production in Chapter 6, "Comparative Statics: The Traditional Methodology"; an extended but easier discussion of the LeChatelier effects in Chapter 7, "The Envelope Theorem and Duality"; and a variety of extensions and emendations throughout the text.

Although all the analysis contained herein derives from topics in microeconomics, the real subject of this book is ra^faeconomics rather than economics itself. That is, we concern ourselves principally with the methodology of positive economics, in particular, the way meaningful theorems are derived in economics. Paul Samuelson explained in his monumental *Foundations of Economic Analysis* (Harvard University Press, 1947) that the meaningful theorems in economics consist not in laying out various equilibrium conditions, which are rarely observable and therefore empirically sterile, but in deriving predictions that the direction of change of some decision variable in response to a change in some observable parameter must be in some particular direction. The statement that consumers equate their marginal rates of substitution to relative prices is not testable unless we can measure indifference curves. By contrast, the law of demand, which merely requires us to be able to measure the direction of change of an observable price and quantity, is a meaningful, i.e., refutable theorem. Thus in this book, in both the new chapters as well as the old, we devote ourselves almost exclusively to exploring the conditions under which models with a maximization hypothesis generate propositions that are at least in principle refutable.

Although the mathematics we use is elementary, it is extremely useful. The late G. H. Hardy wrote in his delightful essay *A Mathematician's Apology* (Cambridge University Press, 1940) that

> It is the dull and elementary parts of applied mathematics, as it is the dull and elementary parts of pure mathematics, that work for good or ill. Time may change all this. No one foresaw the applications of matrices and groups and other purely mathematical theories to modern physics, and it may be that some of the "highbrow" applied mathematics will become useful in as unexpected a way; but the evidence so far points to the conclusion that, in one subject as in the other, it is what is commonplace and dull that counts for practical life.

Moreover,

> The general conclusion, surely, stands out plainly enough. If useful knowledge is, as we agreed provisionally to say, knowledge which is likely now or in the comparatively near future, to contribute to the material comfort of mankind, so that mere intellectual satisfaction is irrelevant, then the great bulk of mathematics is useless.

But this is precisely what an economist would expect! Hardy was observing the law of diminishing marginal product in the application of mathematical tools to science. A large gain in clarity and economy of exposition can be had from the incorporation of elementary algebra and calculus. The gain from adding real analysis and topology, however, is apt to be less. And perhaps, when such arcane fields as complex analysis and algebraic topology are brought to bear on scientific analysis, their marginal product will be found to be approximately zero, fitting Hardy's definition of "useless." (It is amusing to note, though, that number theory, long considered one of the most useless of all mathematical inquiries, has recently found important application in modern cryptography.)

In this book we explore the insights that elementary mathematics affords the study of positive economics. We do not explore these issues to their fullest generality or mathematical rigor. Although generality and rigor are important economic goods, their production, because of the above-mentioned law of diminishing returns, entails increasing marginal costs. Thus we are usually content with intuitive, heuristic proofs of many mathematical propositions. We refer students to standard mathematics texts for rigorous discussions of various theorems we use in this book. We aimed for that unobservable margin where for the bulk of our readers, the marginal benefits of greater rigor and generality equal their respective marginal costs. By example after example we hope to convince the reader that these elementary tools yield interesting and sometimes profound insights into modern economics.

A note to students and instructors: Long experience teaching this material, and the authors' own experiences in learning it, have made it abundantly clear that mastering this material is impossible without doing the problems. So do the problems! The only true indicator of understanding is that you can explain the solution to someone else. An *Instructor's Manual* is available from McGraw-Hill.

Eugene Silberberg
Wing Suen

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CHAPTER 1

COMPARATIVE STATICS AND THE PARADIGM OF ECONOMICS

1.1 INTRODUCTION

Suppose we are in a conversation about social changes that have taken place in the past generation. We might discuss, for example, the substantial increase in the rate of participation of women in the competitive labor market, especially in "nontradi-tional" occupations such as engineering, law, and medicine, the increasing prominence of the "two-earner" family, the increase in the age of first marriage, the rise of "women's liberation," and the like. Suppose now that someone says, "Let me give you an 'economic explanation' of these events." What do you expect to hear? What is meant by the phrase "economic explanation," and what would distinguish it from, say, a sociological or political explanation? For that matter, what do we mean by the term "explanation"?

A list of facts, for example, is not an explanation. Compilations of changes in the weather as seasons pass, or changes in various stock market indices, are not explanations of those events. The stylized data presented in the preceding paragraph are not an explanation of anything; they are only a collection of economic (and sociological) facts, which we typically call "data." The data may be interesting, but they are not "explanations." The term *explanation* means that there is some more general proposition than the observed data for which these facts are special cases. We interpret or understand these facts by applying some general laws or rules by which these events are supposedly guided. For example, physicists "explain" the motion of ordinary objects on the basis of Newton's classical laws of mechanics. An explanation of the previous socioeeonomic data would mean an interpretation of these events in terms of a framework of systematic human behavior, not merely a documentation that these events happened to occur at a particular time. Moreover, we would want to apply that same framework to different sets of facts, allowing the investigator to interpret these other data sets using the same guiding principles. The development of the framework and the specific models employed by economists to explain social phenomena is the subject of this book.

Students who have come this far in economics will undoubtedly have encountered the standard textbook definition of economics that goes something like, "Economics is the science that studies human behavior as a relationship between ends and scarce means which have alternative uses."* This is indeed the substantive content of economics in terms of the class of phenomena generally studied. To many economists (including the authors), however, the most striking aspect of economics is not the subject matter itself, but rather the conceptual framework within which the previously mentioned phenomena are analyzed. After all, sociologists and political scientists are also interested in how scarce resources are allocated and how the decisions of individuals are related to that process. What economists have in common with each other is a methodology, or paradigm, in which *all* problems are analyzed. In fact, what most economists would classify as *noneconomic* problems are precisely those problems that are incapable of being analyzed with what has come to be called the *neoclassical* or *marginalist* paradigm.

The history of science includes many paradigms or schools of thought. The Ptolemaic explanation for planetary motion, in which the earth was placed at the center of the coordinate system (perhaps for theological reasons), was replaced by the Copernican paradigm which moved the origin to the sun. When this was done, the equations of planetary motion were so vastly simplified that the older school was soon replaced (though the Ptolemaic paradigm is essentially maintained in problems of navigation). The Newtonian paradigm of classical mechanics served admirably well in physics, and still does, in fact, in most everyday problems. For study of fundamental processes of nature, however, it has been found to be inadequate and has been replaced by the Einsteinian paradigm of relativity theory.

In economics, the classical school of Smith, Ricardo, and Marx provided explanations of the growth of productive capacity, the gains from specialization and trade (comparative advantage), and the like. One outstanding puzzle persisted: the diamond-water paradox. The classical paradigm, dependent largely on a theory of value based on inputs, was incapable of explaining why water, which is essential to life, is generally available at modest cost, while diamonds, an obvious frivolity, are expensive, even if dug up accidentally in one's backyard (considering the

^Taken from Lionel Robbins' classic monograph, *An Essay on the Nature and Significance of Economic Science,* Macmillan & Co., Ltd., London, 1932, p. 15.

opportunity cost of withholding one from sale).* With the advent of marginal analysis, beginning in the 1870s and continuing in later decades by Jevons, Walras, Marshall, Pareto, and others, the older paradigm was supplanted. Economic problems came to be analyzed more explicitly in terms of individual choice. Values were perceived to be determined by consumers' tastes as well as production costs, and the value placed on goods by consumers was not considered to be "intrinsic," but rather depended on the quantities of that good and other goods available.

The structure of this new paradigm was explored further by Hicks, Allen, Samuelson, and others. As this was done, the usefulness and limitations of the new paradigm became more apparent. It is with these properties that this book is concerned.

1.2 THE MARGINALIST PARADIGM

Let us consider the definition of economics in more depth. Economics, first and foremost, is an *empirical science. Positive* economics is concerned with questions of *fact,* which are in principle either true or false. What *ought* to be, as opposed to what *is,* is a normative study, based on the observer's value judgments. In this text, we shall be concerned only with positive economics, the determination of what *is.* (For expositional ease the term *positive* will generally be dropped.) Two economists, one favoring, say, more transfers of income to the poor, and the other favoring less, should still come to the same conclusions regarding the effects of such transfers. Positive economics consists of propositions that are to be tested against facts, and either confirmed or refuted.

But what *is* economics, and what distinguishes it from other aspects of social science? For that matter, what is social science? *Social science is the study of human behavior.* One particular paradigm of social science, i.e., the conceptual framework under which human behavior is studied, is known as the *theory of choice.* This is the framework that will be adopted throughout this book. Its basic postulate is that individual behavior is fundamentally characterized by individual choices, or decisions.[i]

This fundamental attribute distinguishes social science from the physical sciences. The atoms and molecular structures of physics, chemistry, biology, etc., are not perceived to possess conscious thought. They are, rather, passive adherents to the laws of nature. The choices humans make may be pleasant (e.g., whether to buy a Porsche or a Jaguar) or dismal (e.g., whether to eat navy beans or potatoes for subsistence), but the aspect of choice is asserted to be pervasive.

^Of course, being different commodities with different "quantity" measurements, it is not possible to say that diamonds are *more* expensive than water.

*A complicating feature, not relevant to the present discussion but also peculiar to the social sciences, is that the participants often have a vested interest in the results of the analysis.

Decisions, i.e., choices, are a consequence of the scarcity of goods and services. Without scarcity, whatever social science might exist would be vastly different than the present variety. That goods and services are scarce is a second, though not independent postulate of the theory of choice. Scarcity is an "idea" in our minds. It is not in itself observable. However, we *assert* scarcity because to say that certain goods or services are *not* scarce is to say that we can all—you, me, everybody—have as much as we want of that good at any time, at zero sacrifice to us all. It is hard to imagine such goods. Even air, if it is taken to *mean fresh* air, is not free in this sense; society must in fact sacrifice consumption of other goods, through increased production costs, if the air is to be less polluted.

Scarcity, in turn, depends upon postulates about individual preferences, in particular that people prefer more goods to less. If such were not the case, then goods, though *limited* in supply, would not necessarily be *scarce.*

The fact that goods are scarce means that choices will have to be made somehow regarding both the goods to be produced in the first place and the system for rationing these final goods to consumers, each of whom would in general prefer to have more of those goods rather than less. This problem, which is often taken as the definition of economics, has many aspects. How are consumers' tastes formed, and are those tastes dependent on ("endogenous to") or independent of ("exogenous to") the allocative process? How are decisions made with regard to whether goods shall be allocated via a market process or through the political system? What system of *rules, i.e., property rights,* is to be used in constraining individual choices? The issues generated by the scarcity of goods involve all the social sciences. All are concerned with different aspects of the problem of choice.

We now come to the fundamental conceptualization of the determinants of choice upon which the neoclassical, or marginalist, paradigm is based. We assert that for a wide range of problems, individual choice can be conceived to be determined by the interaction of two distinct classifications of phenomena:

1.80 Tastes, or preferences

1.81 Opportunities, or constraints

Suppose we were to list all variables that were measurable and that we believed affected individual choices; this would constitute the set of constraints on behavior. What sorts of things would appear?

Certainly, the money prices of goods and the money incomes of individuals play a major part. In most everyday decisions to exchange goods and services, prices and income are the major constraints. More fundamental, however, are the constraints imposed by the system of laws and the property rights in a given society. Without these rights, prices and money income would be largely irrelevant. Ordinary exchange is difficult or impossible if the traders have not previously agreed upon who owns what in the first place, and whether contracts entered into are enforceable. Laws also determine various restrictions on trading. During the winter of 1973-1974, gasoline was quoted at a certain price, but in many parts of the country, it was unavailable for exchange. The *price* of the good loses meaning if the good is unavailable at that price. The same situation existed during World War II when goods were price-controlled. Then, the property rights individuals enjoyed over their goods no longer included the right to sell the good at a mutually satisfactory price with the buyer. Hence, the system of laws and the property rights endowed to the participants in a given society are a fundamental part of their opportunity set.

In addition to the preceding, technology and the law of diminishing returns constitute the other important constraints in economic analysis. Together with the system of laws and the property rights, technology determines the production possibilities of a society, i.e., the limits on total consumption.

Suppose now that we had available complete data on the preceding variables for a given individual. Would this be enough information to enable us to predict the choices the person would make, e.g., whether he or she would eat meat or be a vegetarian, or attend classical rather than rock concerts? It is apparent that no matter how complete a listing of constraints we could contemplate, there would still be other *unmeasured* variables that would influence behavior. These other variables are what we refer to as *tastes,* or *preferences.* Typically; they comprise the hypothetical exchanges a person is willing to make at various terms of trade. These hypothetical offers are our subjective evaluations of the relative desirability of goods.

Furthermore, these unmeasured taste variables seem to vary from individual to individual. Some people, for example, would gladly exchange two pounds of coffee for one of tea; others, in the same circumstances, would do the reverse. Even when the constraints facing two individuals are largely the same, i.e., the individuals have equal incomes, shop at the same stores, and are equal under the law, they will usually purchase different bundles of goods and services. Some people live in small houses and drive big cars; others in similar circumstances buy large houses and drive small cars.

We have thus classified the variables affecting choice as being either *constraints,* which are in principle, at least, observable and measurable, or *tastes,* which are not. Prices, for example, are generally posted, or otherwise available; incomes are usually known to people; laws and property rights can be complicated but are at least on the books, and their enforceability can be determined. In contrast, tastes are not in general observable. It is in fact precisely for this reason that we make *assertions,* or *postulates,* about individual tastes. If tastes were observable, assertions about their nature would not be needed.

Observations of a person's consumption habits, i.e., the baskets of goods purchased, do not constitute observations of tastes. Actual consumption depends on opportunities as well as tastes. The generally nonobservable nature of the preferences of individuals requires that they be postulated, or asserted.

Here, then, is the central puzzle. We have seen that tastes apparently vary, and constraints clearly also vary from individual to individual. (U.S. census figures attest to large differences in incomes among individuals in the United States; the same seems to be true in most societies.) How then can any systematic analysis of choice be made under these horrendously complicated circumstances? The answer to this important question to a large extent defines the field of economics.

To answer all questions of choice, even about a well-defined situation, both tastes and opportunities must be included. Unfortunately, this situation cannot be realized in actual practice. However, it is still often possible to analyze problems of choice in a narrower but still fruitful manner. Suppose we assume that whatever people's tastes are, they do not change very much, if at all, during the course of investigation of some problem in social science. Certain decisions will be made by individuals, given those tastes and the opportunities they face. If, now, the *opportunities* faced by those individuals *change, in an observable fashion,* then we can expect the decisions of individuals to somehow *change,* and those *changes in decisions, or choices, can be attributed to the changes in opportunities.* Moreover, if the unmeasured taste variables can be characterized in a systematic way, so that individuals display regularities in behavior, then while it may not be possible to predict the original choices made by individuals, it may still be possible to predict how those choices *change,* when opportunities or constraints *change.*

We therefore impose structure on individual preferences in order to be able to predict responses to changes in constraints. Subject, as always, to possible refutation by empirical testing, economists assert universal postulates of behavior. In particular, we construe individual behavior to be "purposeful." We assert, for example, that all individuals prefer "more" to "less," and that *they attempt to "mitigate the damages" imposed by constraints, i.e., to reduce rather than reinforce the impact of restrictions on their opportunities.* We give operational content to the behavioral postulates typically by expressing the theory (or parts of it) mathematically as a problem of maximizing (or, if convenient, minimizing) some specified objective function subject to specified constraints.t

In terms of methodology, therefore, *economics is that discipline within social science that seeks refutable explanations of changes in human events on the basis of changes in observable constraints, utilizing universal postulates of behavior and technology, and the simplifying assumption that the unmeasured variables ("tastes ") remain constant}* This is the paradigm of economics, a paradigm that at present distinguishes economics from other social sciences.

Notice that economics does *not* thereby assert either that tastes do not matter or that they remain constant for all time. Preferences are, in fact, asserted to affect individual choices, as previously discussed. What the paradigm of economics recognizes is that it is possible to obtain answers regarding *marginal* quantities, i.e., how total quantities *change,* without a specific investigation of individual preferences or how such preferences might be formed.

Constancy of tastes is a *simplifying assumption,* not an article of faith. It is invoked because it allows investigation of responses to changes in constraints. It

^Because minimizing some function is equivalent to maximizing its negative, no generality is lost by using the term maximizing behavior.

^Strictly speaking, all that is necessary for testing theories is that the unmeasured variables be *uncorre-lated* with the observed data.

is of course impossible to be certain that unmeasured variables remain constant. Tastes may change. But to accept that as an explanation of observed events is to abandon the search for an explanation based on systematic, and therefore testable, behavior. Any observation whatsoever is consistent with a theory that asserts that some unmeasured taste variables suddenly, for no apparent reason, changed. The challenge of economics is always to search for explanations based on changes in constraints; explanations based on changes in tastes are to be viewed with skepticism and as indicative of inadequate insight. We leave such explanations to those who, for example, would "explain" the prevalence of relatively large cars in the United States as a peculiar American "love affair" with big cars, rather than as a consequence of a relatively low retail price of gasoline (generally one-third to one-half of the European price) for most of the twentieth century. The switch to economy cars in the 1970s and the return of "high-performance" cars in the 1990s could be random taste changes, but these observations confirm a more general proposition, the law of demand, because the relative price of gasoline rose in the mid-1970s and fell in the 1980s and 1990s. We prefer the more general theory based on responses to changes in the constraints faced by consumers of cars to ad hoc assertions about changes in tastes.t

How would we apply the neoclassical economic paradigm to the data presented in the opening paragraphs of this chapter? We reject out of hand any explanation based on changes in tastes. The assertion that these events occurred because the young adults of the late sixties and early seventies were more radical than their predecessors is an ad hoc hypothesis, i.e., a theory made up simply to suit a particular set of facts, with no capability for application beyond that immediate data set. Such theories are no better than asserting that people do certain things because they do them. Why should the preferences of large numbers of people suddenly have shifted in unison at that time?

In order to provide an economic explanation, we need to look for a wide-ranging constraint that changed during the 1960s, and explain the events that took

t George Stigler and Gary Becker analyzed "fads and fashions," a subject seemingly not amenable to an analysis in which tastes are assumed constant. They argued that the desire to be "fashionable" is constant. Because consumption of fashion takes place over time, the axiom of diminishing marginal values suggests that fashions will change over time. Moreover, the less costly it is to be fashionable, the more frequent the changes will be. This may explain why fashions may change more quickly for clothing than for automobiles. See George Stigler and Gary Becker, "De Gustibus non est Disputandum," *American Economic Review,* 66:76-90, March 1977.

An additional example of the power of the paradigm is provided by Corry Azzi and Ron Ehrenberg, who showed that participation in religion varied in accordance with the law of demand. The relatively higher participation of women, for example, is what would be predicted on the basis of relatively lower wages for women than for men. Relatively low church attendance in the young adult years, followed by increasing attendance with age, is an implication of young adults' typically heavy time investment in human capital, and increasing present value of possible benefits after death. Higher attendance in rural vs. urban areas is easily related to the higher opportunity costs in urban areas due to the greater variety of recreational services available. See Corry Azzi and Ron Ehrenberg, "Household Allocation of Time and Church Attendance," *Journal of Political Economy,* 83:27-56, February 1975.

place in terms of the movement of that constraint. An economic basis for explaining these events is in fact provided by the World War II "baby boom," the unprecedented increase in births that took place in North America after the war.^ Altogether, one-third more children were born between 1946 and 1950 than between 1941 and 1945. (Births continued at a high level until the 1960s.)

Consider first how this would affect marriage prospects 20 years later, i.e., in the late sixties. The baby boomers were, of course, about equally divided by sex. However, women have always tended to marry men slightly older than themselves. When the baby boomers reached young adulthood, the women were faced with a very different constraint than the slightly older generation: There were vastly fewer men in their middle or late twenties (i.e., those born in the early 1940s) than women in their early twenties (i.e., those born in the late 1940s). In fact, for about 20 percent of the young female population, the traditional marriage pattern simply could not be sustained.* Is it any wonder, therefore, that "women's liberation" flourished at this time?§ The old plan of simply getting married and raising children was arithmetically impossible for a large portion of the young female population. Pursuing a career became relatively more attractive than in the past.

In addition to this "marriage squeeze," because there was an unusually large cohort of young adults available in the labor market, entry level wages fell.[1] Is it surprising that this generation was somewhat disenchanted? Moreover, with earnings levels lowered, it would not be surprising that two-earner families would become more common. Because having babies raised the cost of working outside the home, these couples put off childbearing, causing birthrates to plummet in the 1970s.

The low birthrates in the 1970s translated into a relatively small cohort of young adults in the 1990s. For this reason, entry-level wages have been relatively high, exceeding the legal minimum wage in most parts of the country. Also, young women at the close of the century are finding a relatively abundant supply of slightly older males, opposite to what the baby boomers experienced. We should not be surprised, therefore, to find a shift back in the direction of traditional marriage patterns.

This discussion is, of course, intended only as an illustration of economic methodology, not as a complete theory of these events. It is, however, meant to suggest the powerful nature of the economic paradigm. In addition to the usual analyses of market phenomena, events traditionally investigated by noneconomists, perhaps, eventually, even that subtle human capital we tend to call "tastes," may be

^We are grateful to Lee Edlefsen for introducing us to these issues and analyses.

*See Richard Easterlin, *Birth and Fortune,* Basic Books, New York, 1980.

§ Similar *demographics* (population structure) took place in the late 1920s, another period in which women shocked their parents.

'See Finis Welch, "Effects of Cohort Size on Earnings: The Baby Boom Babies' Financial Bust," *Journal of Political Economy,* Part II, 87(5):S65-S98, October 1979.

amenable to analysis with the economic paradigm. Changes in events are explained on the basis of changes in constraints, assuming the unmeasured variables remain constant, and utilizing an assertion of maximizing behavior.

1.3 THEORIES AND REFUTABLE PROPOSITIONS

In the past several pages we have used the terms *theory, propositions,* and *confirm,* as well as other phrases that warrant a closer look. In particular, what is a theory, and what is the role of theories in scientific explanations?

It is sometimes suggested that the way to attack any given problem is to "let the facts speak for themselves." Suppose one wanted to discover why motorists were suddenly waiting in line for gasoline, often for several hours, during the winter of 1973-1974, the so-called energy crisis. The first thing to do, perhaps, is to get some facts. Where will they be found? Perhaps the government documents section of the local university library will be useful. A problem arises. Once there, one suddenly finds oneself up to the ears in facts. The data collected by the United States federal government and other governments fill many rooms. Where should one start? Consider, perhaps, the following list of "facts."

1.82 Many oil-producing nations embargoed oil to the United States in the fall of 1973.

1.83 The gross national product of the United States rose, in money terms, by 11.5 percent from 1972 to 1973.

1.84 Gasoline and heating oils are petroleum distillates.

1.85 Wage and price controls were in effect on the oil industry during that time.

1.86 The average miles per gallon achieved by cars in the United States has decreased due to the growing use of antipollution devices.

1.87 The price of food rose dramatically in this period.

1.88 Rents rose during this time, but not as fast as food prices.

1.89 The price of tomatoes in Lincoln, Nebraska was 39 cents per pound on September 14, 1968.

1.90 Most of the pollution in the New York metropolitan area is due to fixed, rather than moving, sources.

The list goes on indefinitely. There are an infinite number of facts. Most readers will have already decided that, e.g., fact 8 is irrelevant, and most of the infinite number of facts that might have been listed are irrelevant. But why? How was this conclusion reached? Can fact 8 be rejected solely on the basis that *most* of us would agree to reject it? What about facts 4 and 5? There may be less than perfect agreement on the relevance of some of these facts.

Facts, by themselves, do not explain events. Without some set of axioms, propositions, etc., about the nature of the phenomena we are seeking to explain, there is simply no way in which to sort out the relevant from the irrelevant facts. The reader who summarily dismissed fact 8 as irrelevant to the events occurring during the energy crisis must have had some behavioral relations in mind that suggested that the tomato market in 1968 was not a determining factor. Such a notion, however rudimentary, is the start of a *theory.*

The Structure of Theories

A theory, in an empirical science, is a set of explanations or predictions about various objects in the real world. Theories consist of three parts:

1.91 A set of *assertions,* or postulates, denoted $A = \{A_1, \ldots, A_n\}$, concerning the behavior of various *theoretical constructs,* i.e., idealized (perhaps mathematical) concepts, which are ultimately to be related to real-world objects. These postulates are generally universal-type statements, i.e., propositions of the form: All x have the property p. Examples of such propositions in economics are the statements that "firms maximize wealth (or profits)," "consumers maximize utility," and the like. At this point, terms such *as firms, consumers, prices, quantities,* etc., mentioned in these behavioral assertions, or postulates, are ideas yet to be identified. They are thus referred to as theoretical constructs.

1.92 If behavioral assertions about theoretical constructs are to be useful in empirical science, these postulates must be related to real objects. The second part of a theory is therefore a set of *assumptions,* or *test conditions,* denoted $C = \{C_1, \ldots, C_n\}$, under which the behavioral postulates are to be tested. These assumptions include statements to the effect that "such-and-such variable/?, called the *price of bread* in the theoretical assertions, in fact corresponds to the price of bread posted at xyz supermarket on such-and-such date."

Note that we are distinguishing the terms *assertions* and *assumptions.* There has been a protracted debate in economics over the need for realism of assumptions. The confusion can be largely eliminated by clearly distinguishing the behavioral postulates of a theory (the assertions) from the specific test conditions (the assumptions) under which the theory is tested.

If the theory is to be at all useful, the assumptions, or test conditions, must be *observable.* It is impossible to tell whether a theory is performing well or badly if it is not possible to tell whether the theory is even relevant to the objects in question. The postulates A are universal statements about the behavior of abstract objects. They are not observable; therefore, debate as to their realism is irrelevant. Assumptions, on the other hand, are the link between the theoretical constructs and real objects. Assumptions *must* be *realistic,* i.e., if the theory is to be validly tested against a given set of data, the data must conform in essential ways to the theoretical constructs.

Suppose, for example, we wish to test whether a rise in the price of gasoline reduces the quantity of gasoline demanded. It will be observed that until the 1980s, the money price of gasoline has been rising generally since World War II and that gasoline consumption has also been rising. Does this refute the behavioral proposition that higher prices lead to less quantity demanded?

Perhaps the data, specifically the assumptions about prices, are not realistic. Does the reported series of prices really reflect the intended characteristics of the theoretical construct: price of gasoline? A careful statement of the law of demand involves changes in *relative* prices, not absolute money prices, and other things, e.g., incomes and other prices, are supposed to be held fixed. When compensated by price-level changes, the *real* price of gasoline, i.e., the price of gasoline relative to other goods, has indeed been falling, except for the periods of supply interruption, 1973-1974 and 1979-1980, thus tending to *confirm* the law of demand. But in order to test the law of demand with this data, the assumptions about income, prices of closely related goods, etc., must also be realistic, i.e., conform to the essential aspects of the theoretical constructs.

We say *essential aspects* of the theoretical constructs because it is impossible to describe, in a finite amount of time and space, every attribute of a given real object. The importance of *realism* of assumptions is to make sure that the unspecified attributes do not significantly affect the test of the theory. In the foregoing example, money prices were an *unrealistic* measure of gasoline prices; i.e., they did not contain the attributes intended by the theory. The assumptions, or test conditions, of a theory *must,* therefore, be realistic; the assertions, or behavioral postulates, are never realistic because they are unobservable. 3. The third part of a theory comprises the events $E = \{E_1, \ldots, E_n\}$ that are predicted by the theory. The theory says that the behavioral assertions A imply that if the test conditions C are valid (realistic), then certain events E will occur. For example, the usual postulates of consumer behavior (utility maximization with diminishing marginal rates of substitution between commodities), which we shall denote A, imply that if the test conditions C hold, where C includes decreasing relative price of gasoline with real incomes and other prices to be held fixed—that is, these assumptions are in fact *observed* to be true—then the event *E, higher gasoline consumption,* will be observed. Note that *both* the assumptions or test conditions C and the events E must be observable. Otherwise, we can't tell whether the theory is applicable.

The logical structure of theories is thus that the assertions A imply that if C is true, then E will be true. In symbols, this is written

$$A \rightarrow (C \rightarrow E)$$

where the symbol $\rightarrow$ means *implies.* By simple logic, the symbolic statement can also be written

$$(A \cdot C) \rightarrow E$$

That is, the postulates *A and* assumptions C together imply that the events E will be observed.

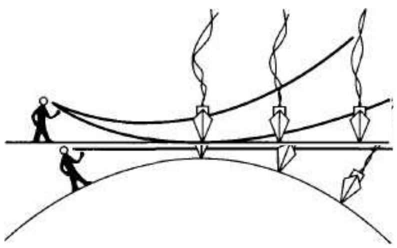
Refutable Propositions

We have spoken casually of *testing theories.* What is it that is being tested, and how does one go about it? In the first place, there is no way to test the postulates *A* directly. Suppose, to take a classic example, one wished to test whether a given firm maximized profits. How would you do it? Suppose the accountants supplied income statements for this year and past years together with the corporate balance sheets. Suppose you found that the firm made $1 million this year. Could you infer from this that the firm made *maximum* profits? Perhaps it could have made $2 million, or $10 million. How would you know?

Maybe we should ask an easier question. Is the firm *minimizing* profits? Certainly not, you say. After all, it made a million dollars. Well, maybe it was in such a good business that there was simply no way not to make less than a million dollars. No, you insist, if the owners of this firm were out to minimize profits, we should expect to see them giving away their goods free, hiring workers at astronomical salaries, throwing sand into the machinery, and indulging in a host of other bizarre behaviors. Precisely. The way one would *infer* that profits were being minimized would be to predict that if such behavior were present, then the given firm would engage in certain predicted events, specified in advance, such as the actions named. Since the object in question is undoubtedly a firm, i.e., the test conditions or assumptions *C* are realistic, and the events predicted by profit-minimization do not occur, the behavioral assertion A, that the firm minimizes profits, is refuted. *But the postulates are refutable only through making logically valid predictions about real, observable events based on those postulates, under assumed test conditions, and then discovering that the predictions are false.* The postulates are not testable in a vacuum. They can only be tested against real facts (events) under assumed, observable test conditions.

We have not, however, shown that firms maximize profits. But, we do know something. It will not be possible to determine whether firms maximize profits on the basis of whether we think that this is a sensible or achievable goal. The way to test the postulate of profit maximization is to derive from that postulate certain behavior that should be observed under certain assumptions. Then, if the events predicted do indeed occur, we shall have evidence as to the validity of the postulate. The theory will be confirmed. But will it *bo, proved?* Alas, no. The nature of logic forbids us to conclude that the postulates *A* are true, even if *C* and *E* are known to be true. This is such a classic error it has a name: It is called the fallacy of *affirming the consequent.* If *A* implies *B,* then if *B* is true, one cannot conclude that *A* is true. For example, "If two triangles are congruent, then they are similar," is a valid proposition. However, if two triangles are known to be similar, one cannot conclude that they are also congruent, as counterexamples are easily demonstrated.

A striking example of why theories cannot be proved is presented in Fig. 1-1. The theory that the earth is round is to be tested by having an observer on the seashore note that when ships come in from afar, first the smoke from the smokestacks is visible, then the stacks, and so on, from the top of the ship on down. Panel *a* shows



(a) *(b)*

FIGURE 1-1
Two Theories of the Shape of the Earth. In Fig. *l-la,* a round earth is postulated. Under the assumption that light waves travel in straight lines, ships coming in from afar become visible from the top down, as they approach the shore. This is confirmed by actual observation. However, this does *not prove* that the earth is round. In Fig. *l-lb,* a flat earth is postulated. However, under the assumption that light waves travel in curves convex to the surface of the earth, the same events are predicted. Therefore, on the basis of this experiment alone, no conclusion can be reached concerning the shape of the earth!

why this is to be expected. It does, in fact, occur every time. However, panel *b* shows that an alternative theory leads to the same events. Here, the earth is flat, but light waves travel in curves convex to the surface of the earth. The same events are predicted. There is no way, on the basis of this experiment, to determine which theory is correct. It is always possible that a new theory will be developed that will explain a given set of events. Hence, theories are in principle, as a matter of logic, unprovable. They can only be confirmed, i.e., found to be consistent with the facts. The more times a theory is confirmed, the more strongly we shall believe in its postulates, but we can never be *sure* that it is true.*

What types of theories are useful in empirical science, then? The only theories that are useful are those that might be wrong, i.e., might be refuted, but are not refuted. A theory that says that it will either rain or not rain tomorrow is no theory at all. It is incapable of being falsified, since the predicted "event" is logically true. A theory that says that if the price of gasoline rises, consumption will either rise or fall is similarly useless and uninteresting, for the same reason. The only theories that are useful are those from which *refutable hypotheses* can be inferred. The theory must assert that some event E will occur and, moreover, it must be possible that E will *not* occur. Such a proposition is, at least in principle, refutable. The facts may refute the theory; for if E is false, then as a matter of logic (A • *C)* is false. (If nonoccurrence of the event E is always attributed to false or unrealistic test conditions or assumptions C, then the theory is likewise nonrefutable.)

Irving M. Copi, *Introduction to Logic,* 4th ed., Macmillan, New York, 1972.

In order to be useful, therefore, the paradigm of economics must consist of refutable propositions. Any other kind of statement is useless. In the various chapters of this book, we shall demonstrate how such refutable hypotheses are derived from behavioral postulates in economics. Perhaps nothing is more readily distinctive about economics than the insistence on a unifying behavioral basis for explanations, in particular, a postulate of maximizing behavior. The need for such a theoretical basis is not controversial; to reject it is to reject economics. The reason such importance is placed on a theoretical basis is that without it, any outcome is admissible; propositions can therefore never be refuted. Economists insist that some events are *not possible,* in the same way that physicists insist that water will never run uphill. Other things constant, a lower price will never induce less consumption of any good; holding other productive inputs constant, marginal products eventually decline. There are to be no exceptions.

1.4 THEORIES VERSUS MODELS; COMPARATIVE STATICS

The testing of a theory usually involves two fairly distinct processes. First, the purely logical aspects of the theory are drawn out. That is, it is shown that the behavioral postulates imply certain behavior for the variables of the theory. Then, at a later stage, the theoretical constructs are applied to real data, and the theory is tested empirically. The first stage of this analysis is what we shall be concerned with in this book. To distinguish the two phases of theorizing, we shall employ a distinction introduced by A. Papandreou* and amplified by M. Bronfenbrenner.* The purely logical aspect of theories will be called a *model.* A model becomes a theory when assumptions relating the theoretical constructs to real objects are added. Models are thus logical systems. They cannot be true or false empirically; rather, they are either logically valid or invalid. A theory can be false either because the underlying model is logically unsound or because the empirical facts refute the theory (or both occur).

The notion of a refutable proposition is preserved, however, even in models. A refutable proposition in a logical system means that when certain *conceptual* test conditions occur, the theoretical variables will have restricted values. Suppose that in a certain model, if a variable denoted *p,* ultimately to mean the price of some good, increases, then another variable*, ultimately to mean the quantity of that good demanded, can validly be inferred to, say, decrease, as a matter of the logic of the model, then a refutable proposition is said to be asserted. The critical thing is that the variable x is to respond in a given manner, and it must be possible for x not to respond in that manner.

^Andreas Papandreou, *Economics as a Science,* J. B. Lippincott Company, Philadelphia, 1958. ^Martin Bronfenbrenner, "A Middlebrow Introduction to Economic Methodology," in S. Krupp (ed.), *The Structure of Economic Science,* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1966.

The logical simulation, usually with mathematics, of the testing of theories in economics is called the *theory of comparative statics.* The word *statics* is an unfortunate misnomer. Nothing really static is implied in the testing of theories. Recall that, in economics, theories are tested on the basis of *changes* in variables, when certain test conditions or assumptions change. The use of the term comparative statics refers to the absence of a prediction about the *rate* of change of variables over time, as opposed to the *direction* of change.

The testing of theories is simulated by dividing the variables into two classes:

1.93 Decision, or choice, variables.

1.94 Parameters, or variables exogenous to the model, i.e., not determined by the actions of the decision maker. The parameters represent the *test conditions* of the theory.

Let us denote the decision or choice variable (or variables) as JC, and the parameters of the model as a. To be useful, the theory must postulate a certain set of choices x as a function of the test conditions a:

$$x = f(a) \qquad (1\text{-}1)$$

That is, given the behavioral postulates A, if certain test conditions C, represented in the model by a, hold, then certain choices JC will be made. Hence, x is functionally dependent on a, as denoted in Eq. (1-1).

As an empirical matter, economists will rarely, if ever, be able to test relations of the form (1-1) directly, i.e., formulate hypotheses about the actual amount of JC chosen for given a. As mentioned earlier, to do this would require full knowledge of tastes as well as opportunities. The neoclassical economic paradigm is therefore based on observations of *marginal* quantities only. These marginal quantities are the responses of JC to *changes* in a.

Mathematically, for "well-behaved" (differentiate) choice functions, it is the properties of the derivative of JC with respect to a, or

^ = /'(«) (1-2)
da

that represent the potentially refutable hypotheses in economics. Most frequently, all that is asserted is a sign for this derivative. For example, in demand theory, prices p are exogenous, i.e., parameters, while quantities demanded JC are choice variables. The law of demand asserts (under the usual qualifications) that $dx/dp < 0$. Because it is possible that $dx/dp > 0$, and since this would contradict the assertions of the model, the statement $dx/dp < 0$ is a potentially refutable hypothesis. *Comparative statics is that mathematical technique by which an economic model is investigated* *to determine if refutable hypotheses are forthcoming.* If not, then actual empirical testing is a waste of time, because no data could ever refute the theory.

1.5 EXAMPLES OF COMPARATIVE STATICS

To illustrate the preceding principles, let us consider three alternative hypotheses about the behavior of firms. Specifically, suppose we were to postulate that:

1.95 Firms maximize profits *n,* where *ix* equals total revenue minus cost.

1.96 Firms maximize some utility function of profits *U(n),* where *U'(ji)*> 0, so that higher profits mean higher utility. Thus, profits are desired not for their own sake, but rather for the utility they provide the firm owner.

1.97 Firms maximize total sales, i.e., total revenue only.

By what means shall these three theories be tested and compared? It is not possible to test theories by introspection. Contemplating whether these postulates sound to us like "reasonable" behavior is not an empirically reliable test. Also, asking firm owners if they behave in these particular ways is similarly unreliable. The only way to test such postulates is to derive from them potentially refutable hypotheses and ultimately to see if actual firms conform to the predictions of the theory.

What sorts of refutable hypotheses emerge from these behavioral assertions? Among the logical implications of profit maximization is the refutable hypothesis that if a per-unit tax is applied to a firm's output, the amount of goods offered for sale will decrease. This hypothesis is refutable because the reverse can be true. We therefore begin our first example by *asserting* that firms maximize profits in order to derive this implication.

Example 1. Let

R(x) = total revenue function (depending on output*) *C(x)* = total cost function

tx = total tax revenue collected, where the per-unit tax rate *t* is a parameter determined by forces beyond the firm's control

If the firm sells its output in a perfectly competitive market, i.e., it is *a. price taker,* then

$$R(x) = px$$

where *p* is the parametrically determined market price of *x.* If the firm is not a perfect competitor, then *p* is determined, along with *x,* via the demand curve, and revenue is simply some function of output, *R(x).*

In the general case, the tax rate *t* represents the only parameter, or test condition, of the model. The first model thus becomes

maximize

$$n(x) = R(x) - \mathrm{CO)} - tx \qquad (1-$$

3)

By simple calculus, the first-order condition for a maximum is

$$R'(x) - C'(x) - t = 0 \qquad (1-4)$$

the prime denoting first derivative.

For a maximum, the sufficient second-order condition is

$$R'' - C'' < 0 \tag{1-5}$$

Condition (1 -4) is the choice function for this firm in implicit form. It states that the firm will choose that level of output such that marginal revenue (MR) equals marginal cost (MC) plus the tax *(t).* If the firm is a perfect competitor, then $R'(x) — p$, and $R''(x) = 0$. Equations (1-4) and (1-5) then become, respectively,

$$p-C'(x)-t = 0 \tag{1-4'}$$

$$-C''(x) < 0 \tag{1-5'}$$

We shall pursue the model from the standpoint of a firm with an unspecified revenue function $R(x)$. Application of the model to the perfectly competitive case will be left as a problem for the student.

Equation (1-4) is a well-known application of "marginal" reasoning. Equation (1-4) states that a firm will produce at a level such that the incremental (marginal) gain in revenues is exactly offset by the incremental cost (including, of course, the tax). This condition, however, does not guarantee a maximum of profits. It is also perfectly consistent with minimizing profits with the same cost and revenue functions, since the same first-order conditions are implied. What we mean to express is that as long as marginal receipts exceed marginal cost, the firm will produce at a higher rate, and if marginal receipts are less than marginal costs, the output will be reduced. This idea is given a precise statement by Eq. (1-5), which says that receipts are increasing at a slower rate than costs. Or, in terms of the marginal revenue and marginal cost curves, Eq. (1-5) says that the marginal cost curve cuts the marginal revenue curve from below.

Notice that we do not assert that the "optimum" output for a firm is where marginal revenue equals marginal cost; this is a value judgment, not a statement about behavior. Likewise, Eq. (1-4) does not represent what this firm does *in equilibrium.* Equation (1-4) is a *necessary* event, logically deduced from the assertion of maximization of profits. If Eq. (1-4) is not observed, it constitutes a refutation of the model, not *disequilibrium* or *nonoptimal* behavior. Thus, we *assert* that firms act as if they are obeying Eqs. (1-4) and (1-5), and on that account we make predictions about their behavior.

To simply assert MR = MC +*1,* however, is not likely to be useful. One is not likely to observe these marginal relationships. Just as tastes are difficult to observe, the total revenue and total cost functions and, hence, their derivatives, will likely not be known. However, a prediction about the response of the firm to a change in the economic environment, i.e., some test condition —in this case, a change in the tax rate—*is,* nonetheless, possible. Even if profit maximization, marginal revenue, and marginal cost are not directly observable, tax rates and quantities sold *are* potentially observable. And profit maximization contains implications about these observable quantities.

How can Eqs. (1-4) and (1-5) be used to obtain predictions about marginal responses? Upon closer observation we notice that Eq. (1-4) is an implicit relationship between x and t. Under certain mathematical conditions this implicit relationship between the variable x and the parameter t can be solved for the explicit choice function:

$$x = x^*(t) \tag{1-6}$$

That is, if we knew the equations of the MR and MC curves, *then as long as the firm can be counted on to always obey the appropriate marginal relations,* no matter what tax rate prevails, we can, in principle, solve for the explicit relationship that states how much output will be produced at each tax rate. Again, although it would be desirable to know the exact form of Eq. (1-6), the economist will not typically have this much information. Hence, predictions about *total* quantities will not generally be forthcoming. We can, nonetheless, make predictions about *marginal* quantities. If Eq. (1-6) is substituted into Eq. (1-4), the *identity*

$$R'\{x^*(t))-C(x^*(t))-t = 0 \qquad (1\text{-}7)$$

results. This is an identity because the left-hand side is 0 for all values of /. It is 0 for all values of t precisely because $x^*(t)$ is that level of output that the firm chooses in order to *make* the left-hand side of (1-7) always equal 0. That is, the firm, by always equating MR to MC plus the tax, for any tax rate, transforms the Eq. (1-4) into the identity (1-7). Because we are interested in what happens to x as t *changes,* the indicated mathematical operation is the differentiation of identity (1-7) with respect to t, keeping Eq. (1-6) in mind. The student must observe that this differentiation makes sense *only if x is a function of t.* Otherwise, the symbol dx/dt has no meaning. It is premature to simply differentiate Eq. (1-4) with respect to t until such functional dependence is formally implied. It is the assertion that the firm will *always* equate at the margin, i.e., obey Eq. (1-4)/or any *tax rate* that allows the specification of Eq. (1-6): the functional dependence of x upon t. The resulting identity, (1-7), *can* be validly differentiated on both sides; Eq. (1-4) cannot be. This step is often left out, yet it is critical from the standpoint of clearly understanding the implied economic relationships as well as mathematical validity^

Performing the indicated differentiation of identity (1-7),

$$R \backslash x \} \frac{\wedge}{dt} \wedge C \{ x) \frac{\wedge}{dt} = Q \qquad (1\text{-}8)$$

Equivalently, assuming $(R'' — C'') \wedge 0$,

$$\frac{dx^*}{*} \quad \frac{1}{K--C-} \qquad (19)$$

Since $R'' — C'' < 0$ by the sufficient second-order condition for profit maximization, this implies

$$\frac{dx^*}{dt} < 0$$

Note well what has been accomplished here. The postulate of profit maximization (not observable), as specified in Eq. (1-3), has led to the refutable proposition that output will decline as the tax rate the firm faces increases. In addition, nothing has been assumed as to the specific functional form of the demand or cost curves, t As an example of the latter, differentiation of both sides of the identity $(x+1)^2 = x^2 + 2x + 1$ is valid; differentiation of both sides of the equation $2x = 6$ yields nonsense. The difference is that the former holds for *all x,* whereas the latter holds only for $x = 3$.

and hence the result holds for all specifications of those functions. A prediction about *changes* in the choice variable, that is, marginal adjustment of output when the parameter facing the decision maker changes, has been rather easily derived, i.e., shown to be implied by a single behavioral assertion. This is the goal of comparative statics; the limitations and abilities of the methodology to accomplish that goal are the subject of this book.

Example 2. Consider now the second previously mentioned behavioral postulate. Let us suppose that profits are desired not for their own sake, but rather for the utility derived from them. Thus, let us now assert that the firm owner maximizes *U(n),* where *U'(n)*> 0, so that increased profits mean increased utility. The function *U(rc)* is some unspecified ordinal measure of the "satisfaction" that this firm owner gains from earning profits. It might seem that since we have replaced a potentially observable quantity, profits, with an unobservable variable, utility, that this theory will be devoid of refutable implications. Let us see. The objective function is now

maximize

$$U(R(x)-C\{x)-tx) = U\{n) \qquad (1\text{-}10)$$

The firm's choice function, as before, is found by setting the derivative of *U(TT)* with respect to *x* equal to 0. Using the chain rule,

$$\frac{dU}{dn}\frac{dn}{dx} \_$$

or

$$U'(n)[R'\{x) - C'\{x) - t] = 0 \qquad (1\text{-}11)$$

Since *U'irc)* > 0, the choice function (1-11) is equivalent to the previous one for simple profit maximization:

$$R'(x)-C'(x)-t = 0 \qquad (1\text{-}4)$$

Since the implicit functions (1-4) and (1-11) are equivalent, their solutions

$$x=x\backslash t) \qquad (1\text{-}12)$$

are identical. Thus, these firms will act identically; they have the same explicit choice functions (1-6) and (1-12) governing the response of output to tax rates. One technicality must not be overlooked, however. We must check that the point of maximum profits is also *maximum,* rather than minimum, utility; i.e., we have to check the second-order conditions for this problem. Otherwise we might be discussing two entirely different points, and the derivatives *dx/dt* at those points would in general differ. The second-order conditions for the two problems are, however, identical: We have, for the first-order condition,

$$\frac{\wedge\ 2}{dx} \qquad (*)] = 0$$

Thus, using the product rule,

$$= U'(7r)[7t''(x)] + [7T^{f}(x)]\{[U''(.7T)][n'(x)]\}$$

Since *TT'(X)* — 0 by the first-order conditions,

Since J7'(7r) > 0, $d^2U(n)/dx^2 < 0$ if and only if $d^2n/dx^2 < 0$; that is, the second-order conditions for the two models are identical.

These two theories of behavior are equivalent in the sense that they yield the same refutable hypotheses. Even if more parameters are introduced into *n(x),* the first- and second-order equations will be identical. Thus, no set of data could even distinguish whether a firm was maximizing profits, or some arbitrary increasing function of profits, *U(n).* We shall never know if the firm is really maximizing *n,* or e^n, or it^3 (not n^1; why?), or whatever. These behavioral postulates all yield the same refutable hypotheses. One is as good as the other.

Example 3. Consider now the last of the three hypotheses about firm behavior, the maximization of total sales. If such a firm were taxed at rate *t,* the objective function would be

maximize

$$4>(x) = R(x) - tx \qquad (1\text{-}14)$$

The implicit choice function of this firm is the first-order condition for a maximum

$$0'(JC) = R'(x) - t = 0 \qquad (1\text{-}15)$$

The sufficient second-order condition for maximizing </>(-*) is

$$4>''(x) = R''(x) < 0 \qquad (1\text{-}16)$$

The *explicit* choice function of this firm is the solution of (1 -15) for output as a function of the tax rate, or

$$x=x^{**}(t) \qquad (1\text{-}17)$$

This choice function will in general indicate a different level of output for any given tax rate than the choice function (1-6) or (1-12). If the revenue function *R(x)* were actually known, then this theory (sales maximization) would be operationally distinguishable from the prior two theories, since different choices are implied. However, if it turns out that *R(x)* is not directly observable (indeed, this is the empirically likely situation), then the only refutable proposition will concern the sign of dx^{**}/dt. This model, like the previous ones, implies a negative sign for this derivative. Substituting (1-17) into (1-15) and differentiating with respect to *t,*

$$rw\frac{dx^{**}}{} = \mathbf{i}$$

or

$$\wedge_a \_\mathbf{J}\_ < \mathbf{O} \qquad (1\text{-}18)$$

$dt \qquad R''(x)$

using the sufficient second-order condition (1-16). Hence, *unless the revenue and cost functions are somehow known, the sales maximization and profit maximization* *postulates are equivalent, in the sense that no observation of* **changes** *in tax rates and* **changes** *in quantities sold will ever distinguish these two theories.* If *R(x)* and *C(x)* are unobservable, and $dx^*/dt < 0$ is implied for *any* *R(x)* and *C(x)* which satisfy the second-order conditions, then the same observations are implied for $C(x) = 0$, i.e., sales maximization. The reader is cautioned against assuming that there is *no* test that could separate these hypotheses. There may be, for example, reasons why the long-term survivability might differ for firms that maximized profits as opposed to sales.

Example 4. Suppose the owner of a firm maximizes "net sensitivity," i.e., sensitivity less taxes, where sensitivity has been reliably (at last) estimated with bivernal data, using generalized five-stage least-squares regression, with of course the usual adjustments for semitruncation and hypercolinearity of the data set, as the function

_ *a* log[tan(^ + *e**^{inx}) + cot(jSx + log(l + *e*^{Px}))]
log(a + *fix)*

where *a* and */3* are positive *S* parameters. How will this firm react to an increase in the tax rate?

The objective function is

maximize

$$S(x) - tx$$

Assuming the first- and second-order conditions hold (don't ask), an explicit choice function $x = x^*(t)$ is implied. The structure of this model is formally identical to the model in Example 3; the results must be identical. It doesn't matter what specific functional form is used in the objective function. The only crucial elements are that:

1.98 The first and second-order conditions hold.

1.99 The tax parameter *t* enters in such a way that when the first-order identity is differ
entiated with respect to *t,* it produces a negative value on the left-hand side of that
identity, and thus a positive entry when it is brought over to the right-hand side.
(When taxes enter the objective function as — *tx,* this procedure yields +1 on the
right-hand side.)

The resulting expression for dx^*/dt will then always consist of a positive term divided by an expression representing the second derivative of the objective function, which is assumed negative by the sufficient second-order conditions. Thus in every such case, we will derive $dx^*/dt < 0$. We don't have to bother differentiating a perhaps messy objective function to get this result.

Example 5. There is nothing in the previous examples that restricts the analysis to noncompetitive firms. For competitive firms, output price/? is taken as given. The firm is a price taker; it cannot influence output price by its own choices regarding output levels. The revenue function, *R(x),* for a competitive firm is simply *px,* price times quantity. Since this is a special case of *R(x),* the previous analysis applies to competitive firms as well: A tax on output will lead to decreases in total output produced.

In this model, however, a new parameter/? appears. Does the postulate of profit maximization imply a refutable hypothesis regarding changes in/?? The objective function is

maximize

$$7t(x) = px-C(x) \qquad (1\text{-}19)$$

The first-order condition for maximization yields the implicit choice function

$$p-C'(x) = 0 \qquad (1\text{-}20)$$

Here, marginal revenue = price/?. Hence, this relation says that the firm will set marginal cost equal to price. However, unless we know the cost function, this information will not be very useful.

The sufficient second-order condition for maximizing *n* is

$$-4 = -C''(x) = -MC'(x) < 0 \qquad (1\text{-}21)$$
$$dx^2$$

That is, the marginal cost function of the firm must be upward-sloping.

The explicit choice function is found by solving (in principle) Eq. (1-20) for the choice variable *x* in terms of the parameter/?:

$$x=x^*(p) \qquad (1\text{-}22)$$

This function is the firm's supply function. It tells how much *x* will be offered for sale at any given price *p.* Strictly speaking, the marginal cost function is *not* the supply function of the firm. In the MC function, output *x* is the *independent* variable, marginal cost (which equals price at the chosen point) is the *dependent* variable. For the supply function, output is the dependent variable, dependent upon price. Thus, the supply function is really the *inverse* of the MC function. How will *x change* when *p changes!* Substituting (1-22) back into (1-20), the identity

$$p-C'(x^*(p)) = 0 \qquad (1\text{-}23)$$

results. The left-hand side is *always* zero, because we are now postulating that the firm will *always* set price equal to marginal cost *for any price.* Hence, this is an identity—the left-hand side vanishes completely. Since the derivative *dx*/dp* is desired, differentiate identity (1-23) with respect to /?, using the chain rule for *C'(x*(p)):*

dp dC'{x)
dx _ dp*
dx dp

or

*dx**
v '

dp and thus, since $C'' \wedge 0$,

$$\wedge \quad \wedge \quad - J - \; > \; 0 \qquad (1\text{-}24)$$
$$dp \qquad C''(x)$$

because $C''\{x) > 0$ by the sufficient second-order condition for a maximum (1-21). Thus, the behavioral postulate of profit maximization yields the refutable hypothesis that if the output price to competitive firms is somehow raised, output levels will increase. The supply function is upward-sloping. Given this mathematical property of the model, the theory can be tested using real data on the assumption that the firms in question correspond to the theoretical construct of the firm used in the model. But empirical testing is worthwhile only because the model yields refutable implications.

PROBLEMS

1.100 Consider the following alleged exception to the law of demand: "As the price of diamonds falls, the quantity of diamonds demanded will also fall since the prestige of owning diamonds will similarly fall." Why is this not an exception to the law of demand? What test condition is being violated? How would one test the law of demand for diamonds? *(Hint:* Do jewelry stores ever lower prices on diamonds? What results?)

1.101 What is the difference between an assertion and an assumption? Which is observable, which is not? Which must be "realistic"?

1.102 Many young people regard their parents and grandparents as rigid and conservative. Recognizing that investment and experimentation in new procedures are costly, explain why one would expect the young to be more likely than the old to adopt new methods (or, why young dogs are more apt to learn new tricks, and why old dogs will more likely perfect the old ones).

1.103 Why do economists limit their analyses to marginal rather than total quantities? Do economists believe that marginal quantities are more useful than the corresponding total quantities?

1.104 What is the difference between a theory and a model?

1.105 Is there a trade-off between the realism of the assumptions of a theory and the *tractability,* i.e., the empirical usefulness, of the theory?

1.106 In regard to Prob. 6, is it necessary to have the latest theory of molecular action to make penicillin and other wonder drugs? How detailed a theory of the firm is necessary to predict the effects of tariffs on a given industry?

1.107 Consider a monopolist whose total cost function is $C = kx^2$ and who faces the demand curve $x — a — bp$.

1.108 What restrictions on the values of the parameters *a, b,* and *k* would you be inclined to assert, a priori?

1.109 Find the explicit function $x = x^*(t)$. Confirm that for the restricted values of *a, b,* and *k* placed in part *(a)* that $x^{*\prime}(t) < 0$, i.e., output decreases as the tax increases.

1.110 What restrictions does the hypothesis of profit maximization place on the parameters *a, b,* and *kl* Are these weaker or stronger than your a priori restrictions?

1.111 Substitute your $x^*(t)$ function into the first-order relation for maximization, and confirm that an identity in t results.

1.112 Confirm that, for this specification of the model, profit maximization alone implies $x^{*\prime}(t) < 0$.

1.113 What is the effect on output and output price of a parallel shift in the demand curve?

9. Show that an increase of a per-unit tax on a perfect competitor will lower that firm's output.

1.114 Show why a monopolist has no supply function. *(Hint:* In Example 1 in this chapter, how would $x = x^*(p)$ be defined?)

1.115 Consider a firm that has as its behavioral postulate the minimization of total costs, irrespective of revenues. How will this theory of the firm differ from those discussed in Examples 1 through 3 in this chapter?

1.116 Consider a firm with "gross profits" $R(y) - C(y)$ and "net profits" $R(y) - C(y) - ty$, where t is a per-unit tax. Prove under profit maximization that if the tax rate rises, both net profits and gross profits will fall.

1.117 Tin (aluminum) cans are manufactured in the shape of right cylindrical cylinders of diameter D and height h. Assume there is no waste cutting out the rectangular piece for the side, but when the circular ends are cut from squares, the corner pieces are discarded.

1.118 Show that the shape of the can that minimizes the (cost of the) metal used for any given volume is h/D = 4/7r ^ 1.27.

1.119 Run to your local supermarket and see if manufacturers utilize this result.

1.120 There seem to be some outstanding anomalies, e.g., tuna fish (too short), soda pop (too long). What factors might explain these anomalies and others you might observe?

1.121 Assume now that the corner pieces for the ends can be recycled at some cost, effectively reducing the amount of metal used by k x *waste,* where $0 < k < 1$. Show that as the waste is reduced, the size of the can approaches $h/D = 1$.

1.122 We, of course, pay for the item itself plus the packaging. How does the value of the item inside the can affect the preceding cost considerations? (See the discussion of the "Alchian and Allen substitution theorem" in Sec. 11.3.)

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Students should have available to them any of the usual basic calculus texts.

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CHAPTER

2

REVIEW OF CALCULUS (ONE VARIABLE)

We assume readers of this book are already familiar with the basic rules of calculus of one variable. We include here only some topics that are particularly useful in economics and that sometimes are not covered in basic calculus courses. We assume differentiability throughout.

2.1 FUNCTIONS, SLOPES, AND ELASTICITY

We write $y = f(x)$ to designate a rule by which some variable x is transformed into some unique other value y. The derivative of /, $dy/dx = f'(x)$, measures how fast the function is changing as x changes. Geometrically, $f'(x)$ is the slope of $f(x)$. The second derivative, $f''(x)$, measures how fast the first derivative is changing. In physics, if $y = f(t)$, for example, measures the height at time t of an object falling toward earth, $f'(t)$ measures the velocity of the object and $f''(t)$ measures its acceleration.

A dimensionless variant of slope is the *percentage* change in the dependent variable due to a percentage change in the independent variable. This quantity is called the *elasticity* of the curve. Suppose $x = f(p)$ is a demand (or supply) curve, where p = price and x = quantity demanded. The elasticity of demand (or supply) is defined as

$$\epsilon = \lim_{\Delta p \to 0} \frac{\Delta x / x}{\Delta p / p} = \lim_{\Delta p \to 0} \frac{p}{x}\frac{\Delta x}{\Delta p} = \frac{p}{x}\frac{dx}{dp} \qquad \text{(2-1)}$$

If $|e| > 1$, the curve is called *elastic;* if $0 < |e| < 1$, the curve is called *inelastic.* Note that supply and demand curves are usually plotted with the *dependent* variable x on the *horizontal* axis. The "slope" dx/dp is thus the reciprocal of the usual slope.

> **Example 1.** Let $x = ap^b$. Show that these functions exhibit constant elasticity $e = b$. Using the definition (2-1),
>
> $$e = \frac{p}{x}\frac{dx}{dp} = \frac{pbap^{b-1}}{ap^b} = b$$
>
> These curves are either always elastic or always inelastic. If $b < 0$, these curves are downward-sloping for all p and are used to represent demand curves. They belong to the class of functions called hyperbolas. If $b > 0$, these curves are upward-sloping and may represent supply curves.
>
> **Example 2.** Consider the linear demand curves $x — a — bp$. Price varies between 0 and a/b. The elasticity at any point is
>
> $$\frac{P}{a - bp} \quad . \quad ,, \quad \frac{-bp}{a - bp} \quad \frac{-p}{a/b - p}$$
>
> Clearly, when $p = 0$, $\epsilon = 0$. As $p \to a/b$, $e \to -\infty$. Also, $\epsilon = — 1$ when $p — a/2b$, the midpoint of the demand curve.

The elasticity of demand is related to the marginal revenue curve. Total revenue is simply price times quantity, or

$$\mathrm{TR} = px$$

Let us write the demand curve as $p = p(x)$, that is, price as a function of quantity. Then marginal revenue MR is defined as the rate of change in total revenue with respect to quantity, or

$$dx$$

Using the product rule

$$dx$$

For demand curves, $e < 0$. Hence, $\mathrm{MR} > 0$ when $e < —1$. Hence, for elastic demand curves, total revenue rises when quantity increases, i.e., when price falls. When demand is inelastic, i.e., when $—1 < \epsilon < 0$, $\mathrm{MR} < 0$. Total revenue falls when quantity increases, i.e., when price falls.

2.2 MAXIMA AND MINIMA

Probably the single most important application of the calculus in economics is its application to finding the maximum or minimum of functions. Most frequently, some postulate of maximizing behavior is made in economics—e.g., firms maximize "profits," consumers maximize "utility," etc. The calculus allows a detailed description of such points of "extrema."

Consider the rather squiggly function depicted in Fig. 2-1. Points *A, B, C, D, E,* and *F* are all points of *relative extrema.* In some neighborhood around these points, they all represent maximum or minimum values of /(JC). The adjective *relative* means that these are *local* extrema only, not the "global" maxima or minima over the whole range of JC. These points all have one thing in common: The slope of / (JC) is 0; i.e., the function is horizontal at all these extrema. *A necessary condition for* /(JC) *to have a local maximum or minimum is that dy/dx* = /'(JC) = 0.

Now consider a relative maximum, say point *C.* Immediately to the left of *C,* the function is rising; that is, /'(JC) > 0, whereas to the right of *C,* /'(JC) < 0. It is for this reason that we know /'(JC) = 0 at point *C.* Moreover, we also know that the slope, /'(JC), is continually falling (going from + to —) as we pass through C. Hence, /"(JC) < 0 at C. We cannot be sure that /"(JC) < 0 at *C;* /"(JC) = 0 is a possibility. However, *if f'(x)* = 0 *at x* = JC_0 *and if f"(xo)* < 0, *then f(x) has a relative maximum at x* = JC_0. *If f'(x_0)* = 0 *and* /"(JCO) > 0, /(JC) *has a relative minimum.* If /'(JCO) = 0, /"(JCO) = 0, then the function may have either a maximum, minimum, or neither at that point.

Around the maximum points *A, C,* and *E* in Fig. 2-1, the function is said to be *concave downward,* or simply *concave.* Around the minimum points *B, D,* and *F,* the function is said to be *convex* (i.e., concave *upward*). For differentiate functions, concavity implies /"(JC) < 0; that is, the slope, /'(JC) is continually nonincreasing. If /"(JC) < 0, then concavity is implied, but concavity allows the possibility that *f"(x)* = 0. Similar remarks hold for convexity. If /"(JC) > 0, then /(JC) is convex, but convexity also allows the possibility of /"(JC) = 0. Example 2 illustrates the possibilities allowed by /"(JC) = 0.

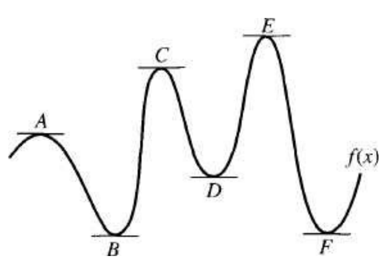


FIGURE 2-1
x Relative minima and maxima (extrema).

Example 1. Consider $y = x^2$. Then /'(*) = *2x, f"{x) = 2.* At $x = 0$, *f(x)* = 0, *f"(x)* > 0. This function has a relative minimum at $x = 0$.

Example 2. Let *y* = -*x*\ Then /'(JC) = -4JC\ /"(JC) = -\2*x*². At x = 0, /'(JC) = *f"(x)* = 0. However, this function /las a relative maximum at *x* = 0, as a sketch of the curve quickly reveals. Likewise, y = +JC⁴ has a relative minimum at JC = 0, with /"(0) = 0.

Example3. Lety = Jc³.Then/'(0) = /"(0) = 0. This function, the "cubic" function, is horizontal at JC = 0, but it has neither a maximum nor a minimum at *x* = 0. The condition *f'(x)* = 0 is a *necessary* condition for a maximum or a minimum (a stationary value); however, /"(JC) < 0, /"(JC) > 0 (note the strict inequalities) are *sufficient* conditions for a relative maximum or minimum, respectively. But these strict inequalities for /"(JC) are not *implied by,* i.e., not necessary conditions for a maximum or minimum.

2.3 CONTINUOUS COMPOUNDING

Suppose you put $1 in a bank account that pays *x* percent interest over the year. At the end of the year, you will have an amount

$$y = 1 + x$$

in the account. Suppose now the bank account pays *x* percent per year, *compounded semiannually*. In this case, the bank pays *(x/2)* percent interest in the first half of the year, and *(x/2)* percent on the increased amount in the second half. Therefore, after 6 months, the account would have

and, with *(x/2)* percent paid on this amount, after 1 year the account would have in

Using similar reasoning, if interest is compounded *quarterly,* after 1 year the account will have

If the account is compounded *n* times during the year *(n* = 365 is common nowadays), the account will grow to

$$y =$$

What is the limit of this expression as *n* —>■ oo? Let us approach the problem in two stages.

(a) Let $x = 1$. We then inquire as to

y = lim *(l + -)* (2-2)

n^oo \nj

Let us expand $(1 + \backslash/n)^n$ by the binomial theorem:

yn = A +
V

2!

3! \ *n*

l + IT + 2! \ *T)* + 3!

Consider the limit of the terms *(n — k)/n* as *n* —▶ oo. Dividing numerator and denominator by *n,*

n — k *k*

n *n*

Clearly

lim -----*)* = 1 — lim - = 1
n^oo \ *n* *J* *n-*oo n*

Moreover, any finite product of such terms tends to 1 as *n* —>• oo.

y = lim y_B = lim (1 +) i + + +
^oo n^oo y ^y 1J 2! 3!

This infinite series converges to an important irrational number known as *e.* To five decimal places,

$$e = 2.71828\cdots$$

(b) Now let us return to the more general

case of

Therefore,

) i

Make the substitution *m — n/x.* For fixed *x,* as *n* —> oo, *m* —>■ oo. Thus, the preceding expression becomes

lim (1 + -) = lim 1 + -) = e^x

using the previous result and the algebra of exponents. Thus,

e^x = lim *(*1 + -

Letting $z_n = [1 + (x/n)]^n$, expanding this expression by the binomial theorem, as before, yields

$$z_n = 1 + n\frac{x}{n} + \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2 + \cdots$$

Using the same reasoning as in the case where $x = 1$,

$$\lim_{n\to\infty} z_n = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2-3}$$

Thus, the exponent e^x is representable by an infinite series. The *convergence* of infinite series to a finite sum is a much explored aspect of mathematics. That the series (2-3) converges to the number e^x is evident from the derivation. Series that do not converge to finite sums (i.e., do not have a unique finite limit) are called *divergent.*

The function $y = e^x$ has an important property. If we differentiate $y = e^x$, term by term (the reader will have to take our word that differentiating this particular series term by term is a valid procedure),

$$\frac{d}{dx}e^x = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots$$

This function is unchanged by differentiation; because of this feature, it occurs frequently in many applications of mathematics.

Let us now return to the original question of compound interest rates. Suppose $1 is placed in an account that pays, say, 5 percent interest compounded every instant of the day. (Actually, daily compounding is minutely close to this limit.) After 1 year, the account will have in it

$$= 1.0513$$

Daily (continuous) compounding will convert 5 percent annual interest to the yearly equivalent of approximately 5.13 percent.

Suppose an amount P is invested at interest rate r, continuously compounded, for a period of t years. The future value FV is

$$\text{FV} = P(e^r)^t = Pe^{rt} \tag{2-4}$$

Also, the *present value* of an amount FV at r percent is, by multiplying through by

$$P = (\text{FV})e^{-rt} \tag{2-5}$$

These formulas provide an analytically easy method of incorporating discounting into problems where time intervals are significant.

We note in passing that the equation $y = \log x$ means the same thing as $x = e^y$. If we differentiate $x = e^y$ implicitly with respect to x,

dx

or

$$\mathbf{dx} \qquad \mathbf{e^y} \tag{2-6}$$

x Thus, for $y = \log x$, $dy/dx = l/x$.

2.4 THE MEAN VALUE THEOREM

Consider Fig. *2-2a. A* differentiable function $y = f(x)$ is shown between the values $x = a$ and* $= b$. Consider the chordjoining the two points *(a, f(a)),* and *(b, f(b)).* The slope of this chord is

$$\frac{fib) - f(a)}{b - a}$$

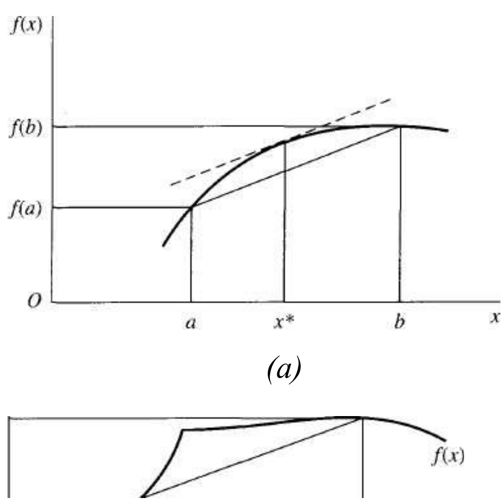


FIGURE 2-2
(a) The mean value theorem, *(b) If fix)* is not differentiable, the existence *of* x^*, $a < x^* < b$, such that $f'(x_0) = [fib) - fia)]/(b - a)$ is not guaranteed.

It is geometrically obvious (though it is not a proof) that at some point x^* between a and b, the slope of $f(x)$ is the same as the slope of this chord, or

,(,., = /W - *f(a)* *b*
— *a*

This statement or the following equivalent one is called the *law of the mean,* or *the mean value theorem:* If *f{x)* is differentiable on the interval $a < x < b$, then there exists an x^*, $a < x^* < b$, such that

$$f(b) = f(a) + \{b-a)f'(x^*) \qquad (2\text{-}7)$$

The reason why $f(x)$ has to be differentiable over the interval is exhibited in Fig. *2-2b.* The mean value theorem is actually a special case of the more general result known as Taylor's theorem. It is to this more general problem that we now turn.

2.5 TAYLOR'S SERIES

It is often of great analytical convenience to approximate a function $f(x)$ by polynomials of the form

$$fix) \% fnix) = a_o + mx + a_1x^l + a_3x^3 - \backslash - - - h\ a_nx^n$$

In particular, let us approximate $f(x)$ around the point $x = 0$. What values of the coefficients *ao,* ... , a_n will best do this? To begin, we should require that $f_n(x) = f(x)$ at $x = 0$. Hence, we need to set

$$\text{flo} = U0) = /(0)$$

Thus, the coefficient a_0 is determined in this fashion to be /(0).

To approximate $f(x)$ even better, let us make the derivatives of $f(x)$ and /,, *(x)* equal, at $x = 0$. We have

$$fnix) = a\backslash + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n''l}$$

$$/_n''(jc) = 2a_2 + 3 \cdot 2a_3x + \cdots + n(n - l)a_nx''^{-2}$$

$$ri^{n)}ix) = nl\ a_n$$

Clearly, when $x = 0$, we get

f I 2 = ^

nl

Having thus determined the coefficients of $f_n(x)$ in this fashion, our approximating polynomial is

2! 3! n!

An important class of functions comprises those for which $f_n(x)$ *converges* to $f(x)$, as n —► oo, that is,

f(x) = /(0) + *f(0)x* + *Q^-x²* + • ■ ■ (2-9)

These functions are called *analytic functions.* The power series representation (2-9) is called Maclaurin's series.

Suppose now we wish to approximate $f(x)$ at some arbitrary point $x = x_0$. In that case, write $f_n(x)$ in terms of powers of *(x — xo):*

fₙ(x) = a₀ + ai(x - x₀) + a₂(x - x₀)² H---h a«(x - x_o)^n

Using the same procedure as before, setting the derivatives of $f(x)$ equal to those of $f_n(x)$ at *x = XQ,* we determine

f(x) = f(x₀) + f(xₒ)(x - x₀) + Qfr-(x - ^o)^2 + • • • (2-10)

In this form, the power series is known as *Taylor's series,* or simply as a Taylor series. The Maclaurin series is a special case, where *xo* = 0.

Example 1. The series developed before, for e^x, is a convergent Taylor series expansion:

Example 2. Find a Taylor series expansion for log(l + *x),* around *x* = 0. (Assume convergence.) We note:

/(0) = l o g l = 0

/'(0) = --------= l a t J t = 0
J *(l+x)*

/"(0) = -*(l+x)~²* = -l a t x =0

/'"(0) = +2(1 + x)"³ = +2 at x = 0

Hence log(l + J C) = X - — + — - — H

A most useful form of a Taylor series expansion for a finite power *n* is a Taylor series with Lagrange's form of the remainder. The finite power series can be made exact (under suitable continuity assumptions) if the last term is evaluated not at x_0 but at some point x^* between x and *xo:*

$$f(x) = /(x_0) + /'(x_o)(x - x_0) +$$

where $x^* = x_0 + 0(JC - x_0)$, $0 < 0 < 1$.

Such an x* between x and x_0 must exist if $/^{("+1)}(x)$ is continuous. Equation (2-11) is one variant of what is known as Taylor's theorem. (The variant is the particular form of the remainder, or last, term.) In this form, Eq. (2-11), the Taylor series expansion, is seen to be a generalization of the mean value theorem. To obtain the mean value theorem, merely terminate (2-11) at $f'(x^*)$.

Applications of Taylor's Series: Derivation of the First- and Second-Order Conditions for a Maximum; Concavity and Convexity

Suppose $f(x)$ has a maximum at Xo- By definition

$$/(*o) > fix)$$

for all x in some neighborhood of x_0. Using the mean value theorem, i.e., a Taylor series terminated at the first-order term,

$$fix_0) - fix) = ix_0 - x)f'ix^*) \qquad (2\text{-}12)$$

for some x* between xo and x. The left-hand side of (2-12) is nonnegative for x near x_0. Therefore, if JC is to the left of xo (i-e., $x < x_0$), $fix^*) > 0$ necessarily, to make the product $ix_0 — x)fix^*) > 0$. For $x > x_Q$, $fix^*) < 0$. Hence, *fix)* is *positive* (or 0) to the left of x_0 and *negative* (or 0) to the right of x_0. If *fix)* is continuous at *xo,* then necessarily it passes through the value 0 at *xo;* i.e.,

$$/'(Jco) = 0$$

Similar reasoning shows that $fix_0) = 0$ is also implied by a minimum at JC_0. Let us now investigate the second-order conditions for a maximum. Consider a Taylor series expansion of *fix)* to the second-order term

$$fix) = /(*_{„}) + fix_0)ix - x_0) + \wedge P^{(x \text{ "" } X_0)2}$$

where, again, $x^* = x_0 + 0(jt — x_o)$, $0 < 0 < 1$. If *fix)* has a maximum at $x = JC_0$, then $fix_0) — 0$. Hence, the preceding equation can be written

$$fix) - /(x_0) = {}^{x}\text{-}f'ix^*)ix - x_0)^2 \qquad (2\text{-}13)$$

If /(JC) has a maximum atjc$_o$, the left-hand side of (2-13), by definition, is nonpositive. Since (JC — JCO)^2 > 0,

$$fix*) < 0$$

By "squeezing" x closer and closer to JC$_0$, we see that /"(JC) < 0 for all points in some neighborhood of JC$_0$; hence, at JC = JC$_0$

$$/"(*o) < 0$$

A maximum point therefore implies $f''(x_0) < 0$. If, however, $f''(x_0) <$ 0, then necessarily /(JC$_0$) > /(JC). Thus, together with /'(JC$_0$) = 0, $f''(x_0) < 0$ is sufficient for a maximum. Similar reasoning shows that at a *minimum* of *f(x), f''(xo)* > 0; if $f''(x_0) > 0$, then a minimum is assured.

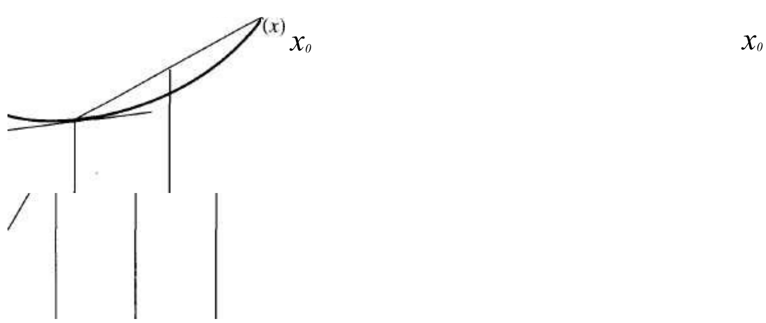
Concave and convex functions. Consider the function depicted in Fig. 2-*3a.* This shape is called *strictly concave.* It can be described by indicating that for any two points JC = JC$_0$ and $x = x\backslash$, say JCO < JCI, the function always lies above the chord joining /(JC$_0$) and /(JCI). That is, suppose JC is some intermediate point

-6)x_x **0 < *0***

Then /(JC) is strictly concave if

If 0 < *0* < 1 and

the function is called *weakly concave,* or simply concave. Convex functions are functions for which the chord connecting any two points on the function lies above



(a) *(b)*

FIGURE 2-3
(a) A concave function, *(b)* A convex function.

the function; an example is shown in Fig. 2-3/?. The terms *strictly* convex and *weakly* convex apply as for concave functions. The weak inequalities allow for straight-line segments in the function. The linear functions *fix) = a + bx* are both (weakly) concave *and* convex.

For differentiate functions, strict concavity can be described by saying that *f(x)* always lies below the tangent line at any point. Consider Fig. *2-3a.* Concavity can be interpreted as saying the slope of the tangent line is greater than that of the chord joining */(XQ)* and *f(x\), if x\ > x₀,* i.e.,

o) ¯

/ for *i > x_0

X\ — XQ /(*i) /(*o) ,

<

for *X* < *XQ*

X\ — XQ

In either case, if both sides are multiplied by *{x* — JC_0), we get, for any *x = x* (if Xi — *XQ* < 0, the inequality reverses sign),

f(x) < /(*„) + *f(x*$_O$*)(x - XQ)* (

or

fix) - fiXQ) - fix$_O$*)ix - XQ)* < 0 (2- 14&)

For concavity (not *strict* concavity), a weak inequality is used in statements (2-14). Using a Taylor series expansion of *fix)* to two terms,

f(x) = *f(x*$_0$*)* + *f'ix*$_O$*)ix - X*$_0$*)* + i/"(x*)(* - *XQ)*^{2}

Bringing the first two terms on the right to the other side, and using Eq. (2-14/?), for concave functions

fix) < 0

since *ix — x*$_0$*)*^{2} > 0. If *x* is squeezed toward JC_0, we see that *f'ixo)* < 0, but *f'ixo)* < 0 is not implied. If, however, /"(xo) < 0, the function must be concave. Similarly, convexity of *fix)* at *x* = x_0 implies *f"ix*$_Q$*)* > 0; if *f"ix*$_0$*)* > 0, then *fix)* is convex.

CHAPTER

3

FUNCTIONS OF SEVERAL VARIABLES

3.1 FUNCTIONS OF SEVERAL VARIABLES

The mathematical examples in Chap. 1 involved only one decision variable. Most often, however, in economic theories, several decision variables are present, all of which simultaneously determine the value of some objective function. Consider, for example, the fundamental proposition in consumer theory that individuals desire many goods simultaneously. This postulate asserts that the satisfaction, or *utility,* derived from consuming some bundle of goods is some function of the consumption levels for each and every good in question. This is denoted mathematically as

$$U = f(xi, x_2, \ldots, * „)$$

where $X\backslash, x_2,$ ■ • • $, x_n$ are the levels of consumption of the n goods. In the theory of production, a function $y = f(L,K)$ (called the *production function)* is typically written which indicates that the level of output depends upon the levels of both labor and capital applied to production. The mathematical notation $y — f(xi, \ldots, x_n)$ is simply a convenient shorthand to denote the inference of a unique value of some dependent variable y from the knowledge of the values of n *independent* variables, denoted jti ,..., x_n. It is a generalization of the notion of a function of one variable,
y - $fix)$.

3.2 LEVEL CURVES: I

Consider a production function $y = f(L, K),$ where y = output, L = labor, and K = capital services. The function/is the numerical rule by which levels of inputs

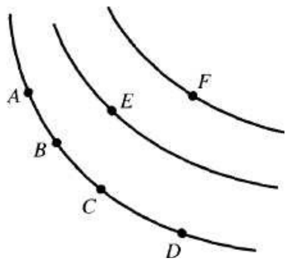


FIGURE 3-1
Level Curves for a Production Function. In this diagram, three separate level curves are drawn (out of the infinity of such curves that exist). Points *A, B, C,* and *D* all represent combinations of labor and capital which yield the same output. They are therefore all on the same level curve, called, in production theory, an *isoquant.* Point *E* represents a higher level of output; point *F* a still higher output level.

are translated into a level of output. With only two independent variables, geometric representation of this function is possible. In Fig. 3-1, all points in the positive quadrant (i.e., points in the Cartesian plane that correspond to positive values of L and K) represent possible input combinations. At each point in the plane, some unique value of the function $f(L, K)$ is implied. For example, at the points *A, B, C,* and *D,* output y is, say, 5, whereas at E, $y = 10$, and at F, $y = 15$.

Economists often have occasion to connect points whose functional values are equal. For example, in Fig. 3-1, the smooth line drawn through the points *A, B, C,* and *D* represents the locus of all points, i.e., the locus of all combinations of labor and capital, for which five units of output result. This curve, called an *isoquant* by economists, is called a *level curve* (in higher dimensions, a level *surface)* by mathematicians. It is a level curve because along such loci, the function (output, here) is neither increasing nor decreasing.^ Another geometric representation of a function of two variables is given in Fig. 3-2.

This is a two-dimensional drawing of a three-dimensional picture. The L axis is perpendicular to the plane of this page. In this diagram, the value of the function y is plotted as the vertical distance above the *LK* plane. This generates a surface in three-dimensional space, whose height represents here the level of output produced. Constant output points of, say, five units would all lie in a horizontal plane (parallel to the *LK* plane) five units above the *LK* plane. The intersection of such a plane with the production surface would yield a curve in that surface all of whose points were five units above the *LK* axes. This level curve, or contour, would be another representation of the five-unit isoquant pictures in Fig. 3-1. In fact, the isoquants in Fig. 3-1 are really projections of the level curves of the surface depicted in Fig. 3-2 into the *LK* plane. Similar level curves are drawn for the theory of consumer

T Those of you familiar with contour maps used in geological surveys (and hiking) will recognize those contours as the level curves of a function denoting the altitude of the terrain.

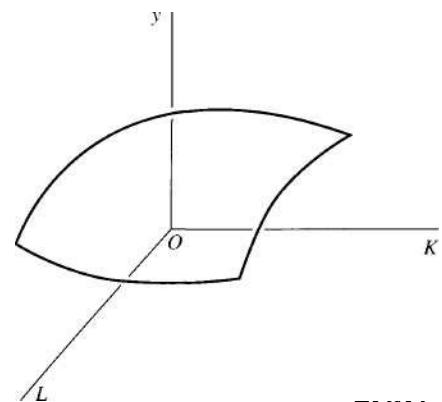


FIGURE 3-2 *A Three-Dimensional Representation of a Function of Two Variables.* This figure depicts a two-dimensional surface in three-dimensional space. The level curves of Fig. 3-1 are projections of the intersection of horizontal planes (at some

value of *y*) and this surface.

behavior. In this context, the level curves represent loci of constant utilities and are called indifference curves. Since these curves play a central role in economic theory, we will have much to say about them in the course of this book.

This three-dimensional representation of a function of two variables, although difficult to draw, provides a useful visualization of the situation. The function is increasing, say, if it is rising vertically as one moves in a given direction, and a maximum of such a function is easily pictured as the "top of the hill." But needless to say, for more than two independent variables, such visual geometry becomes impossible, and, hence, algebraic methods become necessary.

3.3 PARTIAL DERIVATIVES

Consider a consumer's utility function, $U = f(x_1, \ldots, x_n)$, where, again, $x_1, \ldots, x_n$ represent th e levels of consumption of *n* goods. If these *Xj* 's are indeed "goods," i.e., they contribute positively to the consumer's welfare at the margin, then it would be convenient to be able to denote and analyze this effect mathematically. The statement that the *marginal utility* of some good *Xj* is positive means that if JC, is increased by some amount Ax_h *holding the other goods* (the other JC,-'S) *constant,* the resulting change in total utility will be positive. This is exactly the same idea as taking derivatives in the calculus of one variable, with one important qualification: Since there are other variables present, we must specify in addition that these other variables are being held fixed at their previous levels. This type of derivative is called a *partial* derivative, since it refers to changes in the function with respect to changes in only one of several variables. Partial derivatives are denoted with curled <fs: *dy/dxj,* instea

d of the ordinary d's used in the calculus of one variable.

As another example, consider a production function $y = f(L, K)$. The marginal product of, say, labor is the rate of change of output when the labor input is adjusted incrementally, for a specified, constant level of capital input. The marginal product of labor is thus the partial derivative of output with respect to labor (L).

Likewise, the marginal product of capital is the partial derivative of *f(L, K)* with respect to *K*.

Proceeding more formally, consider some function, *y* = *f(x\, ... ,x_n),* evaluated at the point *X* = *x®,* ..., x_n = JC°. Consider how this function changes with adjustments in JCI alone. We define the partial derivative of *f(x*, ..., x_n*)* with respect to *x* as

dy = hm **A/**

The partial derivative, *(df/dxi),* is evaluated at *xi* = *x®,* ..., x_n = *x®* provided the limit exists. The foregoing difference quotient is really an intuitive generalization from the difference quotient used to define the ordinary derivatives of functions of one variable. Analogously, we define

where / = 1, ..., *n* (3-2)

We will use the notation *dy/*dx_t and 3//3x, interchangeably.

When taking partial derivatives, the rule is simply to treat all other variables as constants. The ordinary rules of differentiation are then applied.

Example 1. Suppose a consumer's utility is given by the function

$$U(x_u x_2) = .*_1 \log x_2$$

The marginal utilities are the partial derivatives *dU/dx\, dU/*dx_2. To find *dU/dxu* treat x_2 as constant:

$$\frac{dU}{\wedge} = \log x_2$$

Similarly, to find *dU/*dx_2, treat *x* as a constant:

$$\frac{dU}{dx_2} = \frac{1}{x_2} = \frac{x_x}{x_2}$$

Example 2. Suppose a firm's production function is

$$y = L^a K?$$

where *a,* /3 > 0 are constants. The marginal products of labor and capital are, respectively,

$$MP_L = \frac{}{dL} = aL^{a \sim l} K^{ft}$$

dK

The ordinary rules of differentiation, e.g., the product and quotient rules, apply to partial derivatives as well.

Example 3. Let $y = X\backslash e^{x\{+XI}$. Using the product rule,

$$-\hat{} = {}_{XI}e^{xi+XI2} + e^{XI+X2\prime} = e^{XI+X22}\,(1 + x_{,})\ dx\backslash$$

Also, using the chain rule as in differentiating e^{a+x} ,

$$\hat{}\boldsymbol{L} = \boldsymbol{x_x e^{x\prime+xi}\backslash lx_2)} = \boldsymbol{2x_{IX2}e^{XI+X22}}$$
9X2

As with the case of ordinary derivatives, partial derivatives can be differentiated (partially!) again, yielding *secondpartials.* However, a richer set of second derivatives exists for functions of several variables than for functions of one variable, because partials such as *df/dx* can be differentiated with respect to any of the *n* variables JCI through x_n. We can denote "the partial derivative of $df/dx_{\{}$ with respect to JC_7" as $3(3//9X_{()}/3JC_{;}$, or $d^2\ f/dxjdxi$. Often, however, it is convenient to simply use subscripts to denote differentiation with respect to a variable. Following this tradition, we will write *df/dxf* = f_{ι}, and for higher-order partials, subscripts read from left to right reflect the order of differentiation. That is, *fj* — $3^2//3jc_y$ 3JC,, which, for utility functions, can be interpreted as the rate of change of the marginal utility of good / when the quantity of goody increases.

Example 4. Consider $U\{x\backslash,\ x_2) = Xi\ \log x_2$ again. We previously found

$$U_x = \log x_2$$

Therefore

U-n =

d	=
u	0
9	x_2
d	1
9	x_2
d	

0X2 XJ

Example 5. For the function $y = f(L, K) = L^a K^\wedge$, the first partials were found to be

Hence

$$III = \frac{\sim}{\partial L}$$

Example 6. For $y = f(x_1, x_2) = x_1 e^{x_1 + x_2^2}$, we found

$$dx_2$$

Thus

$$f_{12} = e^{x_1 + x_2^2}(1 + x_1)2x_2 = 2(1$$

$$f_{21} = 2x_2(x_1 e^{x_1 + x_2^2} + e^{x_1 + x_2^2}) = 2(1 + x_1)x_2 e^{x_1 + x_2^2}$$

$$f_{22} = 2x_1 [x_2 e^{x_1 + x_2^2}(2x_2) + e^{x_1 + x_2^2}] = 2x_1 (1 + 2x_2^2)e^{x_1 + x_2^2}$$

Curiously enough, for each of these functions, $f_{12} = f_{21}$ (or, in the notation of Example 6, $f_{LK} = f_{KL}$). The same "cross partial" derivative results no matter in which order the variables are differentiated. This occurrence, in fact, is general for all functions of several variables whose second partials are themselves continuous.

This invariance to the order of differentiation is one of the least intuitive theorems in elementary mathematics. It is sometimes known as Young's theorem.* (Try asking some of your mathematician friends for an intuitive explanation of it!) The result accounts for some surprising relationships that appear in economics. Provided below is, in our opinion, the simplest explanation of invariance to the order of differentiation, for the case of functions in two variables. The generalization to *n* variables is routine. A rigorous discussion of the limit process is not given; hence, what follows is not a formal proof of the matter. It will do for our purposes, however.

*The reference apparently is to W. H. Young, who published a rigorous proof of the theorem in 1909 using the modern mathematical theory of limits. (See Cambridge Tract No. 11, "The Fundamental Theorems of the Differential Calculus," Cambridge University Press, reprinted in 1971 by Hafner Press.) In fact, the result was published by Euler in 1734 ("De Infinitis Curvis Eiusdem Generis...," Commentatio 44 indicis Enestroemiani). It really should be called "Euler's theorem," but that title is reserved for a famous result discussed later in this chapter. We are grateful to Joel Holmes and Neil Theobald for these tidbits.

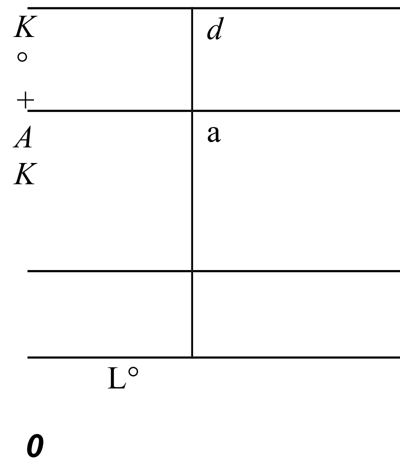


FIGURE 3-3 *Young's Theorem.* In this diagram, *a, b, c,* and *d* represent the *values* of $f(L, K)$ at the four corners of the rectangle.

Theorem. Let *y*

— $f\{x_x,$ have second-order partials that exist and are continuous. Then $/_{12} = /_{21}$.

Discussion. Consider a production function $y = f(L, K)$, where L and K are, respectively, the quantity of labor and capital used in the production process. If the theorem is to hold, then the answers to the following two questions should be identical:

1.123 How much, in the limit, does the marginal product of labor change when an extra unit of capital is added?

1.124 How much, in the limit, does the marginal product of capital change when one adds an extra unit of labor?

(Of course, both of these measurements must be made at the same point.)

In Fig. 3-3, let

$a = /(L°, K°)$

$b = f(L° + AL, K°)$

$c =$

$d = f(L°, K°$

That is, a, b, c, and d are the values of the function / (here, the levels of output) at the corners of the rectangle in the *LK* plane formed by the initial point L°, *K°,* and then changing *L* and *K*by amounts AL and *AK,* respectively, separately and then together.

Let us approximate the second-order partial derivatives by their second differences and see how they compare before any limits are taken. The marginal product of labor L evaluated at (L°, *K°)* is approximately

$$\text{A L} \qquad \frac{b}{a} \text{----} \frac{A}{L} \qquad \textbf{(3-3)}$$

If the amount of capital K is now increased by some amount AK, the marginal product of labor, evaluated at ($L°$, $K° + AK$) is

$$f_L(L\backslash K° + AK) \wedge \frac{f(L°+ AL, K° + AK) - f(L°, K°+ AK)}{ZA\,LJ}$$

$$= \frac{c - d}{AL} \qquad (3\text{-}4)$$

Then, f_{LK}, which measures the *change* in the marginal product of labor when an incremental amount of capital is added, can be found by taking the difference, per increment of capital, between the two marginal products of labor (3-3) and (3-4):

$$f_{LK} \quad \frac{1}{AK} \left(\frac{c}{AL} \quad \frac{d}{AL} \right) \frac{b}{AKAL} + a) \quad d - b + a) \qquad (3\text{-}5)$$

To find the other cross-partial f_{KL} we begin the process by first finding the marginal product of capital, and then asking how that value changes when the quantity of labor changes. Proceeding as before, the marginal product of capital evaluated at
($L°$, $K°$) is

$$J \quad \% \; /(£\blacksquare * \; + A*) - /a°, *°) \over AK = \frac{d-a}{AK}$$

Increasing the amount of labor to $L° + AL$, the marginal product of capital is

$$f_K(L+AL, K) \text{------------} = \sim \qquad (3\text{-}7)$$

Hence, the *change* in the marginal product of capital due to the change in labor is approximately

$$\frac{1}{AL}\left(\frac{fc-b}{AK} \quad \frac{d-a}{AK}\right) \frac{1}{AL\,AK}(c - b - d + a) \qquad (3\text{-}8)$$

Notice that Eqs. (3-5) and (3-8) are identical! That is, the second differences are the same, whether L or K is changed first. The remaining step (and it is a big step) in proving the theorem is to show that, under appropriate mathematical conditions on the function $f(L, K)$, the *limits* as $AL \longrightarrow 0$ and $AK \longrightarrow 0$, are the same, taken in either order. This step is omitted here. The argument is based on an application of the mean value theorem and can be found in most elementary calculus texts.

In general, assuming the function is sufficiently well-behaved (no discontinuities in higher-order derivatives, etc.), the higher-order partial derivatives are also invariant to the order of differentiation. This is derived by simply applying Young's theorem over and over.

Example 7. Consider $y = f(x_1, x_2, x_3)$. Show that $/_{12}3 = /312 = fn\backslash$, etc-Applying Young's theorem to $f\backslash (x\backslash, x_2,$

x_3)

$$/l23 — /''1\ (23) = /''1\ (32) = /l32$$

However, $/_{13} = /_{31}$. Hence

$$/l32 - /312$$

Thus, $/_{123} = /_{3}i_{2}$. Also, since $/_{3(12)} = /_{3(2}i)$,

$$/l23 = /312 = /321$$

In a similar fashion, for $y = f(x\backslash,\ldots, x_n)$,

$$fijk — fjki = \bullet \bullet \bullet$$

3.4 THE CHAIN RULE

In economics, as well as most sciences, one often encounters a sequence of functional relationships. For example, the output of a firm depends upon the input levels chosen by the firm, as specified in the production function. However, the input levels are determined, i.e., functionally related to the factor and output prices. Hence, output is related, indirectly, to factor and output prices. It is therefore meaningful to inquire as to the changes in output that would follow a change in some price, i.e., a partial derivative of output with respect to that price. The chain rule is the mathematical device that expresses the partial derivative of the composite function in terms of the various partial derivatives of the individual functions in the functional sequence. For functions of one variable, if

$$y = f(x) \qquad \text{and} \qquad x = g(t)$$

then the functional dependence of y on t can be

written

$$y = f(8(0) =$$

$h(t)$ It follows by simple algebra that

$$\begin{array}{cc} Ay & Ay \\ Ax\,At & \sim \\ Ax & At \end{array}$$

Taking limits, assuming both $f(x)$ and $git)$ are differentiable, we get the chain rule for functions of one variable,

$$\begin{array}{cc} dy & dy \\ dx & dt \\ dx\,dt & \end{array}$$

Intuitively, suppose $y = 2x$ and $x — 3t$. Then $y = 6t$, and it is clear that dy/dt is the product of dy/dx and dx/dt.

Suppose now that y is a function of two variables, $y — f\{x\backslash,x_2)$. Suppose $x\backslash$ and $X2$ are in turn functions of some other variable t. Let $X\backslash — X\backslash(t)$, $x_2 = x_2(t)$.$^{j\prime}$ Then if t changes, so will, in general, JCI and x_2 and, hence, also y. To express this functional dependence of y on t, we write $y = f(x\backslash(t),X2(t)) — y(t)$. How can $y'(t)$ be expressed in terms of /i, f_2, $x[(t)$ and $x'_2(t)7$ In this case, a change in t produces changes in *both* $x\backslash$ and x_2. The combined effect on y is the sum of the two

t Mathematicians frown on the use of the same symbol to denote a function and the value of that function. It will not get us into trouble, however, and it will reduce the number of symbols that the reader has to keep in mind.

individual chain rule effects for $x\backslash$ and x_2. Thus

$$\frac{dy}{dt} = \frac{dy}{dx_1}\frac{dx_1}{dt} + \frac{dy}{dx_2}\frac{dx_2}{dt} = h\frac{dx_1}{dt} + h\frac{dx_2}{dt} \qquad (3\text{-}10)$$

Suppose now that JCI and x_2 are themselves functions of several variables. For example, let JCI $= g(r,s)$, $x_2 = h(r,s)$. In this case, $y = f(g(r,s),h(r,s)) = F(r, s)$, and we can only speak meaningfully of the *partial* derivatives of y with respect to r and s. The chain rule here is

3y dg dh

/ = /./ + /2ir (3-11)

or or or

with a similar expression holding with respect to the variable s. The only difference between (3-10) and (3-11) is that since r is one of several variables, the appropriate partial notation must be used.

The chain rule generalizes in a straightforward manner to the case where each independent variable is in turn a function of m other independent variables. Let

$y =$

and let

$x =$ *t* z

Then chain rule

dy 9 9 *d*

dt$_k$ *d* ***dt***$_k$ 3 *B*

This also be a

dy - i

f

= 1, . . . , m (3-12)

jfc=1,…,m (3-13)

where the symbol g'_k means dg^1/dt_k.

Example 1. Let $y = f\{x\backslash, x_2)$, and let

$$x_: = x? + h\backslash t$$

$$X_2 = X_2^\circ + \hat{}_2'$$

where /?! and /i$_2$ are arbitrary constants. When $t = 0$, $X\backslash$ = JC°, JC$_2$ = x". As / changes, $X\backslash$ and X2 move along a straight line in the

$X\backslash X_2$ plane. This can be seen by eliminating t from these equations:

$$. \quad h_2 i x_x \boldsymbol{-xi)} \qquad \boldsymbol{h_2} \quad \boldsymbol{(}\ {}_{\mathbf{n}} \qquad {}_n\boldsymbol{h_2}\backslash$$

This is the equation of a straight line with slope h_2/h_u passing through the point (JC{\ $JC_2{}^0$). Writing is equivalent to saying that $f(x\backslash,x_2)$ is evaluated along the straight line (3-14), or, equivalently, (3-15). Using the chain rule,

$$y'(t) = f_1h_1+f_2h_2 \qquad (3\text{-}16)$$

Example 2. Suppose y — log(X[+ x_2), where x, = t, $x_2 = t^2$. This is equivalent to evaluating log (xi + x_2) along the parabola $x_2 = x^2$. Let us find dy/dt by direct substitution and by the chain rule.

(a) By direct substitution

$$y = \backslash og\{t + t^2)$$

Therefore

£ (l+2

$dt \qquad t + t^2$

(b) Using the chain rule,

$dy \qquad dx_l \qquad dx_2$

X, +X$_2$ X, +X$_2$

1

■(1+20

t + t-

as before.

Example 3. Suppose $y=x^2e^{X2}$, with JCJ = log t, $x_2 = t^2$. Find J_y/<i/ by *(a)* direct substitution and by *(b)* the chain rule.

faj Substituting the expressions for $x\backslash$ and x_2, y = (log $t)^2e'$.
Using the product rule for differentiation,

%=/(21og 0- + dog

$t)^2(2te^{t2})$ **dt** ***t***

(b) Using the chain rule,

$dy \; \_ \; f \; ^{dX\{} \qquad _f \; ^{dXl}$

The final expressions are, as they must be, identical by either method.

Second Derivatives by the Chain Rule

Suppose that _y = $f(x\backslash,x_2)$ and $X\backslash = x\backslash(t)$, $x_2 = x_2(t)$. We need to find an expression for d^2y/dt^2, as this second derivative is important for analyzing the sufficient conditions under which a function of several variables achieves a maximum or a minimum position. Using the chain rule,

dy _ dx

$$dy\ \_\quad dx_x$$
$$dx_2 \sim dt\,"$$
$$^{h}U^{+h}\wedge T$$

Then, to find d^2y/dt^2, we have to differentiate this expression again. Do not forget, however, that f_x and f_2 are themselves functions of x_x and x_2, and, hence, functions of *t*. Then, using the product rule,

$$d^2y \qquad d\ dy \qquad d\ dx\backslash \qquad dx\backslash\ d$$
$$dt\ dt \qquad dt\ dt \qquad dt\ dt$$

$$d\ dx_2 \qquad dx_2\ d$$
$$dt\ dt \qquad dt\ dt$$

Now use the chain rule to differentiate /I(JCI(O, *x₂(t)),* et cetera, with respect to *t*. Noting that $dfi/dx\backslash = f_{xx}$, et cetera,

$$d^2y$$

$$dt^2 \quad {}^{Ji}\,dt^2 \quad dt\ \ V \quad dt \quad {}^{J\wedge}\,dt\,) \quad {}^{JL}\,dt^2 \quad dt\ \ V \quad dt$$

${}^{JLA}\,dt$ Regrouping terms, and noting that $f_{x2} = f_{2X}$,

$$d^2v \quad d^2x_x \quad d^2x_2 \quad f\,dx_x\backslash^2 \quad dx_x\,dx_2 \quad d^2$$

$$x_x \quad dx_x\,(\quad dx_x \quad ,, \quad dx_2\backslash \quad d^2x_2 \quad dx_2\,(\quad dx_x \quad dx$$
$$+ \quad / \quad + \quad +\,/ \quad + \quad +$$

Note that this expression is linear in the second derivatives of x_x and x_2 with respect to *t*, and *quadratic* in the first derivatives of Xi and x_2. The appropriate generalization to *n* variables, with 3; = $f(x_x,\ldots, x_n)$ and JC, = *X[(t), i* = 1,..., *n* is obtained in the same manner:

$$\wedge \quad V \quad \wedge \quad V \quad V \quad {}^{dXid\wedge} \qquad (3\text{-}18)$$

$$dt$$

Example 6. Let $y — f(X],x_2)$ and consider the straight lines $x\backslash = x® + h_xt$, $x_2 = x® + h_2t$ once more. From Eq. (3-16),

Therefore

$$(df_xdx_x \quad df_xdx_2\backslash \quad fdf_2dxi \quad df_2\,dx_2$$
$$y^{(t)} = \{d7\wedge + 3\wedge)\,{}^{h\prime +}\,\{-d7_l\text{-}dT + d7_2\wedge T$$

Since $dfi/dx\backslash$ = /n, et cetera, and dx_i/dt = /?,, this expression reduces to

$$fit) = f_nh] + 2f_nh,h_2 + f_{22}h\backslash \qquad (3\text{-}19)$$

For this "parameterization" of $X\backslash$ and x_2 in terms of *t, y"(t)* is a "quadratic form" in $h\backslash$ and h_2.

3.5 LEVEL CURVES: II

Consider again the representation of a function of two variables as presented in Fig. 3-1, with $y = f(L, K)$, a production function. The level curve representing, say, five units of output is simply $f(L, K) = 5$. In general, the level curves of some function $y = f\{x\backslash, x_2)$ are defined by $f(x\backslash, x_2) = yo$, where y_0 is some constant. How do we determine the curvature properties, such as the slope in the $X\backslash X_2$ plane, or the convexity of that level curve?

The equation $f(x\backslash, x_2) = yo$ represents one equation in two *unknowns,* $x\backslash$ and x_2. Under certain mathematical conditions (to be determined below), this equation can be solved for one of the unknowns in terms of the other, say,

$$x_2 = x_2(xi)$$

When this solution is substituted back into the equation from which it was derived, the *identity*

$$f(x_x, x_2(xi)) = y_0$$

results, by definition of a solution. In this identity, x_2 always adjusts to any value of JCI so as to keep $f(x\backslash, x_2(x\backslash))$ always equal to y_0.

The slope of any level curve is simply the derivative $dx_2/dx\backslash$. But it is important to understand that this symbol, $dx_2/dx\backslash$, makes sense only if we have explicitly defined x_2 as a function of $x\backslash$, as we have, in fact, done previously. It is nonsense to speak of derivatives unless one knows what function it is that is being differentiated. Since our function $x_2 = x_2(x\backslash)$ is well defined, $dx_2/dx\backslash$ can be found by differentiating the identity $f(x\backslash, x_2\{x\backslash)) = yo$ with respect to $x\backslash$, using the chain rule. We therefore get

$$\frac{df}{dx\backslash}\frac{dx_x}{dx\backslash} + \frac{df}{dx_2}\frac{dx_2}{dx\backslash} = \frac{dy_0}{dx\backslash} = 0$$

or

Now assuming that $f_2 \neq 0$

$$\frac{dx_2}{} = -\frac{}{f_2} \qquad (3\text{-}20)$$

The slope of a level curve at any point is the ratio of the first partials of the function $y = f\{x\backslash, x_2)$, evaluated, of course, at some particular point on the level curve in question. The condition alluded to previously which allows solution of $f\{x\backslash, x_2) = yo$ for $x_2 — x_2(xi)$ can be seen to be simply that $f_2 \neq 0$. When $f_2 \neq 0$, at some point the derivative $dx_2/dx\backslash$ can be expressed in terms of the partials of the original function, and, hence, the equations $f(x\backslash,x_2) = yo$ and $x_2 — x_2\{x\backslash)$ are equivalent at such points. When $f_2 = 0$, the level curve becomes vertical and its derivative does not exist.

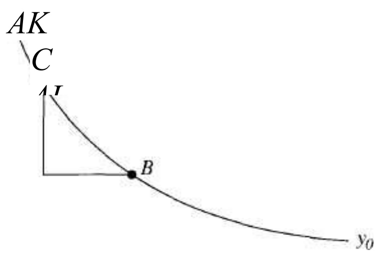


FIGURE 3-4
Movement Along an Isoquant. The move from *A* to *B* can be broken down into a decrease in *K(A* to C), then an increase in *L (C* to *B)* to achieve the same production level. Since *y* is constant, the *decrease* in output going from *A* to *C* (—MP^AAT) equals the *increase* in output going from *C* to *B* (MP_LAL). Thus *Ay* = MP/. AL =0.

What is the meaning of $dx_2/dx\backslash = —f\backslash\ If\text{-}p$. Consider the production function $y = f(L, K)$ again. The level curves are the isoquants of this production function. In Fig. 3-4, consider a movement along an isoquant y_0, from A to *B.* This movement can be conceptually broken down into a vertical movement down to *C,* in which case only *K* is changed by an amount *AK,* and then a horizontal movement from C to *B,* in which only L changes by an amount AL. The output change from A to C is approximately the marginal product of capital, evaluated at A, times the loss of capital, *AK,* or f_K *AK*. In going from C to *B,* since labor is being added, the gain in output is approximately the marginal product of labor (evaluated at *B)* times the gain in labor, or $/_L$AL. Since output is unchanged, by definition of an isoquant, from A to *B,* these quantities must add to 0, or

$$f_K AK=0 \qquad (3\text{-}21)$$

In the limit, as points A and *B* are brought closer and closer together so that AL and *AK* —► 0, Eq. (3-21) is simply an expression that the total differential of $y = f(L, K)$ equals 0, since *y* is unchanged, or

$$dy = \qquad f_K dK=0 \qquad (3\text{-}22)$$

Now if and only if *K* can be expressed as a function of L or $K = K(L)$, as it always can if the isoquant is not vertical, then the total differential may be divided through by *dL,* yielding

or **dL**

dK ~dL

SK

Thus Eq. (3-20), for production functions, measures the willingness of firms to substitute labor for capital, since it measures the ratio of the benefits of the additional labor, f_L, to the output lost due to using less capital, f_K.

In the theory of the consumer, the level curves of a utility function $U = U(x\backslash,$ JC_2), the indifference curves, can be similarly analyzed. The slope of an indifference curve, which expresses the willingness of a consumer to make exchanges, is based on the ratio of perceived gains and losses from such an exchange. Following the above analysis, this slope, or exchange rate, *dx>ildx\,* is equal to — *U1/U2,* the ratio of marginal utility of good 1 to good 2. This ratio, since it expresses an evaluation of giving up some *X2* (a loss of U_2 dx_2) in order to obtain some *X[* (a gain of *U\ dx\)* is called the marginal rate of substitution of x_x for x_2. Since along an indifference curve $dU = 0$, $U_2\ dx_2 =$ — *U\ dx*. Assuming $x_2 = x_2(x\backslash)$ is well defined, $dx_2/dx\backslash$ = —*U1/U2,* the ratio of perceived gains to losses, at the margin.

Convexity of the Level Curves

From the formula $dx_2/dx\backslash$ = —*f\ //*$_2$, if the first partials are both positive, the level curves must be negatively sloped. In production theory, if the marginal products of each factor input are positive, then the isoquants will have a negative slope. An analogous statement concerning the marginal utilities and indifference curves holds for the consumer. Simply stated, a movement to the "northeast" from any factor input combination, say, involves more of both factors. If the marginal products are positive, this must yield an increase in output, and, hence, the new point cannot lie along the same isoquant as the old. The willingness of consumers to make trade-offs—that is, to give up some of one good in order to get more of another good—is evidence that the level curves of utility function (the indifference curves) are negatively sloped. If they were positively sloped, consumers would have to be bribed by one good in order to consume some other good; indeed, one of the "goods" would really be a "bad," yielding negative utility at the margin.

However, in addition to asserting a negative slope of these level curves, economists also insist that these curves are "convex to the origin," as shown in Fig. 3-1.t Why do economists believe this, and how can we represent this convexity mathematically? Strict convexity of these level curves to the origin is a statement that the marginal value of either good (or factor) declines along that curve, as more of that good or factor is obtained, relative to the other. As $x_{\prime}$ is increased, say, the ratio -/1//2 declines in absolute value, meaning that the benefits associated with having greater *x\,* that is, /\, are declining relative to the benefits of having some more x_2, measured by f_2 at the margin. The reason why economists believe this to be empirically correct is that the opposite assumption would imply that consumers would spend all of their income on one good, or that firms would hire only one factor of production. After all, if the marginal benefits of having *x* rose the more *x* one had, why would a person ever stop purchasing *x* in favor of x_2 (assuming it

^The phrase "convex to the origin" is imprecise; the correct characterization is that the utility function is strictly increasing and *quasi-concave.* In two dimensions, this yields the familiar shape described previously. We shall define and explore such functions in

Chap. 6.

was worthwhile to purchase some X\ in the first place)? We are assuming that the consumer or firm is a sufficiently small part of the market to have a negligible effect on the price of *xi.* Convexity of the level curves is asserted because it is the only assertion about preferences or technology that is consistent with the simultaneous use of several goods or inputs, i.e., with the decision to *stop* utilizing some economic good at some point short of exhaustion of one's entire wealth.

Mathematically, convexity of the level curves can be represented, in two-dimensional space, by considering the curve *X2* = *2(*i), the explicit function of the level curve. The negative slope of this curve is indicated by $dx_2/dx\backslash < 0$; convexity by $d^2x_2/dx^2 > 0$. The positive second derivative means that the slope $dx_2/dx\backslash$ is increasing as *x* increases, and this is precisely what is indicated by the level curves in Fig. 3-1. As *x* (or L, there) increases, the slope becomes less and less negative; i.e., it increases. How do we express d^2x_2/dx^2 in terms of the partials of /(JCI, JC_2), from which the level curve is derived? As was seen before,

$$dx_2 \qquad /1(*$$

Note, however, that we have explicitly indicated the independent variables *X* and x_2 with the functional dependence of x_2 on *x* also explicitly shown. To find d^2x_2/dx^2, we must differentiate the right-hand side of (3-23), using the quotient rule, and using the chain rule in the numerator and denominator. Hence,

4

dx{ dft dx2\ fdf2

dx\ dx›dx\) \dx\ dx\ dx›dx\ J\ f² dxx df2 dx2

+ ir~ -j— —Ji I +

— - \ J 2 \ ju + Jn-r~ I - J \ \ J 2 i + J

dxx) \ dx\) \ f2'

However, $dx_2/dx\backslash = —f\backslash/f_2$. Substituting this into the last expression, and noting that/i_2 = *fix,*

$$d^2x_2\_(\qquad f^2f_{22}\backslash \quad 1$$
$$dx_x \qquad \backslash \qquad h \quad / h$$

or

^T = (-fifn + 2/1/2/12 - fifn)— (3-24)

ax\ Si

Note that convexity of the level curve depends in a rather complicated manner on the first and second partials of *f(x\,x₂).* We shall have more to say about this expression and how it is generalized to more than two variables in Chap. 6. But note the following: Suppose *y* = /(JC1? JC2) is a utility function. Then convexity of the indifference curves in no way implies, or is implied by, "diminishing marginal utility," that is, $f_u < 0$, $/_2 2 < 0$. There is a cross-effect $/_{12}$ that must also be considered, and which can outweigh the effects, positive or negative, of the second partials /i i and f_{22} • Hence, diminishing marginal utility and convexity of indifference curves are two entirely independent concepts. And that is how it must be: Convexity of an indifference curve relates to how marginal evaluations change *holding utility* (the dependent variable) *constant.* The concept of diminishing marginal utility refers to changes in total utilities, i.e., movements from one indifference level to another.

Monotonic Transformations and Diminishing Marginal Utility

We need to consider one last chain rule that figures prominently in economic theory. Suppose a consumer's utility function is given by

$$U = U(x_1, x_2)$$

In the modern theory of the consumer, the utility function is just an *ordinal* ranking of preferences. We say that consumers can express that they prefer bundle *A* to bundle *B,* but we do not quantify this any further. We do not, for example, assert that consumers can say that *A* gives them twice the pleasure of *B* so that we could measure their satisfaction with some cardinal scale of "utiles." Cardinality would mean that a consumer could say, "This steak gives me 20 utiles of pleasure, and that potato gives me only 10 utiles," and we would all know what he or she meant, just like we know what a temperature of 90°F or a grade point average of 3.5 means. These are cardinal measures; we use the numbers 1, 2, 3, … to measure such quantities. The ordinal numbers, on the other hand, are just rankings, like one's standing in one's class. Most sporting events and elections are based solely on ordinal rankings—whoever gets the most (or fewest, in the case of golf) points wins. The actual numbers don't matter, just one person's ranking vs. another's.

Ordinality is given precise expression by saying that the utility function given by

$$V\{x_u x_2) = F(U) = F\{U\{x_u x_2)) \qquad (3\text{-}25)$$

where $F'(U) > 0$, conveys the same information as $U(x\backslash, x_2)$. The condition $F'(U) > 0$ means that U and V always move in the same direction. The function Vis called a *monotonically increasing* function of U. [If $F'(U) < 0$, V would be called monotonically decreasing.] Most often, the single term *monotonic* is used to mean *monotonically increasing.*

What the function F does is relabel the level curves of $U,$ giving them new numbers, $V.$ This is a different situation than previously where the independent variables were dependent on some other variable or variables. Here, the dependent variable U (in this case) is given a new value, $F(U) = V.$ The function V is a function of the one variable U which in turn is a function of two variables, JCI and x_2. We can thus ask, since V ultimately depends on *both* $x\backslash$ and $x_2,$ how can we express the partial derivatives of V in terms of partials of U and the derivatives of *F(U)1*

In this instance, the chain rule is actually simpler than in our previous discussion. Notice from (3-25) that when JCI changes, there is no effect on $x_2,$ and likewise, when x_2 changes, there is no effect on

JCI . What we have here is just the simple chain rule for one variable, except that the derivatives of *U* with respect to *x* and x_2 are partial derivatives. For finite changes,

AV AV *AU*
Ax *AU*

Ax Taking limits,

$$V_x = F\backslash U) U_X \qquad ($$

and

$$V_2 = F\backslash U) U_2 \qquad (3\text{-}266)$$

Notice that the slope of the indifference curve is unaffected by relabeling the indifference curves in this way:

dx2 = _ *V_i* = _™i = _*lh*
dx_x *V_2* *F'U_2* *U_2*

What (3-27) reveals is that if some indifference curves were labeled as 1,2, 3, etc., we could just as easily use log 1, log 2, log 3 or e^l, e^2, e^3, etc., and then there would be no implied change in behavior, because the consumer's behavior is defined only in terms of the trade-offs given by the slope of the indifference curve at a given point. What about the second partials of *VI* Differentiating (3-26) again partially with respect to x_x and x_2 yields (using the product rule)

$$V_n = F'U_n + F''U^2_l \qquad (3\text{-}28a)$$

$$V_{22} = F'U_{22} + F''U\backslash \qquad (3\text{-}286)$$

and

$$V12 = V_{2l} = F'U_n + F''U_X U_2 \qquad (3\text{-}29)$$

Equations (3-28) show once more why the phrase "diminishing marginal utility" has no meaning in the context of ordinal utility. Diminishing marginal utility means, for the *U* index, that $U_n < 0$ and $U_{22} < 0$. But notice from (3-28a) that U_n and V_n don't necessarily have the same sign. Although $F' > 0$, F'' ^ 0. So even though *U* < 0, we might, if $F'' > 0$, wind up with $V_n > 0$, i.e., with increasing marginal utility, when the indifference map is relabeled to the *V* index. Moreover, no changes in consumers' trade-offs and therefore no changes in observable behavior occur with this relabeling. Diminishing marginal utility requires an assumption of cardinal utility to have operational meaning.

Consider now a related idea. Suppose I were to say that beer and pretzels are complements for me because my marginal utility of beer increases when I eat some pretzels. Or, my marginal utility of butter decreases when I have some additional margarine, so butter and margarine are substitutes. Is this a good definition of substitutes and complements? Although plausible sounding, such a definition is useless. Stating that a consumer's marginal utility of Xi increases if more x_2 is consumed means the cross-partial $dU\backslash/dx_2 > 0$. But (3-29) shows that the sign of $dU\backslash/dx_2 = U\backslash_2$ is not invariant with a monotonic relabeling of the indifference map. Relabeling, which, again, produces no change in observable behavior, could produce a new utility index V for which the sign of $V\backslash_2$ is opposite that of $U\backslash_2$- With just ordinal utility, we cannot attach meaning to rates of change of the marginal utilities. We will further explore these issues in Chap. 10, which deals in greater detail with the theory of consumer behavior.

PROBLEMS

1. Consider the following three utility functions:
   (i) $U = x_x x_2$ *(ii)* $V = x\backslash x\backslash$ *(Hi)* W = log*! + logx_2
   1.125 Find the marginal utilities of x, and x_2 for each utility function.
   1.126 Find the rates of change of marginal utility of one good with respect to a change in consumption of the other good for each utility function. Verify that, for these functions, the change in the marginal utility of one good due to a change in the other good is the same, no matter which good is chosen first.
   1.127 Find the marginal rate of substitution of Xi for x_2 for each utility function, and show that they are all identical.
   1.128 From the preceding parts of this problem, which value, that derived in *(b)* or in *(c),* would you expect to play a positive role in the theory of consumer behavior?
2. Consider the two utility functions
   (i) $U = x_x e^{xi}$ *(ii)* $V = x_2 +$ log x_x
   1.129 Answer the same questions as in Prob. 1.
   1.130 Verify that three of the four second partials of V are identically 0, whereas for *U,* those three are all =£ 0. Can it be that these two utility functions nonetheless imply identical behavior on the part of the consumer? *(Answer:* Yes! *Moral:* Beware of rate of change of marginal utilities.)
3. Consider the production function $y = L^a K^{1}{\sim}^{a}$, where L = labor, K = capital, y = output, and a is restricted to the values $0 < a < 1$. (This type of production function is called *Cobb-Douglas.)*
   1.131 Find the marginal products of labor and capital, MP_L and MP^, respectively.
   1.132 Find the rates of change of these marginal products due to changes in both labor and capital. Verify that the rate of change of MP^ with respect to K is the same as that of MP/j- with respect to L.
   1.133 Does the law of diminishing marginal productivity hold for this production function?

1.134 For the production function in Prob. 3, show that $f_L L + f_K K = y$. (This is an example of Euler's theorem, which will be explored later.)

1.135 The theorem on invariance of second partials to the order of differentiation breaks down when the second partials are not continuous. Those students who know what *continuous* means to a mathematician should try to make up a function whose second partials *exist* but are not continuous.

1.136 Let $y — L^aK^{1-a}$ represent society's production function. Suppose L and K both grow at constant, though different, rates; i.e., let $L = L_oe^{nt}$, $K = K_oe^{mt}$, where t represents "time." Find dy/dt by direct substitution and by the chain rule.

1.137 Let $U = f(x_u x_2)$ be a utility function, and let $V(x_x, x_2) = F(U)$, where $F'(U) > 0$. (V is a monotonic transformation of U.)

1.138 Show that $V_1/V_2 = U_1/U_2$.

1.139 Find V_{ij} in terms of U_{ij}, i, j — 1,2. Show that in general U_{ij} and V_{ij} need not have the same sign.

1.140 Consider the utility function U — $x\backslash^{/3}xl^{/3}$. The demand curves associated with U are
$xi = M/3pi$, x_2 — $2M/3p_2$, as will be shown later. Find the rates of change of U with
respect to changes in each price and money income. Do the signs of these expressions
agree with your intuition?

1.141 Let $y = f\{x_x, x_2)$ = g(xi — x_2). Let $u = x_x$ — x_2. Show that

$$dy/du = dy/dx_t = -dy/dx_2 \qquad d^2y/du^2 = d^2y/dx^2 = d^2y/dx^2{}_2$$

3.6 HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

In order to efficiently study the structure of many important economic models, it is necessary to first discuss an important class of functions known as *homogeneous* functions. The interest in these functions arose from a problem in the economic theory of distribution. The development of marginal productivity theory by Marshall and others led to the conclusion that factors of production would be paid the value of their marginal products. (This will be studied in the next and subsequent chapters in more detail.) Roughly speaking, factors would be hired until their contribution to the output of the firm just equaled the cost of acquiring additional units of that factor. Letting $y = f\{x\backslash, X2)$ be the firm's production function and letting w, denote the wage of factor x, andp the price of the firm's output, the rule developed was that

$$pMPi = pft = w_t$$

where f_t = 3//3x,. But this analysis was developed in a "partial equilibrium" framework; that is, each factor was analyzed independently. The question then arose, how is it possible to be sure that the firm was capable of making these payments to both factors? All factor payments had to be derived from the output produced by the firm. Would enough output be produced (or perhaps would too much be produced, leaving the excess unclaimed) to be able to pay each unit of each factor the value of its marginal product?

A theorem developed by the great Swiss mathematician Euler (pronounced "Oiler") came to the rescue of this analysis. (It leads to other problems, but those will be deferred.) It turns out that if the production function exhibits constant returns to scale, then the sum of the factor payments will identically equal total output. Mathematically, if each factor JC, is paid w, = pf_t, then the total payment to all JC,- is $wtXi = pfiX_t$. Total payment to both factors is thus

$$pf\backslash*\backslash + Ph^xi = P(f\backslash^xi + fixi)$$

But, as we shall see, constant-returns-to-scale production functions have the convenient property that, identically,

$$/i*i + /2\text{-}X2 = y = f(xi,x_2)$$

Hence, in this case,

$$W1X1 + W_2X_2 = pf\,1X1 + pf_2X_2 = p(/i*i + f2X_2) = Py$$

or, total costs identically equal total revenues, and the product of the firm is exactly "exhausted" in making payments to all the factors.

How is the feature of *constant returns to scale* characterized? This means that if each factor is increased by the same proportion, output will increase by a like proportion. Mathematically, a production function $y — f(x\backslash, \ldots ,x_n)$ exhibits constant returns to scale if

$$f(tx_1,\ldots,tx_n)=tf(x_1,\ldots,x_n) \qquad \text{(3-30)}$$

Note the identity sign: this proportionality of output and inputs must hold for all $x_($'s and all t. If, for example, all inputs are doubled, output will double, starting at any input combination.

The relation (3-30) is a special case of the more general mathematical notion of homogeneity of functions.

Definition 1. A function $f(x\backslash,\ldots, x_n)$ is said to be homogeneous of degree r if and only if

$$f(tx_1,\ldots,tx_n) = t^r f(x_u\ldots,x_n) \qquad \text{(3-31)}$$

That is, changing all arguments of the function by the same proportion t results in a change in the value of the function by an amount f', identically. Note again the identity sign—this is not an equation that holds only at one or a few points; the above relation is to hold for $a\backslash lt,xi,\ldots,x_n$. Constant returns to scale is the special case where a production function is homogeneous of degree 1. Homogeneity of degree 1 is often called *linear homogeneity.*

Example 1. Consider the very famous Cobb-Douglas production function, $y = L^aK^{x\sim a} — f(L,K)$, where L = labor, K = capital. This production function is homogeneous of degree 1; i.e., it exhibits constant returns to scale. Suppose labor and capital are changed by some factor t. Then,

$$f(tL,tK) = \{tL)^a(tK)^{x\sim a} = t^aL^at^{x-a}K^{x-a}$$

Output $f(L, K)$ is affected in exactly the same proportion t as are both inputs.

Consider now another important area in which the notion of homogeneity arises. In the theory of the consumer (also to be discussed later), individuals are presumed to possess demand functions for the goods and services they consume. If $Pi,\ldots, p_n$ represents the money prices of the goods $X\backslash, \ldots, x_n$ that a person actually consumes, and if M represents the consumer's money income, the ordinary demand curves are representable as

$$x_i=x^*(p_1,\ldots,p_n,M), \qquad \text{(3-32)}$$

That is, the quantity consumed of any good x, depends on its price p_t, all other relevant prices, and money income M.

How would we expect the consumer to react to a proportionate change in *all* prices, with the same proportionate change in his or her money income? Although a formal proof must wait until a later chapter, we should expect *no change* in consumption under these conditions. Economists (for good reason) in general assert that only *relative* price changes, not absolute price changes, matter in consumers' decisions.

What is being asserted here, mathematically? We are asserting homogeneity of degree 0 of the above demand equations, i.e.,

$$x^*(tp_1, \ldots, tp_n, tM) = t^0 x^*(p_1, \ldots, p_n, M) = x^*(p_1, \ldots, p_n, M)$$

The functional value is to be unchanged by proportionate change in all the independent variables; this is precisely homogeneity of degree 0. The demands for goods and services are not to depend on the *absolute* levels of prices and income.* The theoretical reasons for asserting this proposition will become clearer in later chapters; our purpose here is only to illustrate and motivate the usefulness of the concept of homogeneity of functions.

Consider now the Cobb-Douglas production function again, $y = L^a K^{1-a} = f(L, K)$. The marginal products of labor and capital are, respectively,

1-a

$$= f_K = (1 - a)L^a K^{-a} = (1 - a)\,(\sim$$

These marginal products exhibit a feature worth noting: They can be written as functions of the *ratios* of the two inputs. They are independent of the absolute value of either input. Only their proportion to one another counts.

Because of this dependence only on ratios, the marginal products of the Cobb-Douglas function are homogeneous of degree 0:

$$MP_L(tL, **) = «\frac{ftK\backslash^{1-a}}{} = a\,(K -$$

Similarly,

$$t\,K\backslash \qquad /K\backslash$$

$$-J = (1 - a)\,(-J = MP_K(L,K)$$

If labor and capital are changed, by the same proportion, say they are both doubled, the marginal products of labor and capital will be unaffected. Geometrically, changing each input by the same proportion means moving along a ray out of the origin,

*There was a time, in the macroeconomics literature, when this homogeneity of demand functions was denied, under the name "money illusion." It was asserted that a completely neutral inflation would lead an economy out of depression; that even though people were not in fact richer, a higher money income (together with proportionately higher money prices) would somehow make people "feel" richer, increasing their consumption expenditures. This line of argument has been largely abandoned.

through the original point. At every point along any such ray, the marginal products of the Cobb-Douglas production function (and others?) are the same.

To what extent, if any, are these results peculiar to the Cobb-Douglas functions; i.e., to what extent do other functions exhibit the same or similar properties? Consider first any function $f(x_1, \ldots, x_n)$ that is homogeneous of degree 0. By definition,

$$f(tx_1, tx_2, \ldots, tx_n) = f(x_1, x_2, \ldots, x_n)$$

Since this holds for *any t,* let $t = 1/x_1$. Then we have

$$f(x_1, x_2, \ldots, x_n) = f\left(1, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) = g\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$$

Similarly, we could let $t = 1/x_i$. What the above shows is that *any* function that is homogeneous of degree 0 is representable as a function of the *ratios* of the independent variables to any one such variable. Hence, that the marginal products of the Cobb-Douglas function were representable as functions of the capital-labor ratios is *not* peculiar to that production function; it will hold for any marginal product functions that are homogeneous of degree 0.

What, then, are the conditions that the marginal products be homogeneous of degree 0? The answer is given, in a more general form, by the following theorem:

Theorem 1. If $f(x_1, x_2, \ldots, x_n)$ is homogeneous of degree r, then the first partials $f_1, \ldots, f_n$ are homogeneous of degree $r - 1$.

Proof. By assumption, $f(tx_1, \ldots, tx_n) = t^r f(x_1, \ldots, x_n)$. Since this is an identity, it is valid to differentiate both sides with respect to x_i:

$$\frac{\partial f}{\partial (tx_i)} \frac{\partial (tx_i)}{\partial x_i} = t^r \frac{\partial f}{\partial x_i}$$

However, $\partial(tx_i)/\partial x_i = t$. Dividing both sides of the identity by t therefore yields

$$\frac{\partial f}{\partial (tx_i)} = t^{r-1} \frac{\partial f}{\partial x_i}$$

But this says that the function f_i evaluated at $(tx_1, \ldots, tx_n)$ equals $t^{r-1} f_i(x_1, \ldots, x_n)$. Hence, f_i is homogeneous of degree $r - 1$.

If $y = f(x_1, \ldots, x_n)$ is any production function exhibiting constant returns to scale, the marginal products are homogeneous of degree 0. That is, the marginal products are the same at every point along any ray through the origin. The Cobb-Douglas function is thus only a special case of this theorem.

Homogeneity of any degree implies that the slopes of the level curves of the function are unchanged along any ray through the origin. This can be shown as follows: Let $y = f(x_1, \ldots, x_n)$ be a production function, for example, that is homogeneous of degree r. The slope of an isoquant in the $x_i x_j$ plane is

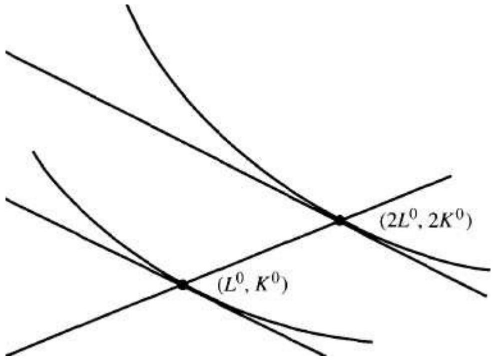


FIGURE 3-5
Invariance of the Slope of Isoquants to a Proportionate Increase in Each Factor. Consider any point (L^0, K^0). Suppose each input is doubled. If the production function is homogenous of any degree, the slope of the isoquant, $-f_L/f_K$, will be the same at $(2L^0, 2K^0)$ as at (L^0, K^0). This property is known as *homotheticity.* The most general functions that exhibit this property can be written $F(f(x_1, \ldots, x_n))$, where $f(x_1, \ldots, x_n)$ is homogenous of any degree and $F' \neq 0$.

But

$f_i(tx_1, \ldots, tx_n) = t^{r-1} f_i(x$

$$\ldots, x_n) = \frac{f_i(tx_1, \ldots, tx_n)}{f_j(tx_1, \ldots, tx_n)} = \frac{t^{r-1} f_i(x_1, \ldots, x_n)}{t^{r-1} f_j(x_1, \ldots, x_n)} = \frac{f_i(x_1, \ldots, x_n)}{f_j(x_1, \ldots, x_n)}$$

Thus, the slope of any isoquant evaluated along a radial expansion of an initial point is identical to the slope at the original point. In other words, the ratios of the marginal products along any ray from the origin remain unchanged for homogeneous functions. The level curves are thus radial blowups or reductions of each other. This situation is depicted in Fig. 3-5.

The following describes a related class of production functions. Let $y = f(x_1, \ldots, x_n)$ be homogeneous of degree r, and let $z = F(y)$, where $F'(y) > 0$. [$F(y)$ is a monotonic transformation of y.] The function $z(x_1, \ldots, x_n)$ is called a *homothetic* function. It is easy to show that homothetic functions also preserve the property that slopes along a radial blowup remain unchanged, i.e., that the slopes of isoquants $z(tx_1, \ldots, tx_n)$ are the same as at $z(x_1, \ldots, x_n)$, and this is left to the student as an exercise. It is less than easy to show, but nonetheless true, that this is the most general class of production functions that have this property.^

Example 2. Consider the function $z = g(L, K) = F(y)$, where $y = L^a K^{1-a}$ and $F(y) = \log y$. Then

$$z = \log L^a K^{1-a} = a \log L \quad (1-a) \log$$

That is, the original function $L^a K^{1-a}$ is transformed by the function "F," in this case "log." We note that $F'(y) = 1/y > 0$, for positive L, K. Now $L^a K^{1-a}$ is homogeneous

^See, e.g., F. W. McElroy, "Returns to Scale, Euler's Theorem, and the Form of Production Functions," *Econometrica,* 37(2):275-279, 1969.

of degree 1, as noted before, but *log(L^aK^l~a)* is *not* a

homogeneous function: *g(tL,tK) = a\logtL + (l -a)logtK*

= a(log/ + logL) + (l -*a)(\logt* + *\logK) -logt* + log*VK'~^a*

^fg(L,K)

However, *g(L, K) — a log L+(l—a) log K is* homothetic: The slope of a level curve i s *-8L* _ *-a/L* _

-a K ~g~7 ~ d-a)/K ~ l - a L

As before, *—gi/gK* is unaffected by changing *K*and *L* by a factor of *t;* the r's cancel in the expression *K/L* and, hence, the slope of the level curves of log *L^a K^{l'a}* are the same along any ray out of the origin. This function is *not* homogeneous, but it *is* homothetic.

Suppose that instead of defining homothetic functions as *F{f{x\, ..., x_n)),* where/ is homogeneous of degree r, that instead we restrict/to be linearly homogeneous; i.e., homogeneous of degree 1. Though it might not seem so at first, this latter definition is just as general as the first definition; i.e., no functions are left out by so doing. The reason is that any homogeneous function of degree *r* can be converted to a linear homogeneous function by taking the rth root of *f(x\, ..., x_n)*. Then, *[f(x\, ..., x_n)]^{1/r}* can be transformed by some function *F.* Thus, since we can always consider *F* to be a composite of two transformations, the first of which takes the rth root of/ and the second, which operates on that, no generality is lost by defining homothetic functions as transformations of *linear* homogeneous functions.

Example 3. Let *y = f(x_x, x_2) = x\X_2.* Here, *f(x\, x_2)* is homogeneous of degree 2. Let

g(x_ux_2) = F(f(x_l,x_2)) = log(jci*$_2$) = log*i +logx$_2$

This function is homothetic but not homogeneous. How could *g(x\, x_2)* be constructed out of a *linear* homogeneous function? Let

g{x_x,x_2) = 21ogUix$_2$)$^{1/2}$

Thus,

where 0 means "take square root" and *F* is log, as before. Then the same function

g(x_x,x_2) = logx_l+logx_2 is constructed as

a transformation of the linear homogeneous function *(x\X_2)^{1/2}*.

We now prove the main theorem of this section.

Theorem 2 (Euler's theorem). Suppose *f(x_l,..., x_n)* is homogeneous of degree *r.* Then

9/ **3/**
- *X] +* ■ ■ • + - -*x_n* =
rf{x\,..., x_n)
dx *ox_n*

Note the identity sign: this is not an equation; rather, it holds for all

x_x, ..., x_n. The two sides are algebraically identical.

Proof. By the definition of homogeneity,

$$f(tx_u \ldots, tx_n) = ff\{x_u \ldots, x_n)$$

Since this identity holds for all values $ofx\backslash, \ldots, x_n$ and t, differentiate both sides with respect to t, using the chain rule:

$$9/ \quad dtx_x \;, \qquad , \quad 9/ \quad d(tx_n) \;,$$

$$. , \bullet \bullet \bullet , *n)$$

$$d(txi) \quad dt \qquad d\{tx_n) \quad dt$$

However, $d(tXi)/dt = x$-,, thus

$$\begin{array}{c} df \qquad\qquad df \\ \text{-}x, + \blacksquare \blacksquare \blacksquare + \text{-}\pm\text{-}x_n = rt^{r\text{-}l}f(x_u \ldots, x_n) \\ \text{O}\;\; x, + \qquad + \\ dtX\backslash \qquad\quad otx_n \end{array}$$

This relation is also an identity that holds for all t and all $x_l, \ldots, x_n\backslash$ in particular, it must hold for $t = 1$. Putting ? = 1 in the preceding identity results in Euler's theorem.

An important special case of homogeneity is that of homogeneity of degree 1, also called *linear homogeneity.* In this case, $r = 1$, and thus the Euler identity yields $Y, f_ix_l\blacksquare = f(x\backslash, \ldots, x_n)$. This is precisely the property that was alluded to in the beginning of this section, concerning constant returns to scale and exhaustion of the product. When r = 1 (linear homogeneity), Euler's theorem says that the sum of the marginal products of each factor times the level of use of that factor exactly and identically adds up to total output. Thus, marginal productivity theory is consistent with itself in that case.

Another interesting case is when $f\{x\backslash, \ldots, x_n)$ is homogeneous of degree 0. Then, Euler's theorem yields

This formula will be used in deriving some properties of demand functions for consumers and firms, both of which exhibit this type of homogeneity.

Example 4. Consider again the Cobb-Douglas function $y = L^aK^{l}\sim^a = f(L, K)$. This function is homogeneous of degree 1, i.e., r = 1. We have $f_L — aL^a\sim^xK^x\sim^a, f_K = (1 — a)L^aK\sim^a$. Then the left-hand side of the Euler identity becomes

$$f_LL + f_KK = aL^{a\text{-}l}K^l\sim^aL + (\backslash \text{ - } a)L^aK\sim^aK$$

$$= (a + 1 \text{ - } a)L^aK^l\sim^a = f(L, K)$$

Thus, $f_LL + f_KK$ is identically $L^aK^l\sim^a$, the original production function.

Example 5. Let $y = x'''x''^2 = f(x,\backslash, x_2)$. Then

$$\begin{array}{c} Ct\backslash — 1 \quad Ct\text{-}y\; r \qquad Ct\backslash \qquad ffo — 1 \\ i = a,x_1^1 \quad x_2^2 \quad f_2 = a_2x_j^jx_2^2 \end{array}$$

/

Then

$$/ \qquad , \quad r \quad ffi\text{-}1 \quad ci \quad . \quad at \quad ai$$

— *l*

$\backslash Xi + J2X2 = <X\backslash X_{l}{}^{l} \quad x_{2}\, x \backslash \ + a_{2}x_{l}{}^{l}x_{2}{}^{z} \quad x_{2}$

$at \quad \text{ff->} \quad , \qquad at \quad a\char`^$

$= a_{l}x_{l}\, x_{2} \ + a_{2}x_{l}\, x_{2}\sim$

$= (\mathrm{a},\ + a_{2})x''^{l}x_{2}{}^{2} = (\mathrm{a},\ + a_{2})f(x_{u}x_{2})$

This function is homogeneous of degree $a_1 + a_2$; hence, that multiple appears on the right-hand side of the Euler identity.

Example 6. Consider a firm with a linear homogeneous production function $y = f(L,K)$. By Euler's theorem,

Dividing by L and rearranging terms gives

Recall that if an average curve $A(x)$ is rising, then the associated marginal curve $M(x)$ lies *above* the average, i.e., $M(x) > A(x)$. Likewise, $A(x)$ is falling if and only if $M(x) < A(x)$. The equation thus says that if the average product of labor is *rising,* the marginal product of capital f_K must be *negative.* Similar manipulation shows that if the average product of capital is rising, the marginal product of labor is negative. The stage of production where AP_L is rising is called *stage I; stage II* occurs when AP_L is falling but $MP_L > 0$; $MP_L < 0$ characterizes *stage III.* The equation shows that for *linear homogeneous* production functions, stage I for labor is stage III for capital, and vice versa.

Example 7. Consider a two-good world with goods x_1 and x_2 that sell at prices p_1, p_2, respectively. Suppose that a consumer with money income M has the following demand function for x_1:

$$- \frac{Mp_2}{p_1}$$

Show that the demand for this good is unaffected by a "balanced" or neutral inflation. Show also that Euler's theorem holds for this function.

Suppose money income M and both prices increase by the same proportion t. Then $x_1(tp_1, tp_2, tM) = tM(tp_2/t^2p_1^2) = Mp_2/p_1^2 = x_1(p_1, p_2, M)$. Hence, the consumer is unaffected by a change in absolute prices alone; i.e., this demand function is homogeneous of degree 0. Now,

$$\frac{\partial x_1}{\partial p_1} = \frac{-2Mp_2}{p_1^3}$$

$$\frac{\partial x_1}{\partial p_2} = \frac{M}{p_1^2}$$

$$\frac{\partial x_1}{\partial M} = \frac{p_2}{p_1^2}$$

Hence,

$$\frac{\partial x_1}{\partial p_1}p_1 + \frac{\partial x_1}{\partial p_2}p_2 + \frac{\partial x_1}{\partial M}M = \frac{-2Mp_2}{p_1^2} + \frac{Mp_2}{p_1^2} + \frac{Mp_2}{p_1^2} = 0$$

In many instances of dealing with homogeneous functions, what is desired is not Euler's theorem per se, but rather its converse. Suppose, for example, the product of a firm was exhausted for any input combination, i.e., we somehow knew that $\sum f_i x_i = f(x_1, \ldots, x_n)$. Would this imply that the function is linear homogeneous? The answer is in the affirmative.

Theorem 3 (The converse of Euler's theorem). Suppose

$$f\backslash*\backslash + hx_2 \text{ H- - -h } f_n x_n =$$

for all jf], … $,x_n$. Then $f(tx_u \ldots, tx_n) = f f(x\backslash, \ldots ,x_n)$; that is, $f(x_u \ldots, x_n)$ is homogeneous of degree $r)$

Proof. (To save notational clutter, we shall prove the case for a function of only two independent variables, $x\backslash, x_2$. The generalization to n variables is routine.) Consider any arbitrary point (x{\ x_2). Construct the function

Differentiating with respect to t yields, using the chain rule,

$$d\pm = 0'(,) = {}_x» /, (,*<>, tx°_2) + x°_2\, h\, (tx°, txl) \qquad (3\text{-}33)$$

By assumption, however, applying $f\backslash X_X + fix_2 = rf(x_u\, x_2)$ at the point

$(tx®, tx®)$

$$/, (txl\ tx°)tx°_x + f_2(tx°, tx°_2)tx^0_2 = rf(tx°_x, tx°_2) \qquad (3\text{-}34)$$

By inspection of Eqs. (3-33) and (3-34),

$$\{\ »\ l\ ) \qquad (3\text{-}35)$$

Equation (3-35) is a differential equation that is easy to solve: We have $z = <p(t), (p'(t) = dz/dt;$ hence (3-35) is equivalent to

$$t\frac{dz}{dt} = rz$$

Grouping each variable,

$$\frac{dz}{z} \,\_\!\sim\, \frac{dt}{t}$$

Integrating both sides yields

$$\frac{dz}{z} = r\ I\text{- - -}\backslash\text{-}C \quad [dt \quad J\ t$$

where C is the constant of integration. But $J(dz/z)$ = log z, $f(dt/t)$ = log t, and letting C = log C for convenience, the solution to (3-35) is

$$\log z = r \log t + \log$$

C or

$$\log z = \log Cf$$

^This theorem technically holds only for positive values of t. Consider the function $f(x\backslash, x_2) — (x\backslash + x|)^{3/2}$. Then $f(tx\backslash, tx_2) = \backslash t\backslash\text{^} f(x\backslash,$

x_2) since the square root is always taken as positive. This type of function would satisfy the proof of Theorem 3; it, however, is not homogeneous for all values of t, but rather just for $t > 0$.

Taking antilogs, the solution of the differential Eq. (3-35) is

$$z = (P(t) = Ct^{r} \qquad (3\text{-}36)$$

That this is a solution to Eq. (3-35) can be verified by substituting this expression into that differential equation. The constant of integration can be evaluated by setting $t = 1$:

$$CV = C = {}_{(}p(l) = f(x^{0}{}_{l}, x°{}_{2})$$

Hence, *(j)(t) — f(tx®, tx%) =* t^{r}*f(x^, x®)*. But this is precisely the definition of homogeneity of degree r! Since (JC°, *x®)* was any point in the *X\X2* plane, the theorem (the converse of Euler's theorem) is proven.

PROBLEMS

1. Show that the following functions are homogeneous and verify that Euler's theorem holds.
   1.142 $f(x_{u}x_{2}) = x\backslash x\backslash$
   1.143 $f(xi,x_{2})$ =xi*$_2$ +*2
   1.144 $f(x_{u}x_{2}) = (_{xl} + x_{2})/(x^{2} - 2x^{2}{}_{2})$
   1.145 $f(xi,x_{2}) = x^{2}j\{x_{x}x_{2} - xl)$
   1.146 $f(x_{u}x_{2}) = xi$
2. Show that the following functions are *homothetic.*
   1.147 y = log*, +logx_2
   1.148 $y = e^{xlX2}$
   1.149 $y = \{x_{x}x_{2})^{2} - x\backslash x_{2}$
   1.150 $y = log(x_{i}x_{2}) + e^{*}M$
   1.151 y = log(xf +$x_{j}x_{2})^{2}$

1.152 Let/(^!, x_2) = *A(axi +(l —a)*$x_{2}y^{/p}$. Show that/(xj, x_2) is homogeneous of degree 1.
(This production function is called a constant elasticity of substitution, or CES, production
function.) Its properties will be investigated in Chap. 9.

1.153 Let *f(x\, x*$_2$*) — F(h(x\, x*$_2$*))* where *h* is homogeneous of degree *r* and *F'* > 0 (/is a
homothetic function). Show that the expansion paths of/are straight lines; i.e., that the
level curves of/have the same slope along any ray out of the origin.

1.154 Let *f{x\, x*$_2$*)* be homogeneous of degree 1. Show that $f_{u}x\backslash + fnx_{2}$ = 0 [by considering
the homogeneity of / (JC, , x_2)].

1.155 Let *f(x \, . . . , x*$_n$*)* be homogeneous of degree *r in the first k variables only,* i.e., *f{tx*, ...,
$tx_{k}, x_{k+}\backslash, \ldots ,x_{n}) = t^{r}f(x\backslash, \ldots, x_{n})$. Show that

SELECTED REFERENCES

In addition to a basic calculus text, students might find the following works useful:

Allen, R. G. D.: *Mathematical Analysis for Economists,* Macmillan & Co., Ltd., London, 1938. Reprinted
by St. Martin's Press Inc., New York, 1967. Courant, R.:
Differential and Integral Calculus, 2d ed., vols. 1 and 2, Interscience

Publishers, Inc., New York, 1936. This is a classic work.

CHAPTER

4

PROFIT MAXIMIZATION

4.1 UNCONSTRAINED MAXIMA AND MINIMA: FIRST-ORDER NECESSARY CONDITIONS

Postulates of purposeful behavior lead naturally to the specification of mathematical models that involve the maximization of some function of several variables. Most often, this maximization takes place subject to test conditions specifying constraints on the movements of the variables in addition to the specifications of values of parameters. The well-known model of utility maximization is an example of such a model: The consumer is asserted to maximize a utility function subject to the condition that he or she not exceed a given budgetary expenditure. There are some important examples, however, of *unconstrained* maximization, such as the model of a profit-maximizing firm (which will be dealt with below). Since the unconstrained case is simpler, we begin the analysis there.

In models with just one independent variable, the first-order condition necessary for $y = f(x)$ to attain a stationary value is $dy/dx = f'(x) = 0$. That is, the line tangent to the curve $f(x)$ must be horizontal at the stationary point. The term *stationary point* rather than *maximum* or *minimum* is appropriate at this juncture. The property of having a horizontal tangent line is common to the functions $y = x^2$, $y = -x^2$, and $y = x^3$ at the point $x = 0$, $y = 0$. The first function has a minimum at the origin, the second, a maximum, and the third, neither. However, it is clear that if the slope of the tangent line is *not* 0 (horizontal), then the function certainly cannot have either a maximum or a minimum. Hence $f'(x) = 0$ is a *necessary* but not sufficient condition for $y = f(x)$ to have a maximum (or minimum) value.

Suppose now that y is a function of two variables, that is, $y = f(x_1, x_2)$. What are the analogous necessary conditions for a maximum of this function? Proceeding intuitively from the case of one variable, it must necessarily be the case that at the point in question, the tangent *plane* must be horizontal. In order for the tangent plane to be horizontal, the first partials *df/dxi*, df/dx_2 must be 0; that is, the function must be level in the *x* and x_2 directions.

Because intuition, especially about the second-order conditions for maximization, is often unreliable, the preceding argument will now be developed more rigorously. Let $y = f(x\backslash, x_2)$, and suppose we wish to consider the behavior of this function at some point x° = *(x®, x®)*.^ Instead of working with the whole function, however, consider the function evaluated along any (differentiable) curve that passes through the point x°. The reason for doing this is that it will enable us to convert a problem in two variables to one involving one variable only, a problem we already know how to solve. All such curves can be represented parametrically by JCI = *x* (/), $x_2 = x_2(t)$, with JCI = *x®, x_2* = *x®* at *t* — 0. That is, as *t* varies in value, *X* and x_2 vary, and hence the pair $[x\backslash (t), x_2(t)]$, denoted *x(t)*, traces out the locus of some curve in the x_ix_2 plane. [Setting .^(O) = *x®, $x_2(0)$ = x®* merely ensures that the curve passes through (JC°, *x®)* for *some* value of *t.]*

Example 1. This parametric representation of a curve in the $X\backslash X_2$ plane was developed in Chap. 3. Again, suppose

$$x\backslash = x^{\varrho}_x + h_x t$$

$$x_2 = x^{\circ}_2 + h_2 t$$

where *h* and h_2 are arbitrary constants. Then these equations represent the straight lines in the $x_x x_2$ plane which pass through *(x°, x®)*. Any such line can be generated by appropriate choice of *h* and h_2.

Example 2. Let

$$x_x = x^{\varrho}_x + t$$

$$x_2 = x^{\circ}_2 e'$$

This parameterization represents an exponential curve. When *t* — 0, *x* = *x®, $x_2 = x_2\backslash$* hence the curve passes through $(x^{\circ}_x$, x°).

Example 3. A parameterization that occurs frequently in the physical sciences is

$$x = a \cos$$

$$6\ y — a$$

$$\sin 6$$

where 0 < *6* < *2n*. This represents the equation of a circle in the *xy* plane, with radius *a* and center at the origin.

t\Ve will often find it convenient to use the vector notation x = *(x\,... ,x_n)*, where the single symbol x denotes multidimensional value.

The function $f(x\backslash,x_2)$ evaluated along some differentiate curve x(f) = (xi(r), $x_2(t)$) is $y(t) = f(x\backslash\{t), x_2(t))$. If $f(x\backslash, x_2)$ is to achieve a maximum value at x = x°, the function evaluated along all such curves must necessarily have a maximum. Hence $y\{t)$ must have a maximum (at t = 0) for all curves $x(t)$. But the condition for this is simply $y'(t) = 0$. Using this chain rule the first-order conditions for a maximum are therefore

However, dy/dt must be 0 for *all* curves (JCI(O, $x_2(t)$) passing through x°; i.e., for *all* values of $dx\backslash/dt$ and dx_2/dt. That is, it must be possible to put any values of dxi/dt, dx_2/dt into this relationship and still obtain $dy/dt = 0$. The only way this can be guaranteed is if $f_j=f_2 = 0$. Hence a necessary condition for $f(x\backslash, x_2)$ to be maximized at *x®, x®* is that the first partials of that function must be 0 at this point. The preceding conditions are, of course, only necessary conditions for y to achieve a stationary point; only the second derivative of $y(t)$ reveals whether *(x®, x®)* is in fact a maximum, a minimum, or neither.

The generalization to the n variable case is direct, and the derivation is identical to the preceding. For $y = f\{x\backslash, x_2, \ldots, x_n)$ to be maximized at x° = (jtj\ ..., *x®)* it is necessary that all the first partial derivatives equal 0; that is, / = 0, z = 1, ..., n.

4.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA: TWO VARIABLES

For functions of one variable, $y = f(x)$, a sufficient condition for $f(x)$ to have a maximum at $x = x°$ is that, together with $f'(x°) = 0$, $f''(x°) < 0$. The condition $f''(x°) < 0$ expresses the notion that the slope is decreasing, e.g., as one walked over the top of a hill, the ground would be first rising, then level at the top, then falling. Alternatively, the function is called *concave downward,* or simply, concave, if $f''\{x) < 0$. If $f(x\backslash,x_2)$ has a maximum at x°, then $y(t) = f(x\backslash(t), x_2(t))$ has a maximum for all curves x(/). Hence it must be the case that at the maximum point, $d^2y/dt^2 = y''(t) < 0$ for all such curves.

The issues here are considerably more subtle than one might may perceive at this point, as the next section will demonstrate. Although $y''\{t) < 0$ is *necessary* for a maximum, it is not sufficient. By expanding $f\{x\backslash, x_2)$ by a Taylor series for functions of two (or, more generally, n variables), it can be shown that if $y''\{t) < 0$ at $t = 0$ (the maximum point), then the function $f\{x\backslash, x_2)$ is strictly concave at (jtpjt^)- Thus, in that case, a maximum will be achieved if $f_l = f_2 = 0$. This analysis will be presented in the appendix to this chapter.

Let us then evaluate $y''(t)$. Using the chain and product rules on Eq. (4-1), one obtains (this was derived explicitly in Chap. 3)

$$\frac{d^2y}{dt^2} = f_1\frac{d^2x_1}{dt^2} + f_2\frac{d^2x_2}{dt^2} + f_{11}\left(\frac{dx_1}{dt}\right)^2 + 2f_{12}\frac{dx_1}{dt}\frac{dx_2}{dt} + f_{22}\left(\frac{dx_2}{dt}\right)^2$$

However, this is evaluated at $(x_1, x_2) = (x_1^\circ, x_2^\circ)$, a stationary point; hence $f_1 = f_2 = 0$. Letting $h_1 = dx_1/dt$, $h_2 = dx_2/dt$ for notational convenience, the condition that $d^2y/dt^2 < 0$ for all curves passing through (x_1°, x_2°) means that

$$f_{11}h_1^2 + 2f_{12}h_1h_2 + f_{22}h_2^2 < 0 \tag{4-2}$$

for all values of h_1 and h_2 (except $h_1 = h_2 = 0$). This inequality, since it must hold for all nontrivial h_1, h_2 (i.e., not both equal to 0), imposes restrictions on the signs and relative magnitudes of the second-order partials.

It is apparent from expression (4-2) that both f_{11} and f_{22} must be negative: Let $h_2 = 0$ and h_1 be any number and suppose f_{11} is positive. Then $d^2y/dt^2 = f_{11}h_1^2 > 0$, violating the sufficient conditions for a maximum. Interchanging all the subscripts gives the desired restriction on f_{22}, as the formulation is completely symmetrical. Thus, in order to have $d^2y/dt^2 < 0$ at $x^\circ = (x_1^\circ, x_2^\circ)$, it is necessary that

$$f_{11}(x^\circ) < 0 \qquad \text{and} \qquad f_{22}(x^\circ) < 0$$

However, these conditions, which one might have guessed at by considering the one-variable case, are *not*, by themselves, sufficient for $f(x_1, x_2)$ to have a maximum. We have yet to consider the role of the cross-partial f_{12} in this analysis. An additional restriction on the f_{ij}'s is required to ensure $d^2y/dt^2 < 0$ for all nontrivial h_1 and h_2. It can be derived by using the technique known as *completing the square*.

Consider the expression $x^2 + 2bx$. If the term b^2 is both added and subtracted, the identity $x^2 + 2bx = (x + b)^2 - b^2$ results. Take Eq. (4-2) and factor out f_{11}:

$$f_{11}\left(h_1^2 + 2\frac{f_{12}}{f_{11}}h_1h_2 + \frac{f_{22}}{f_{11}}h_2^2\right) < 0$$

The first two terms in parentheses are quadratic in h_1 in the same sense as the preceding algebraic example. Completing the square in h_1 is accomplished by adding and subtracting $(f_{12}h_2/f_{11})^2$ in the parentheses. This yields

$$f_{11}\left[\left(h_1 + \frac{f_{12}h_2}{f_{11}}\right)^2 + \left(\frac{h_2}{f_{11}}\right)^2\left(f_{11}f_{22} - f_{12}^2\right)\right] < 0$$

Since $f_{11} < 0$, in order to guarantee $d^2y/dt^2 < 0$, the square-bracketed term must be positive. However, the first term in the bracket is a squared term and hence is always positive anyway. In order to guarantee that $d^2y/dt^2 < 0$ *for all values of h_1 and h_2*, we must also require that the second term, in particular $f_{11}f_{22} - f_{12}^2$, be positive.

To sum up, then, suppose *f(x\, x_2)* has a stationary point x = x°, that is, the first-order necessary conditions for an extremum occur:

$$/i(x°) = /2(x°)=0 \qquad (4\text{-}3)$$

If, in addition,

$$/n < 0 \qquad \text{and} \qquad fufn \sim fn > 0 \qquad \text{evaluated at x°} \qquad (4\text{-}4)$$

a maximum position is assured. Note that if (4-4) is satisfied, f_{22} < 0 is implied. It is also important to note that condition (4-4) imposes a restriction only on the relative magnitude of $/_{12}$; it does not imply anything about the sign of this second partial. The sign of $f\backslash_2$ is thus irrelevant in determining whether a function has a maximum or minimum.

For *f(xi (t), x_2(t))* to achieve a *minimum* at x° = *(x®, x®)* the same first-order conditions (4-3) must, of course, be met. The analogous sufficient second-order conditions, i.e., guaranteeing d^2y/dt^2 > 0, are

$$/ii>0, \qquad /_{22}>0 \qquad \text{and} \qquad /n/22 - /,^2{}_2 > 0 \qquad (4\text{-}5)$$

where all partials are evaluated at x°. *Note that the term* /n/22 — $f\backslash_2$ *is positive for both minima and maxima.* If this term is found to be negative, then the surface has a "saddle" shape at x°: It rises in one direction and falls in another, similar to the point in the center of a saddle.

One last precautionary note must be mentioned. These second-order conditions are *sufficient* conditions for a maximum or minimum; the strict inequalities (4-4) and (4-5) are *not* implied by maxima and minima. For example, the function 3; = —x^4 has a maximum at the origin, yet its second derivative is 0 there. Likewise $y = x^3$ has neither a maximum nor a minimum at x = 0, yet its second derivative is also 0 there. Hence, if one or more of the relations in (4-4) or (4-5) hold as *equalities,* the observer is unable at that juncture to determine the shape of the function at that point. The general rule, which will not be proved here, is if d^2y/dt^2 = 0 for some *x(t),* one must calculate the higher-order derivatives d^3y/dt^3, d^4y/dt^4, et cetera. Then if the first occurrence of d^ny/dt^n < 0 for all curves *x(t)* is an *even* order *n,* then the function has a maximum (minimum, if > 0), whereas if that first occurrence happens for an odd number *n,* neither a maximum nor a minimum is achieved. To make matters worse, however, there are functions, for example, $y = e{\sim}^{l/x}$, which have a minimum, say, at some point (here, x = 0), and yet the derivatives of all finite orders are 0 at that point (for this function, at x = 0). We shall ignore all such "nonregular" situations in which the ordinary sufficient conditions for an extremum do not hold; we will confine our attention only to "regular" extrema.

It can be shown that the second-order conditions (4-4) are sufficient for a function to be concave (downward) at points other than a stationary value. Likewise, (4-5) guarantees that the function is convex (i.e., concave upward) at any point. Proof of these propositions will be deferred to the appendix.

Example 1. Suppose $f(x_1,x_2)$ has a maximum at some point. Then the sufficient second-order conditions are, again,

$$fnh] + 2f_{12}h_xh_2 + f_{22}h^2_2 <0 \qquad (4\text{-}2)$$

for all nontrivial values of $h\backslash$ and h_2. Since this holds for *all* values of h_1 and h_2, suppose we let $h\backslash$ — 1, $h_2 = \pm1$. Then this condition implies

$$/,,+/22\pm2/_{12}<0$$

or

Since /[1 and f_{22} are both negative,

$$1/,,+/_{22}|>2|/_{12}| \qquad (4\text{-}6)$$

is implied by the sufficient second-order conditions for a maximum.

Example 2. Suppose $f\{x\backslash, x_2)$ is strictly concave at some point. The sufficient condition for concavity is again Eq. (4-2),

$$fnh\backslash + 2f_nh_xh_2 + f_{22}h^2_2 <0 \qquad (4\text{-}2)$$

Now let $h\backslash$ —f_2, h_2 = —$f\backslash$. Then (4-2) implies

$$/n/_2{}^2-2/_{12}/_1/_2 + /_{22}/_1{}^2<0 \qquad (4\text{-}7)$$

This was the condition developed in Chap. 3 [Eq. (3-24)] for the level curves to be convex to the origin. Hence concavity implies level curves having this property. The converse, however, is false.

Example 3 (Monopolistic price discrimination). In a practice called price discrimination, monopolists are sometimes able to charge different consumers different prices for the same service. When government regulation gave certain airlines near-monopoly rights over certain routes, the airline industry discovered that some of its customers were businesspersons, eager to make some meeting in a distant city for a day or two, while other of its customers were more likely families planning vacations far in advance of their trips. The business travelers' demands were likely less elastic than those of the families, and the airlines sought to exploit that difference. The airlines found that self-selection would occur by giving discounts for tickets that required a Saturday stayover, something businesspersons rarely did. (These practices persist in the absence of route regulation, apparently due to the scarcity of gates at airports, which effectively restricts entry in many markets.)

A discriminating monopolist faces two market demands, $p\backslash = p_x(x_x)$ and $p_2 = p_2\{x_2)$. Its objective is to

maximize

$$it = R_x(x_x) + R_2(x_2) - C(x_x + x_2)$$

where $R\backslash\{x\backslash)$ and /?$_2$(x$_2$) are the total revenue functions $p_1(x\backslash)x_1$ and $p_2(x_2)x_2$, respectively. The derivatives of the total revenue functions are the marginal revenues MRI(JCI) and MR$_2$(x$_2$), respectively.

Note also that total cost C is really a function of one variable, $x = X\backslash + x_2$, and that $C\backslash = C_2 = C'(x)$ = MC(x). The necessary first-order conditions are

$$\text{TTI} = \text{MR,(jd)} - \text{MC}(x) = 0 \qquad \text{(4-8a)}$$

$$\text{TT}_2 = \text{MR}_2(x_2) - \text{MC}(x) = 0 \qquad \text{(4-86)}$$

These conditions imply that maximum profits occur when the firm sets the marginal revenues in each market equal to each other and equal to marginal cost:

$$\text{MR,(x!)} = \text{MR}_2(\text{JC}_2) = \text{MC}(x)$$

The intuition should be clear. If the marginal revenue is $100 in market 1 and only $50 in market 2, the firm would increase profits by shifting sales from market 2 to market 1. The common value of marginal revenue must then equal marginal cost; if MR > MC, the firm could increase profits by increasing output, and so forth. The sufficient second-order conditions are

$$Tt_u = \text{MR',(J:,)} - \text{MC}'(x) < 0 \qquad \textit{(4-9a)}$$

$$jx_{22} = \text{MR;}(x_2) - \text{MC}'(x) < 0 \qquad \text{(4-9/?)}$$

and

$$\text{*117T}_{22} - \text{7TJ2} > 0 \qquad \text{(4-9C)}$$

Relations *(4-9a)* and *(4-9b)* state that the MC curve must cut each MR curve from below, else, in each market, the firm would increase profits by increasing output beyond where MR = MC. We leave it as an exercise to show that (4-9c) means that the MC curve must cut the lateral sum of the MR curves from below.

Consider now the first-order conditions (4-8). Marginal revenue is related to the elasticity of the demand curve by the formula MR = $p(\backslash + \backslash/e)$ (see Sec. 2.1). Using this formula in each market, $p_{;}(l + l/€\backslash) = p_2(l + \text{l}/e_2)$, or

The demands in both markets must be elastic (e < —1) (why?). It is apparent that if, say, the elasticity in market 1 (families) is —4 while the elasticity in market 2 (the business travelers) is —2, the price in market 1 will be lower for the families (with these numbers, $p_{\ell} = (2/3)p_2$. Not surprisingly, the monopolist charges a lower price in the market with the higher elasticity.

PROBLEMS

1. For each of the following functions, find the stationary point and determine whether that point is a relative maximum, minimum, or saddle point of $f(x\backslash, x_2)$.

   1.156 $f(x_1, x_2) = x^* - 4x_l x_2 + 2xj$

   1.157 $f(x\backslash, x_2) = — 4*! - 6x_2 + x\backslash — x_x x_2 + 2x\backslash$

1.158 $f(x\, x_2) = \2x\ — 4JC_2 — 2x\ + 2x_xx_2 — x\$

2. Using Eq. (4-2), show that the sufficient conditions for $f\{x\, x_2)$ to achieve a minimum at x° are the relations (4-5).

1.159 Consider the production function y — L^aK^. Show that this function is strictly concave
(downward) for all values of L and ATif $0 < a < 1$, $0 <$ /J < 1 and if a + / $3 < 1$. What
shape does the function have for a + /3 = 1 ?

1.160 Show that the production function $y = \log L''K$^ is concave for all
a, *fi* > 0.

1.161 Let $y = f(x_u x_2)$ and let $z = F(y) = F(f(x_1,x_2)) = g(x_u x_2)$. Show that if $F' > 0$,
then g has a stationary point at *(x®, x%)* when and only when/is stationary there. Under
what conditions will/ have a maximum when and only when g has a maximum?

4.3 AN EXTENDED FOOTNOTE

In the previous section, sufficient conditions for the maximization of a function of two variables were derived via an artifice that reduced the problem to one dimension, or one variable. It is true that if a function has a maximum at some point, then all curves lying in the surface depicted by that function and passing through the maximum point must themselves have a maximum at that point. In that case, therefore, $y''(t) < 0$ for all such curves. Various plausible-sounding converses of this proposition, however, are *not,* in general, true. For example, suppose the function *f{x\, x₂)* possesses a maximum when evaluated along all possible polynomial curves, for any values of the coefficients a\, ... , a_n, b\, ..., b_n, for any finite n:

$$a_x t + a_2 t^2 \text{ H-----h } a_n t^n$$

$$x_2 =$$

Even if (JCI (t), $x_2(t)$) has a maximum at $t = 0$ when evaluated along this wide range of curves, the function $f(x\backslash, x_2)$ itself need *not* have a maximum at *x®, x®.*

To illustrate this phenomenon, suppose the curves $(xi\ (t), x_2(t))$ are limited to *straight lines* passing through (*°, **JC°).** That is, consider the curves in the surface $y = f(x\backslash, x_2)$ formed by the intersection of that surface and vertical (perpendicular to the JC1X2 plane) planes. Then it is *not* the case that if all those curves have a maximum, then the function itself has a maximum, as the following counterexample, developed by the mathematician Peano, shows: Consider the function

$$y=(x_2-x^2)(x_2-2x^2)$$

depicted graphically in Fig. 4-1. This function has the value 0 along the curves $x_2 = x^2$, and along $x_2 = 2x^2$, both of which are parabolas in the $X\backslash x_2$ space. In particular, $y = 0$ at the origin. The pluses and minuses shown in the diagram reflect the value of the function in the given section of the $X\backslash X_2$ space. For any point below the lower parabola, $x_2 < x^2$, and hence $x_2 < 2x^2$ (the point is also below the upper parabola). Hence y is the product of two negative numbers and is thus positive. Likewise, above the upper parabola, $x_2 > 2x^2$; hence $x_2 > x\backslash$ and therefore $y = (+)(+) > 0$. In between the two parabolas, $x_2 >$ jc^butx₂ $< 2x^2$, hence $y = (+)(—) < 0$. Note how any neighborhood containing the origin possesses both positive and negative values of *y*. Therefore, the function cannot attain either a maximum or minimum at the origin. That is, since some values are greater than 0 and some less than 0 around the

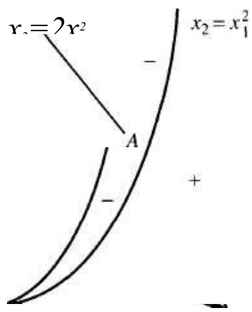


FIGURE 4-1
The Function y — *(x₂* — 2x^)(x2 — *x*). This function exhibits the interesting property that when evaluated along all straight lines through the origin, the function has a minimum (of 0). However, the function itself clearly does *not* have either a minimum or a maximum at the origin since in any neighborhood of the origin, this function takes on both positive and negative values.

origin, neither a maximum nor minimum can be achieved there. Rather, something analogous to a saddle point occurs. However, consider the function evaluated along any straight line through the origin, e.g., line *AA* in Fig. 4-1. After passing through the upper parabola, the function, along this line, changes from positive to 0 (at the origin) to positive again, implying that the origin is a minimum value of *y, evaluated along this* or any such line. However, the function itself, as we just have shown, does *not* have a minimum at the origin. Thus it is *not* the case that if a function attains a maximum (or minimum) evaluated along all straight lines going through some point that the function necessarily attains a maximum (minimum) there. It is possible to construct functions such that even if *y (t)* has a maximum for all polynomial curves in the $X\backslash X_2$ plane, the function itself does not have a maximum^ Exactly what class of functions *x(t)* for which a valid converse is obtainable seems to be unresolved.

4.4 AN APPLICATION OF MAXIMIZING BEHAVIOR: THE PROFIT-MAXIMIZING FIRM

The tools developed in the previous sections will now be applied to analyze the comparative statics of a profit-maximizing firm that sells its output *y* at constant unit price *p* and purchases two inputs *xi* and *X2* at constant unit factor prices *w* and w_2, respectively. That is, the firm in question is the textbook prototype, facing competitive input and output markets. The production process of the firm will be summarized by the *production function, y — f(xi,x₂)*- The production function

. Hancock, *Theory of Maxima and Minima,* Dover Publications, Inc., New York, 1960.

will be interpreted here as a technological statement of the maximum output that can be obtained through the combining of two inputs, or factors, x_1 and x_2. The objective function of this firm is total revenue minus total cost *(profits)*. We assert that the firm maximizes this function, i.e.,

maximize

$$\pi = pf(x_1, x_2) - w_1x_1 - w_2x_2 \tag{4-11}$$

The test conditions of this model are the particular values of the input prices w_1, w_2, and output price p. The objective of the model is to be able to state refutable propositions concerning observable behavior, e.g., changes in the levels of inputs used, as the test conditions change, i.e., as factor or output prices change. The first-order conditions for profit maximization are

$$\pi_1 = \frac{\partial \pi}{\partial x_1} = pf_1 - w_1 = 0 \tag{4-12a}$$

and

$$\pi_2 = \frac{\partial \pi}{\partial x_2} = pf_2 - w_2 = 0 \tag{4-12b}$$

Sufficient conditions for a maximum position are

$$\pi_{11} < 0 \quad \pi_{22} < 0 \quad \text{and} \quad \pi_{11}\pi_{22} - \pi_{12}^2 > 0 \tag{4-13}$$

Since $\pi_{ij} = pf_{ij}$, these second-order conditions reduce to

$$f_{11} < 0 \quad f_{22} < 0 \tag{4-14}$$

and

$$f_{11}f_{22} - f_{12}^2 > 0 \tag{4-15}$$

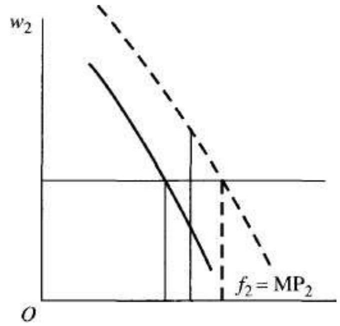
What is the economic interpretation of these conditions? Equations (4-12) say that a profit-maximizing firm will employ resources up to the point where the marginal contribution of each factor to producing revenues, pf_i, the value of the marginal product of factor i, is equal to the cost of acquiring additional units of that factor, w_i. These are *necessarily* implied by profit maximization; however, to ensure that the resulting factor employment pertains to *maximum* rather than minimum profits, conditions (4-14) and (4-15) are needed. Conditions (4-14) are statements of the

student should be wary of the terms *firm* and *profits*. With regard to the former, the concept has not been defined here, and there is, in fact, considerable debate in the profession as to exactly what firms are, why they exist at all, and what their boundaries are. With regard to profits, the model leaves unspecified who has claims to the supposed excess of revenues over cost. Alternatively, if $x\backslash$ and $X2$ are indeed the only two factors, in whose interest is it to maximize the expression in Eq. (4-11)? In spite of these shortcomings, since the model does yield refutable hypotheses, as we shall see shortly, it is potentially interesting. It might be referred to as a "black box" theory of the firm.

law of diminishing returns. That such a law is involved is easily seen. [Remember, though, conditions (4-13) are sufficient, not necessary—a maximum position is consistent with these relations holding as equalities.] Assuming it was worthwhile to hire one unit of that factor in the first place, if the value of the marginal product of that factor was increasing, the firm would hire that factor without bound, since the input would be generating more income than it was getting paid. Hence a finite maximum position is inconsistent with increasing marginal productivity.

However, diminishing marginal productivity in each factor does *not,* by itself, guarantee that a maximum profit position will be achieved. Condition (4-15) is also required.

This relation, though less intuitive than diminishing marginal productivity, arises from the fact that changes in one factor affect the marginal products of the other factors as well as its own marginal product, and the overall effect on all marginal products must be akin to diminishing marginal productivity. Suppose, for example, that/12 = /21 = *dM?i/dx*$_2$ = 3MP$_2$/3xi is very large, in absolute terms, relative to /n = 3MP]/3JCI and /22 = 3MP$_2$/3x2- That is, suppose a change in *x\,* say, affects the marginal product of factor 2 much more than the marginal product of factor 1. Then consider the consequences of an increase of one unit of *X{*. In Fig. 4-2, if *fn* = /21 > 0, MPi initially declines; however MP$_2$ shifts upward by a considerable amount, causing the firm to purchase many additional units of x_2. However, these additional units of x_2 have an effect on MPi. Since f_{12} = 3MPi/3x$_2$ > 0, MPi also shifts up, by a relatively large amount. The final result, then, is that an increase in JCI can lead to an *increase* in MPi, if the cross-effects are large enough. Hence the original factor employment levels, though characterized by diminishing marginal productivity in each ar. An increase



factor, do not nonetheless describe a profit maximum position, since it is clearly profitable in this case to increase the usage of both x_1 and x_2 together. In the case where f_{12} is negative and large relative to f_{11} and f_{22}, the analysis is simil

in x_1 causes a relatively large fall in MP_2, a fall

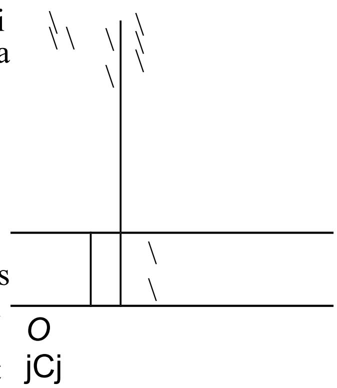


FIGURE 4-2
The Law of Diminishing Returns. The fact that $f_{11} < 0$, $f_{22} < 0$ alone is *not* sufficient to guarantee a finite profit-maximum position. The cross-effects between the two factors must be considered. If x_1 is increased, MP2 might shift out, say, a great deal, shifting MP! out resulting in a net *increase* in MPi even though $f_{11} = \partial MP_1 / \partial x_1 < 0$. This will occur if f_{12} is large. Hence the condition $f_{11} f_{22} - f_{12}^2$ - 0 is also needed to achieve a finite profit maximum.

in x_2 and hence a relatively large *increase* in MP^ (Remember, $3MP_1/3JC_2 <$

0!) In this case, increasing one factor and decreasing the other (together) will increase profits.

Let us return now to the marginal relations (4-12). The purpose of formulating this model is not simply to assert the implied marginal reasoning; that is a rather sterile endeavor. The purpose of this analysis is to be able to formulate refutable hypotheses as to how firms react to changes in the parameters they face; in particular, in this case, to changes in factor and output prices. To this end, we now consi der the comparative statics of this model.

and

x

The first-order conditions in complete form are

$Pfi(xi,x$

$pfi(.x\,x$

These are two implicit relations in essentially five unknowns: $X\,x_2,w\,$ w_2, *and* p. Under the "right" conditions (to be discussed in what follows) it is possible to solve for two of these values in terms of the other three. In particular, we can solve for the choice functions

(

Equations (4-16) represent the factor demand curves. These relations indicate the amount of each factor that will be hired, according to this model, as a function of the factor prices and product price; they are the choice functions of this model. Assuming that it is possible to solve for Eqs. (4-16), it becomes meaningful to ask questions regarding the signs of the following six partial derivatives,

which comprise the comparative statics of the profit-maximizing model:

-1 /) *dx** *dx*

3wi

dw_2

dp

$dw\backslash$ dw_2 $p))$ - wi =0

dp and

These partials indicate the marginal changes in factor employment due to given price changes. It is important to keep in mind that in order to write down these relations and interpret them in some meaningful fashion, the explicit functions x^* must be well defined. Also note that the preceding factor demand curves are not the marginal product curves. The marginal product functions $f\backslash$ and f_2 are expressed in terms of the factor inputs, while the factor demand curves are expressed in terms of prices, and dependent upon the behavioral assertion of the model.

Substituting Eqs. (4-16) back into Eqs. (4-12) produces the following *identities:*

Recall the monopolist tax example of Chap. 1, where the solution $x = x^*(t)$ of the first-order relation (which set marginal revenue equal to marginal cost plus the tax) was then substituted back into that relation, yielding an *identity* in the tax rate t. For the same reasons, the relations (4-18) are identities in the prices *wi,* w_2, and p. The factor demand functions xj* and x| are precisely those levels of x_1 and x_2 that the entrepreneur employs to keep the value of the marginal products of each factor equal to the wage of each factor, for any prices.

Hence, the assertion that the firm *always* obeys Eqs. (4-12), for any prices, converts those equations to the identities (4-18). Being identities, the relations (4-18) can be differentiated implicitly with respect to the various prices, producing relations that allow solutions for the partial derivatives (4-17). The general procedure is exactly the same as in the monopolist example. However, in this example, two first-order relations are present instead of only one, and that fact makes the algebra more difficult.

Before we do the differentiation, note that if the firm's production functions were in fact known, then one could actually solve for the factor demand curves explicitly. In that case we could know the *total* quantities involved in this model, a happy state of affairs. The factor demand curves (4-16) could be differentiated directly to yield the partial derivatives (4-17). However, the economist is not likely to have this much information. Nonetheless, it is still possible to state refutable hypotheses concerning *marginal* quantities, through implicit differentiation of the identities (4-18).

Differentiating (4-1*8a)* and (4-18fr) partially, with respect to *w* \, using the chain rule (remembering that *f*\ is a function of *x*\ and x_2, which are in turn functions of wi, w>2, andp, etc.),

$$p\,\frac{9/2}{dx\backslash}\frac{dx^*}{dw\backslash} + p\,\frac{df_2}{dx_2}\frac{dx^*}{dw\backslash}$$

Using subscript notation, these can be written

(4-19fl)

Phi 1^- + P/22^- = 0 (4-1%)

aw 1 *ow*

Although the identities (4-19) look complicated, they are a good deal simpler in form than (4-18). Whereas the first-order relations (4-18) are in general complicated algebraic expressions, *(4-\9a)* and (4-.19Z?) are simple *linear* relations in the unknowns *dx*/dw* and *dxydwi.* That is, *(4-\9a)* and (4-19Z?) are of the same form as the elementary system of two simultaneous linear equations in two unknowns. The coefficients of the unknowns are the functions *pfu, pfn,* etc., but the system is still simple in that no products, or squares, of the terms *dx*/dw\,* etc., are involved. And this is fortunate, since the goal of this analysis is to solve for those terms, i.e., find expressions for the partials of the form *dx*/d\Vj.*

To solve for *dx*/dw\,* for example, multiply (4-19a) by $/_{22}$ and (4-1%) by /12 and subtract (4-19Z?) from (4-19a). This yields, after some factoring (remember

22

IZ/ dw_x ^{J}Now, if *f* i $/_{22}$ — $/^2{}_2$

^ 0, that term can be divided on both sides, yielding

(4-20*) *dxj_* = *f22.*

dwi p(fnf22-

fn) In like fashion, one obtains

(4-206) 9*$_2$* -/21

= $^{9\,w}$ i

p(f\\f2 2 - fh)

To obtain the responses of the firm to changes in w_2, differentiate Eqs. (4-18) with respect to w_2. Noting that w_2 enters only the second equation explicitly, the system of comparative statics relations becomes

dxf 9x?

0W2 0W2

dxf *dxX*

A/21---r P/22- = A

9^2 $9w_2$

Solving these equations as before yields

— = -7—————TT (4-20c)

$9w_2$ *P\f\\ f22 - fn)*

-^ = —, - -——---TT (4-20^)

dw_2 *P\f\\ f22 - f\2)*

Note that sufficient condition (4-15), *f* \ $/_{22}$ — $/,^2{}_2$ > 0, is enough to guarantee /11/22 — $/i^2{}_2$ 7^ 0 and hence allow solution for these partials *(4-20a-d).* This is not mere coincidence; it is in fact an application of the "implicit function theorem" in mathematics that will be dealt with more generally in Chap. 5. The condition /11 /22 — $/i^2{}_2$ ^ 0 is precisely the mathematical condition to allow solution *(locally,* not everywhere) for the factor demand curves *x*(w\,* w_2*, p)* in the first place. The relevance of that term is brought out in the situation for the partial derivatives.

tin accordance with general custom, we will use the equality rather than the identity sign when the special emphasis is not required.

What refutable hypotheses emerge from this analysis? Condition (4-15) implies that the denominators of *(4-20a-d)* are all positive. Condition (4-14), f_{n}, $f_{22} < 0$, (diminishing marginal productivity) makes the numerators of (4-20a) and (4-20J) negative. Hence, *the regular (sufficient) conditions for maximum profits imply that the factor demand curves must be downward-sloping in their respective factor prices.* The model implies that changes in a factor price will result in a change in the usage of that factor in the opposite direction.

What about the cross-effects *dx*/dw₂, dx^/dw^*. The most remarkable aspect of these two expressions is that they are always equal, by inspection of (4-20Z?) and (4-20c), noting that $f\backslash_{2} = f_{2}\backslash$. This *reciprocity relation,*

$$dx^*$$
$$dxl\ dw_2$$
$$dw\backslash$$

is representative of a number of such relations that appear in economics, as well as in the physical sciences, when maximizing principles are involved. As is obvious from the forms of these expressions, however, the reciprocity relations are no less intuitive than the mathematical theorem from which they originate—the invariance of cross-partial derivations to the order of differentiation.

Beyond the equality of these cross-effects, there is little else to say about them. The sign of $f\backslash_{2}$ is not implied by the maximization hypothesis; hence the sign of *dx*/dwj, i =fc j* is similarly not implied. No refutable proposition emerges about these terms from the profit maximization model. All observed events relating, say, to the change in labor employment when the rental rate on capital increases are consistent with the previous model.

Suppose now it is desired to find expressions relating to the effects of changes in the output price *p.* The procedure here is identical up through relations (4-18). Then, we differentiate those identities partially with respect to *p,* producing

$$-f\backslash \qquad (4\text{-}21a)$$
$$dp \qquad dp$$

$$Ph\backslash\ \wedge + Pfi2\sim\wedge = -h \qquad (4\text{-}21\&)$$
$$3/7 \qquad dp$$

remembering that the product rule is called for in differentiating the terms $pf\backslash$, pf_2. Solving these equations for *dx*/dp* and *dx^/dp* yields

(4-22a)

$$\circ_{x}\backslash \qquad —JIJ22 + J2J12$$
$$dp \qquad P\{fnfi2-$$
$$fn)\ dx^*$$

$$-/2/11+/1/12$$
$$dp \qquad P\{fnfi2-f\backslash i)$$

It can be seen that no refutable implications emerge from these expressions. An increase in output price can lead to an increase or a decrease in the use of either factor, since the sign of/i$_2$ is unknown. (Note that if/12 > 0 is assumed, *dx*/dp* > 0 and *dx$_2$/dp* > 0.) It is possible to show, however, that it cannot be the case that *both*

$dx^*/dp < 0$ and $dx\^/dp < 0$ simultaneously. An increase in output price cannot lead to less use of both factors. The proof of this is left as an exercise.

The Supply Function

It is also possible to ask how *output* varies when a parameter changes. Since _y =

$f\{x\backslash,x_2),$

$$y^* = f(x^*,x^*_2)$$

where v* is the profit-maximizing level of output.

The factor demand curves are functions of the prices,

$$Xj = x^*(w\backslash, w_2, p) \quad i =$$

1, 2 Substituting these functions into $f(x^*, x\%)$

yields

$$y^* =/(^*r(wi,w_2,/?),^*{}_2^*(wi,W2,p)) = y^*(wi,w_2,p) \qquad (4\text{-}23)$$

Equation (4-23) represents the supply function of this firm. It shows how output is related (1) to output price *p,* and (2) to the factor prices. Though the supply curve is commonly drawn only against output price *p,* factor prices must also enter the function, since factor costs obviously affect the level of output a firm will choose to produce.

How will output be affected by an increase in output price? To answer this, differentiate (4-23) with respect *top* using the chain rule,

dp **dx\ dp** **dx_2**

dp or

dp **dp** **dp** (4.24)

Now, substitute Eqs. (4-22) into this expression.
This yields

$$\frac{By^*}{dp} = \frac{-/i^2/_{22} + 2/_{12}/i/_2 \sim /_2{}^2/n}{p\{fnfi2 — fh)} \qquad (4\text{-}25)$$

The denominator of this expression is positive by the sufficient second-order conditions. We also can infer, from Eq. (4-7), that the numerator is also positive. Therefore,

$$\frac{dy^*}{dp} \; \text{-f-} > 0 \qquad (4\text{-}26)$$

This says that the sufficient second-order conditions for profit maximization imply that the supply curve, as usually drawn, must be upward-sloping. It also provides an explanation as to why it cannot be the case that both dx^*/dp and dxj/dp are negative. If *p* increases, output will increase. It is impossible, with positive marginal products, to produce more output with less of both factors.

It is also possible to derive some *reciprocity* relationships with regard to the output supply and factor demand functions. In particular, one can show

$$\frac{\partial y^*}{\partial w_i} = -\frac{\partial x_i^*}{\partial p} \qquad i = 1, 2 \tag{4-27}$$

The signs of these expressions are indeterminate; however, this curious reciprocity result is valid. Its proof is left as an exercise.

The tools used in this analysis include the solution of simultaneous linear equations. For this reason, the next chapter is on the theory of matrices and determinants. It will be of great advantage to be able to have a general way of expressing the solutions of such equation systems, instead of laboriously working through each expression separately.

4.5 HOMOGENEITY OF THE DEMAND AND SUPPLY FUNCTIONS; ELASTICITIES

Suppose the economy were to experience a perfectly neutral inflation, i.e., input and output prices all increasing in the same proportion, say 10 percent. Since relative prices would not have changed, it would be important that the model predict that no decisions would be changed in response to this. In other words, the factor demand functions and the supply function should be homogeneous of degree 0 in all prices. Is this the case?

The factor demand functions are the simultaneous solutions to the first-order conditions

$$pf_i(x_1, x_2) - w_i = 0$$

Suppose w_1, w_2, and p all change in the same proportion, i.e., these prices become tw_1, tw_2, and tp, where t is some scalar factor. The factor demand functions are now evaluated at these new prices: $x_1^*(tw_1, tw_2, tp)$, $x_2^*(tw_1, tw_2, tp)$. These functions are the solutions to the first-order equations at the new prices:

$$(tp)f_1(x_1, x_2) - (tw_1) = 0$$
$$(tp)f_2(x_1, x_2) - (tw_2) = 0$$

But these equations are clearly equivalent to the original ones; all that has happened algebraically is that the equations have been multiplied through by t. Since the equations from which the two solutions are derived are algebraically identical, the solutions must also be identical. That is,

$$x_j^*(tw_1, tw_2, tp) = x_j^*(w_1, w_2, p) \qquad j = 1, 2$$

In this model, therefore, the factor-demand functions are necessarily homogeneous of degree 0. [It quickly follows that the supply function $y^*(w_1, w_2, p)$ must also be homogeneous of degree 0; its proof is left as an exercise.]

Notice that the preceding proof in no way depends on any assumption about the functional form of the production function. In particular, to head off a frequently made error, it is not the case that the production function must be homogeneous of some degree. The demand functions are not the partial derivatives of the production function. They are the simultaneous solutions to the first-order equations. The result follows because those first-order equations are linear in *W[, w_2, and p.* When each of those parameters is increased in the same proportion, the factor of proportionality cancels out of the first-order equations, leaving the system unchanged.

Elasticities

The properties of the factor demand functions *x*(w* 1, w_2, /?) are often stated in terms of dimensionless elasticity expressions instead of the slopes (partial derivatives). These elasticities are defined as

^ * - (4-28)

The elasticity e,-y represents the (limit of the) percentage change in the use of factor *X{* per percentage change in price of factory. When / = *j,* this is called the *own elasticity* of factor demand; when *i =fc j,* it is called a *cross-elasticity.* Taking limits and simplifying the compound fraction,

« , = ^ (4-29)

This is the definition we shall use throughout. In like fashion, one can define the *output price elasticity* of factor demand as the percentage change in the utilization of a factor per percentage change in output price (holding factor prices constant), or

$$\epsilon_{ip} = \lim_{Ap \to 0} \frac{Axi/x;}{Ap/p} = \frac{pdx^*}{xfdp} \quad (4\text{-}30)$$

Elasticities are dimensionless expressions, as can be seen by inspection: the units all cancel. To a mathematician, they are logarithmic derivatives. For example, letting u_t = log*;, *Vj* = log w_{jf}

du_t *dxi/xi* *Wjdxi*

The notation changes appropriately for partial derivatives. Many economists prefer to deal with elasticities; others prefer the slopes (unadorned partial derivatives). It is mainly a matter of taste.

By applying Euler's theorem to the factor demand functions (JCI in the example that follows), we can derive some relationships concerning the elasticities and cross-elasticities of demand:

axi \ I ox[\ fox, .

Dividing through by *x* yields

with a similar expression holding for x_2. In general, for models with *n* factors of production,

y_{ϵ} $+_{\epsilon}$ $=_0$ i = 1, . . . , n (4-31)

4.6 THE LONG RUN AND THE SHORT RUN: AN EXAMPLE OF THE LE CHATELIER PRINCIPLE

It is commonplace to assert that certain factors of production are "fixed" over certain time intervals, e.g., that capital inputs cannot be varied over the short run. In fact, of course, these statements are incorrect; virtually anything can be changed, even quickly, if the benefits of doing so are great enough. Yet it does seem that certain inputs are more easily varied, i.e., less costly to vary than others. The extreme abstraction of this is to simply assert that for all intents and purposes, one factor is fixed. (A government edict fixing some level of input would suffice, if ignoring such edict carried with it a sufficiently long jail sentence.) How would a profit-maximizing firm react to changes in the wage of one factor *x* when it found that it could not vary the level of x_2 employed? Would the factor demand curve for *X* be more elastic or less elastic than previously?

Suppose x_2 is held fixed at x_2 = *x®*. The profit function then becomes

max7r = *pf(x\, x®)* — *w\Xi* — w_2*2-

In this case, there is only one decision variable: *x*. Hence the first-order condition for maximization is simply

ni=pfx{$x_u$xl)-w_x=0 (4-32)

and the sufficient second-order condition is

*ii = p/ii< 0 (4-33)

We are dealing with a one-variable problem with, now, *four* parameters, *w*, w_2, *p*, and *x®*■ The factor demand curve, obtained from Eq. (4-32), is

x^xKwuPtx") (4-34)

where *x* stands for short-run demand. Note, however, that w_2 does not enter this factor demand curve. With x_2 fixed, w_2*x®* is a fixed cost, and thus w_2 is irrelevant for the choice of $x\backslash$ in the short run. The slope of the short-run factor demand curve is $dx\backslash/dw\backslash$. To obtain an expression for this partial, substitute, as before, $x\backslash$ into Eq. (4-32), yielding the identity

Differenti ; iden with W yie

$$\frac{f \quad {}^{dx}l}{\ } \quad 1$$

or

$$\frac{dx\backslash}{dw_l} \; \frac{\ }{pf_u} \quad \text{- -} \qquad (4\text{-}35)$$

Thus, the short-run factor demand curve is downward-sloping. How does this slope compare with $dx*/dw\backslash = dx^\wedge/dwi$ (xf for long-run demand) derived in (4-20a)? Taking the difference,

$$\frac{dx[}{dwi} \quad \frac{dx\backslash}{dwi} \quad \frac{f_{22}}{p(fnf22\sim fn)} \quad 1$$

Pfu Combining terms yields

$$\frac{r)x^L}{9wi} \quad \frac{i)x^s}{9wi} \quad \frac{f^2}{pfu\backslash fuf2\,2 - fn)}$$

a determinately *negative* expression due to the second-order conditions (4-15) and (4-33). Since both $dx^\wedge/dwi$ and $dx\backslash/dw\backslash$ are negative, (4-36) says that the change in JCI due to a change in its price is larger, in absolute value, when x_2 is variable (the long run) than when x_2 is fixed (the short run). This result is sometimes referred to as the *second law of demand.* It is in agreement with intuition—if the price of labor, say, were to increase relative to capital's price, the firm would attempt to substitute out of labor. The degree to which it could do this, however, would be impaired if it could not at the same time increase the amount of capital employed. Hence the model implies that over longer periods of time, as the other factor becomes "unstuck," the demand for the less-costly-to-change factor will become more elastic. Incidentally, the usual factor demand diagrams are drawn with the dependent variable $x\backslash$ on the *horizontal* axis; in that case the long-run factor demand curves appear flatter than the short-run curves. Also note that this comparison makes sense only if the level of x_2 employed is the same in both cases. That is, the preceding is a local theorem, holding only at the point where the short- and long-run demand curves intersect, i.e., at the common values of x_2. At any finite distance from this intersection, the long-run demand curve might actually be less elastic than the short-run curve.

The result contained in this section is commonly believed to be empirically true, simply as a matter of assertion. It is interesting and noteworthy that this type of behavior is in fact mathematically implied by a maximization hypothesis. These types of relations are sometimes referred to as Le Chatelier effects, after the similar tendency of thermodynamic systems to exhibit the same types of responses. Some generalizations of this phenomenon and its relation to "envelope" theorems will be presented in Chap. 7.

A **More Fundamental Look at the Le Chatelier Principle**

Although the above algebra proves that when the level of one factor, say, x_2, is held fixed at its profit-maximizing level, the resulting short-run factor demand curve is less elastic than the long-run curve at that point, the proof provides no insight into the fundamental relationship between the long- and short-run factor demands. *If a consistent relationship exists between the partial derivatives of two separate demand functions, it must be the case that some fundamental identity exists that relates the two demands to each other.*

In the instant case, consider what would convert the short-run demand to the long-run demand. We would accomplish this by letting x_2 adjust to the change in *w* i instead of holding it fixed. In fact, we can *define* the long-run factor demand in terms of the short-run demand by letting x_2 (the "fixed" factor) adjust to its profit-maximizing levels as *w* changes:

$$x*(wi,w_2,p) = x\backslash(w_u\, p,X2(wi,w_2,\, p)) \qquad (4\text{-}37)$$

This identity is the fundamental relationship between the short- and long-run factor demands. Using this identity, we can demonstrate and explain the Le Chatelier results with much greater clarity. The right-hand side of Eq. (4-37) is known as a *conditional demand.*^

The relation (4-37) is an identity; it holds for all *w* \, w_2, and/7. We can therefore validly differentiate it with respect to any of those arguments. In particular, differentiate with respect to *w\,* noting that on the right-hand side of (4-37), *w* enters once explicitly by itself, and another time as an argument of *x%* '■

1 " ' + (^Jr 1 (*P*-) (4-38)
dw *d\V* *\dx₂)* \

Inspect the notation in the chain rule part of the right-hand side of (4-38) carefully: *x* is a function of *x%* (not JC|); the functional dependence of x_2 on w_2 is defined by the long-run demand *x%.*

Equation (4-38) reveals that the slopes of the short- and long-run factor demand functions differ by a term representing the product of two effects: the change in x_2 resulting from a change in vvi, and the change in *xi* that would be induced by a (parametric) change in x_2. This product is easily seen to represent the marginal

^This approach was first developed by Robert Pollak, for the case of consumer demands. See his "Conditional Demand Functions and Consumption Theory," *Quarterly Journal of Economics,* 83:60-78, February 1969.

effect on *x* of allowing x_2 to vary as wi changes. The important question is, can this latter term be signed?

It should seem plausible that *dx^/dx®* and *dx^/dw* have opposite signs. From reciprocity, *dx^/dwi* = *dx\/dw₂*. Increasing $x_2^{\prime}$ parametrically accomplishes directly what a *decrease* in w_2 would induce. We can verify this algebraically as follows. Differentiating (4-37) with respect to w_2

(4) ((4-39)

dw₂ *\dX2J \dw₂*

Since $dx2^*/dw_2 < 0$, *dx\/dw₂* and *dx\/dx* are of opposite sign. Using Eq. (4-39) to eliminate *dx\/dx* from Eq. (4-38), and using reciprocity,

9x1 s M + (94/3"i) (4-40)

dw *dw* *dx₂/dw₂*

Since the last term must be negative, Eq. (4-40) says that *dx*/dwi* is more negative than *dx\/dw\,* the Le Chatelier result. More importantly, it illuminates the fundamental relationship between the long- and short-run factor demand functions. A similar analysis can be used to show that the long-run output supply function is more elastic than the short-run function. The fundamental identity is

y*(H>i,H>₂, *P)* = /(Wl, *P,X2(W_U W₂, p))* (4-41)

Differentiating with respect to *p,*

ajL *(ajL)* (M) (4-42)

dp *dp* *\dx^J* V *dp*

By differentiating (4-41) with respect to w_2 and using a reciprocity condition, it can be shown that $dy^*/dp > dy^s/dp$. The proof is left as an exercise.

We shall employ this technique throughout this book. In so doing, many expressions that were once difficult to prove become transparently simple.

To sum up, it has again been possible to state refutable propositions about some *marginal* quantities, in spite of the scarcity of information contained in the model. Should further information be used, e.g., the specific functional form of the production function, or, less grandiosely, independent measures of the sign of the cross-effect f_{i2}, additional restrictions can be placed on the signs of the partial derivatives of the factor demand functions.

PROBLEMS

1.162 Show that no refutable implications emerge from the profit maximization model with
regard to the effects of changes in output price on factor inputs. Show, however, that it
cannot be the case that both factors decrease when output price is increased.

1.163 Show that the rate of change of output with respect to a factor price change is equal to the
negative of the rate of change of that factor with respect to output

price, i.e., Eq. (4-27).

1.164 (Very messy, but you should probably do this once in your life.) Consider the production function $y — x''^{}_1x''_2$. Find the factor demand curves and the comparative statics of a profit-maximizing firm with this production function. Be sure to review Prob. 3, Sec. 4.2, first. Show that for this firm, the sign of the cross-effect term, *dx^/dwi,* is negative.

1.165 There are several definitions of complementary and substitute factors in the literature, among which are:

(i) "Factor 1 is a substitute (complement) for factor 2 if the marginal product of factor 1 decreases (increases) as factor 2 is increased."

(ii) "Factor lisa substitute (complement) for factor 2 if the quantity of factor 1 employed increases when the price of factor 2 increases (decreases)."

1.166 Show that both of these definitions are *symmetric;* i.e., if factor 1 is a substitute for factor 2, then factor 2 can't be a complement to factor 1.

1.167 Show that these two definitions are equivalent in the two-factor, profit maximization model.

1.168 Do you think that these two definitions will be equivalent in a model with three or more factors? Why?

5. Consider again Example 3, Sec. 4.2, wherein a monopolist sells his or her output in two separate markets. Suppose a per-unit tax t is placed on output sold in the first market.

1.169 Show that an increase in t will reduce the output sold in market 1.

1.170 What does the maximization hypothesis *alone* imply about the response of output in the second market to an increase in *ft*

1.171 Show that it is possible that an increase in the tax on market 1 can lead to an *increase* in total output $x^*(t) = x^*(t) + x_2(t)$, even assuming the usual sufficient second-order conditions. Under what circumstances (slopes of the marginal cost and marginal revenue functions) does this occur? (This possibility is known as the *Hotelling taxation paradox,* after Harold Hotelling, an early pioneer of modern economics and statistics, who first explored it.)

1.172 Suppose the output in market 2 were held fixed at the previously profit-maximizing level, by government regulation. Show that the response in output in market 1 to a tax increase is less in absolute terms in the regulated situation than in the unregulated situation. Provide an intuitive explanation for this.

1.173 The Le Chatelier results of Sec. 4.4 (also Prob. 5) hold, regardless of whether the two factors are complementary or substitutes. Explain the phenomenon intuitively for the case of complementary factors.

1.174 A monopolist sells his or her output in two markets, with revenue functions $R\(yi)$, $Riiyi)$, respectively. Total cost is a function of total output, $y = y_x + y_2$. The same per-unit tax, t, is levied on output sold in *both* markets.

1.175 Find $dy*/dt$, dy_2/dt, and $dy*/dt$, where $y*$ is the profit-maximizing level of output in market i and $y* = y* + y^$. Which, if any, of these partials have a sign implied by profit maximization?

1.176 Suppose output v_2 is held fixed. Find $(dy*/dt)_{y2}$. Does $(dy*/dt)_{y2}$ have a determinate sign?

8. Consider the following two models of a discriminating monopolist subject to a tax in one market:

(i) max /?i(xi) + $R_2(x_2)$ - $C(x_x + x_2)$ - tx_x *(ii)* max $R(x_x, x_2)$ — $C(x\,x_2)$ — $tx\$

In model *(i)*, cost is a function only of total output, whereas in *(ii)*, cost and revenue are more complicated (and general) functions of both outputs. The tax rate / is a parameter. What are the observable similarities and differences between these two models?

9. Consider a profit-maximizing firm with the production function $y = f(x\, x_2)$, facing output price p and factor prices $w\$ and w_2. Suppose this firm is taxed according to the total cost of factor 2, i.e., tax = tw_2x_2.
   1.177 Derive the factor demand functions; i.e., show where they come from, etc. Are these choice functions homogeneous of any degree in any of the parameters?
   1.178 Show that if the tax rate rises, the firm will use less of factor 2.
   1.179 Show that $dx*/dt = w_2\, dx^/dw^$
   1.180 Suppose that factor 1 is held fixed at its profit-maximizing level. Show that the response of factor 2 to a change in the tax rate is less in absolute value than before.
10. Consider a monopolistic firm that hires two inputs x_x and x_2 in competitive factor markets at wages $w\$ and vv_2, respectively. The firm's revenue function is expressible in terms of the inputs as $R(x_u\, x_2)$. Assuming profit maximization,
   1.181 Indicate the derivation of the factor demand functions. Are these factor demands homogeneous of some degree in wages?
   1.182 Show that the factor demand curves are downward-sloping in their own prices.
   1.183 Is a refutable hypothesis forthcoming as to how the total revenue of this firm would change with regard to a change in a factor price?
11. Consider a profit-maximizing U.S. monopolistic firm that produces some good y at two different plants, with (total) cost functions $C\$ (yi), $C_2(y_2)$. The total revenue function of this firm is $R(y)$, where y = y, 4- y_2. Plant 2 is located in Canada, and output from that plant is subject to a U.S. tariff (tax) in the amount of t per unit produced.
   1.184 What is implied, if anything, about the slopes of the marginal revenue and marginal cost curves in this model?
   1.185 What refutable comparative statics implications are forthcoming, if any?
   1.186 Suppose this firm was not a monopolist, but rather, sold its total output in a compet itive market at price p. What differences would exist in the observable implications of the model in the competitive versus the monopolistic case?
   1.187 Suppose this competitive output price rose. Will the output in each plant increase?
   1.188 Returning now to the monopolistic case, suppose this monopolist decided to raise the price charged to consumers. What effect would this have on the output of each plant... hey, wait a minute ... does this make any sense?
   1.189 Suppose the total revenue received by this (monopolistic) firm depends in some complicated way on outputs in both plants, rather than simply on the sum of those two outputs. What observable differences, if any, are implied by this change in

assumptions?

1.190 Suppose that output at the U.S. plant (yi) is held fixed at the previously profit-maximizing level and the tax on Canadian output is increased. How does the resulting magnitude of the response in production at the Canadian plant compare with the response when U.S. output is unconstrained? (Again, assume the monopoly case.)

1.191 Prove, using Eq. (4-42), that the long-run supply curve of a competitive firm is more elastic than the supply curve in which one factor is held fixed at a previously profit-maximizing level.

1.192 Consider a profit-maximizing firm with production function y $= f(x_u\ x_2)$ that sells its output competitively at price *p*. The firm obtains input x_x at a competitively determined unit wage $w\backslash$, but the firm faces an upward-sloping supply function for x_2 given by w_2 = H>2 + kx_2, where w_u w°, *p,* and *k* are positive parameters.

(a) Derive the first- and (sufficient) second-order conditions and explain the derivation of the explicit choice functions implied in this model. Characterize each of these

choice functions as a demand function, a supply function, or neither, and explain. Is the "law of diminishing marginal product" implied for each factor?

1.193 Derive the comparative statics results available for the parameter w i. What refutable implications are forthcoming, if any?

1.194 How will the use of x_2 by this firm respond to an increase in w_2?

1.195 Are the explicit choice functions homogeneous of some degree in some or all of the parameters? Prove that they either are or are not. What relation, if any, does homogeneity of factor demand or other similarly derived functions have, when it appears, to the homogeneity of the production function?

1.196 Derive the comparative statics results for w° indicating which if any represent a refutable implication, and prove a "reciprocity" result involving the parameters w" and w i.

1.197 Suppose now that the firm is a monopolist in the output market, facing a demand curve $p = p(y)$, with total revenue $R(x_u x_2) — p(y)f(x\backslash, x_2)$. What observable differences, if any, with regard to a firm's responses to changes in factor prices would exist between this monopolistic model and the previous model of profit maximization in a competitive output market?

1.198 Returning to the competitive output market model, suppose x_2 is held fixed at its previous profit-maximizing level. Show how the "short-run" choice function for $x\backslash$, jc*(w!, p, $x\%$) is derived, and prove that it is downward-sloping inw|.

(h) The supply function of this firm can be defined in the long and short run as y^* (w i, w ", p, k) and $y^s(w_u\ p,\ x\%)$, respectively. Show how these supply functions are derived and then explain clearly the identity

$$y^*(w_u\ w^\circ_2,\ p,\ k) = /(wi, p,\ \mathrm{x}^*(\mathrm{w},,\ \mathrm{w}^\circ,\ p,\ k))$$

Use this result to show that the long-run supply function is more elastic than the short-run supply function.

14. Consider a profit-maximizing firm that employs one input x and produces two outputs y_x and y_2 according to the production frontier $f\{y_x,\ y_2) = x$. It sells its outputs at prices Px and p_2, respectively, and purchases the input x at price w. The firm obtains input x at a competitively determined unit wage w and sells output y_2 in a competitive market at price p_2. However, the firm faces a downward-sloping demand curve for y_l, given by $Px = p\backslash — ky_x$, where p°_x, p_2, w, and k are positive parameters.

1.199 Derive the first and (sufficient) second-order conditions, and explain the derivation of the explicit choice functions implied in this model. Characterize each of these choice functions as a demand function, a supply function, or neither, and explain.

1.200 Derive the comparative statics results available for the parameter p_2. What refutable implications are forthcoming, if any?

1.201 Are the explicit choice functions homogeneous of some degree in some or all of the parameters? Explain. If so, derive a relationship of elasticities for these functions. What relation, if any, does homogeneity of the explicit choice functions have, when it appears, to the homogeneity of the production relationship?

1.202 Derive the comparative statics relations for the parameter p°_x and interpret these results. What refutable implications, if any, appear?

1.203 How will the use of $y\backslash$ by this firm respond to an increase in p_{ℓ} ?

1.204 Derive a "reciprocity" result involving the parameters p° and p_2.

1.205 Suppose now that the firm is a monopsonist in the input market; i.e., as it purchases more x, it bids up the wage w. Assume the firm faces a supply curve $w = w(x)$, with total cost $C(y_x, y_2) — w(f(y_u\ y_2))f(yi,$ ^2)- What differences with regard to changes in output in response to a change in an output price would exist, if any, in the *observable* implications of such a model and the model of profit maximization in a competitive input market?

(h) Returning to the competitive input market model, suppose Vi is held fixed at its previous profit-maximizing level, denoted y°. Show how the "short-run" choice function for y_2, y_2*(P2, w,* y°) is derived, and show that it is upward-sloping in p_2.

(i) Explain clearly the identity

$$y_2(p\wedge Pi,w,k) = y^s_2(p_2,w,y^*(p^0_1,p_2,w,k))$$

Use this result to show that the long-run choice function for y_2 is more elastic than the short-run function.

4.7 ANALYSIS OF FINITE CHANGES: A DIGRESSION

The downward slope of the factor demand curves can be derived without the use of calculus, on the basis of simple algebra. Suppose that at some factor price vector (Wp w_2*),* the input vector that maximizes profits is (JC°, x_2). This means that if some other input levels *(x{*, JC^) were employed at the factor prices (vv°, w_2*),* profits would not be as high. Algebraically, then,

pf(r° Y°\ - w°r° W° Y° — yt;°r° > nf(y PJ(x\V2) - w\x\ PJ\X2-i^2)

However, there must be *some* factor price vector *(w*, *w\)* at which the input levels *(x\,x\)* would be the profit-maximizing levels to employ. Since *{x\,x\)* leads to maximum profits at (w{, *w\),* any other level of inputs, in particular *(x®,* x_2*),* will not do as well. Hence,

> 2/ 11 2 2 —

If these two inequalities are added together, all the production function terms cancel, leaving (after multiplication through by —1):

WJXJ -f- w_2x_2 -\- vVjjf^ -|- w_2x_2 ^ W|Xj -|- w_2x_2 -\- VVJXJ -f- w_2x_2

If the terms on the right-hand side are brought over to the left and the w, 's factored, the result is

However, this can be factored again, using the terms (JC^ — JCJ[1]), et cetera [note that

JC{* — *x* = — *(x* — jcj[1])], yielding

W - H-!) (*» - *x\)* + *(w° - w\) {4 - x >) < 0* (4-43)

Suppose now that only one factor price, say wi, changed. Then Eq. (4-43) becomes

or

(Awi)(A*i) < 0 (4-

44)

Equation (4-44) says that the changes in factor utilization will move oppositely to changes in factor price; i.e., the law of demand applies to these factors. Note that if the profit maximization point is unique, the weak inequalities can be replaced with strict inequalities.

This is the type of algebra that underlies the theory of revealed preference, to be discussed later. Curiously enough, this analysis cannot be used to show the second law of demand, that (factor) demands will become more elastic as more factors are allowed to vary. As was stated in Sec. 4.4, that theorem was a strictly local phenomenon, holding only at a point. The previous analysis, which makes use of finite changes, turns out to be insufficiently powerful to analyze the Le Chatelier effects, i.e., the *second* law of demand.

APPENDIX

TAYLOR SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

In Chap. 2, we indicated that it is sometimes possible to represent a function of one variable x by an infinite power series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \cdots \tag{4A-1}$$

It is, however, always possible to represent a function in a finite power series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x^*)(x - x_0)^n}{n!} \tag{4A-2}$$

where x^* lies between x_0 and x, that is; $x^* = x_0 + \theta(x - x_0)$ where $0 < \theta < 1$. These formulas were used to derive the necessary and sufficient conditions for a maximum (or minimum) at $y = f(x)$.

Let us generalize these formulas to the case of, first, two independent variables; that is, $y = f(x_1, x_2)$. This is accomplished by an artifice similar to the derivation of the maximum conditions in the text. Consider $f(x_1, x_2)$ evaluated at some point $x^\circ = (x_1^\circ, x_2^\circ)$, that is, $f(x_1^\circ, x_2^\circ)$. Let us now move to a new point, $(x_1^\circ + h_1, x_2^\circ + h_2)$, where we can consider $h_1 = \Delta x_1$, $h_2 = \Delta x_2$. If we let

$$y(t) = f(x_1^\circ + h_1 t, x_2^\circ + h_2 t) \tag{4A-3}$$

then when $t = 0$, $f(x_1, x_2) = f(x_1^\circ, x_2^\circ)$, and when $t = 1$, $f(x_1, x_2) = f(x_1^\circ + h_1, x_2^\circ + h_2)$. If h_1 and h_2 take on arbitrary values, any point in the x_1x_2 plane can be reached. We can therefore derive a Taylor series for $f(x_1, x_2)$ by writing one for $y(t)$, around the point $t = 0$. In terms of finite sums,

$$y(t) = y(0) + y'(0)t + \frac{y''(0)t^2}{2!} + \cdots + \frac{y^{(n)}(t^*)t^n}{n!} \tag{4A-4}$$

where 0 < \t*\ < |/|. Setting *t* — 1, we have

/(0) = /i (*?, $*_2$>i + f_2{xl $x°_2$)h

i=\ *i*=\

Therefore, Eq. (4A-4) becomes

f(y° _1_ h rUU- f(y° y°\ -L V^ f h

4- ^ ^ ^ ' , 4-

/ ^j + H i , x_2 + n_2) — f {x_i , x_2) + Z^ *'"' ""----2^-------

m!

where the last term is an m-sum of mth partials times a product of the appropriate *m hi*'s. The value of **x** = (xi, x_2) at which the last term is evaluated is some *x** between **x** and x°, i.e., where

$$x^* = x? + 6\ (x_i - x?) \quad i = 1,2 \qquad (4A\text{-}6)$$

with 0 < *0* < 1. Formula (4A-5) generalizes in an obvious fashion to functions of *n* variables. Then the sums run from 1 through *n* instead of merely from 1 to 2.

Concavity and the Maximum Conditions

FIRST-ORDER NECESSARY CONDITIONS. We can derive the first-order conditions for maximizing _y = *f(xi,* x_2) atjCpXj by considering (4A-5) with the last term being the linear term. In that case, we have the mean value theorem for *f(x*, x_2):

f(x» + $h_u x°_2$ + h_2) - f(xl $x°_2$) = M^)hi + $/_2$(x*)/i_2 (4A-7)

If *f(xi,*x_2) has a maximum at *f{*x^Q_t, x_2), then the left-hand side of Eq. (4A-7) is necessarily nonpositive (negative for a unique maximum) for all *h* i, h_2 (not both 0). Letting h_2 = 0 first, we see that

/i(*f,$*_2$*)<0 *hi*

>0 and

This can happen (if /i is continuous) only if *fi(x®, x_2)* — 0. Similarly, we deduce f_2 = 0. This procedure generalizes to the case of *n* variables in an obvious fashion.

THE SECOND-ORDER CONDITIONS; CONCAVITY. If *f(x_u,x_2)* is a *concave* function at a stationary value, then *f(x\,x_2)* has a maximum there. A concave function of two (or *n)* variables is defined as in Chap. 2 for one variable. A function *f(x\, x_2)* is concave if it lies above (or on) the chord joining any two points.

If JC° = *(x®,x_2)* an d -x¹ = *{x\,x\)* are an y tw0 Pom ts in the *X\x_2* plane, x^l = *tx°* + (1 — *t)x^l*, 0 < *t* < 1 represents all points on the straight line joining *x°* and JC¹. Algebraically, then, /($x_{1?}$ *x_2)* is concave if *for any x°,x^l,*

f(tx° + (1 - 0*¹) > *tf(x°)* + (1 - *t)f(x^l)* 0 < f < 1

If the strict inequality holds (for 0 < *t* < 1), implying no "flat" sections, the function is said to be *strictly concave. Convex* and *strictly convex* functions are defined analogously, with the direction of the inequality sign reversed. These definitions all generalize in an obvious way for functions of *n* variables; simply let *x°* and x^l represent vectors in *n*-space.

For differentiate functions, concave functions lie below (or on) the tangent plane. Letting

$$y(t) = f(x^o_1+h_1t, x^\circ_2+h_2t) \qquad \text{(4A-3)}$$

as before, and recalling Eqs. (2-14) in Chap. 2, strict concavity implies

$$y(t)-y(O)-y'(O)t <0 \qquad \text{(4A-8)}$$

for all nontrivial *h\,h_2*. Applying (4A-8) with *t* = 1, x_{-t} = x(° + *hi, i* = 1, 2,

/(*i, *₂) - /(*?,*?) - /1 W*2>i - *fi(xlx°₂)h₂* < 0 (4A-9)

Taking the Taylor series expansion (4A-5) to the second-order term and rearranging slightly yields

f($x_U$$X_2$) - *f(xlx°₂)* - /!(*?, *₂O)/H - *f_2(xlx°)h_2* = *J2Y,* (4A-10)

From (4A-9),

for all h_t, *hj* not both 0. Hence, strict concavity implies (4A-11). If the *hj* 's are made smaller and smaller, *fij(x*, Xj)* converges toward *fij(x®, x_2)*. We can deduce that concavity at *X®, x_2* implies that

,-(*?,*₂°)My<0 (4A-

12)

for all h_i, h_j, but *not* that this expression is *strictly* negative at $(x^®, x^®)$. If this doubles um is strictly negative, then $f(x_1, x_2)$ must be strictly concave. Similar remarks hold for *convex* functions.

If $f(x_1, x_2)$ has an extremum at (x_1^*, x_2^*), then $f_1 = f_2 = 0$ there. Equation (4A-10) then reveals how the second partials are related to a maximum or minimum position. Again, all the results of this section generalize to functions of n variables by simply having the sums in expressions (4A-5), (4A-9), (4A-10), etc., run from 1 to n, instead of just from 1 to 2.

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CHAPTER

5

MATRICES AND DETERMINANTS

5.1 MATRICES

Most economic models involve the simultaneous interaction of several variables. We have seen, for the case of the profit-maximizing firm with two inputs, that the comparative statics of the model depended on solving two simultaneous linear equations. This occurrence is indeed general; for models with *n* variables, systems of *n* simultaneous linear equations need to be solved. For this reason, we shall take a short departure in this chapter and study the algebra of such systems. We will then show how this algebra can simplify the comparative statics of economic models.

Let us begin with the simplest system of simultaneous equations, two equations in two unknowns. Denote these equations as

$$a_n x_x + a_n x_2 = bi$$

Notice the double-subscript notation for the coefficients. This permits easy identification of these numbers. The element a(appears in the zth *row* (horizontal) and7th *column* (vertical). Here, / *andj* take on the values 1,2; in general, they will run from 1 through *n*.

A very convenient notation that is extensively used in virtually all sciences involves separating out the coefficients (the a,-/s) from the unknowns (the JC,'S) and writing Eqs. (5-1) thus:

fa_n ***a_l2***
y#21 ^22

This is known as *matrix notation;* the rectangular arrays of numbers are called *matrices* (plural of matrix). In general, the system of m equations in n unknowns

$$+ \ldots + a_{1n} x_n = b_1$$
$$+ \cdots + a_{2n} X_n - \quad 2 \qquad \wedge$$

$$a_{m1} X_1 \; \text{-p"} \; a_{m2} x_2 \; \text{"T} \bullet \bullet \bullet \text{ ~r } a_{mn} x_n - o_m$$

is written in matrix form as

$$\left(\begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \right)$$

$a \backslash_2$ ■ ■ ■

$a \backslash_n$

«22 a_{2n}

(5-4)

$\backslash a_{m1} \backslash \quad a_{m2}$

The system (5-4) is just another way of writing Eqs. (5-3). This system involves "multiplication" of an m x n *(m* rows, n columns) matrix by an n x 1 matrix, forming another m x 1 matrix on the right-hand side. In general for any coefficient b_i from (5-3),

E' $= bi \quad i = 1, \ldots, m$ (5-5)

Notice that to arrive at any particular b_i, the elements of the **ith** row of the (a,-y) matrix are multiplied, term by term, with the elements of the *(XJ)* matrix, which consists of only one column, and those products are then summed. In this manner, general matrix multiplication is defined. Consider the matrix "product"

$$an \cdots a_{1n} \backslash \; / b_{11} \; ■ ■ ■ \; b_{1r} \backslash \; / c_{11} \cdots c_{1r} \backslash$$
$$\qquad h \qquad = \quad : \qquad \textbf{(5-6)}$$
$$a_{mn} \Big) \; \backslash b_{nr} \qquad b_{nr} / \quad \backslash C_{m1} \backslash \quad C_{mr} /$$

or, simply,

$$AB = C$$

Any element c_{ij} of the C matrix is defined to be

$$= c_{ij} \qquad i = 1, \ldots, m \quad j = 1, \ldots, r \qquad \textbf{(5-7)}$$

$k=\backslash$

That is, the element in the /th row andyth column of C is defined to be the sum of the products, term by term, of the elements in the /th row of A and they'th column of B. This definition is therefore valid only if the number of *columns* of A equals the number of *rows* of B. Otherwise, the definition yields nonsense.

Example 1

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & -1 \end{pmatrix}$$

\ — ! !

Here, a 2 x 3 matrix is multiplied by a 3 x 2 matrix. It results in a 2 x 2 matrix.

Example 2

? -1

A matrix with only one column is sometimes called a column vector or column matrix; a matrix with only one row is sometimes called a row vector or row matrix.

Example 3

Notice that it *matters* if matrices are multiplied on the left or on the right; different matrices result.

Consider any two *n* vectors,

$$a = (a_1, \ldots, a_n) \qquad b = (b_1, \ldots, b_n)$$

The scalar product ab (variously called the *dot product* or *inner product,* sometimes written a • b) is defined to be

$$ab =$$

The matrix product AB can be seen to be defined in terms of the scalar product of the *row* vectors of A and *column* vectors of B.

The algebra of matrices will be relegated to the appendix of this chapter. We are concerned here only with a way of systematically representing the solution of simultaneous equations.

5.2 DETERMINANTS, CRAMER'S RULE

Let us return to the two equation, two unknown system (5-1):

$$\begin{aligned} &+a_{12}x_2 = b_1 \\ &+ a_{22}x_2 = b_2 \end{aligned} \tag{5-1}$$

To solve these equations for x_1, we multiply the first equation by a_{22} and the second equation by a_{12} and subtract the second equation from the first:

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - b_2a_{12}$$ If

then

$$\frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \tag{5-8}$$

Similarly, to solve for x_2, multiply the first equation by 021 and the second by $a\backslash\backslash$, then subtract the first equation from the second:

(fl 11022 —

If, again

2
2

then

x

Let us now define something called a 2 x 2 *determinant,* or a *determinant of order 2.* Suppose

$$\begin{matrix} \mathbf{0} & \mathbf{6} \\ c & d \end{matrix}$$

is any *square* 2 x 2 matrix. The determinant of this square matrix, written with straight vertical lines around the matrix, is defined to be

a b c d =

That is, the product of the upper right and lower left elements is subtracted from the product of the upper left and lower right elements.*

In terms

of determinants, the solutions (5-8) and (5-9) can be written

$$x_2 = \frac{\begin{vmatrix} 0 & b_2 \\ 2 & \end{vmatrix}}{\begin{vmatrix} 0 & a \\ 0 & 02 \end{vmatrix}}$$

$$\frac{\begin{vmatrix} b & 0_{12} \\ b & 02 \end{vmatrix}}{\begin{vmatrix} 0 & 0, \\ 0 & a_{22} \end{vmatrix}}$$

Notice that the determinant in the denominator of these expres sions is the determinant of the matrix of coefficients, (0,;). In the numerators, for the solution for *x*\, the *first* column of the *a*;*j*1 determinant is replaced with the *b*;:'s, whereas for x_2, the *second* column is replaced by the £;'s. This formula is known as *Cramer's rule.* It is the generalization of this rule to *n* variables that we shall investigate. Notice that the solutions for *x*i and x_2 exist only if

011 012
021 022

What is the geometric significance of this condition? Equations (5-1) represent two straight lines in the *X*\X_2 plane. These equations will not have any solution at all if

^Throughout this text, matrices and vectors will be indicated by boldface type. The determinant of a square matrix A will be indicated by the symbol |A| or A.

the lines are parallel; if the lines are not only parallel but coincident, an infinity (all points on the common line) of solutions results.

These two lines will be parallel if they have the same slope. Solving each equation for x_2, Eqs. (5-1) are equivalent to

If the slopes are the same, then an $an\ b_2$

$x_2 = — c$

or

$\underline{an}\ an$

$a_n \quad a_n a_2\backslash \quad a_{22}$

$= a_n a_2 2 - a_{xl} a_1 \backslash = 0$

Hence the inability to solve Eqs. (5-1) because |A| = 0 occurs because the lines are parallel or coincident.

Consider now a system of three equations in three unknowns:

$) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

/flu a 12
a_u (a_{2X} a_{22}
a_{23} \$a_3 i$ a_{32}
a_{33} (5-11)

Define the determinant of order
3 as

$\begin{vmatrix} a & a & a & 3 \\ a & a & a_{23} \\ a & a & a_{33} \end{vmatrix}$

$n \begin{vmatrix} a_{22} & a \\ a_{32} & a \end{vmatrix} \quad \begin{vmatrix} a & a_2 \\ a & a_3 \end{vmatrix} \quad a\backslash \begin{vmatrix} a & a_2 \\ a & a_3 \end{vmatrix}$

M

(5-12)

The determinant D_3 is defined in terms of certain second-order determinants. All in all, six terms involving the products of three elements are involved, with particular signs. Notice that the determinant multiplied by a_n is the determinant that remains from D_3 when row 1 and column 1 are deleted. In like fashion, the determinant multiplied by a_{i2} is the determinant that remains from D_3 when row 1 and column 2 are deleted (the row and column that *an* appears in), and similarly for the last determinant.

We define the *minor of $a_i j$* as that determinant that remains when row / and column *j* are deleted from the original determinant.

In the above definition of D_3, the elements of the first row are multiplied by their respective minors, but one such minor comes in with a negative sign.

Define the *cofactor of aij,* written A_{tj}, as $(— l)^{l+j}$ times the minor of a,;. [Sometimes the term *signed cofactor* is used. This is redundant, though perhaps useful to emphasize the signing element $(— l)^{l+j}$.] In terms of cofactors, D_3 can be written

$$D_3 = a_nA_n + a_{l2}A_{l2} + a_{i3}A_{l3} \qquad (5\text{-}13)$$

Expanding this expression, i.e., Eq. (5-12),

$$D_3 = a_Ua_{22}a_{33} - «11«23«32 \sim 012021033 + 012023031 + «13«21«32 — «13«22«31 \qquad (5\text{-}14)$$

In each triple, the three elements come from different rows and columns. No row or column is ever repeated. (If you are a chess player, the triples represent all possible ways three castles, or rooks, can be placed on a 3 x 3 chessboard such that they cannot capture one another.) Equation (5-14) can be factored in another way, e.g.,

$$D_3 = -012(021033 - 023031) + 022(011033 - 013031) - a_{32}(a_na_23 - a_{l3}a_{2l}) \qquad (5\text{-}15)$$

But this factorization can be written in terms of the elements and cofactors of column 2. By inspection, from (5-15)

$$D_3 = a_{X2}A_{X2} + 022\text{^}22 + a_{32}A_{32} \qquad (5\text{-}16)$$

[Notice that A_{12} and A_{32} both have negative signing factors, since $(—1)^{1+2} = (—1)^{3+2} = — 1$. This doesn't mean that Ai_2 or A_{32} are necessarily negative; just that the *minors* of a_{x2} and a_{32} are multiplied by —1.]

This algebra indicates that D_3 can be defined as the sum of the products of the elements of *any* row or *any* column times their respective cofactors. That is, D_3 can be written as

$$D_3 =]T_{7=1}^{3} a_{ij}Aij = Y_{i=1}^{3} 0*7\ A_u \qquad (5\text{-}17)$$

In the first sum, the determinant is expanded using the elements and cofactors of row *i;* in the second sum, column *j* is used. Either way, the same number results. This result can be proved for determinants of order 3 by simply finding all six sums and verifying the result. More importantly, it is the generality of this result that is useful. Determinants of higher order can be defined in terms of lower-order ones. That is,

$$\begin{matrix} 011 & 012 & 013 & 014 \\ 021 & 022 & 023 & 024 \\ 031 & 032 & 033 & 034 \\ a_4\backslash & a_{42} & a_{43} & £?44 \end{matrix}$$

where $A_{(;}$ is the (signed) cofactor of element $a\text{^}$, that is, $(—l)^{l+j}$ times the third-order determinant that remains when row i and column j are deleted from D_4. The generalization of Eq. (5-17) will be stated now, without proof.

Theorem 1. Let D_n be any nth-order determinant of a square matrix $A = (a_{ij})$. Then

$$D_n = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij} \qquad (5\text{-}1)$$

where A_{ij} is the cofactor of element a_{ij}.

We shall now state and briefly sketch the proofs of the important elementary properties of determinants, culminating in Cramer's rule.

Theorem 2. If all the elements in any row (column) of D_n are 0, then $D_n = 0$. ***Proof.*** Expand D_n by that given row (column), and the sum of many 0s is 0.

Theorem 3. If D'_n is obtained from D_n by interchanging any two rows (columns), then $D'_n = -D_n$. A rigorous proof will not be given. However, it is clear that the same terms are involved in D'_n as in D_n since all the n-tuples are chosen with one element from each row and column, with no repeats. Only the signing factor $(-1)^{i+j}$ can be affected. If row 1 is interchanged, say, with row 2, then expanding D_n by the second row means that the signing factor will be $(-1)^{2+j}$ instead of $(-1)^{1+j}$. Hence, the sign of D_n will reverse. If row 1 and row 3 are interchanged, we can consider this as three separate steps: interchange rows 1 and 2, then 1 and 3, and then 3 and 2. This *odd* number of reversals changes the sign of D_n. The result in fact follows, as the theorem indicates, for an arbitrary interchange of rows, or an arbitrary interchange of columns.

Theorem 4. If D'_n is obtained from D_n by multiplying any row (column) by some scalar (number) k, then $D'_n = kD_n$.

Proof. Take that given row (column) and expand D_n by the cofactors of that row or column. Then k appears in each term, and, by factoring it out, the result is obtained.

Theorem 5. If D_n has 2 rows (columns) that are identical, then $D_n = 0$.

Proof. If any two rows, in particular the two identical ones, are interchanged, then by Theorem 3 the value of the resulting determinant is opposite in sign but has the same absolute value as the original determinant. But since the determinant has exactly the same elements after interchange as before, the value of the determinant must be identical. The only value that satisfies this relationship of $D_n = -D_n$ is $D_n = 0$.

Corollary. If one row (column) is proportional to another row (column), then $D_n = 0$. *Proof.* Factor out the constant of proportionality and then use Theorem 5.

Theorem 6. Suppose each element of the Mi row, a_{kj}, is equal to $a_{kj} = b_{kj} + c_{kj}$, the sum of two terms. Then let D'_n be the determinant formed by using the elements b_{kj} in the *kth* row and D''_n be the determinant formed using c_{kj} as elements in row k. Then $D_n =$

Proof. Expanding D_n by the elements and cofactors of row k,

$$D_n = (b_{k1} + c_{k1})A_{k1} + \cdots + (b_{kn} + c_{kn})A_{kn}$$

Theorem 7. If D'_n is formed by adding, term by term, a multiple of any row (column) of D_n to another row (column) of D_n, then $D'_n = D_n$.

Proof. Consider for example the 3 x 3 determinant

flu a_n a

021 022 0

Multiply the elements in row 1 by some number k and add this product, term by term, to row 2. Then

ka ka_{i2} 013

fl31 033

By Theorem 6

$D'_3 =$ 0 0 0 0 0 0 0 0 0 «11 012 013 011 012 013 031 032 033

By Theorem 5, this latter determinant equals 0. Thus $D'_3 = D_3$. The proof is general, of course, for any two rows (or columns), for any size determinant.

Theorem 8. If the elements of any row (column) are multiplied by the respective cofactors of some other row (column), the resulting sum is zero. This process is called *expansion by alien cofactors.*

Proof. This is equivalent to expanding a determinant that has two identical rows. Consider again the 3 x 3 determinant of the previous theorem. The theorem asserts, for example, that

$a_2\backslash A\backslash\backslash + a_{22}A_n +$ = 0

This is the expansion of the determinant

021 022
023 021
022 023
031
032
033

by row 1 or row 2. But this determinant is 0 by Theorem 5. The generalization to any D_n is straightforward.

Theorem 9 (Cramer's rule). Consider a system of *n* linear equations in *n* unknowns,

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \text{'flu} \quad \begin{matrix} a_{1n} \\ \\ a_{nn} \end{matrix} \Bigg) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

If the determinant |A| of the coefficient matrix |A| = *(a^)* is nonzero, then a unique solution exists for each x_t. In particular, the solution for each JC, may be expressed as the quotient of two determinants; the denominator is always the determinant |A|, while the numerator is that determinant formed when column *i* in |A| is replaced by the column of b_t 's. For example,

a_{l2} a_{Xn}

b_n a_{n2} a_{nn}

0ii b_x

x_2 = $a_n\backslash$ b_n

Proof. We shall demonstrate Cramer's rule for the three-equation case only. Consider such a system:

$a_nx_x + a_{l2}x_2$ + 013*3 = b_x021*1 + $a_{22}x_2$

$+ a_{23}x_3 = b_2$ (5-19)

033*3 $= b_3$

In general, these equations are solved by multiplying through by various numbers, adding or subtracting one equation from another, etc. The theory of determinants gives us some handy numbers to work with.

Let us solve for $x\backslash$. Multiply the first equation through by A_n, the cofactor of an; multiply the second and third equations, respectively, by $A_2\backslash$ and $A_{3!}$. Then add the three resulting equations together. After factoring out the x,'s, this yields

$(a_nA_n + a_{22}A_{2l} + a_{32}A_{3l})x_2$ $(a_{l3}A_n + a_{23}A_{2X} +$

$a_{33}A_{3l})x_3 = b_tA_n + b_2A_{2X}$ b_3A_{3l} (5-20)

The first set of parentheses in (5-20) equals the determinant |A|, since it is the sum of the elements of the first column of |A| times their respective cof actors. The second and third sets of parentheses, however, represent an expansion by alien cofactors. There, the elements of the second or third column are multiplied by the cofactors of the first column and summed. By Theorem 8, these terms sum to 0. Hence Eq. (5-20) reduces to

$$=b_lA_n + b_2A_{2l} + b_3A_3$$

$a_n \quad a_{X3}$

$X\backslash \; =$ (5-21)

In like fashion, x_2 is obtained by multiplying the first, second, and third equations in (5-19) by $A\backslash 2$, A_22, and A_{32}, respectively, and summing. Then the coefficients of $x\backslash$ and x_3 are 0, and the b_t 's multiply the respective cofactors of the second column. The same procedure obtains the general result, as stated in the theorem.

5.3 THE IMPLICIT FUNCTION THEOREM

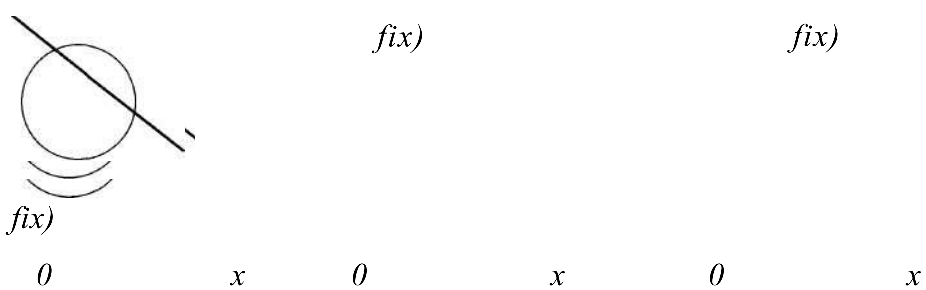
We have referred at several instances to the problem of "solving" the implicit first-order equations

for the explicit relations

$$x \;\; a) \;\; 0$$
$$fli \;\; x \;\; a \;\; 0 \qquad (5\text{-}22)$$
$$X \;\; {}_2 \;\;)$$
$$xx = \, !$$
$$-- \quad *($$
$$x_7 = \, ? \qquad (5\text{-}23)$$

where $x\backslash$ and x_2 are the choice variables and a represents the parameters of the model.

Sufficient conditions under which this procedure is valid are known as the *implicit function theorem.* One should be wary, incidentally, of a "nontheorem" that appears every now and then. This nontheorem asserts that if there are n equations and n unknowns, a unique solution results. This proposition is valid only in the case of *linear* equations whose coefficient matrix has a nonzero determinant. Figures 5-la, *b,* and *c* demonstrate why the theorem cannot be applied to nonlinear functions.



Two solutions *(a)* One solution *(b)* No solutions *(c)*

FIGURE 5-1
In general, with nonlinear functions, no general assertions are possible regarding the number of solutions to n equations and n unknowns.

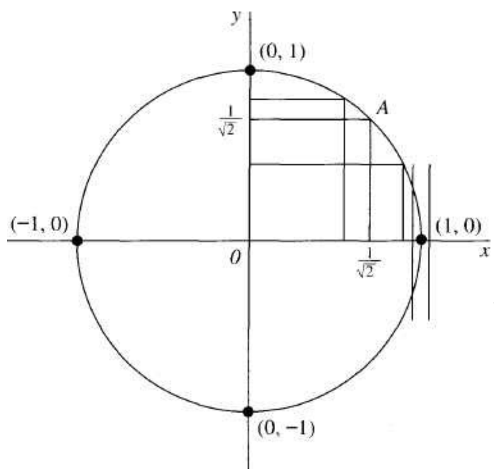


FIGURE 5-2
The Implicit Function Theorem. Around any point where the circle is not vertical, a unique *y* exists for any *x*. However, around *x* — +1 or *x* — — 1, *two* values of *y* are associated with anyjc, no matter how small the interval is made around that *x* value. If the function is *not* vertical, an explicit solution *y* = /(JC) exists for an explicit relation *g{x, y)* = 0. However, *dg/dy j*= 0, while sufficient, is not necessary.

The implicit function theorem is narrower in scope than the above nontheorem. Suppose Eqs. (5-22) have a unique simultaneous solution at some point *(x®,* x°, a⁰). Under what conditions can the implicit relations (5-22) be written as the explicit relations (5-23)?

To answer this, consider first the simplest case of one equation in two unknowns, e.g., the unit circle, depicted in Fig. 5-2,

$$\text{JC}^2 + y^2 = 1 \qquad (5\text{-}24)$$

For this function to be written as some explicit function, *y* = /(JC), *a unique y* must be associated with any JC, around a certain point. Of course, (5-24) can be solved for *y* as

$$y = \pm(\backslash\text{-}x^2)^{x\,\prime 2}$$

The function as written here is technically not a function at all; for each JC, *two* values of *y* are given, instead of a unique *y.* However, such is not the case for solutions around individual points on the unit circle. Consider some point, *A, x* = 1 */V2, y* = l/\/2. In some *neighborhood* around *x* = 1/V2, a unique value of *y* is associated. That is, *around x* = 1/V2, *y* = l/\/2, the explicit functional relation

_ ,2x1/2 (5-25)

is valid. The implicit function (5-24) admits an explicit solution *around the point A,* not necessarily for all JC.

The situation is different, however, at the intercepts of the unit circle and the *x* axis, the points (—1, 0) and (1, 0). At either of these two points, no matter how small the interval is made around the point, any value of JC will be associated with *two* values of *y.* The implicit relation (5-24) does *not* admit of an explicit solution *y* = /(JC). An explicit solution of *x* on _y, that is, *x* = *g(y),* does exist—for any value of *y* around (1, 0) or (—1, 0), a unique value of JC is implied [however, not at the points (0, 1) and (0,-1)].

It can be seen that the reason why the implicit equation $x^2 + y^2 = 1$ does not admit of a unique solution $y = f(x)$ at (1, 0) and (—1, 0) is that at these points, the function *turns back* on itself. Moving counterclockwise around the circle, as y increases through the value 0, on the right semicircle, x first increases and then decreases. (On the left semicircle, moving clockwise, x decreases and then increases.) At the points (1,0) and (—1,0) the implicit function $x^2 + y^2 = 1$ is *vertical,* that is, $dy/dx \to \pm\infty$. As long as the function is not vertical, the implicit relation yields a well-defined explicit solution $y = f(x)$.

We can see how the preceding analysis relates to the ability to do comparative statics, in one-variable models. Consider one implicit choice equation, which might be the first-order equation of some objective function:

$$h(y,a)=0 \tag{5-26}$$

To find dy/da, an explicit solution of (5-26) must be assumed:

$$y = y^*(a) \tag{5-27}$$

Substituting (5-27) into (5-26), the identity

$$h(y^*(a),a)=0 \tag{5-28}$$

results. Differentiating with respect to a,

$$h_y \frac{dy^*}{da} + h_a = 0 \tag{5-29}$$

In order to solve (5-29) for dy^*/da,

$$h_y \neq 0 \tag{5-30}$$

must be assumed. This amounts to assuming that the function $h(y,a)$ is not vertical (a plotted horizontally, y vertically).

In maximization models, the *sufficient* second-order conditions guarantee the existence of the explicit solutions (5-27). In these models, the implicit relation (5-26) is already the first partial of some objective function, $f(y,a)$. That is, (5-26) is

$$f_y(y, a) = h(y,a) = 0$$

The condition that $h_y \neq 0$ is guaranteed by the *sufficient* second-order condition for a maximum,

$$f_{yy} = h_y < 0$$

It should be noted that whereas $h_y \neq 0$ is sufficient to be able to write $y = y^*(a)$, it is not necessary. There are some functions for which $h_y = 0$ at some point, and it is still possible to write $y = y^*(a)$. For example, consider the function

$$y^3 - a = 0$$

The solution to this equation, depicted in Fig. 5-3, is

$$y = a^{1/3}$$

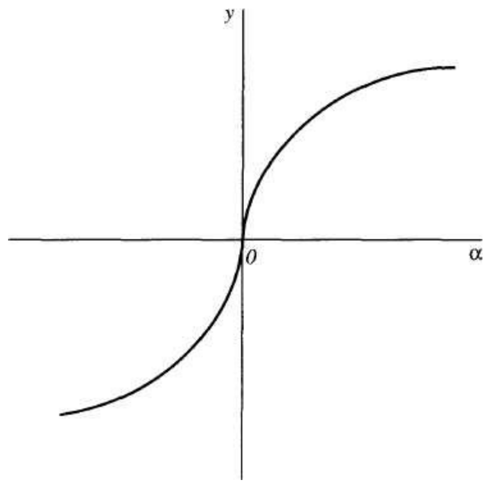


FIGURE 5-3
*The Function y = a*1//3. This function illustrates why the condition h_y *j*= 0 is sufficient but not necessary for writing an implicit function in explicit form. This function becomes vertical at the origin, yet it is still possible to define *y* as a single-valued function of a, because a1//3 does not turn back on itself. If h_y ^ 0, the explicit formulation is *always* possible; if h_y = 0, it *may* not be.

Although *dy /da* —>• oo as *a* —> 0, it is still the case that a unique *y* is associated with any *a* around *a* = 0; the function, while vertical at *a* = 0, does not turn back on itself there.

In models with two equations and two choice variables, the situation is algebraically more complicated, but conceptually similar. Consider the system (5-22) again, but let us just assume that these are just two equations in three unknowns, *X\,*x_2*,* and *a,* without assuming for the moment that there exists an *f(x*, x_2*, a)* for which /i = *df/dxi,* f_2 = *df/*dx_2. A *sufficient* condition that Eqs. (5-22) admit the explicit solution (5-23) at some point is that neither of the explicit functions (5-23) become vertical, for any a, if a is one of many parameters. Let us try to solve for *dx*/da* and 3*2/30:.

Differentiating Eqs. (5-22), we get

9/L 9/L
3XI 3x2 *da* —*f\a ~fla* (5-31)
9/2_ *Bh_* *dx**$_2$
dx dx_2 *\~da*

A necessary and sufficient condition for solving for *dx*/da* and *dx^/da* uniquely is that the determinant

J = **L** dx_2
ML (5-32)
dx_2

This determinant, whose rows are the first partials of the equations to be solved, is called a *Jacobian* determinant. If / ^ 0, the partials *dx*/da* are well defined, and in fact are so because the explicit equations x_t = *x*(a)* are well defined. That is, / ^ 0 is precisely the sufficient condition that allows solution of the simultaneous equations (5-22) for the explicit equations (5-23). This is the generalization of relation (5-30) for one equation.

Condition (5-32) is implied by the *sufficient* second-order conditions for a maximum. In maximization models, *f\{x\, x_2, a)* and *f_2(xi, x_2,* «) are *df/dxi, df/dx_2*. Therefore, *dfi/dxi = fu,* etc., and the Jacobian is

J = /ll

From the sufficient second-order conditions, 7^0,
since *J* > 0. For models with *n* equations

$$f_1(x_u, \ldots, x_n, a) = 0$$

(5-33)

$$f_n(xi, \ldots, x_n, a)$$

= *O* a sufficient condition for explicit

solutions

$$x_t = x^*\{a) \qquad (5\text{-}34)$$

to exist at some point is that the Jacobian of (5-33) be nonvanishing

there:

dx_n

(5-35)

Bfn *Bfn*
dx_n

PROBLEMS

(1. Evaluate the following determinants

a)

1	2		
-1	-3		
-2	-1	-4	-3

1 **2**
0 1
-1 0

(

1.206	1	1
1.207	0	1
1	1	0

2. Suppose a square matrix is *triangular;* i.e., all elements below the diagonal are 0:

/flu a 12 ∎ ∎ • a_{1n}

0 a_{22} a_{2n}

A =

V 0 0 *aj*

Show that |A| = $a_{11}a_{22}, \ldots, a_{nn}$, the product of the diagonal elements.

3. Consider the system of *n* equations in *n* unknowns

$$Ax = b$$

where the vector b consists of 0s in all entries except a 1 in some rowj. Assuming |A| =/= 0, show that the solutions can be represented as

* =]A[U = 1................"

where A,i is the cofactor of element a,,-.

APPENDIX

SIMPLE MATRIX OPERATIONS

A matrix, again, is any rectangular array of numbers:

*fa*n • • • c

a_{mn}/

This matrix has m rows and *n* columns. Suppose some other matrix B has *n* rows and r columns (B must have the same number of rows as A has columns):

/bn • • • **b**

B= ;

\bn**i** **b**nrl

The matrix product C = AB is defined to be the m x *r*

matrix ***c*** \ , \ ***I ' a*** n ∎ ∎ ∎

a i n \ / ***b*** n •

*d*m\

where any element c,y of C is defined to be

ykj

k=l

Schematically, each element of any *row* of A is multiplied, term by term, by the elements of some *column* of B (as shown by the direction of the arrows above) and the result is summed. Note that while the product **AB** may be well defined, **BA** may not be, since the number of columns in the left-hand matrix must equal the number of rows in the right-hand matrix in a matrix product. In general, even for square matrices, *matrix multiplication is not commutative,* i.e., in general,

AB^BA

The associative and distributive laws do hold, however. If A is m x n, B is n x r, and C is r x p, then **ABC** is ra x p and the following laws are valid:

Associative law (AB)C = A(BC)
If **A** is m x n, **B** and **C** are n x p,
then: **Distributive law A(B + C) =**
AB + AC For the associative law, we
simply note

$h=\backslash$ $k=\backslash$ $k=\backslash$ $h=\backslash$

For the distributive law,

$a_{ik}(b_{kj} + c_{kj}) =$
$k=\backslash$ $k=\backslash$ $k=\backslash$

The *transpose* of any matrix, A', is the matrix A with its rows and columns interchanged. That is,

The transpose of a product is the product of the transposed matrices, in the reverse order:

(AB)' = B A To

prove this, let $c_{,7}$ be an element of (AB)'. By

definition,

$k=l$

An element of B'A' is

$k=\backslash$

identical to the former sum.

A matrix is called *symmetric* if it equals its transpose; that is,

A = A'

That is, for every element a_{ij}, $a_ij = a$^. The rows and columns can be interchanged leaving the same matrix. This is a very important class of matrices in economics. The matrices encountered in maximization models are the second partials of some objective function, $f(x i, \ldots, x_n)$. By Young's theorem, fa = /},-. Hence, these matrices are symmetric.

The Rank of a Matrix

Consider *anmxn* matrix (a,-y) and consider each of its rows, Ai, ..., A_m, separately. Each row /,

represents a point in Euclidean n-space. It is important to discuss the "dimensionality" of these *m* points; i.e., do they all lie on a single line (one dimension), a plane (two dimensions), etc.? Algebraically, if these *m* vectors lie in an ra-dimensional space, then it is not possible to write any vector A as a linear combination of the others. In other words, if

$$\text{fcjAi H---h } k_mA_m = 0$$

where the k_t are scalars (ordinary numbers), then all the k_t 's must be zero. In this case, Ai, ..., A_m are said to be *linearly independent.*

For any given matrix A, the maximum number of linearly independent row vectors in A is called the *rank* of A. If A has *m* rows and *n* columns, and *n* > m, then the maximum possible rank of A is *m.* It is not obvious, but true that the number of linearly independent *column* vectors of A equals the number of linearly independent *row* vectors. Thus the rank of a matrix is the maximum number of linear independent vectors in A, formed from either the rows or the columns of **A.**

Example 1. The vectors Ai = (1, 0, 0), A_2 = (0,1, 0), A_3 = (0, 0, 1) are linearly independent.

$$\text{fc,Ai} + k_2A_2 + \text{£}_3A_3 = (\text{fci}, k_2, k_3)$$

= 0 if and only if $k_j = k_2 = \&_3 = 0$. The matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

therefore has rank 3.

Example 2. Let Ai = (1, 1, 0), A_2 = (1, 0, 1), A_3 = (1,-1, 2). These vectors are linearly *dependent.* Here, $A_3 = 2A_2$ — A,, or

$$A, - 2A_2 + A_3 = 0$$

Any one of these vectors can be written as a linear combination of the other two, but not less than two. The matrix

/I 1
0\ A= 1
0 1
\1 -1
2/ therefore has rank 2.

A set of *m* linearly independent vectors Ai, ..., A_m is said to form a *basis* for Euclidean ra-space. Any vector b in that space can be written as a linear combination of Ai, ..., A_m, that is,

where the fc/'s are scalars.

Consider a system of *n* equations in *n* unknowns,

$$a_{1n} \Bigg) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

or, in matrix notation, Ax = b. If the rank of A is less than *n,* then some row of A is a linear combination of other rows. But this is the procedure for solving the above system for the JC'S. If rank (A) < *n,* then at least one equation is derivable from the others, i.e., there are really less than *n* independent equations in *n* unknowns. In this case, no unique solution exists. We saw in the chapter that simultaneous equations admitted a unique solution if the determinant of A, |A|, was nonzero. An important result of matrix theory is thus:

Theorem. If **A** is a square *n x n* matrix, then the rank of A is *n* if and only if |A| ^ 0.

Discussion.This algebra is the basis of the nonvanishing Jacobian determinant of the implicit function theorem. Briefly, if rank (A) < *n,* then some row (or column) is a linear combination of the other rows (columns). By repeated application of the corollary to Theorem 5 in the chapter proper, |A| — 0. Conversely, if |A| = 0, some row of A is either 0 or a linear combination of the other rows, and hence A],..., A_n are linearly dependent. A more formal proof of this part can be found in any standard linear algebra text.

A square *nxn* matrix A that has a rank *n* is called *nonsingular.* If rank (A) < *n,* A is called *singular.*

The Inverse of a Matrix

In ordinary arithmetic, the inverse of a number *x* is its reciprocal, *l/x.* The inverse of a number *x* is that number *y* which makes the product *xy* = 1. In matrix algebra, the unity element for square *n x n* matrices is the *identity matrix* I where,

/I 0 ••• 0\
0 1 0

1/

That is, I is a square *n x n* matrix with Is on the main diagonal, and 0s elsewhere. Formally, if *(a,j)* is the identity matrix, then a_{ij} = 1 if/ = *j,* a_{ij} = 0 if / ^ *j.* It can be verified that for any square matrix A,

$$AI = IA = A$$

Thus the identity matrix I corresponds to the number 1 in ordinary arithmetic. Is there a *reciprocal* matrix B, for some matrix A such that

$$AB = I$$

If so, we call B the *inverse* of A, denoted A^{-1}.

The problem of finding the inverse of a matrix is equivalent to solving

$$Ax = b$$

for a unique x, where A is an $n \times n$ square matrix. If A^{-1} exists, premultiply the equation by A^{-1}, yielding

$$x = A^{-1}b$$

This is the simultaneous solution for x. This solution exists if and only if $|A| \neq 0$. This is correspondingly the condition that A^{-1} exists, that is, A must be nonsingular, or have rank *n*.

Assuming $|A| \neq 0$, consider the following matrix, A*, called the *adjoint* of A:

$$A^* = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$

The adjoint, A*, is formed from the cofactors of the a_{ij}'s, *transposed.* Consider the matrix product AA*:

$$\begin{pmatrix} \cdots & a_{1n} \\ & \\ \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & \cdots \\ & \\ A_{1n} & \cdots \end{pmatrix} = |A|I$$

Any element of AA* off the main diagonal is formed by the product of the elements of some row of A and the cofactors of some other row; these products sum to zero by the theorem on alien cofactors. The diagonal elements of A A*, however, are formed from the sums of products of a row of A and the cofactors of that row; this sums to |A|. Hence

$$AA^* = |A|I$$

The inverse of A, A^{-1}, is thus $(1/|A|)A^*$, or
$^1 A^{-1} =$

$\backslash|A| \qquad |A$

By inspection, it can be seen that if $AA^{-1} = I$, then $A^{-1}A = I$ also; that is, the left or right inverse of A is the same A^{-1}. Also, A^{-1} is unique. Suppose there exists some B such that

$$\mathbf{AB} =$$

Premultiplying by $\mathbf{A}^{-1}$
A AB = IB = B = A

It is also true that

$$= B$$

The proof of this is left as an exercise.

Orthogonality

Two vectors are called orthogonal if their scalar product is 0.

Example 1. The vectors $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$, and $E_3 = (0, 0, 1)$ are all mutually orthogonal.

Example 2. Let $\mathbf{a} = (2, -1, 1)$, $\mathbf{b} = (-1, -1, 1)$. Then ab = 0; thus **a** and b are orthogonal.

Orthogonal vectors must be linearly independent. Suppose a square matrix A is made up of row vectors $a_1, \ldots, a_n$ which are mutually orthogonal, and whose Euclidean "length" is unity:

A is called an *orthogonal* matrix. It can be quickly verified that the transpose of A, A', is the inverse of A, i.e.,

$$AA = I$$

PROBLEMS

1. Find the rank of the following matrices. For which does $|A| \neq 0$?

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & -2 \\ -2 & 2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & 3 \end{pmatrix}$$

1.208 Prove that $(AB)^{-1} = B^{-1}A^{-1}$, if A and B are two square nonsingular matrices.

1.209 Prove that $(A^{-1})^{-1} = A$, that is, the inverse of the inverse is the original matrix.

1.210 Show that $(A')^{-1} = (A^{-1})'$, i.e., the transpose of the inverse is the inverse of the transpose.

1.211 Show that if A is $n \times n$, and h is an $n \times 1$ column vector, then

$$h'Ah =$$

The expression $h'Ah$ is called a quadratic form. These expressions appear in the theory of maxima and minima.

1.212 Show that if $h'Ah < 0$ for any vectors $h \neq 0$, then (among other things) the diagonal elements of A are all negative; that is, $a_{ii} < 0$, $i = 1, \ldots, n$.

1.213 Prove that if A is an orthogonal matrix, $A'A = I$; that is, $A' = A^{-1}$.

1.214 Prove that if the rows of a square matrix A are orthogonal and have unit length, the columns likewise have these properties.

SELECTED REFERENCES

The implicit function theorem can be found in any advanced calculus text. Classic references are:

Apostol, T.: *Mathematical Analysis,* Addison-Wesley Publishing Co., Inc., Reading, MA, 1957.

Courant, R.: *Differential and Integral Calculus,* 2d ed., vols. 1 and 2, Interscience Publishers, Inc., New York, 1936.

Matrices and determinants are the subject of any linear, or matrix, algebra text. Perhaps the clearest and most useful for economists is:

Hadley, G.: *Linear Algebra,* Addison-Wesley Publishing Co., Inc., Reading, MA, 1961.

Samuelson, P. A.: *Foundations of Economic Analysis,* Harvard University Press, Cambridge, MA, 1947.
The first systematic exposition of the application of the implicit function theorem in economic methodology.

CHAPTER

6

COMPARATIVE STATICS: THE TRADITIONAL METHODOLOGY

6.1 INTRODUCTION; PROFIT MAXIMIZATION ONCE MORE

In this chapter we shall begin the general comparative statics analysis of economic models that contain an explicit maximization hypothesis. The focus, as always, will be on discovering the structure that must be imposed on the models so that useful, i.e., refutable, hypotheses are implied. A very powerful methodology, *duality theory,* has been developed for some important models such as profit maximization, constrained cost minimization, and utility maximization subject to a budget constraint. These new methods provide a vast simplification and clarification of the traditional methodology for those models; we shall explore them in the next chapter. In order to analyze models other than the three just mentioned, however, and to better appreciate the newer methods, it is still necessary to understand the traditional methodology. It is to that task that we now turn.

Comparative statics of economic models involving more than one variable requires the solution to simultaneous linear equations in the partial derivatives of the choice variables with respect to the parameters. We shall employ elementary matrix manipulations and Cramer's rule in order to systematically write down the solutions to the first-order equations. In that way, the structure of these models can be most efficiently explored.

Consider again the profit-maximizing firm analyzed in Chap. 4, and recall Eqs. (4-19):

Ph 3w>i

*dx**

9wi *dw* (4-19)

In matrix form these equations appear as

Pfn *pf* (6-1)

Pf21 n r

$\left(\frac{\partial x_2^*}{\partial w_1}\right)$ *

9wi

Pf n	A/2 2	whe re	*H* =
H	*H*		

Pfu *pfn* *Pf21* *Pf22*

(6-2)

This is Eq. (4-20a), which was derived by algebraic manipulations. Notice that the term 1 on the right-hand side of (6-1) will always appear in column *i,* in the solution for *dx* /dw j.* If the numerator is expanded by that column, it is immediately apparent that Eqs. *(4-20a-d)* can be written as

dw *H* *ij = 1 , 2* (6-3)

where *Hji* is the cofactor (signed, of course) of the element in the yth row and *ith* column. In this model, H_n = *pfn, H22 = pfu, Hn — H₂\ =* —*pfn-* Notice, too, that *H* = p^2*(fwf22 — ffyi* an<^ mat *H > 0,* from the second-order conditions (4-15). This is in fact indicative of a trend; determinants will play a crucial role in the theory of maxima and minima.

In like fashion, Eqs. (4-21), dealing with changes in the factor utilizations due to output price changes, can be written

(Pfu) */dxl\ dp*

*dx**

-h (6-4)

where the expression *{pf_ij)* stands for the 2 x 2 matrix in the left-hand side of (6-1). It is obvious from Cramer's rule that the solutions for *dx*/dp* and *dx^/dp* will involve the "off-diagonal" terms of *pf\₂* and / 7/21. Since the sign of these (equal) terms is not implied by maximization, we immediately suspect that no sign will emerge for

dx^*/dp, etc., and hence no refutable hypotheses concerning the responses of inputs to output price changes will emerge.

The two-factor, profit-maximizing firm is an example of a maximization model with two choice variables. The most general form of such models is[†]

maximize

$$f(x_1, x_2, \alpha) \tag{6-5}$$

where the choice variables are x_1 and x_2 and α is a parameter, or perhaps a vector of parameters, $\alpha = (\alpha_1, \ldots, \alpha_m)$. The first-order necessary conditions implied by (6-5), usually called the *equilibrium conditions,* are

$$\begin{aligned} f_1(x_1, x_2, \alpha) &= 0 \\ f_2(x_1, x_2, \alpha) &= 0 \end{aligned} \tag{6-6}$$

The sufficient second-order conditions are

$$f_{11} < 0 \qquad f_{22} < 0 \qquad f_{11}f_{22} - f_{12}^2 > 0 \tag{6-7}$$

Equations (6-6) are two equations in three variables, x_1, x_2, and α. The sufficient second-order conditions imply, by the implicit function theorem, that these equations can be solved for the explicit choice functions

$$\begin{aligned} x_1 &= x_1^*(\alpha) \\ x_2 &= x_2^*(\alpha) \end{aligned} \tag{6-8}$$

It should always be remembered that Eqs. (6-8) are the *simultaneous* solutions of (6-6). As the parameter α changes, *both* x_1 and x_2 will in general change. Substituting (6-8) back into (6-6), the identities from which the comparative statics are derivable are obtained:

$$\begin{aligned} f_1(x_1^*(\alpha), x_2^*(\alpha), \alpha) &\equiv 0 \\ f_2(x_1^*(\alpha), x_2^*(\alpha), \alpha) &\equiv 0 \end{aligned} \tag{6-9}$$

Differentiating this system with respect to α, the following system is obtained.

$$\begin{aligned} f_{11}\frac{\partial x_1^*}{\partial \alpha} + f_{12}\frac{\partial x_2^*}{\partial \alpha} + f_{1\alpha} &\equiv 0 \\ f_{21}\frac{\partial x_1^*}{\partial \alpha} + f_{22}\frac{\partial x_2^*}{\partial \alpha} + f_{2\alpha} &\equiv 0 \end{aligned} \tag{6-10}$$

In matrix form, this system is

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x_1^*}{\partial \alpha} \\ \dfrac{\partial x_2^*}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} -f_{1\alpha} \\ -f_{2\alpha} \end{pmatrix}$$

^The function/ here refers to the whole maximand, not just the production function part of the previous objective function.

Solving by Cramer's rule,

a

$$\frac{dx^*}{da}$$

$$\frac{dx^*}{3a}$$

mif

H =

Equations (6-12) represent the most general comparative statics relations for unconstrained maximization models with two choice variables. Not surprisingly at this level of generality, no refutable hypotheses are implied. Certain information is available, though. The denominators H in Eqs. (6-12) are positive. In addition, /n> /22 are negative. This information is provided by the sufficient conditions for a maximum.

The other information that is available is provided by the actual structure of the model.

Specifically, to be useful, a model must be constructed so that the effects of the parameters on the objective function, and hence the first-order equations, will in general be known. That is, f_{1a} and f_{2a} will have an assumed sign, or else the model is simply not specified well enough to yield any results. In the preceding profit maximization model, for the factor prices (recall, / in that model designates only the production function, not the whole objective function),

$$fa = Kiw \quad i = -1 \qquad ($$

6 - and

$n\backslash_{wi} = n_{lw\{} = 0$

The parameter wi, for example, appears only in the first first-order equation, $Ti\backslash = 0$. That is, $/_{2a} = 0$, in Eqs. (6-10). For that reason, the term involving the cross-partial $/_{12}$ in Eq. (6-12a) is 0. Since $7i_{iwi} = —1$, the result *dx\/dw* < 0 is obtained for the profit maximization model.

Similarly, for vt>2, /2a = — 1, *f\a* = 0. Hence, in Eq. (6-12Z?), the only remaining term on the right-hand side is *—f^fn/H.* From the second-order conditions, $dx2/dw_2 < 0$ is implied.

The situation is different for the parameter/), output price. Output price enters *both* first-order equations (6-10). Therefore, the indeterminate cross-term $/_{12}$ appears in the expressions for *dx*/dp* and *dx^/dp.* As a result, no refutable hypotheses emerge for this parameter with regard to each input.

The preceding analysis suggests that refutable comparative statics theorems will be forthcoming in a maximization model only if a given parameter

ente
rs
one

and only one first-order equation. This result, known as the *conjugate pairs theorem,* will be shown in greater generality in the succeeding sections. From Eqs. (6-12), if some parameter a_i enters only the *ith* first-order equation, then $\partial x_i^*/\partial a_i$ and f_{ia_i} must have the same sign. This can be expressed as

$$f_{ia_i}\frac{\partial x_i^*}{\partial a_i} > 0 \qquad (6\text{-}15)$$

Virtually all of the comparative statics results in economics are specific instances of Eq. (6-15), where some parameter a_i enters only the ith first-order equation.

6.2 GENERALIZATION TO *n* VARIABLES

Let us now investigate how the two-factor, profit maximization model is generalized to *n* factors. We must first derive the first- and second-order conditions for an unconstrained maximum (and minimum). We will then use the profit maximization model to motivate and illustrate the general methodology of comparative statics.

First-Order Necessary Conditions

As we noted in Chap. 4, the necessary first-order conditions for $y = f(x_1, \ldots, x_n)$ to have a stationary value are that all the first partials of f equal zero; that is, $f_i = 0$, $i = 1, \ldots, n$. This is a straightforward and intuitive generalization of the two-variable case. The second-order conditions, however, are a bit more complex.

Second-Order Sufficient Conditions

Using a Taylor series approach, as was done in the Appendix to Chap. 4, it can be shown that a sufficient condition for $y = f(x_1, \ldots, x_n)$ to have a maximum at some stationary value is that for all curves, $y(t) = f(x_1(t), \ldots, x_n(t))$, $y''(t) < 0$. Using the chain rule, this sufficient condition is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt} < 0$$

for all dx_i/dt, dx_j/dt not all equal to 0.

A square matrix (a_{ij}) which has the property that

for all nontrivial (not all 0) h_i, h_j is said to be *negative definite.* (If the strict inequality is replaced by "< 0," the matrix is called *negative semidefinite.)* Similarly, (a_{ij}) positive definite (semidefinite) means that the sum in (6-17) is strictly positive (nonnegative) for all nontrivial h_i, h_j. Thus, if at a point where $f_i = 0$, $i = 1, \ldots, n,$ the matrix of second partials of f (called the *Hessian* matrix) is negative definite, then $f(x_1, \ldots, x_n)$ has a maximum there. If the Hessian matrix is positive definite there, a minimum exists. If the Hessian is negative semidefinite, then/ definitely does *not* have a minimum, but it is not possible to say whether / has a maximum or some sort of saddle point at the stationary value. An expression of the form (6-17), in matrix form h Ah, is *called* a *quadratic form.*

Geometrically, negative definiteness of the Hessian matrix

//n • • • *fu*

H= i

V/nl *fnn'*

ensures that the function / will be *strictly concave* (downward). If H is positive definite,/is strictly convex.

> **Example.** Consider the function *y = (x₂ — x^)(x₂ — 2x])* depicted in Fig. 4-1 of Chap. 4. This is a function that has a minimum at the origin when evaluated along all straight lines through the origin, yet the function itself does *not* have a minimum there. The Hessian matrix of second partials is
>
> /24*?-6*₂
>
> -6,,
>
> V - 6x, 2
>
> At the origin, this matrix is
>
> This matrix is clearly positive *semidefinite:*
>
> **2 2**
>
> When *hi* — anything, *h₂* — 0, this quadratic form *Q* = 0; when *h₂* ^ 0, *Q* > 0.

In the two-variable case, *y — f(x₁, x₂),* the sufficient second-order conditions for a maximum, (6-16), imply that *fₙ* < 0, *f₂i < 0,* and *fnfn — fn* > ^₅ as was shown in Chap. 4. Note that this last expression can be stated as the determinant of the cross-partials of the objective function,

/n *fn* >Q

/21 /22

Note also that the conditions ***f*** 1, /₂₂ < 0 relate to the diagonal elements of that determinant. The theory of determinants allows a very simple statement of the sufficient second-order conditions for _y = *f{x\,* ..., *xₙ)* to have a maximum. First, consider the following construction:

> **Definition.** Let A_n be some nth-order determinant. By a "principal minor of order *k"* of A_n we mean that determinant which remains of A_n when any *n — k* rows and the *same numbered columns* are eliminated from A_n.

For example, if some row, row *i,* is eliminated, then to form a principal minor of order *n* — 1, *column i* must be eliminated. Since there are *n* choices of rows (and their corresponding columns) to eliminate, there are clearly n principal minors of order n — 1 of A_n. If, say, rows 1 and 3 and columns 1 and 3 are eliminat ed, then a principal minor of order n — 2 remains.

There are (~) = $n(n$ — l)/2! of these, and in general = $n\backslash/kl(n — k)\backslash$ principal minors of order k [or order $(n — k)]$. Note that the first-order principal minors of A_n are simply the diagonal elements of A_n, and the second-order principal minors are the set of 2 x 2 determinants that look like

The resemblance of this determinant to the 2 x 2 determinant of cross-partials of a function $f(x\backslash, x_2)$ provides the motivation for the following theorem.

Theorem. Consider a function $y — f\{x_x, \ldots ,x_n)$ that has a stationary value at x = x°. Consider the Hessian matrix of cross-partials of/, (/;). Then if all of the principal minors of |(/),)| of order k have sign (—1)*, for all k = 1, ..., n $(k — n$ yields the whole determinant, |(/i;)|) at x = x°, then $f(x\backslash, \ldots, x_n)$ has a maximum at x = x°. If all the principal minors of |(/)7)| are positive, for all k = 1, ..., n, at x = x°, then $f(x\backslash, \ldots ,x_n)$ has a minimum value at x = x°. If any of the principal minors has a sign strictly opposite to that stated above, the function has a saddle point at x = x°. If some or all of the principal minors are 0 and the rest have the appropriate sign given in the preceding conditions, then it is not possible to indicate the shape of the function at x = x°. (This corresponds to the 0 second-derivative situation in the calculus of functions of one variable.)

The theorem as stated is the form in which we shall actually use the result. However, it is somewhat overstated. Consider the "naturally ordered" principal minors of an n x n Hessian,

l/i /1 33 /

/1 /

Recall that in the two-variable case, /11 < 0 and $f\backslash \backslash f_{22}$ — $f\backslash_{2}$ > 0 implies $f_{2}2$ < 0. In fact, if all of these naturally ordered principal minors have the appropriate sign for a maximum or minimum of $f\{x\backslash, ..., x_n)$, then all of the other principal minors have the appropriate sign. Thus, the theorem as stated is in some sense "too strong"; i.e., more is assumed than is necessary, but we shall need the sufficient condition that *all* principal minors of order k have sign (—1)* for a maximum, or that they are *all* positive for a minimum.

There are several inelegant proofs of this theorem, one by completing a rather gigantic square a la the proof used in Chap. 4, and an elegant proof based on matrix theory, a proof that is beyond the level of this book.^ Hence, no proof will be offered.

eorge Hadley, *Linear Algebra,* Addison-Wesley Publishing Co., Inc., Reading, MA, 1961.

It is hoped that the discussion of the two-variable case will have at least made the theorem not implausible.

Profit Maximization: *n* Factors

Consider the profit-maximizing firm with n factors of production. The objective function, again, is

maximize

K

T

*

The firm equates the value of marginal product to the wage at every margin, i.e., for every factor or input. This is a straightforward generalization of the two-variable case. These equations represent n equations in the n decision variables $x_1, \ldots, x_n$ and $n + 1$ parameters $w_1, \ldots, w_n, p$. If the Jacobian determinant is nonzero, i.e.,

$$J = d$$

(

6- (6 -

then at this stationary value, these equations can be solved for the explicit choice functions, i.e., the factor demand curves,

X

The sufficient conditions for a maximum are that the principal minors of $(\pi_{ij}) = (pf_{ij})$ alternate in sign, i.e., have sign $(-1)^k$, $k = 1, \ldots, n$. Since $p > 0$, this is equivalent to saying that the principal minors of the matrix of second partials of the production function,

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & & & f_{2n} \\ \vdots & & & \vdots \\ f_{n1} & & \cdots & f_{nn} \end{pmatrix}$$

alternate in sign. Specifically, this means that, among other things, the diagonal terms are all negative, that is, $f_{ii} < 0$, $i = 1, \ldots, n$. This says that there is diminishing marginal productivity in each factor. In addition, all $n(n-1)/2$ second-order determinants

$$\begin{vmatrix} f_{ii} & f_{ij} \\ f_{ji} & f_{jj} \end{vmatrix} > 0$$

The "own-effects" dominate cross-effects in the sense that $fnfj\ j - fjj > 0$, $i, j = 1, \ldots, n$, $i =fc\ j$. Then there are all the remaining principal minors to consider; these are not easily given intuitive explanations.

The sufficient second-order conditions say that in a neighborhood of a maxi-mum point, the objective function (in this example

, this is equivalent to the production function) must be strictly concave (downward). The conditions $f_{ii} < 0$ ensure that the function is concave in all the two-dimensional planes whose axes are y and some JC, . The second-order principal minors relate to concavity in all possible three-dimensional subspaces y, x,, Xj. But concavity in all of these lower-order dimensions is not sufficient to guarantee concavity in higher dimensions; hence, all the orders of principal minors, including the whole Hessian determinant itself, must be checked for the appropriate sign.

In terms of solving for the factor demand curves, the *sufficient* second-order conditions guarantee that this is possible. The «th-order principal minor, i.e., the determinant of the entire (7T,7) matrix, has sign (—1)" ^= 0 by these sufficient conditions. But this determinant is precisely the Jacobian of the system (6-18); hence, applying the implicit function theorem, the choice functions (6-20) are derivable from (6-18).

Substituting the choice functions (6-20) back into (6-18) yields the identities

$$pf_i(x^*, \dots, x^*) - w_i = 0 \quad i = \qquad (6\text{-}21)$$

To find the responses of the system to a change in some factor price w_j, differentiate (6-21) with respect to W_j. This yields the system of equations

pfl dx*

dx dXn dw

In matrix notation, this system is written

P f l P f l n d x t \ d w ■ (6-22)

\ P v v

where the 1 on the right-hand side appears in row j. Solving for dx^*/dw_j by Cramer's rule involves putting the right-hand column in column i of the $|(p_{ij})|$ determinant, in the numerator, i.e.,

$$Pfu \quad 0 \quad pf_{ln}$$

$$Pfn\backslash \quad 0 \quad pf_{nn} \qquad H \tag{6-23}$$

where $H = \backslash pfc |$, the Jacobian determinant of second partials of n. Expanding the numerator by the cofactors of column i,

$$dWj \qquad H \tag{6-24}$$

where Hji is the cofactor of the element in row j and column / of H.

In general, H has sign (— \)^n by the sufficient second-order conditions for a maximum. For $i =\pounds j$, however, the sign of H_ij is not implied by the maximum conditions. Thus, in general, no refutable implications emerge for the response of any factor to a change in the price of some *other* factor. However, when $i = j$,

$$H \tag{6-25}$$

i.e The cofactor Ha is a principal minor; by the maximum conditions " \ it has sign (— 1) ., opposite to the sign of H. Thus,

$$— — < 0 \quad H \quad \bullet \quad 1 \qquad l = 1, \ldots, n \tag{6-26}$$

As in the two-factor case, the model does yield a refutable hypothesis concerning the slope of each factor demand curve. The response of any factor to a change in its *own* price is in the opposite direction to the change in its price. Finally, from the symmetry of H, using Eq. (6-24),

$$dwj \quad H \qquad dx^* \qquad H \tag{6-27}$$

The reciprocity conditions thus generalize in a straightforward fashion to the n-factor case. Since the parameter p enters each first-order equation (6-18), no refutable hypotheses emerge for the responses of factor inputs to output price changes. The matrix system of comparative statics relations obtained from differentiating (6-18) with respect *top* are [compare Eqs. (4-21), Chap. 4]:

$$\begin{matrix} Pfx & Pfln & \backslash \ /dx^*\backslash \ dp & -/, \\ Pfn & Pfnn & UAn \ \mathrm{V}\ dp\ ) & -fn \end{matrix} \tag{6-28}$$

Solving by Cramer's rule for *dx*/dp,*

-A = _ *SJ2 ILJL* ^ o (6-29)

It can be shown that if/? increases, then at least one factor must increase, but this is precious little information.

Finally, the supply function of this competitive firm is defined as

$$y = f(x^*(w, p), \ldots, x^*(w, p)) = y^*(w_1, \ldots, w_n, p)$$

where w is the vector of factor prices *(w\\, ..., w_n)*. It can be shown that

*By**
> 0 (6-30)
dp

i = l, . . . , n (6-31)

and

*dy** _
*dx**
dWi *dp*

We shall leave these results to a later chapter, as they are difficult to obtain by the present methods and outrageously simple by methods involving what is known as the *envelope theorem,* which will be discussed later.

We now state an interesting theorem without proof and apply it to the profit maximization model.

> **Theorem.** Let *H* be an *n* x *n* negative definite matrix (whose diagonal elements are all necessarily negative) and whose off-diagonal elements are all positive. Then the inverse matrix H^{-1} consists entirely of negative entries.

This theorem is evident upon inspection for the 2 x 2 and 3 x 3 cases; however, we have found no simple proof for the general case. The proof is an application of what are known as the Perron-Frobenius theorems. We refer the reader to A. Takayama's text^ for discussion and proof of these propositions.

Consider the application of this theorem to the profit maximization model. For changes in some wage w_j, we get the matrix equation (6-22) above. Let *b* be the column vector on the right-hand side of this equation; it consists of zeros in every row except row *j,* in which the element +1 appears. The solution to this equation, in matrix form, is *dxi/dwj* = $H^{-1}b$. Since every element of H^{-1} is negative and *b* is either 1 or 0, *dxi/dwj* < 0, *i, j* = 1, ..., *n.* In the two-variable model, we showed that the sign of *dxi/dw*$_2$ is the same as the sign of —*fn*- With only two factors, technical complementarity (f_{12} > 0) is the same as complementarity defined in terms of the change in the use of one factor as the price of the other factor changes.

^*Mathematical Economics,* 2d ed., Cambridge Press, Cambridge, England, 1985, pp. 392ff.

However, if more than two factors are present, one cannot infer that if, say, /n > 0, then *dxi/dwi* < 0; the signs of the other cross-partials of the production function matter. The above theorem shows, however, that if *all* the factors are complements in the sense of f_{ij} > 0, then dx_i/dwj < 0 for all the factors.

Likewise, consider Eq. (6-28) for the responses to a change in output price. Assuming the marginal products of each factor are positive, the solution of this equation is the matrix product of *H~* which has only negative elements, and the column vector of the negatives of the marginal products of each factor. It therefore follows that *dxi/dp* > 0 for all factors; i.e., there are no inferior factors with these assumptions.

6.3 THE THEORY OF CONSTRAINED MAXIMA AND MINIMA: FIRST-ORDER NECESSARY CONDITIONS

In most of the maximization problems encountered in economics, a separate, additional equation appears that constrains the values of the decision variables to some subspace of all real values, i.e., some subspace of what is referred to as Euclidean n-space. For example, in the theory of the consumer, individuals are posited to maximize a utility function, $U(x\backslash, x_2)$, subject to a constraint that dictates that the consumer not exceed a certain total budgetary expenditure. This problem can be stated more formally as

maximize

$$U(xi,x_2) = U \qquad (6\text{-}32)$$

subject to

$$+ p_2x_2 = M \qquad (6\text{-}33)$$

where JCI and *X2* are the amounts of two goods consumed, $p\backslash$ and p_2 are their respective prices, and *M* is total money income. This problem can be solved simply by solving for one of the decision variables, say x_2, from the constraint and inserting that solution into the objective function. In that case, an unconstrained problem of one less dimension results: From (6-33),

$$JC(JC) \quad x \quad + \qquad (6\text{-}34)$$

$$2(I) \quad \prime \quad Pi \quad Pi$$

Since once $x\backslash$ is known, x_2 is known also from the preceding, the problem reduces to maximizing $U(x\backslash,x_2(xi))$ over the one decision variable JCI. This yields

$$\frac{dU}{dx\backslash} \quad \frac{3U}{dx\backslash} \quad \frac{dU}{dx_2}\frac{dx_2}{dx} = u_x + u_2 — Pi$$

$= 0$

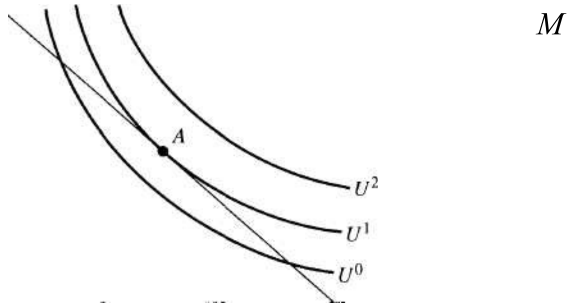


FIGURE 6-1 *Utility Maximization*. In this diagram, three indifference levels are drawn, with $U^2 > U^1 > U^0$. The line *MM* represents a con-sumer'

s budget constraint. The constrained utility maximum occurs at point A, where the indifference curve is tangent to (has the same slope as) the budget constraint. The second-order conditions for a maximum say that the level curves of the utility function, i.e., the indifference curves, must be convex to the origin; i.e., the utility function must be "quasi-concave" (in addition to strictly increasing).

or

$$\frac{U_1}{U_2} = \frac{p_1}{p_2} \tag{6-35}$$

This is the familiar tangency condition that the marginal rate of substitution ($-U_1/U_2$, the rate at which a consumer is *willing* to trade off x_2 for x_1) is equal to the opportunity to do so in the market ($-p_1/p_2$, the slope of the budget line). The condition is illustrated in Fig. 6-1. Under the right curvature conditions on the utility function (to be guaranteed by the appropriate second-order conditions), point *A* clearly represents the maximum achievable utility if the consumer is constrained to consume some consumption bundle along the budget line *MM*. The more general constrained maximum problem,

maximize $f(x_1$

subject to

$$g(x_1, \ldots, x_n) = 0$$

can be solved in the same way, i.e., by direct substitution, reducing the problem to an unconstrained one in $n - 1$ dimensions. However, a highly elegant solution that preserves the symmetry of the problem, known as the method of Lagrange multipliers (after the French mathematician Lagrange), will be given instead. The proof of proceeds along the lines developed earlier for unconstrained maxima.

Consider the behavior of the function $f(x\backslash, \dots ,x_n)$ along some differentiate curve $x(t) = (x\backslash(t), \dots, x_n(t))\backslash$ that is, consider $y(t)$ — $f(x\backslash(t), \dots, x_n\{t))$. If $y'(t) = 0$ and $y''(t) < 0$ for every feasible curve $x(t)$, then $f(x\backslash, \dots, x_n)$ has a maximum at that point. However, in this case, $x(t)$ cannot represent *all* curves in n-space. Only those curves that lie in the constraint are admissible. This smaller family of curves comprises those curves for which $g(x\backslash(t),\dots, x_n(t)) = 0$. Notice the identity sign—we mean to ensure that $g(x\backslash, \dots, x_n)$ is *Ofor every point along a* *given curve x(t),* not just for some points. The problem can be stated as follows: maximize

$$\tag{6-36}$$

subject to

$$g(x_1(t), \ldots, x_n(t)) = 0 \tag{6-37}$$

Setting $y'(t) = 0$ yields

$$f_1 \frac{dx_1}{dt} + \cdots + f_n \frac{dx_n}{dt} = 0 \tag{6-38}$$

for all values of the dx_i/dt that satisfy the constraint. What restriction does $g(x_1(t), \ldots, x_n(t)) = 0$ place on these values? Differentiating g with respect to t yields

$$g_1 \frac{dx_1}{dt} + \cdots + g_n \frac{dx_n}{dt} = 0 \tag{6-39}$$

In the unconstrained case, the expression (6-38) was zero for *all* dx_i/dt; thus, in that case $f_i = 0$, $i = 1, \ldots, n$, was necessary for a maximum. Here, however, (6-38) *and* (6-39) must hold simultaneously. Hence, the values of dx_i/dt are not completely unrestricted. However, assuming $f_1 \neq 0$, we can write, from (6-38),

$$\frac{dx_1}{dt} = -\frac{f_2}{f_1}\frac{dx_2}{dt} - \cdots - \frac{f_n}{f_1}\frac{dx_n}{dt} \tag{6-40}$$

Similarly, from (6-39),

$$\frac{dx_1}{dt} = -\frac{g_2}{g_1}\frac{dx_2}{dt} - \cdots - \frac{g_n}{g_1}\frac{dx_n}{dt} \tag{6-41}$$

Subtracting (6-40) from (6-41) yields, after rearranging,

$$\left(\frac{g_2}{g_1} - \frac{f_2}{f_1}\right)\frac{dx_2}{dt} + \cdots + \left(\frac{g_n}{g_1} - \frac{f_n}{f_1}\right)\frac{dx_n}{dt} = 0 \tag{6-42}$$

and, what is more, this expression must be 0 for *all* $dx_2/dt, \ldots, dx_n/dt$. By eliminating one of the dx_i/dt's, the remaining dx_i/dt's can have unrestricted values. If $f_1 \neq 0$, $g_1 \neq 0$, then for any values whatsoever of $dx_2/dt, \ldots, dx_n/dt$, a judicious choice of dx_1/dt will allow (6-38) and (6-39) to hold. But since (6-42) holds for any values at all of $dx_2/dt, \ldots, dx_n/dt$, it must be true that the coefficients in parentheses are all 0; i.e., *fi/fi — gi/g\,* *i* = 2,. . . , «. In the case where all of the *f_i, g_i* are not 0, these conditions can be expressed simply as

$$4 = \sim \quad iJ = U \ldots, n \qquad (6\text{-}43)$$

These *n* — 1 conditions say that the level curves of the objective function have to be parallel to the level curves of the constraint. This is the familiar tangency condition, illustrated by the preceding utility maximization problem. The *n* — \ conditions (6-43) and the constraint (6-37) itself constitute the complete set of first-order conditions for a constrained maximum problem with one constraint. Of course, these first-order conditions are necessary for any stationary value—maximum, minimum, or saddle shape.

The above conditions can be given an elegant and useful formulation by constructing a new function *!£* called a Lagrangian, where

$$i£ = fix_u \ldots, x_n) + Xg(x_u \ldots, x_n)$$

The variable A is simply a new, independent variable and is called a Lagrange multiplier. ^ Note that *X* always equals / for values of *x\,* ..., x_n that satisfy the constraint. Thus, i£ can be expected to have a stationary value when/does. Indeed, taking the partials of *X* with respect to *X*,..., x_n and A and setting them equal to 0 yields

$$2_{)i} = g(x_1, \ldots, x_n) = 0$$

Eliminating A. from the first *n* equations of (6-44) (by bringing Xg_i over to the right-hand side and dividing one equation by another) yields

$$\frac{/\bullet}{fj} = \frac{8i}{Sj}$$

precisely the first-order conditions for a constrained maximum. Hence, the Lagrangian function provides an easy mnemonic for writing the first-order conditions for constrained maximum problems. However, we shall see that this is a most useful construction for the second-order conditions also, and, in the theory of comparative statics, the Lagrange multiplier A often has an interesting economic interpretation.

Example. Consider again the utility maximization problem analyzed at the beginning of this section. The Lagrangian for this problem is

$$££ = U(x\backslash, x_2) + X(M — p\backslash X\backslash —$$

P2X2) Differentiating *X* with respect to *i, x_2, and A yields

$$X_X = U_X - Xp_x = 0 \qquad \text{(6-45a)}$$

$$£_2 = U_2 - \hat{}Pi = 0 \qquad \text{(6-456)}$$

$$££_A = M - p_x x_x - p_2 x_2 = 0 \qquad \text{(6-45c)}$$

The partial $\&_k$ is simply the budget constraint again since *!£* is linear in A. The variable A can be eliminated from *(6-45a)* and (6-45£>) by bringing Xp_x, Xp_2 over to the

TIt is of no consequence whether one writes *!£* = / + *kg* or *i£,* = *f* — *kg;* this merely changes the sign of the Lagrange multiplier.

right-hand side and then dividing one equation by the other. This yields *U\/U₂ = P\/P2,* the tangency conditions (6-35) arrived at by direct substitution.

There are many problems in economics in which more than one constraint appears. For example, a famous general equilibrium model is that of the "small country" which maximizes the value of its output with fixed world prices, subject to constraints which say that the amount of each of several factors of production used cannot exceed a given amount. The general mathematical structure of maximization problems with *r* constraints is

maximize

$$f(x_l, \ldots, x_a) = y \qquad (6\text{-}46)$$

subject to

$$g^l(xi, \ldots, *„) = 0$$

$$! \qquad (6\text{-}47)$$

$$g^r(xi, \ldots, x_n) = 0$$

These are *r* equations where, of necessity, $r < n$. (Why?)

The first-order conditions for this problem can be found by generalizing the Lagrange multiplier method previously derived. Multiplying each constraint by its own Lagrange multiplier A/', form the Lagrangian

X = f{x_u ..., *„) + A.¹*¹ (*i. ..., *„) + ■ • ■ + Vy (*i. ■■-. *,.) (6-48)

Then the first partials of i£ with respect to the *n* 4- *r* variables x,, A/ give the correct first-order conditions:

% = f_i + l}g} + '~ + Xg^r_i = 0 / = 1, . . . , n (6-49)

£j = g^J = O j = l, . . . , r (6-50)

where *gj* means *dg^j*/3x,-. The proof of this can be obtained only by more advanced methods; it is given in the next section.

6.4 CONSTRAINED MAXIMIZATION WITH MORE THAN ONE CONSTRAINT: A DIGRESSION*

Consider the maximization

problem maximize

^ In order to understand this section, the student must be familiar with some concepts of linear algebra, such as rank of a matrix, etc., developed in the Appendix to Chap. 5. We are indebted to Ron Heiner for demonstrating this approach to the problem to us.

subject to

$$g^1(x_1,\ldots,x_n)=0$$

$$g^r(x_1,\ldots,x_n) = 0$$

Letting $x_i = x_i(t)$, $i = 1, \ldots, n$, as before, the first-order conditions for a maximum (or any stationary value) are

$$\frac{dy}{dt} = f_1\frac{dx_1}{dt} + \cdots + f_n\frac{dx_n}{dt} = 0 \qquad (6\text{-}51)$$

for any $dx_1/dt, \ldots, dx_n/dt$ satisfying

(6-52)

where $g^j_i = \partial g^j/\partial x_i$.

For any function $y = f(x_1, \ldots, x_n)$, the *gradient of f,* written $\nabla \mathbf{f}$, is a vector composed of the first partials of f:

$$\nabla \mathbf{f} = (f_1, \ldots, f_n)$$

The differential of f can be written

$$dy = \nabla \mathbf{f}\, \mathbf{dx}$$

where $\mathbf{dx} = (dx_1, \ldots, dx_n)$. Along a level surface, $dy = 0$, and hence ∇f is orthogonal to the direction of the tangent hyperplane. The gradient of f, $\nabla \mathbf{f}$, thus represents the direction of maximum increase of $f(x_1,\ldots, x_n)$.

Note that Eq. (6-51) is the scalar product of the gradient of f ∇f, and the vector $\mathbf{h} = (h_1, \ldots, h_n) = (dx_1/dt, \ldots, dx_n/dt)$. Likewise, Eqs. (6-52) are the scalar products of the gradients of the g^j functions, ∇g^j, and h. Let ∇g denote the $r \times n$ matrix whose rows are, respectively, $\nabla g^1, \ldots, \nabla g^r$. Then Eqs. (6-51) and (6-52) can be written, respectively,

$$\nabla \mathbf{f} \cdot \mathbf{h} = 0 \qquad (6\text{-}53)$$

for all $\mathbf{h} \neq 0$ satisfying

$$(\nabla \mathbf{g})\mathbf{h} = 0 \qquad (6\text{-}54)$$

Assume now that the matrix ∇g has rank r, equal to the number of constraints. This says that the constraints are independent, i.e., there are no redundant constraints. If the rank of ∇g was less than r, say $r - 1$, then one constraint could be dropped and the subspace in which the dx_i/dt could range would not be affected. It is as if a ration-point constraint were imposed with the ration prices proportional to the original money prices. In that case, the additional rationing constraint would either be redundant to or inconsistent with the original budget constraint.

Assuming rank Vg = *r,* the rows of Vg, that is, the gradient vectors Vg^7 — C?i' • • •' £«)» *j* — 1, • • •, r, form a basis for an r-dimensional subspace E_r of E_n, Euclidean n-space. From (6-54), the admissible vectors h are all orthogonal to E_r; hence, they must all lie in the remaining *n* — *r* dimensional space, E'_r. However, from (6-53), Vf is orthogonal to all those h's and hence to E'_r. Hence, Vf must lie in E_r. Since the vectors Vg^7 form a basis for E_r, Vf can be written as a unique linear combination of those vectors, or

$$\text{Vf} = k^l\ \text{Vg}^1 + \bullet \bullet \bullet + \text{V Vg}^r \qquad (6\text{-}55)$$

However, this is equivalent to setting the partial derivatives of the Lagrangian expression i£ = / — *Y2 ^8^j* ^{w}i^m respect to *X\, ..., x_n* equal to 0.

6.5 SECOND-ORDER CONDITIONS

In the past two sections, the first-order necessary conditions for a function to achieve a stationary value subject to constraints were derived. Those conditions are implied whenever the function has a maximum, a minimum, or a saddle shape (a minimum in some directions and a maximum in others). We now seek to state sufficient conditions under which the type of stationary position can be specified. The discussion will be largely limited to the two-variable case, with the general theorems stated at the end of this section.

Consider the two-variable problem

maximize

$$f(x_u x_2) = y$$

subject to

$$g(x_l, x_2) = 0$$

The Lagrangian function is *!£(xi, x_2,* A.) = *f(x\, x_2)* + *kg(x\, x_2)*. The first-order conditions are, again,

£ $ £ > (6-56)

dt at at

for all *dxi/dt, dx_2/dt* satisfying

$$g1\wedge + g2\wedge = O \qquad (6\text{-}57)$$

dt dt

These conditions imply that 5£i = *f* + *kgi* = 0, *!£$_2$* = f_2 + Xg_2 = 0. Sufficient conditions for these equations to represent a relative *maximum* are that $d^2y/dt^2 < 0$, for all *dxi/dt, dx_2/dt* satisfying (6-57). Similarly, $d^2y/dt^2 > 0$ under those conditions implies a *minimum.* How can these conditions be put into a more useful form? Differentiating (6-56) again with respect to *t,* the sufficient second-order conditio

$$\frac{d^2y}{dt^2} \quad \frac{d^2x\backslash}{dt^2} \quad h\,\frac{d^2x_2}{dt^2} \quad r\ \left(\frac{dx}{}A^2 + \right. U \backslash d\,t\,) \quad \frac{dx_x}{J[1} \quad \frac{dx_2}{d\,t} \quad f \qquad 2\ 1 \quad 1 <$$

(6-

subject to

dt dt

Since (6-57) is an identity, differentiate it again with respect to *t,* remembering that g_x and g_2 are functions of *Xi(t),* $x_2(t)$. This yields

$$d^2x\backslash \quad d^2x_2 \quad (dx\backslash\backslash^2 \quad 2 \quad dx\backslash dx_2 \quad (dx_2\backslash^2 \qquad (6\text{-}59)$$

Now multiply (6-59) through by *k,* the Lagrange multiplier, and add to Eq. (6-58). Since this amounts to adding 0,

$$\frac{d^2y}{dt^2} = (/i + \hat{}\ i)\frac{d^2x_x}{dt^2} + (/2 + \hat{}\ 2)\frac{d^2x_2}{dt^2} + (/n + \left(\frac{dx_x}{dt}\right)^2$$

$$+ 2(/_{12} + Hn)\hat{}\ \hat{}\ + (/22 + \hat{}\ 22)\ f\ \hat{}\ ) < 0 \qquad (6\text{-}60)$$

$$dt \quad dt \qquad \backslash\ dt\ J$$

subject to (6-57). However, from the first-order conditions, i£i = $f\backslash + kg\backslash = 0$, $\hat{}_2 = f_2$ -\-kg_2 = 0. Also, /n +Ag_n is simply $££_u$, and likewise ^12 = /12 + ^12, etc. If we simplify the notation a bit and write Ai = $dx\backslash/dt$, $h_2 = dx_2/dt$, then the sufficient second-order conditions for a maximum are that

$$X_nh\backslash + 2X_nh_xh_2 + £6_{22}\hat{} < 0 \qquad (6\text{-}61)$$

for all *h],* h_2 not both equal to 0, such that

$$gihi+g_2h_2 = 0 \qquad (6\text{-}62)$$

For the case of *n* variables and one constraint, the derivations proceed along similar lines, producing

$$\hat{}jhihj<0 \qquad (6\text{-}63)$$

$$i=\backslash\ j=\backslash$$

for all *hj, hj* such that

$$\overset{n}{Y,8ihi=0} \qquad (6\text{-}64)$$

In this case the matrix of terms (if,7) is said to be *negative definite subject to constraint.*

Equations (6-61) and (6-62) can be combined into one useful expression: From (6-62),

Si
= —*hi* —
82

Substituting this into (6-61) yields

82 **< 0**
82

Or, by multiplying by *g*,

-2^28182 2 < 0 (6-65)

for any value of *h* 1 ^ 0. This implies that the expression in the parentheses must itself be < 0. How can that expression be conveniently remembered? It turns out, fortuitously, that the expression in parentheses in (6-65) is precisely the negative of the determinant

H = gl *82*
0 (6-66)
8 ***82***

as can be immediately verified by expansion of *H. Hence, a sufficient condition for f(x\, X2) to have a maximum subject to g(x\, Xj)* = 0 *is, together with the first-order relations, that H* > 0. Likewise, for a minimum subject to constraint, the sufficient second-order condition is that *H* < 0. Also, *H* = 0 corresponds to the case where the second derivatives $d^2y/dt^2 = 0$; hence no statement can be made regarding the type of stationary value in question. Note that *d*²^*£/dx\dX* = ££$_u$ = g_x = *&xi* and *!£*$_2$*x* = *&X2* = *gi,* and *!£*$_X$*x* — 0, since A. enters theLagrangian if = / + *kg* linearly. Hence, *H* is simply the determinant of the matrix of cross-partials of if with respect to JCI, x_2, and *X,* that is,

TJ ***D***
21

For the n-variable case, the situation is more complicated, but the rules are analogous to the unconstrained case. The Lagrangian is if = *f{x\,... ,x*$_n$*)* + *kg{xi, ... ,x*$_n$*).* Consider the matrix of cross-partials of if with respect to x_x,..., x_n *and X,* noting, as before, that if,^ = *gi,* if AX = 0:

Si\

H =
if
V Si **0/**

This matrix is commonly referred to as a *bordered Hessian* matrix, noting how the first partials of the constraint function g *border* the cross-partials of !£ with respect to JCI, ..., x_n.

Consider the following construction: By a *"border-preserving* principal minor of order *k"* of the preceding matrix, we mean that determinant which remains when any *n* — *k* rows and the same numbered columns are deleted, *with the special added proviso that the border itself not be deleted.* Hence, the deletions that can occur must only come from rows 1 through *n,* not row or column *n* + 1. [Note that a border-preserving principal minor of order & i s a (£ + 1) x (£ + 1) determinant.]

The second-order sufficient conditions are then:

Theorem. Together with the first-order conditions ££, = 0, *i* = 1,..., *n* and ££x = *g* = 0, if all the border- preserving principal minors of //of order *k* have sign *(—l)*^{k}*,k* = 2, ... *,n,* then a maximum position is obtained. If all the border-preserving principal minors are negative, *k = 2,... ,n,* then a minimum is obtained.''

Suppose, even more generally, that there are r constraints involved. The Lagrangian function is ££ = /(JCI, *... ,xn) + Yl)=\ ^j8j(xii • •• > xn)-* The bordered Hessian matrix of this Langrangian is

'i n 8 i

8l

0 0/

The sufficient conditions here state that for a *minimum,* the border-preserving principal

minors of order *k* > *r* (which again must involve deletions only from rows 1 through *n)* have sign (—1)^r, whe

*In fact, if only the "naturally ordered" principal minors have this property, then *all* of the border-preserving principal minors have that property.

TABLE 6-1
Second-order conditions: Sign of all size *m* x *m* (border-preserving) principal minors

					Constra			
Conditi	0			1		***r***		
Maxim um	*m* =	1	.. , *n*	***m* = 3**		***m* —**	**1**	**., *n* + *r***
Minimu m	(- 1)	o	**+ 1**	(- D¹	**= -1**	(- D		

The Geometry of Constrained Maximization

We visualize an unconstrained maximization in three dimensions as the top of a hill; the surface must be concave there. Constrained maxima (or minima) are somewhat more subtle. Consider the problem in two variables:

maximize

$$f(xi,x_2) =$$

y subject to

$$g(x_u x_2) = k$$

The constraint $g\{x\backslash,x_2) = k$ represents a curve in the $X\backslash X_2$ plane; we typically think of it as a "frontier," i.e., some sort of boundary that constrains consumption or production. Assume that the first partials $g\backslash$ and g_2 are positive so that the frontier has a negative slope $(—g\backslash /g_2)$, and increases in k move the frontier "northeast" in the $x\backslash x_2$ plane. Three such frontiers are represented in Fig. 6-2: in panel *(a),* the frontier is concave, in panel *(b)* it is linear, and in panel *(c)* it is convex.

Assume that the first partials of $f(x\backslash, x_2)$ are also positive so that the level curves of/are likewise negatively sloping $(—f\backslash/f_2)$, and increasing values of/are associated with level curves that are increasingly distant from the origin. It is visually obvious that if the constrained maximum occurs at some interior point along the frontier (i.e., not at a corner, where the constraint intersects an axis), the maximum occurs where a level curve of $f(x\backslash, x_2)$ is tangent to the frontier. This is the algebraic condition $—f\backslash/f_2 = —gi/g2$, derived earlier. However, this tangency condition is implied by both a maximum and a minimum. If this condition is to represent a maximum, the level curves of the objective function must be either less concave than the constraint frontier, as shown in panel *(a),* or more convex than the frontier, as shown in panels *(b)* and *(c).*

If the constraint is linear, the level curves must appear "convex to the origin," the classic shape attributed to consumer's indifference curves and production iso-quants. However, this characterization is in fact imprecise. The essential property required of the objective function to guarantee a constrained maximum subject to a linear constraint is that $f(x\backslash, x_2)$ be strictly increasing and *quasi-concave.* This latter characteristic is defined as follows.

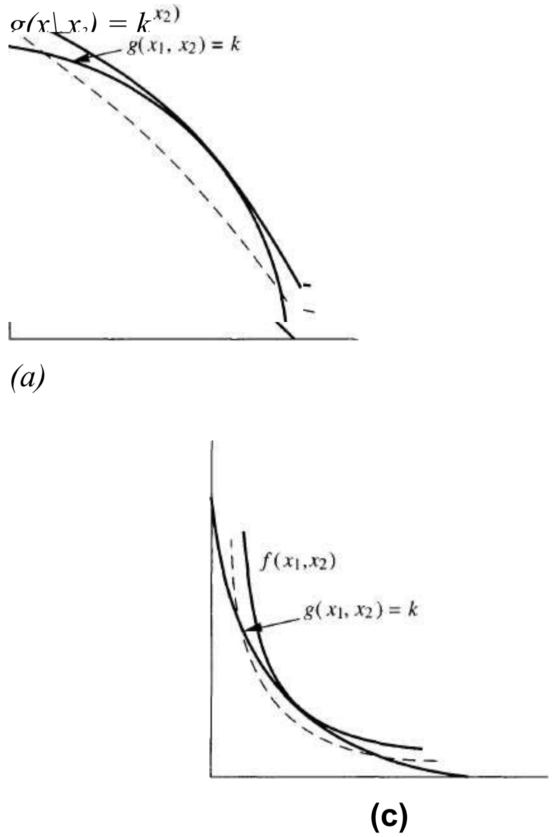
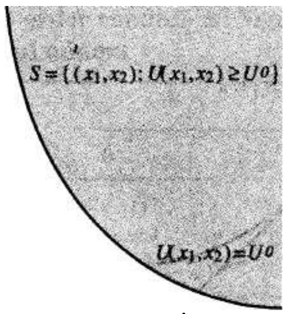


FIGURE 6-2
Constrained Maximization. Constrained maximization requires, for increasing functions, that the level curves of the objective function be either less concave or more convex than the level curves of the constraint. If the constraint is linear, as in panel *(b),* or convex, as in (c), the level curves of the objective function must be "convex to the origin"; i.e., the objective function must be quasi-concave.

Consider a typical indifference curve $U°$ as shown in Fig. 6-3. Consider the set, call it *S,* of points that are at least as preferred as a point on $U°$, shown as the shaded area. This set has the property that if any two points in *S* are connected by a straight line, the entire line also lies in *S. A* set with this property is called a *convex set* (not to be confused with a convex function). (As an example of a set that is not convex, consider the set of consumption bundles that are *less* preferred than those on $U°$.) Algebraically, if x° = (JC°, jc^andx^1 = (xj, JC]) are any two points in the $X\backslash x_2$ plane, $x' = t\backslash° + (1 — t)x^l$, $0 < t < 1$ represents all points on the straight line joining x° and x^1. *A function is called quasi-concave if the set of points for which the function takes on values greater than or equal to some arbitrary value comprises a convex set.* That is, $U(x\backslash,x_2)$ is quasi-concave if $U(x^l) > $ t/(x°) implies $U(tx° + (1 - Ox^1) > U(x°)$, $0 < t < 1$. (The definition is generalized in an obvious way for functions of *n* variables.) We note in passing that if the function *decreases* as the distance from the origin increases, quasi-concavity produces level curves that are "concave to the origin."



xi

FIGURE 6-3
Quasi-Concavity. A function is said to be quasi-concave if the set of points for which the function takes on values greater than or equal to some arbitrary amount, say, U^0, is a convex set. These points are represented by the shaded area. This is the property generally assumed for utility and production functions.

Recall from Chap. 2 that a concave function is one for which $f(tx^0 + (1-t)x^1) > tf(x^0) + (1-t)f(x^1)$, $0<t<1$. Concavity clearly implies quasi-concavity: assuming $f(x^1) > f(x^0)$, $f(tx^0 + (1-t)x^1) > tf(x^0) + (1-t)f(x^1) > tf(x^0) + (1-t)f(x^0) = f(x^0)$. The converse, however, is not true. Quasi-concavity is a weaker restriction than concavity. Concavity is required for an unconstrained maximum; quasi-concavity is all that is required for maximization subject to a linear constraint. In the preceding theorem, the second-order conditions dealing with the signs of the border-preserving principal minors define algebraically the geometric properties of the objective and constraint functions required for a constrained maximum (or minimum). If the constraint is linear, these second-order conditions for a maximum can be used to define algebraically the property of quasi-concavity of the objective function. (This requires the additional step of using the first-order conditions to replace the first partials of g with those of f in the bordering row and column.) If a linear objective function is *minimized* subject to constraint, these second-order conditions likewise describe quasi-concavity of the constraint function. This situation is encountered in Chap. 8, dealing with the minimization of cost subject to an output constraint. These concepts will be applied in the following chapters. Lastly, it is true, but not easy to prove, that if a function $f(x)$, $x = (x_1, \ldots, x_n)$ is quasi-concave and linear homogeneous, it is (weakly) concave. Also, if f is strictly quasi-concave and homogeneous of degree r, $0 < r < 1$, it is strictly concave. The proofs are left as exercises.

Example. Consider again the basic consumer theory model, maximize $U(x_1, x_2)$ subject to $p_1x_1 + p_2x_2 = M$. (See Fig. 6-1 again.) Assuming more is preferred to less, the ordinal indifference levels must be indexed such that $U^2 > U^1 > U^0$. The condition that a point of tangency of an indifference curve and the budget constraint actually represents a maximum rather than a minimum of utility subject to a linear budget constraint is that the utility function be strictly increasing and quasi-concave. In this two-variable model, these conditions imply the usual shape, "convex to the origin." These assumptions compose the law of diminishing marginal rate of substitution, i.e., in two dimensions, that the slope of the level (indifference) curve increases (becomes less negative) as x_x increases. We showed in Chap. 3 that the algebraic expression of this shape is [see Sec. 3.5, Eq. (3-24)]

$$u$$

$$>0 , J_2 X_2 = \-UjUn$$

If this is to be positive, the square-bracketed term must be positive, assuming that $U_2 > 0$, i.e., the consumer is not sated in good 2. But by inspection, the term in brackets is equal to the following determinant, which must therefore itself be positive:

$$H' = \begin{vmatrix} U_{11} & U_{12} & -U_1 \\ U_{21} & U_{22} & -U_2 \\ -U_1 & -U_2 & 0 \end{vmatrix} > 0 \qquad (6\text{-}68)$$

However, from the first-order conditions for utility maximization (6-45), $U_1 = \lambda p_1$, $U_2 = \lambda p_2$. Substituting this into H' and then dividing the last row and column by λ (and, hence, H' by λ^2, which is positive), the condition $H' > 0$ is equivalent to

$$H = \begin{vmatrix} U_{11} & U_{12} & -p_1 \\ U_{21} & U_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} > 0 \qquad (6\text{-}69)$$

But H is seen to be the determinant of the bordered Hessian matrix, the cross-partials of $\mathcal{L}$ with respect to x_1, x_2, and λ. This is in accordance with the general theorem of this section.

6.6 GENERAL METHODOLOGY

At the beginning of this chapter, we considered the general economic model that was characterized by being an *unconstrained* maximization. Let us now explore models that have a constraint as an added feature.

Consider some economic agent that behaves in accordance with the following general model:

m f

s g

where x_1 and x_2 are the decision variables and a is some parameter (or vector of parameters) over which the agent has no control. What will be the response to autonomous changes in the environment, i.e., to changes in the parameter a?

The first-order conditions for a maximum are derived by setting the partials of the Lagrangian function $\mathcal{L} = f(x_1, x_2, a) + \lambda g(x_1, x_2, a)$, with respect to x_1, x_2, and λ, equal to zero:

$,$ $x_2, a)$ $g(xi, x_2, a$ $) = 0$	$, x_2,$ $a) =$ $0 \;\; ,$ $x_2, a)$ $= 0$	(6- 72)

Equations (6-72) represent three equations in the four unknowns *X*, x_2, *k,* and *a.* Assuming the implicit function theorem (as was discussed previously) is applicable, these equations can be solved, in principle at least, for the choice functions

$$xi = x^*(a)$$
$$x_2 = x^*(ci) \qquad (6\text{-}73)$$
$$k = k^*(a)$$

Substituting these values back into Eqs. (6-72) from which they were derived yields the *identities*

$$f_2(xl\ x^*,\ a) + k^*{}_{g2}(xl\ x^*_2,\ \mathrm{a}) = 0\ g(x^*,x^*,a) \qquad (6\text{-}74)$$

=0

Since we are interested in *changes* in the x*'s (i.e., marginal values) as *a* changes, we differentiate (6-74) with respect to *a,* using the chain rule. The first equation then yields

d A.*g_n *dk"*

3a 3a da 3a *da*

Noting that ££n = $/_u$ + k^*g_n, ... ,this equation can be more conveniently written

(6-75)

'l₂ '12"

3a 3a 3a

Similarly, differentiating the second and third equations of (6-74) yields

-3a' T I" *g2~* = ~"= (6-76)

d 3a 3a

\+ *dx"* (6-77)

In matrix notation, this system of three linear equations can be written

(6-78) *g*

3J **-21 =^22 *gl***

C* *gl* *g2* 0

d **V 3a /**

Notice that the coefficient matrix on the left of (6-78) is the matrix of second partials of the Lagrangian function. In unconstrained maximization models, this coefficient matrix was the matrix of second partials of the objective function. The manipulation of the model is formally identical in the constrained and unconstrained cases; the only difference is the conditions imposed on the principal minors of the coefficient matrix by the sufficient second-order conditions. The reason why the coefficient matrix comes out to be the second partials of ££ is that identities (6-74) are precisely the first partials of ££,

(6-79)

[Notice that X^* does not appear in $!£_x = g(x^*, x\%, a) = 0$.] Differentiating the first identity with respect to a yields

dx"

EEQ

3a 3a
3a

This is precisely Eq. (6-75), noting again that $X_{xx} = g\backslash$. In like fashion, Eqs. (6-76) and (6-77) are derivable directly from $X_2 = 0$, $X_x = 0$.

Since the Jacobian determinant J needed to ensure solution of Eqs. (6-72) for the explicit choice functions (6-73) is formed from the matrix of first partials of (6-72), J is in fact the determinant of second partials of the Lagrangian ££ with respect to JCI, x_2, and X, that is, the determinant of the coefficient matrix in (6-78). This determinant is denoted by H below. The *sufficient* second-order conditions imply, among other things, that this determinant is nonzero, and thus the explicit relations (6-73) are valid. And this determinant forms the denominator in the solution by Cramer's rule for dx^*/da and dX^*/da. Let us now proceed, in the same manner as for the unconstrained models.

Solving for 3jtj*/3a by Cramer's rule,

(where H is the bordered Hessian determinant of the coefficient matrix. Solutions for Qjj/Oa and *dk*/da* are, likewise,

(6-81)

9a - 2a H 9 Si 82 - 0 22 H - 2a 9 H H S e

d x a? 21 g1 cp =^2 a 3 H *2laH* H cp

d	°	g2	?	–	^:	
X	*	-*I*	a	^1a#	2:	(

(6-82)

3a ***H*** ***H*** ***H*** ***H*** ***H***

It is clear that at this level of generality, no prediction as to the sign of *dx*/da* or *dX*/da* is forthcoming. There simply is not enough information in the system. All we know is that the denominators in these expressions are positive, but we have no information regarding the numerators. The signs of the off-diagonal cof actors are not implied by the maximum conditions.

Suppose now that the parameter *a* did not appear in either the second or third first-order relations (6-72). Then i£$_2$« = 0 and $g_a = 0$, and

d x * d a < \ a ^ 2 2 8 2 8 2 O H

The partial dx^*/da now has a predictable sign: Since $H > 0$ and $H_n < 0$, by the second-order conditions (here, H_n = —*g*\ < 0 always), 3x*/3a will have the same sign as the direction of "disturbance" of the first equation. That is, if an increase in *a* has the effect of shifting the marginal curve i£i to the right (*£*\ₐ > 0), then the response will be to increase the utilization of *X*\. Hence, if it is possible to make statements like, "an increase in income will *shift* a demand curve to the right," or "a change in technology will lower (shift down) such and such marginal cost curve," then if that income or technology parameter enters only one first-order relation, it will in general be possible to predict the direction of change of the associated variable (the one for which that first-order equation is the first partial of the Lagrangian). More succinctly, if a enters the *ith* first-order equation only, then *dx*/da* and %$_a$ have the same sign. However, since $g_a = 0$, 5£$_{ia}$ = f_{ia} + $Xg_{ia} = f_{ia}$, and thus, just as in the case of maximization models without constraints, *dxi/da* and f_{ia} must have the same sign, or

f (

This result holds for the case of *n* variables as well as for just two variables; its precise statement is given in the problems following. The result follows because of the conditions on the principal minors imposed by the second-order conditions for a constrained maximum.

In the case of 3A.*/3a, however, a sign is *never* implied by the sufficient second-order conditions alone, no matter how the parameter *a* enters the first-order equations. Suppose, for example, *a* enters only the constraint, i.e., the third first-order equation. Then — X_a = — X_{2a} = 0, and

3
a ***H*** (

6 - The cofactor $//_{33}$, while a principal minor, is not a *border-preserving* principal minor. The border row and column of *H* are deleted when forming $//_{33}$. Hence, no sign is implied for *dX*/da.* If *a* enters any of the other equations, then the off-diagonal cofactors *H^* and $//_{32}$ will enter the expressions. These expressions are likewise not signed by the maximum conditions.

For the same reasons, it is apparent that any time the parameter *a* enters the constraint, off-diagonal cofactors will be present in the expressions for *dxj/da*. Thus no refutable implications are possible in models for a parameter that appears in the constraint.

Example. To illustrate the principles just developed, let us return to the profit maximization model, slightly modified. Consider a firm with production y = $f\{x_x,x_2)$ selling output *y* at price *p*. The firm hires input x_x at wage w_x; x_2, however, represents the entrepreneur's input and is fixed at some level $x°$. The firm seeks to maximize net rents *R*, the difference between total revenue and the total factor cost of *X*. Algebraically, the model is

maximize

x_x,x_2

$$R = pf\{x_u x_2) - w_x x_x$$

subject to

$$x_2 — x_2$$

Although we have essentially solved this model in Chap. 4, by directly substituting the constraint into the objective function, we shall analyze it here as a constrained maximization model. Even though in this particular example the constraint says that x_2 is fixed, we treat x_2 as a variable, maintaining the structure of the Lagrangian analysis. Using the Lagrangian

$$££ = pf(x_u x_2) - W]X_X +$$

A(*2 - x_2) the first-order conditions are

$$<£, = pf_x(x_x,x_2) - \text{w}, = 0 \qquad ($$

$$\$2 = pfiixi,\ x_2) - X = 0 \qquad (6\text{-}866)$$

$$X_x = x°_2 - x_2 = Q \qquad (6\text{-}86c)$$

Equation (6-86a) says that the firm will hire x_x until the value of its marginal product of that factor equals its wage, as previously derived. Equation (6-866) identifies the Lagrange multiplier *X* as the value of the marginal product of the entrepreneurial input. Whereas the wage of factor 1 is exogenously set by the competitive labor market, the wage of factor 2 is endogenously "imputed." If a competitive market existed for entrepreneurial services, another firm would be willing to pay *X* for this owner's services.

The sufficient second-order condition is that the bordered Hessian determinant formed from the second partials of iS is positive:

$$H = \begin{vmatrix} Pfn & pfn & 0 \\ \mathbf{P/21} & \mathbf{A/22} & \mathbf{-1} \\ 0\ -\ 1 & 0 & \end{vmatrix} > 0 \qquad (6\text{-}87)$$

Evaluating this determinant (say, by the third row, which has two zeros in it) yields *pfn* < 0. Note that no restriction is placed on f_{22}; since only x_x is really variable (even though we treat x_2 as variable in the constrained model), the only margin on which the firm adjusts is how much x_1 to hire. Only diminishing marginal product of x_1 is thus required for an interior maximum.

Assuming the sufficient second-order condition holds, the first-order equations can be solved simultaneously for the explicit choice functions:

$$x_2 = x^*(w_1, p, x$$

(

These choice functions represent the factor demands for $x\backslash$ and x_2 (trivial, in the case of x_2; x_2 — $X j)$ and the profit-maximizing imputed value of entrepreneurial input. Multiplying Eq. (6-86a) by x^*, (6-86/?) by x_2, and adding,

$$P(f\backslash x^* + fix_2) = w_1 x^* + X^* x^*$$

(

8 If the production function is homogeneous of degree 1 (constant returns to scale), then from Euler's theorem, the left-hand side of this identity is *py**. In that case, (6-89) can be interpreted as Total Revenue = Total Cost, where the total factor cost of x_2 is its imputed opportunity cost X^*x_2. Thus with constant returns to scale, the product is "exhausted"; i.e., the revenue received by the firm is exactly accounted for by the total factor cost. Incidentally, (6-89) is an identity in vv$_t$, x'_2\ and *p,* not in *X*\ and x_2. This relation holds only for values of the factors satisfying the first-order equations, assuming the sufficient second-order conditions are also satisfied.

Let us now investigate the comparative statics of this model. Note that the parameter w_1 enters only the objective function, whereas x_2 enters the constraint. Substituting the solutions (6-88) back into the first-order equations yields the identities

$$pf_1(x_1^*, x_2^*) - w_1 = 0$$

$$pf_2($$

$$x_1 - x_2 = 0$$

Since the parameter w 1 enters only the first first-order equation, we expect therefore to be able to derive a refutable implication for this parameter. The parameter x_2, on the other hand, appears in the constraint; we expect no refutable implication for this parameter. Differentiating these identities first with respect to w, produces the matrix equation

Pfu \ \
Pfv. 0 d
w
\
P/2 d
1
A/2 x
2 2
-
d
\
V
\
d
x
*
-

6 -91) 92 a)

V

Solving for dx^*/dw_u

dH <

w

Also, as expected, since x_2 is fixed,

and $8X^*$ _ H H

As we showed earlier, a si

gn is never implied for rates of change of the La-grange multiplier with respect to any parameter. However, Eq. (6-92c) shows that if the marginal product of $X\$ increases with an increase in the entrepreneurial input (meaning, in the two-factor case, that the two factors are complements), the imputed marginal value of the entrepreneurial input moves in the opposite direction as the wage of $X\$. (If elevators are fixed in supply, an increase in the wages of elevator operators will lower the imputed marginal value of elevators.)

Differentiating Eqs. (6-90) with respect to x_2 produces the matrix equation

Pfn **0** \
Pfv. $dx°_2$

Phi
Phi
-1

(6-93)

0 $\underline{dX*}$ 1
-1
V" /

Solving,

"^3i / *Pfn* – ~*H*~ ~ -*pf*$_u$ (6-94a)

Also, since x_2 = *x?*

*dx***Jx J* *-H*$_{32}$ / *-pf*$_L$ *H* ~ *-pfi* = 1 (6-94^7)

and

*dX** *-H,* *H* - *Pfu* (6-94c)

Note the curious "reciprocity" result *dx*/dxj* — 3A*/3w, , since H_{j3} = $H_{3\,}$. We shall have more to say about these types of relations in the next chapter. Note also

To sum up, for parameters entering only the objective function, refutable implications are possible. Because such a parameter, w_1, enters one and only one first-order condition, a sign can be determined for dx^*/dw_1. For parameters entering the constraint, such as x_2 in this model, refutable implications are not possible on the basis of the maximization hypothesis alone, though additional assumptions may yield useful propositions.

PROBLEMS

1. Consider the constrained maximum problem

maximize

$$, \ldots, x_n, a_u \ldots, a_m) = y$$

subject to

$$g(x_u \ldots, x_n, a_l, \ldots, a_m) = 0$$

Prove that if some parameter a, enters the fth first-order relation and that equation only, then

$$\%_a \text{—} > 0$$

1.215 Prove the same result if there is more than one constraint.
1.216 Show that diminishing marginal utility in each good neither implies nor is implied by
convexity of the indifference curves.
1.217 Find the maximum or minimum values of the following functions $f(x_u x_2)$ subject to
the constraints $g(x\backslash, x_2) = 0$ by the method of direct substitution and by Lagrange mul
tipliers. Be sure to check the second-order conditions to see if a maximum or minimum
(if either) is achieved.

1.218 $f\{x_x, x_2) = x_x x_2;\ g(x_x ,x_2) = 2- (x_{\{} + x_2)$.
1.219 $f\{x_u x_2)=x_x+x_2 \backslash g\{x_x,x_2) = \backslash -x_x x_2$.
1.220 $f(xi,x_2) = X\backslash X_2\backslash g\{x\backslash,x_2)$ — M — $piX\backslash$ — p_2x_2, where $p\backslash,p_2$, and Mare parameters.
1.221 $f(x_{\{},x_2) = p_x x\backslash + p_2x_2;g(x_u x_2) = U° - xix_2$.

1.222 Show that the second-order conditions for Probs. *4(a)* and *A(b)* are equivalent; also that
the second-order conditions for Probs. *4(c)* and *4(d)* are equivalent.
1.223 Consider the class of models

maximize

$$y = f(x_u x_2) + < xx_Y$$

subject to

where $x\backslash$ and x_2 are choice variables and *a* and *fi* are parameters. Using the Lagrangian

$$_x, x_2) + axi + X\{g(x\backslash, x_2) + Px_2)$$

1.224 Prove that $dx^*/da > 0$ but that no refutable comparative

statics result is available
for p.

1.225 Prove that $dx\backslash/dp = X^*(dx^*/da) + x^*(dX^*/da)$.

7. Consider a general maximization problem

maximize

$$y = f(x_u x_2, a)$$

subject to

$$g(x_u x_2) = k$$

where $X\backslash$ and x_2 are choice variables, and a and k are parameters. Using the Lagrangian

1.226 Prove that $f_{Xa}\{dx^*Jdk) + f_{2a}(dx^*/dk) = dX^*/da$.
1.227 What functional forms of the objective function and constraint would lead to the simple reciprocity result $dx^*/dk = dk^*/da!$

8. Consider a firm that hires two inputs x_ℓ and x_2 at factor prices w_ℓ and w_2, respectively. If this firm is one of many identical firms, then in the long run, the profit-maximizing position will be at the minimum of its average cost (AC) curve. Analyze the comparative statics of this firm in the long run by asserting the behavioral postulate

minimize

$$- \frac{WjXj + W_2X_2}{f(x_u x_2)}$$

where $y — f(x_x, x_2)$ is the firm's production function.

1.228 Show that the first-order necessary conditions for min AC are w, — AC* /i = 0, / = 1, 2, where AC* is min AC. Interpret.
1.229 Show that the sufficient second-order conditions for min AC are the same as for profit maximization in the short run (fixed-output price), that is,

$$/n < 0 \quad /_{22}<0 \quad f_{ij}f22\text{-}fn>0$$

(Hint: In differentiating the product AC*/-, remember that 3AC*/3x, = 0 by the first-order conditions.)
1.230 Find all partials of the form dx^*/dwj. (Remember that w_x and vv$_2$ appear in AC.) Show that $dx^*/dwi < 0$ is *not* implied by this model, nor is dx^*/dwj = 9x*/3w$_\ell$.
1.231 Show that $f_xx^* + f_2x\% = y^*$. Is this Euler's theorem? (If it is, you have just proved that all production functions are linear homogeneous!)

9. Consider a firm with the production function $y = f(x_u x_2)$, which sells its output in a competitive output market at price p. It is, however, a monopsonist in the input market, i.e., it faces rising factor supply curves, in which the unit factor prices w_x and vv$_2$ rise with increasing factor usage, that is, Wi = $k\backslash X\backslash$, $w_2 = k_2x_2$. The firm is asserted to be a profit maximizer.

1.232 How might one represent algebraically a decrease in the supply of factor 1 ?

1.233 If the supply of JCI decreases, will the use of factor 1 decrease? Demonstrate.

1.234 What will happen to the usage of factor 2 if the supply of $x\backslash$ decreases?

1.235 Explain, in about one sentence, why factor demand curves for this firm do *not* exist.

1.236 Suppose the government holds the firm's use of x_2 constant at the previous profit-maximizing level. If the supply of $X\backslash$ decreases, will the use of $x\backslash$ change by more or less, absolutely, than previously?

10. Prove the propositions stated at the end of Sec. 6.5 that if a function/(x), x = $(x_1, \ldots, x_n)$ is quasi-concave and linear homogeneous, it is (weakly) concave, and if / is strictly quasi-concave and homogeneous of degree r, $0 < r < 1$, it is strictly concave.

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CHAPTER 7

THE ENVELOPE THEOREM AND DUALITY

7.1 HISTORY OF THE PROBLEM

In the early 1930s, a very distinguished economist, Jacob Viner, was analyzing the behavior of firms in the short and long run. Viner defined the "short run" as a time period in which one factor of production, presumably capital, was fixed, while the other factor, labor, was variable. He posited a series of short-run cost curves, whose minimum points (for successively larger capital inputs) first fall and then rise. Viner reasoned that if both inputs were variable, the resulting "long-run" average cost would always be less than or equal to the corresponding short-run cost. He therefore concluded that the long-run average cost curve should be drawn as an "envelope" to all the short-run curves. The eventual diagram, pictured in Fig. 7-1, now appears in virtually all intermediate price theory texts.

However, Viner also was puzzled by the fact that the resulting long-run curve did not pass through the minimum points of the short-run curves, since reducing unit costs seemed to increase available profits. Moreover, at the points of tangency, the slopes of the long-run and short-run curves were the same, indicating that average cost was falling (or rising) at the same rate, irrespective of whether capital was being held constant. Viner therefore apparently asked his draftsman, Wong, to draw a long-run average cost curve that was both an envelope curve to the short-run curves and that also passed through the minimum points of the short-run curves. When Wong indicated the impossibility of this joint occurrence, Viner opted to draw the long-run average cost curve through the minimum points of the short-run average cost curves,

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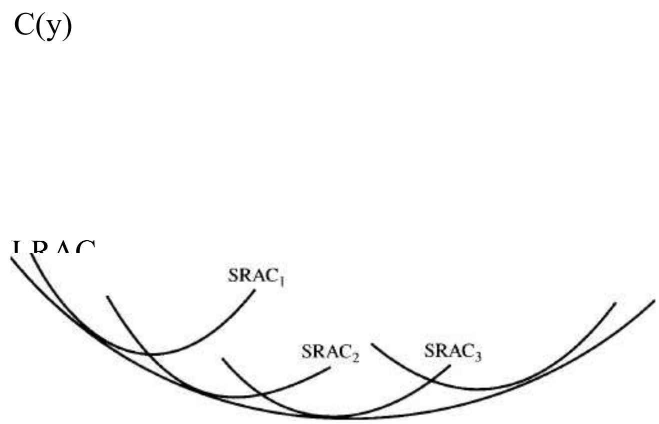

FIGURE 7-1
The modern Viner-Wong diagram shows the long-run average cost curve as an *envelope* to the short-run average cost curves.

rather than as an envelope curved The egos of many succeeding economists have been soothed by that decision.

The problem was soon analyzed algebraically by Paul Samuelson, who demonstrated the correctness of the tangency of such long- and short-run curves.* However, it remained somewhat of a puzzle that the rate of change of an objective function should be the same whether or not one variable is held constant. Perhaps most surprising, as economists investigated this puzzle further, was the discovery that the relationships that underlie this "envelope theorem" also reveal the basic theorems about the existence of refutable comparative statics theorems. It is to this larger issue that we now turn.

7.2 THE PROFIT FUNCTION

Samuelson began his analysis as follows. Consider a general maximization model with two decision variables, *xi* and x_2, and one parameter, a:

Jacob Viner, "Cost Curves and Supply Curves," *Zeitschrift fur Nationalokonomie,* 3:1931. Reprinted in *AEA Readings in Price Theory,* Irwin, Homewood, IL, 1952.

*See Paul Samuelson, *Foundations of Economic Analysis,* Harvard University Press, Cambridge, MA, 1947.

maximize

$$y = f(xi,x_2,a)$$

(The generalization to *n* variables is trivial; we will later consider models with multiple parameters.) The first-order necessary conditions are, of course, *f* — f_2 = 0; assuming the sufficient second-order conditions hold, the explicit choice functions x_t = *x*(a)* are derived as the solutions to the first-order equations. If we now substitute these solutions *into the objective function,* we obtain the function

(f)(a) = f(x(a),x*(a),a)* (7-1)

The function </> *(a)* is the value of the objective function / when the x, 's that maximize /(for given *a)* are used. Therefore, *<p (oc)* represents the maximum value of /, for any specified *a. We call* 0(a) *the indirect objective function.*

How does *(p* vary (as compared to /) when *a* varies? Differentiating with respect to *a,*

dxf *dxX*
</><*(«) /l ir + /2 + /«
dot *aa*

However, from the first-order conditions, f_x = f_2 = 0; hence the first two terms on the right-hand side vanish. Therefore,

<t>$_a$(a) = f_a **(7-2)**

Equation (7-2) says that as *a* changes, the rate of change of *the maximum value of f* where x_t and x_2 vary optimally as *a* varies, equals the rate of change of / as *a* varies, holding *X* and x_2 constant! This result has puzzled many economists long after the publication of Viner's original article.

Before we study the geometry of Eq. (7-2), let us verify the result for the profit maximization model. The explicit choice functions (factor demand functions) that result from the hypothesis, maximize *n = pf(X[,X2) — WiX\\ —w$_2$x$_2$* are, again, *X* = JC*(H>I, *W$_2$, p), X$_2$* = JC|(WI, *w$_2$, p).* If these profit-maximizing levels of input are substituted into the objective function, the resulting profit level, by definition, must be the maximum profits attainable at those factor and output prices. Algebraically,

7r*(w,, *w$_2$, p) = pfixf, x$) - wix* - w$_2$x** (7-3)

The function *n*(wi, w$_2$, p)* is called the *profit function;* it is the indirect objective function for this model. Its value is always the maximum value of profits for given Wi, *w$_2$, and p.*

How do profits vary when, say, *w* changes? One could simply differentiate the objective function with respect to *w\\,* holding not only other prices constant, but the input levels JCI and x_2 constant as well. In that case, we would find

dn

No assumption of profit maximization is invoked here. This relation simply says, for example, that if a firm employed 100 workers, and if wages increased by, say, \$1, profits would start to decrease (note the minus sign) by \$100 (100 workers times \$1, the change in the wage rate). However, a profit-maximizing firm would start to reduce the number of its workers as wages increased. If we want to evaluate how *maximum* profit varies when w_x changes, we must differentiate the *indirect* profit function. Differentiating (7-3) with respect to *w* \,

*dn** (*dx** *dx** \ *dx** „ *dx**
----- = *P*\ *f*\ — + *fi* — -W| —-— *x** — w_2 —-
dw\ \ *d**V*\ *dw**J* *d**V*\ *dw* i

Combining the terms involving *dx*/dw\,* etc., yields

*dn** {
(/)
d**V **\dw\/** **\ow**

However, the terms in parentheses on the right-hand side are zero at profit-maximizing values of JC[and x_2. Therefore,

*dn** , *dn*

where the latter term must be evaluated at *x**. Equation (7-4) says that starting at some profit-maximizing input levels, the *instantaneous* rate of change of profits with respect to a change in a factor price is the same whether or not the factors are held fixed or whether they in principle can vary as that factor price changes. Moreover, the value of this instantaneous rate of change is simply the negative of the factor demand function for JCI, *x*\ = *x*{w\, w_2, p),* evaluated at the particular prices for which the input levels are in fact profit-maximizing.

We can get a better understanding of what is going on here by considering the geometry more closely. Suppose the factor and output prices have the specific values Wp w>2, *P*°- Some values of JC* and *x* *%* are implied:

Let us vary *w*\ only, holding w_2 and/? fixed at the above values, and observe how the level of profit varies. In particular, we shall initially hold *x*\ and x_2 fixed at *x*\ and x_2. In Fig. 7-2, the "constrained" profit function

n{w₁,w°₂,p⁰,x⁰₁,x!>)=p⁰f(x⁰₁x°)-w₁x⁰₁-w⁰₂x⁰₂ **(7-5)**

shows the level of profits as *w*\ varies, holding everything else constant, i.e., for given w_2 and *p*°, with *x*\ — JC_P x_2 = JC_2 . [Note that every variable in Eq. (7-5) has a superscript 0 except wj.] Note also that *n(yv\,* w_2, *p°,* JC°, JC_2) is a linear function in wi. Its slope is *dn/dwi* = — JC{\

Now consider where the profit function *n*(w*\, vv_2, *p°)* lies in relation to this line. Since *TZ*{W* , w_2, *p°)* is by definition the *maximum* profits for given factor and output prices, it must in general lie above the straight line defined by *TT{W\ , w®, p°,* JC°, JC_2).

However, when wi = wf, exactly the correct input levels are used, since *x*® and x_2 were *defined* as the profit-maximizing input levels when *w*\ = wf. Thus,

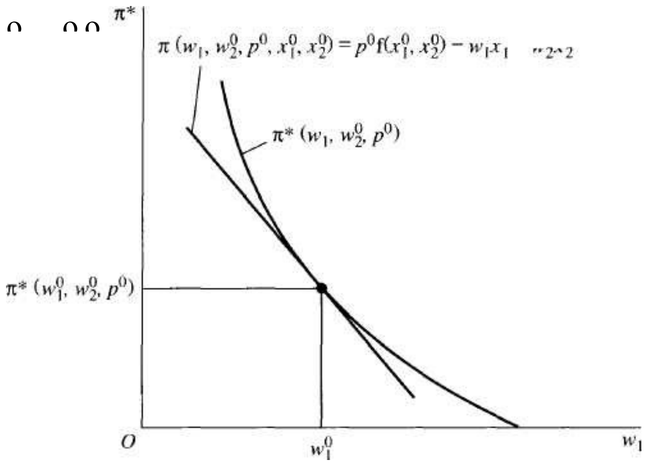

FIGURE 7-2
The profit function *TI*(W\\,* w", *p°)* and the profit function *n(w\\, w®, p°,* JC°, *x®),* where xj* and x° are those levels that maximize profits when Wi =wf.

at *W] =w®, TT*(W\\, w®, p°) =JT(W\\,* W^, *p°, x®, x®).* When *w* ^wj^1, the input levels *x®* and *x®* are "wrong," i.e., non-profit-maximizing. Hence *n' (w$_t$, w®, p°) > n(w\\, w®, p°,* xf, *x®) on both sides* of *w°$_t$*. But observe the geometric consequences of this in Fig. 7-2. Assuming *TT** and *n* are both differentiate, *JT** and *n* must be tangent to each other at w,. Tangency means that *TT** and *n* have the same slope at wf. This is precisely Eq. (7-4), *dn*/dw\\ = diz/dwi — —x*.*

Suppose we had started at some other level of *w \\,* say w}. In that case we would have held *Xi* and *x$_2$* fixed at the levels implied by that wage, *x* = Jt*(wJ, w^, *p°), x^ = X2 (w\\, w®, p°).* The resulting constrained profit function would be some other straight line tangent to *TT** at this different value of *w* their common slope at this pointwouldbe *—x*(w\\, w®, p°).* We can see the reason for the name "envelope" theorem: the profit function *TT * (w* 1, *w$_2$, p)* is the envelope of all the possible constrained profit lines as w 1 is varied.

However, we have more information than just the equality of slope of *TT* and *TT*.* Since *TT** lies above *TT* on both sides of w,, *TT*(W\\, w®, p°)* must be more convex (or less concave) than *TT(W\\ , w®, p°, x®, x®).* But in this model, *TT* is linear, and therefore *TT ** (w 1, W2, *p°)* must be convex in *w* j, as shown in Fig. 7-2. That the indirect function is convex (we assume *strictly* convex) has major consequences for the comparative statics of this model. Convexity in *W* means *8^{2}TT*/dw* > 0. But from Eq. (7-4), *dn*/dw\\ = —x*(w\\, w$_2$, p).* Differentiating both sides therefore yields

$$3^2TT^* dw^2 \quad dx \quad >0 \qquad (7\text{-}6)$$

Since in this model the factor demand function JC*(WI, *W$_2$, p)* is in fact the negative of the first partial of *TT*(W\\, W$_2$, p)* with respect to *w\\,* the slope of the factor demand function (its first partial with respect to *w\)* is the negative second partial derivative of 7T* with respect to *wi.* Since this second partial of *TT** is positive (nonnegative), the slope of the factor demand function must be negative. Thus (in this model at least), the curvature of the indirect objective function (the profit function, here) directly implies an important comparative statics result.

By symmetry, it follows obviously that *n*(wi*, w_2, *p)* is convex in w_2, yielding the same comparative statics result for that factor. It is also the case that *TT* * *(w \,* w_2, *p)* is convex in output price/?, and that therefore $d^2 7T^*/dp^2 = dy^*/dp > 0$. The proof and geometrical explanation of this are left as an exercise. We now turn to an examination of the general maximization model. Can the preceding results be derived without resort to visual geometry?

7.3 GENERAL COMPARATIVE STATICS ANALYSIS: UNCONSTRAINED MODELS

Consider any two-variable model, maximize $y = f(x\backslash, x_2, a)$, where $x\backslash$ and x_2 are the choice variables and, for the moment, a is a single parameter representing some constraint on the maximizing agent's behavior. The first-order equations are $f_1 = f_2 = 0$. By solving the first-order equations simultaneously, assuming unique solutions, explicit choice functions Xi = $x^*(a)$, x_2 = x^ioc) are implied. Again, the refutable propositions consist of the implications of maximization regarding the directions of change in some or all JC,-'S as *a* changes. The "indirect objective function" is, again, 0*(a)* = *f(x*(a), x%(a), a)*. By definition, 0(a) gives the maximum value of /for given *a*. At what rates do 0(a) and *f(x,a)* vary (both first- and second-order rates of change) as *a* changes?

In Fig. 7-3, </>(«) is plotted for various a's. For an arbitrary a^0 some *x®* = *x*(a°)* and *x®* = *x%* *(a^0)* are implied.

Consider the behavior of *f(x\\, x₂, a)* when *x* and *x₂* are held fixed at *x®* and JC° as opposed to when they are variable. Since 0(a) is the *maximum* value of/for given *a,* in general, / < 0. When *a* = *a°,* the "correct" X/'s are chosen, and therefore 0(a) = *f(x\\, x₂, a)* at that one point. On both sides of a^0, the "wrong" (i.e., nonmaximizing) JC/'S are used, and thus by definition,

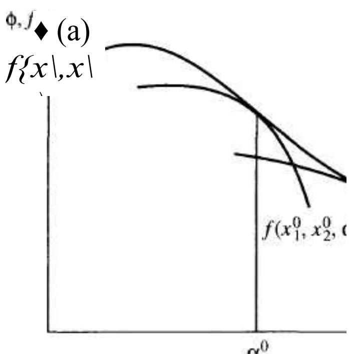

FIGURE 7-3
The indirect objective function *4>(a)* is an envelope curve to the direct objective functions for various a's.

$f(x^{®}, x_2, a) < 0(a)$ in any neighborhood around a^0. Unless/has some sort of nondifferentiable corner at a^0, 0 and / must be tangent at a^0, and, moreover

, / must be either more concave or less convex than 0 there. Since this must happen for arbitrary *a,* similar tangencies occur at other values of a. It is apparent from the diagram that 0 *(a)* is the *envelope* of the $f(x\backslash, x_2,$ a)'s for each *a.* How do we derive these properties algebraically?

Consider a new function, the difference between the actual and the maximum value of / for given *a,*

$$F(x_u x_2, a) = f\{x_x, x_2, a) - <f>(a)$$

called the *primal-dual* objective function. Since / < 0 for *x* ^=*x** and / = 0 for $x_t = x^*$, Fhas a maximum (of zero) when JC, = *xf(a)J* Moreover, we can consider $F(xi, x_2, a)$ as a function of *three* independent variables, $x\backslash$, x_2, and *a.* That is, just as for a given *a* there are values of JCI and x_2 that maximize/ for *given xi* and x_2, there is some value of *a* which makes those JC,-'S the "correct" (i.e., maximizing) values. For example, for a given amount of labor and capital, there is some set of factor and output prices for which those input levels would be the profit-maximizing values.

This maximum position of $F(x_u x_2, a)$ can be described by the usual first- and second-order conditions. The first-order conditions are that $f(x\backslash, x_2, a)$ — 0(a) has zero partial derivatives with respect to the original choice variables x_1 and x_2, and also a:

$$= f_i = 0 \quad i = 1, 2$$

and

$$= f_a - \phi_a = 0$$

Equations (7-7) are simply the original maximum conditions. Equation (7-8) is the "envelope" result, $\phi_a = f_a$. These first-order conditions hold whenever $x_i = x^*(a)$, $i = 1,2$.

The sufficient second-order conditions state that the Hessian matrix of second partials of $F(x_1, x_2, a)$ (with respect to x_1, x_2, and a) is negative definite, or that its principal minors alternate in sign. By inspection, $F_{11} = f_{11}$, etc., and $F_{aa} = f_{aa} - \phi_{aa}$. Thus,

$$H = \begin{vmatrix} F & F_1 & F \\ F & F_{2}i & F \\ F & F & t \end{vmatrix} = \Bigg|$$

7-

If we think of x_1 and x_2 as an "efficient" allocation of resources, and x_1 and $X2$ as any other allocation, then this says that efficient allocation occurs when "waste" equals zero.

These second-order conditions include the original ones (/i 1 < 0, f_n fn ~ fn > 0, etc.) in the top left corner. In addition, the sufficient second-order conditions also imply F_{aa} < 0, or f_{aa} — (p_{aa} < 0. *Moreover, it is from this inequality that all known comparative statics results (in maximization models) flow.*

The first-order envelope result (7-8), with the functional dependence noted, is *cj)$_a$(a) = f_a(x*(a), x%(a), a).* Differentiating both sides with respect to *a* yields

raa**A,**— *Jaxi* **f** ,, *dxt* **L** **L f** *dxZ* **£** **i** **f**
——> —*J01X2* ∘ — —~ *J aa*
da *da*

From the sufficient second-order conditions, therefore, and using Young's theorem,

3JC,* *dxX*
(t>aa ~ faa = /la ~-h *fla* — > °
da *da*

This analysis is readily generalized to the ^-variable case, producing the condition

n *dx**
LO

(710)

Equation (7-10) is the general and fundamental comparative statics equation for all unconstrained maximization models. As it stands, however, it is too general to be of much use. In order for a model to have refutable implications, it must contain more structure than just a general maximization problem. Suppose therefore that some *a* enters only one first-order condition *f* = 0, i.e., f_{ja} = 0 for *j =fci.* Then Eq. (7-10) reduces to a single term,

fia^d~>0 (7-H)
da

This is Samuelson's famous "conjugate pairs" result. In maximization models, if some parameter *a* enters only the *ith* first-order equation, the response of the rth choice variable x_t to a change in that parameter is in the same direction as the effect *a* has on the first-order equation.

The significance of this theorem lies in its application to some important models. For example, in the profit maximization model, the parameter *w* enters only the first first-order equation *ii* = *pf* — *W* — 0; it enters with a negative sign: *dii\ldw* — — 1. Thus, the conjugate pairs theorem states that the response of *x** to an increase in Wi will be negative, and similarly for *x^.* The theorem also applies to the constrained cost minimization model, as we shall presently see.

In the more general case where *x* is a vector of decision variables *(xi, ... ,x_n),* and *a* is a vector of parameters *a* = *(a\, ..., a_m),* the second-order conditions for maximizing *F(x,a) = f(x,a)* — </>(«) with respect to *a* are that the matrix F_{aa} = f_{aa} — *4>$_{aa}$* is negative semidefinite. The usual comparative statics results follow from the negativity of the diagonal elements of this matrix. However, a richer set of theorems is also available from the other properties of negative semidefinite matrices: The principal minors of the terms in $f_{aa} — 4>_{aa}$ alternate in sign.

The envelope theorem also reveals the origins of the nonintuitive "reciprocity" conditions that appear in maximization models. Recall that in the profit maximization model, we derived 3x*/3w2 = *dx^/dw*. This result can be more clearly shown by first noting that each factor demand is the negative first partial of *JT** with respect to its factor price, i.e., *TT** = — *x*(w\, w₂, p), n₂ — —x₂(w\,w₂,p)*. Applying Young's theorem on invariance of cross-partials to the order of differentiation to *7T*(w\, w₂, p)* therefore yields *n*₂ = — dx*/dw₂ = —dx₂/dw\ = n₂ₗ*. Thus this curious result is no more curious than Young's theorem itself.

All reciprocity theorems are in fact simply the application of Young's theorem to the indirect objective function. Suppose there are two parameters *a* and *P* so that the model is maximize _y = /(*i, *x₂, a, P)*. The implied choice functions are then *Xi = x*(a, P), i* = 1, 2, and the indirect objective function is 0(a, *P) = f(x*(a, P), x₂(a, P), a, P)*. Then noting that </>«(«, *P) = fₐ,*

3JC* 3^2

p ^ ~ ^*al*~*dp*~ + -^2"3^ + ^

Similarly,

*dx** *dxX*

Since *(j>ₐp = (j)pₐ,*

(7-12)

J *dp* *J* *dp* *J p* 3a

J p 3a For the general case of *n* decision variables,

'ᵖ~fa (7-13)

However, these relations are most interesting when each parameter enters only one first-order equation. In that case, Eq. (7-13) reduces to one term on each side, as in the profit maximization model.

7.4 MODELS WITH CONSTRAINTS

Most models in economics involve one or more side constraints. A particularly important model, for example, is

minimize

C = *2_, ᵂiˣi*

subject to

If/is a production function of *n* inputs, *X*, ..., *xₙ,* and the w,'s are factor prices, this famous model, which we shall presently analyze in detail, describes achieving some output level *y°* at minimum cost.

The extension of the results for unconstrained maximization models to models involving one or more side conditions (constraints) depends critically on whether the parameters enter only the objective function or whether they enter the constraints also (or exclusively). Note that in the preceding cost minimization model, the prices enter only the objective function, whereas the specified output level enters only the constraint. We shall show that if the parameters enter only the objective function, the comparative statics results are the same as for unconstrained models. However, if a parameter enters a constraint, as that parameter changes, the constraint space also changes, destroying the relation *(p$_{aa}$ > f$_{a0l}$*. Let us investigate these more general models.

The traditional derivation of the envelope theorem for models with one constraint proceeds as follows.

Consider

maximize

$$f(x i, \ldots, x_n, a)$$

= v subject to

$$g(x_u \ldots, x_n, a) = 0$$ The Lagrangian

is $X = f +$ A.g. Setting the first partials of ££ equal to 0,

$$\%=fi+Xgi=0 \quad i = l, \ldots, n \qquad (7\text{-}14)$$

$$2* = * = 0 \qquad (7\text{-}15)$$

Solving these equations for

$x_t = x*(a) \quad i = 1, \ldots, n \; X = X*(a)$ we define

(7-16)

as before. Here, *(f>(a)* is the maximum value of v for any or, for JC$_($'S that satisfy the constraint.

How does </>(«) change when *a* changes? Differentiating (7-16) with respect too;

Here, however, f_t ^= 0. Differentiating the constraint

$$g(x*(a)_1 \ldots, x:(q), a)$$

with respect to *a,*

$$dx*$$

Multiply Eq. (7-18) by *k*, and add to Eq. (7-17). (This adds zero to that expression.) Then

$$dx^* \qquad \wedge \qquad dx^*$$

$$dx^*$$

$$*\text{-}\wedge \qquad da$$

Using the first-order conditions (7-14),

$$(7\text{-}19)$$

where i£a is the partial derivative of the Lagrangian function with respect to *a*, holding the *Xi* 's fixed. Thus, in evaluating the response of the indirect objective function to a change in a parameter in a *constrained* maximization model, the Lagrangian function plays an analogous role to the objective function in an unconstrained model. We can derive the envelope theorem for constrained maximization models more conveniently using primal-dual analysis. It is still the case in these models that *(j)(a) > f(x\, ... ,xn, a)*, but in this case, the variables must also satisfy the constraint. The primal-dual model is therefore

maximize

$$f(xi, \ldots, *_{,,}, a)$$

-<f>(a) subject to

$$g(x_u, \ldots, x_n, a) = 0$$

treating a as a (vector of) decision variables as well as the x, 's. The Lagrangian for the primal-dual problem is

Setting the first partials of ££ with respect to the x_t 's and *X* equal to zero produces the ordinary first-order equations (7-14) and (7-15) for a constrained maximum; setting the first partial of *5£* with respect to *a* equal to zero produces the envelope relation (7-19) above.

Comparative Statics: Primal-Dual Analysis

We now investigate, using primal-dual analysis, the conditions under which refutable propositions appear in constrained maximization models. We already know from traditional methods developed in Chap. 6 that no refutable propositions appear for parameters that appear in the constraint. We refer the reader to Silberberg's 1974 comparative statics paper for the general results. We can demonstrate the nature of the more likely useful results using the following simple model. Consider maximize

$$f(xi,x_2,a) = y$$

subject to

$$g(x_1, x_2, P) = 0$$

In this model, a single parameter *a* enters the objective function only, and another parameter, *p,* enters the constraint only. Using Lagrangian techniques, the first-order equations are solved for the explicit choice equations

$$x\backslash = x^*(a,P)$$

$$x_2 = x_2(a,P)$$

Substituting these solutions into the objective function yields the maximum value of $f(x\backslash, x_2, a)$ for given *a* and *p,* for JCI and x_2 that satisfy the constraint:

Since 0(a, *P)* is the maximum value of / for given *a* and *p, (j>(a, P)* > $f(x\backslash, x_2, a.)$ for any JC,'S that satisfy the constraint. Thus, the function $F(x\backslash, x_2$, a, $P) = f(x\backslash,x_2, a)$ — *4>(a, P)* has a maximum (of zero) for any jt/'s that satisfy the constraint. However, $F(x\backslash, x_2,a, P)$ is a function of four independent variables, one of which, *a,* does not enter the constraint. Therefore, starting with values of $x\backslash$, x_2, and *P* which satisfy the constraint, and holding them fixed at those values, the constraint does not further impinge on the choice of *a* that maximizes $F(xi, x_2, a, P)$. The constraint affects the values of Xi and x_2 that can be chosen, but not the maximizing value of *a. In the a dimension(s), therefore,* $F(x\backslash, x_2, a, P)$ *has an unconstrained maximum.* (Consider, for example, what happens when some good, say, air, enters a person's utility function, but not the budget constraint, there being no price paid for breathing. In that case, we breathe until the marginal utility of air is zero; i.e., we consume in the manner of an unconstrained maximum in that dimension.) The Lagrangian for this primal-dual problem is

$$X = f(x_u \ldots, *_{,,}, «) - </>(a, P) + kg\{x_u \ldots, x_n, P)$$

The envelope relations are obtained by setting the first partials of i£ with respect to *a* and *p* equal to zero, yielding

$$fa \sim 0« = 0 \qquad (7\text{-}20a)$$

$$O \qquad (7\text{-}206)$$

Equation (7-20a) is just Eq. (7-8), the same envelope relation for unconstrained models. Moreover, since this primal-dual model is an unconstrained maximum in *a,* $F_{aa} = f_{aa} — (p_{aa} < 0$, assuming, as always, the sufficient second-order conditions.

The fundamental comparative statics result (7-10) follows as before:

$$\frac{\mathrm{P}}{da} \quad \frac{/2«\wedge>0}{da} \qquad (7\text{-}10)$$

If a represents a vector of parameters that enter the objective function only, then the matrix of terms $(f_{aa}$ — $(f)_{aa})$ must be negative semidefinite; Eq. (7-10) then follows from the fact that the diagonal elements are nonpositive.

No such easy relationships exist with regard to changes in /3. To best see this, try to construct a diagram like Fig. 7-3 for the parameter *fi.* Plot /3 on the horizontal axis and/and *(f)(a,* /3) on the vertical axis. Hold *a* constant throughout. At some value /3°, *x*® = *x*(a°,* yS°), x° = x|(a°, /3°) are implied. The next step is to vary the parameter in question, holding *xi* and x_2 constant. However, it is impossible to do that for *fi.* In the first place, since *fi* is not a variable in the objective function /, it is impossible to plot / against *j3.* Second, if *x* and x_2 are held constant, /3 cannot be changed without violating the constraint! Thus the procedure for showing the greater relative concavity of / vs. *(p* breaks down for parameters entering the constraint: One cannot change only one variable in an equation without destroying the equality. As a result, no refutable hypotheses are implied by the maximization hypothesis alone, for parameters that enter the constraint.

In the case where /? is a vector of two or more parameters *{f}\, ..., fi_m),* it is possible to hold *X\,* x_2 and *a* constant and characterize the /3/s that solve the primal-dual problem. Since the original objective function does not contain any of the *fij's,* the primal-dual problem reduces to

maximize

subject to

where *x* = (JCI, x_2) (or, for that matter, a general ^-dimensional vector of decision variables). Of course, maximizing —(/>(«, */3)* is the same as minimizing 0 *(a,* /3); thus in this case, the indirect objective function is *convex in the (5 parameters, subject to constraint,* i.e., in the parameters that enter the constraint exclusively. If the constraint is linear in the $y6_y$'s, then the indirect objective function must be *quasi-convex* in these parameters (though linearity is not a necessary condition for quasi-convexity).

Example. In the important consumer model, utility of goods is maximized subject to a linear budget constraint:

maximize

$$U(x_1, x_2)$$

subject to

$$p_2 x_2 = M$$

Using Lagrangian methods, the implied choice functions are the Marshallian demands $x_t = x^*(p_u\ p_2,$ M), i = 1,2. Substituting these functions into the objective function yields the *indirect utility function* $U^*(p\backslash, p_2, M) = U(x^*(p\backslash, p_2, M), x\hat{}ipi, p_2,$ M)). The primal-dual problem is thus

maximize

$$U(x_u x_2)-$$

$U^*(p_u p_2, M)$ subject to

$$+ p_2 x_2 = M$$

where the maximization runs over $X\backslash, x_2,$ *and* the parameters $p_x, p_2,$ and M. Since all the parameters are in the constraint exclusively, the maximization problem with respect to the prices and money income is simply

maximize

$$-U^*(p_l, p_2, M)$$

subject to

$$+ p_2 x_2 = M$$

This says that choosing goods x_x and x_2 so as to maximize utility (subject to the budget constraint) is equivalent to choosing prices and money income so as to minimize the indirect utility function, also, of course, subject to the budget constraint. Since the budget constraint is *linear* in prices and money income, this implies that the indirect utility function is quasi-convex in prices and money income. The result generalizes immediately to the case of n goods.

Reciprocity relations can be derived in these models using the envelope relations (7-20). Writing these relations as identities and showing the functional dependencies using the explicit choice functions, we have

$$<p_a(a, p) = f_a(x^*(a, 0), x^*(a, P), a) \qquad (l\text{-}2\backslash a)$$

$$P) = \mathbf{r}\ (\mathbf{a}, P)g_{fi}(x;(a, P), x^*(a, p), P) \qquad (7\text{-}21^*)$$

Identity (7-2la) is the same as in the case of models without constraints, because the a parameters enter only the objective function. For two such parameters *oil* and a_2, we derive the reciprocity conditions displayed in Eqs. (7-12) and (7-13) in the same manner as before. In addition, since $4>_a p = (j>p_a,$ we derive, using the product as

$$da \qquad da$$

An additional set of reciprocity relations is available in the case of two parameters Pi and p_2 that both enter the constraint only; these relationships necessarily involve the partial derivatives of k^* as well as the chain rule on the right-hand side of (7-21 ft),

$$ft, |: \qquad (7\_22)$$
$$3$$

derivations as an exercise for the student. At this level of generality, these reciprocity relations are not very interesting, but in many more specialized models, (7-22) reduces to interesting expressions. Last, very general reciprocity relations can be derived in models in which the parameters enter both the objective function and the constraint, but there are no known instances of any interesting ones.

An Important Special Case

Most of the useful models encountered in economics involve expressions that are linear in at least some of the parameters, typically the prices of goods or factors. Consider, therefore, models in which the objective function involves the expression

maximize

$$y = f(x, a) = 6(x_u \ldots, x_n) +]£«,■*,■ \qquad (^{7\text{"}23})$$

subject to

$$g(x_u \ldots, x_n, /3) = 0 \qquad (7\text{-}24)$$

where $x = (x\backslash, \ldots, x_n)$, the vector of decision variables, $a = (a_u \ldots, a_n)$, and /3 is any vector of parameters entering the constraint only. Parameters that enter the constraint are assumed to be absent from the objective function.

Denote the indirect objective function $(p(a, f3)$. We know from the preceding analysis that the function $f(x, a)$ — 0(or, $/3)$ must be concave in a and that the matrix $(f_{aa}$ — (p_{aa}) must therefore be negative semidefinite. The parameters fi and the functional form of g are irrelevant, as long as the first- and second-order conditions are satisfied. However, since / is linear in the a_t 's, $f_{aa} = 0$, and thus / has no effect on the curvature of the primal-dual function. Therefore, for these models, —0 is concave (or, alternatively, </> is convex) in a, and the matrix [—$<p_{aa}\backslash$ must be symmetric (by Young's theorem) and negative semidefinite (or, [0_{aa}] is positive semidefinite). In the case of *minimization* models with these properties, $(p(a)$ is *concave,* and [0_{aa}] is a negative semidefinite matrix.

Even more important than these curvature properties are the implications for deriving useful comparative statics theorems. By the envelope theorem, $<j>_a. = f_a. = x^*$ in these models. Therefore, the matrix [0_{aa}] consists of the terms $dxf/dcij$. From symmetry, $dx^*/doij = dx^*/da_t$. The properties of positive semidefinite matrices include nonnegative diagonal terms, i.e., 3x*/3a, > 0, and positive principal minors of higher order. These results comprise the useful theorems in economics.

The profit function derived above exhibited these properties (but note that the a,-'s are the negative prices). In the next chapter we will study the cost minimization model, which has a similar structure. We shall show that the cost function associated with production functions with the usual properties must be concave, and the demand functions implied by that model are negatively sloped.

Interpretation of the Lagrange Multiplier

The Lagrange multiplier A. has been carried along thus far mainly as a convenient way of stating the first- and second-order conditions for maximization. In fact, the main reason for the use of Lagrangian techniques in economics (and also other sciences) is that A often has an interesting interpretation of its own. Consider the constrained maximization model

maximize

$$f(xi, x_2) = y$$

subject to

$$g(x_u x_2) = k$$

Usually we set the constraint equation equal to zero; here, it equals some arbitrary value *k*. By stating the constraint in this manner, we can consider parametric changes in the value of the *g* function. Using the Lagrangian

$$_u\ x_2) + X(k - g(x_x,\ x_2))$$

the usual first-order equations are

$$\#i = fx\ (*i,\ x_2) - kgi\ (*i,\ x_2) = 0 \qquad (7\text{-}25a)$$

$$\hat{}2 = fi(xux_2) - k_{g2}(x_u x_2) = 0 \qquad (7\text{-}256)$$

$$2i = k - g(_{xl}, x_2) = 0 \qquad (7\text{-}25c)$$

From Eqs. (7-25a) and (7-256),

$$A = \frac{}{g\backslash} = \frac{}{82} \qquad (7\text{-}26)$$

However, a more revealing expression for A can be obtained using the envelope theorem.

By solving Eqs. (7-25) simultaneously, we obtain the explicit choice functions *x*(k), x^k),* and *k*(k)*. Substituting these solutions into $f(x\backslash,\ x_2)$ yields the indirect objective function

By the envelope theorem for constrained maximization models, Eq. (7-19),

$$<t>k(k) = \frac{d}{dk} = X\backslash k) \qquad (7\text{-}27)$$

That is, the Lagrange multiplier A. equals the rate of change of the maximum (or minimum, as the case may be) value of the objective function with respect to parametric changes in the value of the constraint.

Consider Eq. (7-19) again, *(j)*$_a$ *{a) =* /« + *kg*$_a$. We can understand this relation by using (7-27). Think of (7-19) as

3 0 / 3 a

9 / 3 a

dj _ *d* *g* _ *d* *g* *d*

a

When a parameter that enters both the objective function and the constraint changes, it produces two separate effects. First, the objective function is affected directly, as indicated by the term *df/da.* In addition, the value of the constraint is affected, by the amount *dg/da.* This is then converted to units of the objective function/ by multiplying by *k(— df/dg).* The sum of these two effects is the total impact of a change in *a* on the maximum value of *y.*

A common application of Eq. (7-27) concerns models in which the objective function is some sort of value of output function, which is maximized subject to a resource constrained to some level *k.* If an additional increment of resource, *Ak,* became available, output would increase by some amount A_y* ^ *k*Ak;* in other words, A.* *is the marginal value of that resource.* In a competitive economy, firms would be willing to pay *k** for each increment in the resource. In the mathematical programming literature, *k** is called a *shadow price* of the resource. In a model in which output of society is maximized subject to constraints of parametr ic labor and capital constraints, the Lagrange multipliers associated with those constraints impute shadow factor prices, i.e., a wage and rental rate to labor and capital. In the next chapter, in a model in which total cost is minimized subject to producing some parametric output level, k^* measures the change in total cost if output is changed, i.e., marginal cost. We shall explore these relationships in the chapters following.

Consider again the model

maximize

subject to

$$g(x_1, x_2) = k$$

Since the parameter k enters the constraint, we know that in general, the sign of dk^*/dk is indeterminate. However, in some important models, additional assumptions provide a sign for this term. Differentiating Eqs. (7-25) with respect to k,

'd ol

-?? ol

dx (7-28)

\ i

- g 2 o \ l

k

Solving for dk^*/dk,

dk # * 3 ~d 3 k H

where //is the bordered Hessian determinant of the Lagrangian ££. From the sufficient second-order conditions, $H > 0$. Suppose now that/and g are strictly increasing functions so that $X^* > 0$ (why?). Suppose in addition that / is a concave and g is a convex function. Then — g must be concave, and thus i£ is concave. In this case, then, $//_{33}$ = i£n£ $£_2$2 — ^12 > 0> ^{an}d thus $dX^*/dk < 0$. If g is linear, these conditions are met as long as / is concave. It is also possible to show, via primal-dual methods, that if $dX^*/dk < 0$, i£ must be a strictly concave function; the proof is left as an exercise.

These results generalize in a straightforward manner to maximization models with multiple constraints,

maximize

$$fix) = y$$

subject to

$$g(x) < k$$

where x-(xi,... $,x_n)$, $g(x) = g^j(x_u \ldots, x_n)$, and $k = (ki, \ldots, k_m)$, j — 1,... ,m. The choice functions $x = x^*(k)$ and the Lagrange multipliers $X^*(k)$ implied by this model are obtained by simultaneous solution of the first-order Lagrangian conditions, assuming the sufficient second-order conditions hold. The indirect objective function is *(f)(k) — f(x*(k))*. By the envelope theorem, $X^{j*}(k) = d(f)/dk^j$, the marginal value of relaxing thejth "resource constraint" k^j, measured by the resulting increase in the value of the objective function. *If f(x) is concave and $g^j(x)$ is convex for j* = 1, ..., m, *(j)(k) is concave in k, and thus (4>kk) = (dX*/dk) is negative semidefinite.* Since the diagonal elements of *((pkk)* would then be nonpositive, this implies that $dX^*/dk < 0$. In many important models, the constraints are linear; such a specification satisfies the conditions of this theorem.

The proof relies on the definitions of concave and convex functions. Let k^l and k^2 be two arbitrary values of the k vectors, and denote the implied choice vectors as $JC^1 = x^*(k^l)$, $x^2 = x^*(k^2)$. Let $k^t = tk^l + (l-t)k^2$, $x' = tx^l + (1-t)x^2$, $0 < t < 1$. By convexity of the constraints,

$$g(x') < tg(x^l) + (1 - t)g(x^2) < tk^l + (1 - t)k^2 = k'$$

Therefore, x^l is a feasible choice for, or solution to, this model; it satisfies the constraints when $k = k'$.

Since /(JC) is concave,

$$/(*') > tf(x^l) + (1 - t)f(x^2) = t<Kk^l) + (1 - t)cP(k^2)$$

But by the definition of </>, *(p{k<) > f(x').* Therefore,

Therefore, 0 *(k)* is concave in *k.* Assuming differentiability, the Hessian matrix $(p_{kk}$ is of course negative semidefinite, yielding the usual comparative statics results in those cases. An important application of this result occurs in the "small country" models of international trade, where total output of an economy is maximized subject to endowment constraints. The factor prices are the associated Lagrange multipliers of those endowment constraints. If the production functions are concave, this theorem implies that an increase in the endowment of some factor cannot increase that factor's price. This model will be developed more fully in the chapters on general equilibrium.

Le Chatelier Effects

We now consider the responses of decision variables to a change in some parameter when an additional just-binding constraint is added to the model. We investigated these Le Chatelier effects in Chap. 4 for the profit maximization model. We showed in that model that if one factor is held constant at its profit-maximizing level, then in a neighborhood of that equilibrium the demand for the remaining factor becomes less elastic. We now consider more general models, and, as always, we are most interested in discovering the structure of models that yield predictable differences in the responses of the choice variables to parameter changes when a just-binding constraint is added. Since no refutable results are available in models in which parameters enter the constraint as well as the objective function, we limit the discussion to models in which some parameters *a* enter the objective function only and other parameters *(3* enter the constraint only. To save notational clutter, we shall use vector notation throughout this section. Thus, recapitulating, consider

maximize

$$y = f\{x,a)$$

subject to

where x = (xi,..., x_n) and *a* and */3* are vectors of parameters that appear only in the objective function and constraint, respectively. Assuming the first-order necessary and second-order sufficient conditions hold, we derive the explicit choice functions *x*(a, f3)* and A.*(a, /3). The indirect objective function is *<p(a, ft) — f(x*(a, fi), a)*. Since the expression *f(x, a) — cf)(a,* /3) has an unconstrained maximum in *a,* we were able to derive the general comparative statics result for any particular scalar *a*

Suppose now an additional constraint, *h(x)* = 0, that is consistent with the original equilibrium is added to the model. That is, defining $x°$ = *x*(a°, ft⁰),* we require that *h(x°)* — 0. We say this constraint *is just binding,* because it does not disturb the original maximum position. However, it does affect the rates of change of the decision variables as the parameter changes. Let us denote the new choice functions, which are solutions to the original first-order conditions and *h (x)* = 0 also,

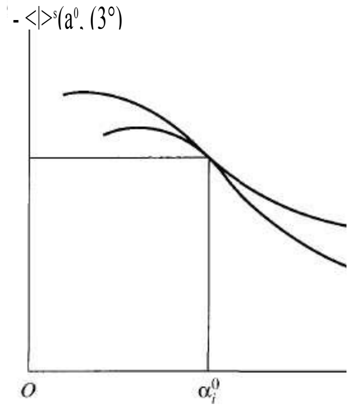

FIGURE 7-4
The indirect objective functions 0(a, /?) and *4>s(a, fi)* plotted against *a, (ps(a, fi)* being the indirect objective function when the just binding constraint *h(x)* = 0 is added. The constraint *h(x)* = 0 is added so as not to disturb the solution *x*° = jc*(a°, /3°). By this construction, *cp(a, fi)* = *4>s{a, fi)* when *a* = a0, and *(f>(a, fi)* > *4>s(a, fi)* in any neighborhood of a0. Therefore, the function *F(a, fi)* — </>(a, *ft)* — *(ps{a, fi)* has an unconstrained minimum with respect to *a* at a0. It follows that 0(a, *fi)* is tangent to 0s(a, /3) at a0, and *(f){a, fi)* is relatively more convex or less concave than 0s(a, *fi)* in a neighborhood of a0. This implies that *<j>aiai* > 0f ,a. in a

neighborhood of a^0.

as *x^s(a, P)* *(s* for "short run"), and the new indirect objective function as *(f>^s ((x, fi)*. We show these curves in Fig. 7-4. By construction, when *a* = a^0 and *fi* = *fi°*, *(p* = *cf)^s*, but for *a* =£ *a^0* or *fi* / *fi°*,

0 > *(f>^s*. Equivalently, the function *F(a, fi)* = *<p* — *<p^s* has an unconstrained minimum value (of zero) at *(a^0, fi°)* with respect to *both a* and *fi* as long as these parameters are not in the auxiliary constraint. Assuming differentiability of these functions, this means that *(j)^s* is relatively more concave than 0. The implied necessary first-order conditions are

= (f > p . 4 > s p = 0

((

2 %

The necessary second-order condition is that the matrix $F_{\alpha\beta}$ of second partials with respect to α and β is positive semidefinite. This condition implies that the submatrices $F_{\alpha\alpha}$ and $F_{\beta\beta}$ are positive semidefinite as well, and thus the diagonal elements of those matrices are nonnegative. Thus for any particular scalar parameter α,

0 (7-30)

Using the analysis leading up to (7-10), this yields

V da

d > (

a

±

f

aJ

x

Although (7-31) summarizes the available comparative statics Le Chatelier results for the *a* parameters, the most useful results occur when the conditions of the conjugate pairs theorem hold, i.e., when some particular *a* enters only the / th first-order equation. In that case, (7-31) reduces to one term, yielding

$$f_{ia}\text{-}\wedge > f_{ia}\text{-}\wedge > O \qquad (7\text{-}32)$$

Since f_{ia} can be negative, we cannot simply cancel this term out. However, since *dx*/da* and *dx^ /da* have the same sign as f_{ia}, the response of x_t to a change in *a* is always greater in *absolute value* in the absence of an auxiliary constraint:

$$\left| \frac{d}{\ } \right| > \left| \frac{d}{d} \right| \qquad (7\text{-}33)$$

The Le Chatelier results are usually stated in terms of the effects of holding one of the choice variables constant. We see here that this is unnecessarily restrictive. The only important restriction on the auxiliary constraint is that it cannot incorporate the parameters in question. The Le Chatelier results thus hold for constraints more complicated than simply x_n — x°_n. Moreover, the process can be repeated as

The *f}* parameters generally do not yield a simple result such as (7-32), since an expression in the Lagrange multiplier is always present. Consider, however, the important special case of models in which the constraint takes the form *g(x)* = *k.* Define the Lagrangian for this model as *SE* = *f(x, a)* + *X(k - g(x))* and assume unique interior solutions *x*(a, k)* and *X*(a, k).* Let *<f>(a, k)* be the indirect objective function. From (7-27), $<j>_k$ = *X*(a, k).* We know from general comparative statics analysis that *dX*/dk* ^ 0. Curiously enough, however, a systematic prediction is available for the Le Chatelier effects.

Add an additional nonbinding constraint *h(x)* = 0 as before. Let $4>^s$*(a, k)* be the indirect objective function when this new constraint is added, and let X^s *(a, k)* be the resulting solution for the Lagrange multiplier for the constraint g(jc) = *k.* The function 0 — $(p^s$ has an unconstrained minimum with respect to *k.* The necessary first-order conditions are *(pk* — $(p^s_k$ = 0, i.e., that *X** = X^s. The second-order condition says that $4>_{kk}$ — $<p^s_{kk}$ > 0, and so

$$\text{™} > \wedge \qquad (7\text{-}34)$$

$$\sim dk$$

$$dk \sim dk$$

Thus even at this rather general level, even though both terms in (7-34) are unsigned by maximization, it is still the case that a smaller change in *X* occurs when *k* changes when an auxiliary constraint is added to the model. In the next chapter we study the cost minimization model; the Lagrange multiplier turns out to be the marginal cost function. This result says that even though minimization does not imply a sign for the slope of the marginal cost function, it is nonetheless true that the marginal cost function either rises faster or falls slower in the short run than in the long run.

PROBLEMS

1. Consider maximization models with the specification maximize

$$y = f(x_u x_2, a)$$

subject to

$g(x_u x_2) = k$ with Lagrangian $!£ =$ $f(xi, x_2, a) + X[k — g(xi, x_2)]$, where x i and x_2 are choice variables and a and k are parameters.

1.237 Define 0(ot, k) = maximum value of y for given a and k in this model. On a graph with a on the horizontal axis and 0 and /on the vertical axis, explain *geometrically* the envelope results 0_a = /„ and $(j>_{aa} > f_{aa}$.

1.238 On a similar graph, explain why it is *not* possible to carry out a similar procedure for the parameter k. How does this result relate to the appearance of refutable comparative statics theorems in economics?

1.239 Using the results of *(a)*, prove that

$$/la \, T\frac{3x,}{aa} + />a\frac{dx_2}{da} > 0$$

(d) Using the primal-dual methodology, prove algebraically the envelope theorem results:

(l) $0a = fa$
(ii) $<p_{aa} > f_{aa}$
(Hi) $(f)_k$ - k^*

1.240 Prove that $f_{la}(dx^*Jdk) + /\hat{}(dx\hat{}/dk) = dX^*/da$.

1.241 Assume that the objective function / measures the net value of some activity, and the constraint represents a restriction on some resource. Using result *(Hi)* in part *(d)*, explain why the Lagrange multiplier imputes a shadow price to the resource, i.e., a marginal value of that resource in terms of the objective specified in the model. Also, in these models, what can be said, if anything, about how this marginal evaluation of the resource changes as the constraint eases, i.e., as k increases?

1.242 Suppose now that the objective function is linear in a, i.e., $f\{x\backslash, x_2, a) = h\{x\backslash, x_2) +$ $ax\backslash$. Prove that 0(a) is convex in a, and, assuming the sufficient second-order con ditions hold, $(p_{aa} > 0$.

2. Consider models with the specification maximize

$$y = f(x_x, x_2) + h(x_u a)$$

subject to

$$g(x_u x_2, P) = 0$$

where x_t and x_2 are choice variables and a and /3 are parameters that enter only the functions shown.

(a) Derive a refutable comparative statics result for *a,* and show that no such result exists

(b) Let 0 *(a, /3)* — maximum value of _y for given *a* and *ft* in this model. Using the primal-dual methodology, prove the envelope theorem results:
(i) <p_a =h_a(x^*_1,*a*) *(ii)* <*pp* = *k**gp*(*x**, *x*\\, /3),
where *X** is the Lagrange multiplier.

(c) Prove the "reciprocity" theorem

1.243 On a graph with _y on the vertical axis and a on the horizontal axis, sketch possible curves 4>(a, fi) and f(x®, JC°) + h(x°,a) wherex®, x® and fi° are some fixed values of those variables. Demonstrate graphically that </>,, = h_a and also that $(p_{aa} > h_{aa}$.

1.244 Explain why it is not possible to carry out a similar procedure for the parameter p, and thus why no refutable comparative statics theorems are available for this parameter from maximization alone.

3. Consider the model,

minimize

$$\frac{+ W_2X_2}{y}$$

where $X\backslash$ and x_2 are factor inputs, W[and w_2 are factor prices, and y = $g\{x\backslash,x_2)$ is a production function. Let AC*(wi, w_2) be the minimum average cost for given factor prices.

1.245 Explain how the factor demands $x^*(w_u\ w_2)$ and the indirect objective function are derived. Prove that the factor demands are homogeneous of degree 0 and that AC* is homogeneous of degree 1 in the factor prices.

1.246 On a graph with AC and AC* on the vertical axis, and w_t on the horizontal axis, plot a typical AC and AC*. Show graphically that AC* is necessarily concave in $W\backslash$ (and, of course, w_2 also.)

1.247 What is the slope of AC* at any given w i ?

1.248 Using this graphical analysis, show that 3(x*/y*)/3w, < 0.

1.249 Show that the elasticity of demand for factor 1 is less than the elasticity of output supply with respect to $w\backslash$.

1.250 Set up the primal-dual model, minimize AC — AC*, and derive the above results algebraically.

1.251 Contrast the factor demands derived from this model, $x^*(w\backslash,\ w_2)$, with the factor demands $xf(w_x,\ w_2,\ p)$ derived from, maximize $pf(x_u\ x_2)$ — $w\backslash X\backslash$ — w_2x_2, where output price p is parametric. Display the first-order conditions for both models, and explain the relation between the models by explaining the following identity, where p^* = AC*(wi, vv₂):

$$X^*(\backslash V_U W_2) = X\backslash \{W_x,\ W_2,\ p^*(\backslash V_l,\ W_2))$$

(h) From this identity, show that the elasticity of demand for $x\backslash$ derived from min AC, [(wi/x*)(3x*/3wi)] is equal to the elasticity of demand derived from profit maximization, plus an output effect which equals the share spent on $x\backslash$ times the output

price elasticity of X\.

4. Consider a profit-maximizing firm employing two factors. Define the short run as the condition where the firm behaves as if it were under a total expenditure constraint; i.e., in the short run, *total expenditures are fixed* (at the long-run profit-maximizing level). The long run is the situation where no additional constraints are placed on the firm.

1.252 Are these short-run demands necessarily downward-sloping?

1.253 Show that the short-run factor demand curves for this model are not necessarily less elastic than the long-run factor demand curves. Why does this anomalous result arise for this model?

(c) Show that if a factor is *inferior* in terms of its response to a change in total expenditure, the slope of the long-run factor demand is necessarily more negative than the short-run demand for that factor. 5. Consider models with the specification

maximize

$$y = /Ui,$$

. . . , * „) subject to

$$g(x_u \ldots, x_n) = k$$

Let *(p (k)* — maximum value of / for given *k.* Assuming an interior solution exists, prove that if /and *g* are both homogeneous of the same degree *r,* then *<fi{k)* is linear in *k,* i.e., *4>{k) = ak,* where *a* is an arbitrary constant, and thus the Lagrange multiplier for such models is a constant.

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CHAPTER

8

THE DERIVATION OF COST FUNCTIONS

8.1 THE COST FUNCTION

We begin this chapter with a discussion of a mathematical construct that has been an important part of the economics literature relating to firm and industry behavior, the *cost function* of a profit- (wealth-) maximizing firm. Specifically, we would like to determine the properties of a function that specifies the total cost of producing any given level of output. Since total costs will obviously be affected by the prices of the inputs that the firm hires, the cost function must be written

$$C = C^*(y, w_1, \ldots, w_n) \tag{8-1}$$

where y is the output level and $w_1, \ldots, w_n$ are the prices of the factors $x_1, \ldots, x_n$, respectively. (The factor prices are assumed here to be constant, for convenience.) The existence of a function as just specified, however, must be predicated on assertions concerning the behavior of firms. If, for example, firms acted randomly, then there would be no unique cost associated with a given output level and factor price vector. Even without the assumption of randomness, there are multiple ways in which a firm could combine given inputs, many of which would produce different levels of output. Each of these different input arrangements would produce a different level of cost, and hence a function such as Eq. (8-1) would not be well defined. Thus, in order to be able to assert the existence of a well-defined cost function, it is necessary, at the very least, to have previously asserted a *theory* of the firm. In doing so, we explicitly recognize that the cost of production depends on what the firm's owners or managers intend to do (the theoretical assertions) and what their constraints are, such as the production function itself, the rules of contracting, and, in some contexts, the factor prices. A wealth-maximizing firm is apt to have a different cost function than a "socialist cooperative" type of firm, which seeks to maximize, say, output per laborer in the firm. Not only are the objective functions of these two firm types different (different behavioral assertions), but if the latter firm is located in in two different countries, the property rights and contracting rules are likely to differ. Thus, even with identical production functions, the cost functions of these firms would differ. And even though production functions might be regarded as strictly technological relationships (hardly likely, since legal frameworks and contracting costs affect output levels), the cost function can never be so regarded. The cost function always depends on the objectives of the firm.

We assert that the predominant firm behavior can be characterized as *wealth-maximizing,* and we derive the cost functions of a firm on this basis. Wealth maximization and the implied resulting cost function are merely assertions. Their usefulness depends on the degree to which refutable propositions emerge from this theory. Even if confirmed, those refutable propositions may also be derivable from other hypotheses about firm behavior, and hence we should not expect to be able to "prove" that firms maximize wealth.

Consider, then, the assertion that firms maximize the quantity π, where

$$\pi = pf(x_1,\ldots,x_n) - \sum_{i=1}^{n} w_i x_i \tag{8-2}$$

This quantity, π, is, of course, not *wealth,* which is a stock concept. Rather, π is the *flow* quantity *profits.* The present, or capitalized, value of π is wealth. In our present model, in which costs of adjustment do not appear, maximizing π necessarily maximizes wealth. How is the cost function (8-1), $C = C^*(y, w_1,\ldots, w_n)$, to be derived? Note that output y is entered as a *parameter* in the cost function. However, the profit-maximizing firm treats y as a decision variable, not as a parameter. That is, output is jointly determined along with inputs as a function of factor and output prices. The factor demand curves for the profit-maximizing firm are $x_i = x_i^*(w_1, \ldots, w_n, p)$. Nowhere does y enter as an argument in these functions. Rather, $y = y^*(w_1,\ldots ,w_n, p)$ defines the *supply curve* of such a firm. This latter function shows how much output will be produced for various output (and also input) prices. The cost function specified in Eq. (8-1) implies that we can observe changes in cost C when an experimental condition, output, is varied autonomously, *holding factor prices constant.* But a profit-maximizing firm never varies output autonomously; output y is changed only when some factor price or output price changes. Hence, the model specified as Eq. (8-2), maximization of profits, cannot be directly used to derive the cost function of a firm.

Cost functions must be derived from models in which output y enters as a parameter. That is, we have to assert that a firm is behaving in a particular way, with regard to the production of some arbitrary level of output y°, where the superscript is added to indicate that this is a parametric value. If, however, it is asserted that the firm in question is a wealth or profit maximizer, then it necessarily follows that such a firm must produce output at the *minimum possible cost.* For any given output, total revenue, py, is fixed. The difference between total revenue and total cost can be a maximum only if the total cost of producing that output level is as small as possible. Hence, the only assertion concerning cost which is consistent with profit-maximizing behavior is

minimize

$$C = \sum_{i=1}^{n} w_i x_i \tag{8-3a}$$

subject to

$$f(x_1, \ldots, x_n) = y^\circ \tag{8-3b}$$

where, again, y° is a parametrically assigned output level.

We can show this result algebraically. Rewrite the two-variable profit maximization model as a *constrained* problem:

maximize

$$py - w_1x_1 - w_2x_2$$

subject to

$$f(x_1, x_2) = y$$

Here, we treat x_1, x_2, and y as three independent variables, linked by the constraint. In Chap. 4 we analyzed this model by immediately substituting the constraint into the objective function.

We could, of course, form the appropriate Lagrangian and set all the first partials equal to zero and solve simultaneously as usual. However, we can also proceed in a *stepwise* manner: First, hold y constant at some arbitrary level y° and maximize with respect to x_1 and x_2 only. This will involve solutions in terms of w_1, w_2 and y°. Then, as a last step, we can substitute these solutions for x_1 and x_2 into the objective function and constraint and maximize with respect to y. Assuming there is a unique global maximum to this problem, we would necessarily get to the same maximum as before. Thus, holding $y = y^\circ$, the first stage of the problem becomes

maximize
x_1, x_2

$$py^\circ - w_1x_1 - w_2x_2$$

subject to

$$f(x_1, x_2) = y^\circ$$

Since py° is a constant, it drops out in the differentiation with respect to x_1 and x_2. The remaining terms are the negative of the objective function (8-3a) for cost minimization. Since maximizing some quantity is equivalent to minimizing its negative, this model is clearly equivalent to (8-3). Thus, profit maximization has embedded in it the implication of cost minimization at the profit-maximizing output. We leave it as an exercise (Prob. 5 at the end of this chapter) to show that the last stage of this stepwise maximization, with respect to *y,* requires that the particular output level the firm chooses must be the one for which marginal cost equals output price. (It also turns out that certain results, especially those that refer to output *y,* such as $dy^*/dp > 0$, are more easily shown when the constraint is explicitly maintained.)

Returning to (8-3«) and (8-3Z?), assuming that $f(x_1, \ldots, x_n)$ is sufficiently well behaved mathematically so that the first- and second-order conditions for a constrained minimum are valid, this model yields, by solution of the first-order Lagrangian equations, the *observable* relations

$$Xj = Xj\,(W_1, \ldots, w_n, y) \qquad i - 1, \ldots, n \qquad \backslash o\text{-}\hat{})$$

Equations (8-4) would be the factor demand curves of a profit-maximizing firm only if that firm were really operating under a constraint that held output constant. It must be noted that these demand curves are *not* the same relations derived in Chap. 4 for a profit-maximizing firm, that is, JC, = x^* (w j, ..., w_n, *p).* Those factor demands are functions of output/?*rice* in addition to factor prices; the factor demand relations (8-4) are functions of output *level* (and factor prices). They are different functions, since they involve different independent variables. It must always be kept in mind which function—i.e., which underlying model—is being considered.

The purpose of specifying these relations is to define the *indirect cost function* (generally referred to as simply the *cost function)*

$$C = \qquad = C^*(w_1, \cdots, \qquad (8\text{-}5)$$

The cost function $C^*(w_1, \ldots, w_n, y^\circ)$ is constructed by substituting those values of the inputs at which the cost of producing ;y° is minimized into the general expression for total cost, ^w,x,. Hence, C^* must be the minimum cost associated with the parametric values $w_1, \ldots, w_n, y^\circ$ (see Fig. 8-1). To reduce notational clutter, we will now drop the superscript 0 from the parameter *y.*

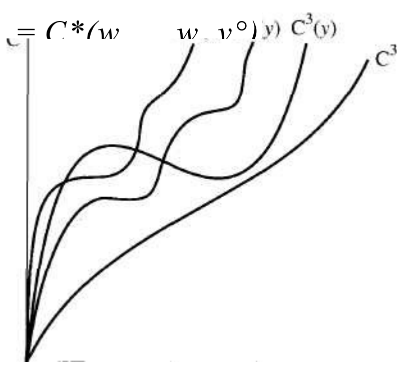

FIGURE 8-1
The cost function is the minimum cost associated with an output level y° and factor prices $w_1, \ldots, w_n$. It is the only cost that is relevant to the behavior of the wealth-maximizing firms. Other behavioral postulates might imply differing cost structures, such as the functions $C^1(y)$, $C^2(y)$, and C^3(>0 illustrated above.

8.2 MARGINAL COST

The marginal cost of a given output level is, loosely speaking, the rate of change of total cost with respect to a change in output. That is, marginal cost is the response of the firm measured by total cost (an event) to a change in a constraint (the level of output). It is tempting to define marginal cost MC as simply

$$\frac{\partial C}{\partial y} = \frac{\partial(\Sigma w_i x_i)}{\partial y}$$

To do so, however, would be to ignore the discussion of the previous section on the meaning of a cost function.

As written, *cost* = *C* = *Y1* w/*; is *not* a function of output *y*. It is a function of the inputs *xi* , ..., x_n and factor prices *w* i, ..., w_n only. It makes no sense mathematically to differentiate a function with respect to a nonexistent argument. The mathematics is telling us something: The cost function has not yet been adequately defined.

As indicated in the last section, there are many ways of combining inputs, and only one of those ways is relevant to us here. Only the *cost-minimizing* combination of inputs that produces a given level of output 3; is relevant. Additionally, marginal cost is not just some arbitrar y increase in total cost that results from an increase in output level; it is the *minimum* increase in cost associated with an increase in output level. Since the function C*(wi, ..., w_n, *y)* defined in Eq. (8-5) gives these minimum costs at any output (and factor price) levels, marginal cost is properly defined in terms of C* as

MC = 9 ***dy***

8-

This partial derivative, being well defined, also shows what is being held constant—the ceteris paribus conditions—

when a marginal cost schedule is drawn. As commonly drawn (see Fig. 8-2) in two dimensions, the marginal cost function depends

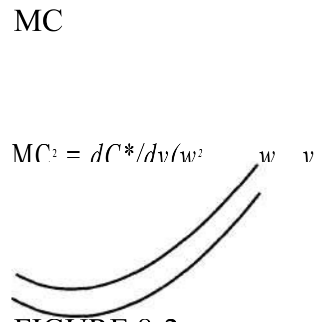

FIGURE 8-2
The Marginal Cost Function. The marginal cost function, being the partial derivative of $C^*(w_1, \ldots, w_n, y)$ with respect to output y, is itself a function of those same arguments $w_1, \ldots, w_n, y$. Shown above are two marginal cost functions, for two different values of w_1. It is not possible to determine from the above graph whether

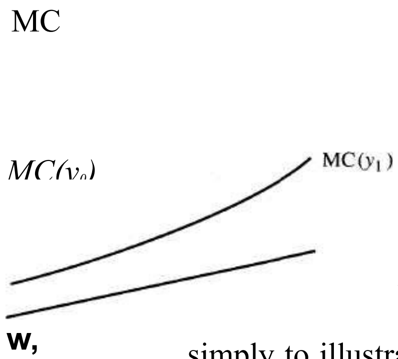

FIGURE 8-3
Marginal Cost as a Function of a Factor Price for Specific Levels of Output. This curve, which has no common name, is drawn simply to illustrate the many-dimensional aspect of marginal cost. Its slope, 9MC/3wi, shown as positive here, is in fact indeterminate.

on the values of the factor prices; i.e., MC can shift up or down when a factor price changes. A change in factor price represents a shift in the MC curve in Fig. 8-2 only because MC there has been drawn as a function of *y* only, holding all the w, 's constant. It is also possible to draw MC as a function of, say, *W*, holding $y, w_2, \ldots, w_n$ constant, resulting in such a curve as is drawn in Fig. 8-3. This curve has no common name, but, as we shall see later, its slope, 3MC/9wi, can be either positive or negative; i.e., its sign is not implied by wealth maximization. On this graph, changes in *y,* as well as the other factor prices, would *shift* the curve. In the next few sections we will explore the implications of wealth maximization and cost minimization on these marginal cost functions. In addition, we shall discuss the relationships between marginal and average cost.

8.3 AVERAGE COST

A frequently discussed function, the *average cost function* AC is defined as

$$AC = \frac{C^*(wi,\ldots,w_n,y)}{y} = AC^*(w_u\ldots,w_n,y) \qquad \textbf{(8-7)}$$

Again, AC must be defined in terms of the *minimum* cost achievable at any output and factor price level, as given by C*(wi,... $, w_n, y)$. As with the marginal cost function, a behavioral postulate is a logical necessity for a proper definition of the average cost functions. Since AC* is a function of factor prices and output, the partial derivatives 3AC*/9w;, / = 1, ..., *n* and 9AC*/9;y are well defined. That is, we can meaningfully inquire as to the changes in average cost when output and factor prices vary. In the usual diagram, Fig. 8-4, average cost is plotted against output _y. Its familiar U shape is *not* implied solely by cost minimization, as we shall see later. Changes in factor price will shift the average cost as drawn in Fig. 8-4. As will be shown later, an increase in a factor price can only increase a firm's average

AC

AC

FIGURE 8-4 *The Average Cost Function.* The average cost function is drawn in its usually assumed U-shaped form. Since jo is the only parameter allowed to vary, changes in factor prices will shift this curve. In fact, *dA*

$\partial C^*/\partial w_i > 0$, for all factor prices; i.e., an increase in factor price must increase a firm's average cost at that output level. It would be possible to plot AC^* explicitly against, say, w_1 holding $w_2, \ldots, w_n$ and y_0 constant. That curve would necessarily have a positive slope.

costs (though this is not true for marginal costs!), as a moment's reflection clearly reveals. Otherwise, a firm could always make a larger profit by agreeing to pay more to some factors of production, say, labor. This would be readily agreed upon. Clearly, all empirical evidence refutes this particular harmony of interests. At some point, firms must begin to run short of revenues and regard increasing factor costs as profit-lowering.

8.4 A GENERAL RELATIONSHIP BETWEEN AVERAGE AND MARGINAL COSTS

By definition

$$\frac{C^*(w_1, \ldots, w_n, y)}{y}$$

Since this

is a mathematical identity, it is valid to differentiate both sides with respect to any of the arguments. Differentiating with respect to v yields (using the quotient rule on the right-hand side)

(8-8)
$$\frac{\partial \mathrm{AC}^*}{dy} = \frac{[y(dC^*/dy) - C^*]}{y^2}$$

Noting that $dC^*/dy = \mathrm{MC}^*$, and rearranging terms slightly, (8-9) gives

$$\mathrm{MC}^* = \mathrm{AC}^* + \frac{}{dy}\, y$$

This is a general relation between marginal and average quantities. (It holds as well for average and marginal products, etc.) It is useful for understanding the nature of these magnitudes.

Marginal cost is *not* the cost of producing the "last" unit of output. The cost of producing the last unit of output is the same as the cost of producing the first or any other unit of output and is, in fact, the *average* cost of output. Marginal cost (in the finite sense) is the increase (or decrease) in cost resulting from the production of an extra increment of output, which is not the same thing as the "cost of the last unit." The decision to produce additional output entails the greater utilization of factor inputs. In most cases (the exception being firms whose productive process is characterized by constant returns to scale, i.e., linear homogeneity), this greater utilization will involve losses (or possibly gains) in input efficiency. When factor proportions and intensities are changed, the marginal productivities of the factors change because of the law of diminishing returns, therefore affecting the per unit cost of output. The effects of these complicated interrelationships are summarized in Eq. (8-9). Note what the equation says: Marginal cost is equal to average cost plus an adjustment factor. This latter effect is the *damage* (or gain, in the case of falling marginal costs) to all the factors of production caused by the increase in output, which causes the cost for each unit of output to increase (or decrease, for falling MC). This total "external" damage equals dAC^*/dy, multiplied by the number of units involved, y. That is to say, marginal cost differs from average cost by the per-unit effect on costs of higher output multiplied by the number of units so affected (total output). The very reason why marginal quantities are usually more useful concepts than average quantitites is that the average quantities ignore, whereas the marginal quantities have incorporated within them, the interrelationships of all the relevant economic variables, in this case, the factor inputs.

The distinction between average and marginal cost is perhaps most clearly seen by considering a famous problem in economics, that of road congestion, first analyzed in 1924 by a distinguished theorist at the University of Chicago, Frank Knight. If a freeway is uncongested, then when an additional car enters, there is no effect on the average speed, or travel time, of the other cars already on the freeway. Suppose all trips on an uncrowded freeway take | hour; in this case, the average time and the marginal time both equal 30 minutes. Suppose, however, there are already 10 cars on a section of the freeway, and when the eleventh car enters the roadway, some congestion occurs, slowing everyone's travel time by 2 minutes. Then, although the *average* travel time is now 32 minutes, this is not the marginal time cost imposed by the eleventh car. The marginal time cost of adding the eleventh car is its own 32 minutes of travel time *plus* the 2 extra minutes imposed on each of the previous 10 cars, or $32 + 10(2) = 52$ minutes. Equation (8-9) expresses this relation in continuous time.

The "economic problem" of freeway congestion exists because consumers of the freeway are unable to pay for the full consequences of using the freeway— freeways are called "freeways" because no fee is charged for their use. Since the marginal cost of using a congested freeway exceeds the price charged, we have "too much" freeway use. Frank Knight pointed out that if the road were privately owned, profit maximization would lead to efficient use. The toll that can be collected is the difference between the time value of using the (presumably slower) sidestreets and the freeway. If sidestreet travel is constant at some level whose value is p, then the private owner will maximize $T = x(p — \mathrm{AC}(x))$, where AC(JC) is the average cost, in dollars, of the time spent on the freeway. The first-order condition for this model is simply p = MC(i), as in ordinary profit maximization. However, in this model, this equation means that the freeway will then be utilized efficiently, since cars do not take the freeway when their marginal opportunity cost to society exceeds the marginal value of using their alternative transport mode.

8.5 THE COST MINIMIZATION PROBLEM

We now turn explicitly to the mathematical model from which all cost curves for wealth-maximizing firms are derived:

minimize

subject to

$$f(x_u \ldots, x_n) = y \qquad (8\text{-}11)$$

where the w,'s are unit (constant) factor prices, $f(x\backslash, \ldots, x_n)$ is the production function of the firm, and y is a parametric value of output. This model, referred to as the *cost minimization model,* asserts that firms will minimize the total cost *J2* w,X/ of producing any arbitrarily specified output level. Let us develop the empirical implications of this assertion.

To keep things manageable, we will develop the two-variable case of this model first. That is, assume that the firm employs two factors $x\backslash$ and x_2 only. Since this is a problem of constrained minimization, form the Lagrangian function

5£ = wiJti + w_2x_2 + $X.[y - f(x_u x_2)]$ (8-12)

where the X is the Langrange multiplier. Differentiating i£ with respect to $X\backslash$, x_2, and A yields the first-order conditions for a minimum:

SBi = w, -A./i =0 (8-13*d)*

$\$_2 = w_2 - Xf_2 = 0$ (8-136)

°*&k* — / — *J* V-M ? X_2) — U \\(*J-LJC)*

The sufficient second-order condition for an interior minimum is that the following bordered Hessian determinant be *negative* (this determinant is, of course, simply the determinant of the matrix formed by the second partials of the Lagrangian i£ with respect to $X\backslash$, x_2, and A):

A/n — A/12 —*fi*

H = -A/21 -A/22 < 0 (8-14)

-/

-*fi* ~*fi* 0

The elements of this determinant are, row by row, the *first* partials of the first-order equations (8-13), which makes them the second partials of the Lagrangian function i£.

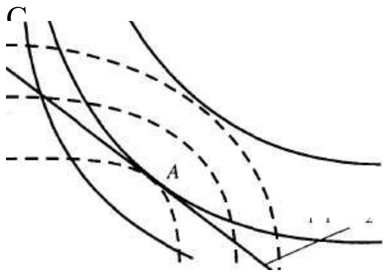

FIGURE 8-5
The Tangency Solution to the Cost Minimization Problem. Assuming that marginal products are positive, i.e., that output increases as one increases either input (movement in a "northeasterly" direction), production isoquants must be convex to the origin in order that a tangency point be a *minimum* cost for a specified output. If the isoquants were shaped like the dashed curves here, *A* would clearly represent a maximum cost, i.e., the most inefficient way to product output v.

These algebraic conditions for a minimum can be interpreted geometrically. In Fig. 8-5, the level curve *f(x\, x_2) = y,* the constraint in the cost minimization problem, defines a locus of input combinations that yield the output *y.* Economists call these level curves *isoquants.* (See Chap. 3 for additional review.) The slope of these isoquants at any point is, again, found by differentiating the identity *f(x\, x_2(x\)) = y* implicitly with respect to *X*. This yields

or

dx_2dx

h (8-15)

assuming f_2 ^ 0, i.e., that the isoquant is not vertical at this point. The slope of the isoquant is the negative ratio of the marginal products of input 1 to input 2. Thus, —/1//2 defines a particular *direction* in the *X\X_2* plane.

On the other hand, the objective function C = WiXi -\-w_2x_2 also defines a direction. For any specific value of C, say C°, the objective function is a linear curve, i.e., a straight line in the *x\x_2* plane. Its slope is dx_2/dx\ = —w_1/w_2.

If A. is eliminated from Eqs. (8-13a) and (8-13Z?) by moving *Xfi* and *Xf_2* to the right-hand side and dividing one equation by the other, one gets

w_2 ***h*** (8-16)

That is, the ratio of wages equals the ratio of marginal products for the two factors. This is a straightforward application of the maximization theorems presented in Chap. 6. That this tangency is necessary for a minimum-cost solution is evident from Fig. 8-5. Lower costs are associated with *isocost* lines (i.e., curves of equal cost, C = W1X1 + w_2x_2) that are closer to the origin. The minimum-cost problem says: Pick the isocost line closest to the origin but that still allows output *y* to be achieved. The furthest point toward the origin that C = *W\Xi* + w_2x_2 can be pushed and still make contact with the isoquant *f(x\,x_2) = y* is clearly the tangency point A.

These second-order conditions (8-14) say that production function is quasi-concave; i.e., in this two-dimensional case, th e isoquants are "convex to the origin." If the isoquants were shaped like the broken curves in Fig. 8-5, i.e., concave to the origin, the tangency would clearly not represent a minimum-cost solution. Costs could be lowered by proceeding to where such an isoquant intersected one of the axes. This would not be a tangency solution, but, rather a "corner solution."

There is a major empirical reason for believing that production isoquants are not shaped like the broken curves in Fig. 8-5 but rather are convex to the origin as originally drawn, and as implied by the second-order conditions (8-14). The empirical reason for believing in such convexity of the isoquants is that if they were otherwise, we would observe firms employing only one factor of production. With isoquants concave to the origin in all dimensions, the minimum-cost solution would be at the intersection of the isoquant with one axis. (In an unlikely case, two or more such intersections could occur at the same cost, providing multiple solutions.)

Since wealth-maximizing firms are cost minimizers, only one factor, that which gave the minimum-cost solution, would be hired. Additionally, consider how the solution would change if a factor price changed. In Fig. 8-6, as the factor price *w* is lowered from its original value of wf, the minimum-cost solution remains at corner *B,* with the firm showing no response to the decreased factor price. It hires *x%* amount of x_2. Then, at some critical value of Wi, say *w\,* the isocost line would cut through both corners, i.e., both intersections of the isoquant with the axes. The firm would then be indifferent to hiring *2 of JC_2 or *x*® of JCI ; i.e., the firm's costs are identical with corner *A* and corner

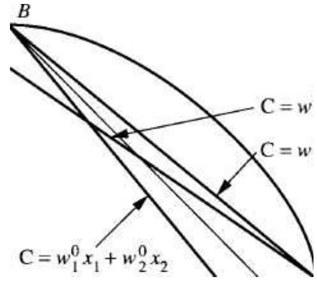

FIGURE 8-6 *The Empirical Implications of Quasi-Convex Production Functions.* With the isoquants concave to the origin and wages

given by wj and *w®,* the minimum-cost solution is at point *B,* where only *X2* is hired. The demand for *x* is 0. As *w* is lowered, the cost line pivots around *B,* the intersection with JCI axis moving outward toward A. When A is reached, a multiple solution exists; the firm is indifferent between hiring *x®* of JCI, or *x®* of *X2*- When *w* is now made arbitrarily smaller, the demand for *X2* falls to 0, and the demand for *X]* jumps discontinuously to *x®.* The demand for *x* remains constant at *x®* for all further lowering of its wage *w\.* The demand curve for *x* is thus vertical at that level.

o

FIGURE 8-7
The Demand for x_1, If Isoquants Were Concave to the Origin. The reason for rejecting concave-to-the-origin isoquants is that they imply empirical behavior inconsistent with the facts. In particular, such firms would show no response to factor-price changes except as critical wage levels (here, w_1). The demand curve would consist of two vertical sections. This behavior is not observed by real-world firms.

B. As soon as w_1 is lowered below w_1, even just a trifle, the firm would suddenly switch over completely to x_1 at the level x_1^0 given by the intersection at *A*. The firm would show no response to further lowering of w_1. This scenario implies that the demand curves for the two factors will be in vertical sections as depicted in Fig. 8-7. There will be no response to some factor price changes, and violent responses (when the firm switches corners) to others. Now we simply do not see this combined intransigence and discontinuous hiring of factors in the real world. Rather, firms respond gradually to factor price changes, with larger responses accompanying larger price changes. This observed behavior is inconsistent with isoquants concave to the origin and hence we can assert with confidence the quasi-concavity of production functions. Indeed, as mentally changing the slope of the isocost line in Fig. 8-5 will indicate, factor price changes imply continuous, or smooth responses in factor hiring for the case of convex-to-the-origin isoquants.

One might believe on *intuitive* or *introspective* grounds that isoquants are convex to the origin. It may seem plausible that along any isoquant, the slope $-f_1/f_2$ decreases in absolute value as more x_1 is hired. That is, the marginal product of x_1 relative to that of x_2 falls as more x_1 is hired. This is often called the *law of diminishing marginal technical rate of substitution.* It is not the same as the law of diminishing returns, discussed earlier in Chap. 3, which asserts $f_{11} < 0$. *Neither one of these two "laws" implies the other.*

If all this is plausible to you, all well and good. But intuition is *not* a good enough reason for believing in the convexity of isoq uants. None of us has ever seen an isoquant, and we are not ever likely to do so. The only reason for believing, with some confidence, in such convexity is that the reverse situation implies firm behavior that is inconsistent with the facts of the empirical world, i.e., the situation of intransigence and discontinuous response to factor price changes.

Having established that an interior tangency point is the only sensible solution to the cost minimization problem; i.e., not in contradiction with the facts, and obvious as it may be from the geometry of Fig. 8-5 that the tangency point A is the minimum-cost solution, it is still interesting and useful to go on to ask: What is it about the decision-making process of the firm that leads to this type of solution? That is, what does such a solution (a tangency) imply about the nature of minimum-cost decision making? To answer this question, it is necessary to examine again the meanings of the slopes of level curves, in this case $-f_1/f_2$ and $-w_1/w_2$. (The reader should review, if necessary, the sections of Chap. 3 dealing with level curves and surfaces.)

In Fig. 8-8, consider point *A'* on the isoquant *f(x\, x₂) = y.* The cost lines C = *w\X* + *W2X2* have a slope — *W\/\V2* that is less in absolute value than the slope of *f(x\, x₂) = y* at *A',* (—/1//2). Suppose the firm decided to produce the same

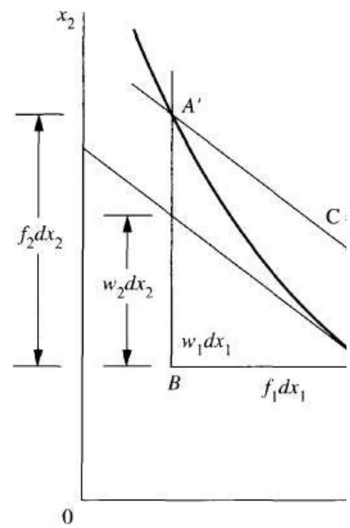

FIGURE 8-8
Tangency and Nontangency Points. It is obvious from the geometry that *A'* cannot be a minimum-cost solution. In terms of the economics of cost-minimizing firms, however, point *A'* indicates that the firm's *willingness* to trade away some *X2* to get some JCI, as measured by the slope of the isoquant at *A',* is unequal to the firm's *opportunities* for doing so, as measured by the isocost line C = *w\X* + W2*2- At point *A'*, the value, internally, to the firm of additional units of *x* is measured by /1//2, the slope of the isoquant *at A'*. This internal value is grea

ter than the market cost of exchanging some *X2* to get more *x*\, or *w* 1 /*V2*- It therefore is cost-saving to move from *A′* to *A*. Similar reasoning would apply if *A′* were to the right of A; then, *xi* would be the desirable factor at market prices.

output y at point A, hiring more Xi by an amount $dx\backslash$ and less x_2 by an amount dx_2. We can view the move from A' to A conceptually as one from A' to B and then from B to A. The decrease in output caused by hiring less x_2 is the marginal product of x_2, MP_2, which is approximately f_2 evaluated at A multiplied by the decrease in x_2, or f_2 dx_2. In Fig. 8-8, this is the decline in output due to a movement from A' to B. In moving from B to A, $x\backslash$ is increased by an amount $dx\backslash$, and that extra JCI has a marginal product approximately equal to $f\backslash$ evaluated at A, and hence the gain in output going from B to A is $f\backslash$ $dx\backslash$. Since the output at A is the same as the output at A', both being y, it must be the case that $f\backslash\, dx\backslash + f_2\, dx_2 = 0$. If the point A' is moved arbitrarily close to A so that the approximation becomes better and better, this simply becomes the statement that along any level curve, the total differential $dy = f\backslash\, dx\backslash + f_2\, dx_2$ equals 0. Assuming x_2 can be written as a function of JCI, the slope of the isoquant again is $dx_2/dx\backslash$ = —$f\backslash/f_2$. But we can now understand what this relationship means to a firm. The slope of an isoquant represents how much x_2 can be given up per unit $X\backslash$ added in order to keep output constant. This output-preserving ratio of inputs must equal the ratio of the gain in output (MP! = fi) to the loss in output ($MP_2 = f_2$) that occurs per unit changes in the inputs. This slope, $f\backslash /f_2$, therefore measures the values of $x\backslash$ to the firm, internally, in terms of x_2.

Suppose, for example, that the marginal product of xi is 10, while the marginal product of x_2 is 5. Then $f\backslash/f_2$ = 2. Then clearly for "small" changes at least it will be possible to decrease x_2 by 2 units for every unit of increased JCI . The ratio $f\backslash /f_2$, equal to 2 here, measures the rate at which one factor x_2 can be displaced by additions of the other factor JCI, keeping output constant. It is the marginal technical rate of substitution.

Now consider point A' again in Fig. 8-8. Suppose that w_x/w_2 — 1; that is, the slope of the isocost line equals — 1. This means that the factor market allows input JCI to be substituted for input x_2 at equal cost. That is, for every added unit of $x\backslash$, exactly one unit of x_2 has to be given up in order to maintain the same expenditure level. But we have seen, in our numerical example, that the firm can give up *two* units of x_2, add one unit of $x\backslash$, and have the same output. Therefore, a savings of the cost of one unit of x_2 is obtained by moving toward A. Hence, A' cannot be a minimum-cost solution.

In general, the slope of the isoquant measures the firm's *willingness* to trade one input for the other (substitute JCI for x_2). The slope of the isocost lines represents the *opportunities* afforded by the factor market for doing so. When the firm is willing to trade one factor for another at terms of trade different from the factor market, cost saving is possible. This is the meaning of a tangency solution. It doesn't matter if the original point is to the left or right of A on the isoquant $f(x\backslash,x_2) = y°$. In a more general sense, the gains from exchange (exchange with a general market at fixed prices as well as exchange with other individuals) are not exhausted unless one's *willingness* to trade, e.g., as measured by a firm's output-preserving marginal rate of factor substitution, equals the available *opportunities* for such trading, e.g., as measured by the cost of exchanging one factor for another. For firms, such efficient factor combinations are summarized by the condition that $fi/f_2 = W\backslash/w_2$.

8.6 THE FACTOR DEMAND CURVES

Let us now return to the first-order Eqs. (8-13), which are, again,

W1-A./i=0 (8-13a)

w_2-A./$_2$ = 0 (8-13Z?)

y-f(x$_u$x$_2$)=O (8-13c)

The sufficient second-order condition is that the determinant of the matrix of second partials of i£ with respect to *x*, *x_2,* and A,, which is in fact the matrix formed by the first partials of (8-13a), (8-13ft), and (8-13c) with respect to those variables (these equations already being the first partials of ££), be negative. This determinant, again, is

H = -Xfn — A./12 —/1
■A/21 — A/22 —fi < 0 (8-14)
-h -h 0

The implicit function theorem, discussed in Chap. 5, says that if the determinant of the first partials of a system of equations is nonzero, those equations can be solved, locally (in principle—not, perhaps, easily) for those variables being differentiated as explicit functions of the remaining variables (here the parameters) of the system. The determinant *H* is such a determinant and is nonzero, in fact negative, by the sufficient second-order conditions. Hence, Eqs. (8-13) can be solved for JCI , *X2,* and A. in terms of the parameters *Wi,w_2,* and *y,* yielding

Xi = x(w\\,* u>2, *y)* *(8-17a)*

x_2=xZ(wi,w_2,y) (8-17ft)

k = k(w$_u$w$_2$,y)* (8-17c)

Equations (8-17a) and (8-17ft) represent the factor demand curves *when output is held constant,* previously discussed as Eqs. (8-4). Note the parameter *y* in these equations. If, say, *x* is plotted on a two-dimensional graph with its wage represented on the other axis, the resulting plot will be a curve [actually, a one-dimensional projection of Eq. (8-17a)] along which w_2 and 3; are constant. These curves, therefore, do not represent the factor demand curves of a firm engaged in unrestricted profit maximization, in which case output would be variable and output *price* (for the competitive case) would be parametric.

Interpretation of the Lagrange Multiplier

Equation (8-17c) gives A. as a function of *w\\, w_2,* and *y.* But what is A.? This new variable was concocted as an artifice—as a convenient way of stating a constrained minimization problem. Does A. have any meaningful economic interpretation?

Indeed, we can show that A, or more correctly A*(wi,w2, *y),* is identically the marginal cost function of the firm!

The first clue to this interpretation of A* can be gleaned from the first-order Eq. (8-13). Solving for A* yields

■ = *-j- = -}* (8-18)

Also, by multiplying (8-13a) by *x\,* (8-13Z?) by x_2, and adding, one obtains

A*/i** + r/2JC* = $w_x x\backslash + w_2 x^*$

Factoring out A*, and noting that *W\x* + $w_2 x\backslash$ = C*,

Note the "units" of W1//1 and w_2/f_2. Say that, for example, JCJ is "labor," x_2 is "capital." The wage rate *w* is measured in dollars per laborer; the marginal product of labor has units output per laborer. Hence, the expression *w* i //i has the units dollars per output, since the labor units cancel. The measure dollars per output in fact comprises the units of marginal cost, though it also comprises the units of average cost.

What, then, is the meaning of the following extended equality?

W\ Wo C*

** = *-j-* = *7}* = , « ^ , * (8-19)

/i *h* ***fix* + hx***

The firm is at its cost-minimizing input mix. Suppose it were to increase its input of *xi,* say, labor, by a small amount *Ax\.* The total cost would rise by an amount (W>I)(AJCI). Output would also rise, by an amount (MPi)(Ajti) = (/i)(Axj). Hence, A. = *Wi/fi* = (WI)(AJCI)/(/I)(AJCI) represents the incremental cost of increasing output through the use of one input, here *Xi,* or labor. Similarly, A = W2//2 = *(w2)(Ax_2)/(f2)(A.x_2)* represents the incremental cost of additional output when the other input, x_2, say capital, is increased. The equality of these two incremental costs, as indicated by Eq. (8-18) means that a necessary condition for cost minimization is that the incremental cost of additional output must be the same *at all margins,* i.e., for each independent decision variable. This common incremental cost of output is the *marginal cost of output.* Equation (8-18) says that the firm is indifferent, at the margin, to hiring additional labor or capital—the net costs of doing so are identical for each input.

This is, of course, what must be true at a minimum-cost point; for suppose that the firm could achieve a lower incremental cost of output by hiring labor, say, rather than capital. In that case, total costs could clearly be lowered by shifting resources away from capital and toward labor. Only when costs are equalized at all the margins can a minimum-cost solution be achieved, and this common marginal cost is equal to A*.

What about the last equality in Eq. (8-19), A* = *C*/(fix^ +* $f_2 x^*$*)7* This is a more difficult expression to interpret. Consider that A* = *w\/f* — *W\x\/f\x\.* Whereas *w\/f* refers explicitly to *per-unit* changes in input

1, $w\backslash x\backslash/f\backslash x\backslash$ applies that marginal factor cost (wjjc*) and benefits *(fix*)* to the total input level. Likewise, *X = w₂xyf₂X2,* the cost of all units of factor 2 per marginal contribution of that factor multiplied by the total factor usage, is also marginal cost, since the incremental costs must be the same at every margin. Then, by elementary algebra^

j *

This expression says that not only is marginal cost the same at every margin, it is also the same if a combination of *both (every,* in the multifactor case) factors is changed. Marginal cost is the same at "either or both" margins.

The foregoing was intended as an intuitive explanation of why the function *X = X*(w\,w₂, y)* might reasonably be regarded as the marginal cost function. While intuitively plausible (and ultimately sound), the approach is deficient in terms of our original definition of marginal cost. Specifically

$$\mathrm{MC} = \frac{ac^*}{dy}$$

where *C*(\V[, w₂, y)* is the (indirect) cost function. It remains to be proved explicitly that *X* = dC*/dy.* We shall now do so. By definition

C* = *W\x*(w\, w₂, y) + w₂x2*(w\, w₂, y)* (8-20)

That is, the minimum cost for any output level *y* (and factor prices *w\, w₂)* is obtained by substituting into the expression for total cost, C = *w\X\ + w₂x₂,* the values of the inputs that are derived from the cost minimization problem. These are the relations *xₜ = x*{w\, w₂, y), i* = 1, 2 (Eqs. *8-lla,b).* Thus, differentiating C* partially with respect to *y,*

$$\frac{ac^*}{ay} = w_1\frac{dx?}{dy} + w_2\frac{dx^*}{ay} \qquad (8\text{-}21)$$

However, from the first-order relations, *W\ = X*fi, w₂ = X*f₂.* Substituting these values into Eq. (8-21), and factoring out *X*,*

f) (8-22)

If in fact *X* = dC*/dy,* the term in parentheses in Eq. (8-22) must equal 1. How can this be shown? Consider the last equation of the first-order conditions (actually the constraint):

$$y - f(x_u x_2) = 0$$

Ht is valid to add the numerators and denominators, respectively, of fractions that are equal. Thus, if *a/b = c/d* (implying *ad = be),* then *a/b = c/d = (a + c)/(b + d),* as can be quickly verified.

When the solutions to the first-order relations, the factor demand curves holding output constant, Eqs. (8-1 *la, b)* are substituted back into those first-order conditions, Eqs. (8-13) become identities. In particular,

$$y - f(x\backslash(wi,W2,y),xZ(w\backslash,w_2, y)) = 0$$

That is, x^* and $x\backslash$ *always* lie on the isoquant of output level y for any $w\backslash, w_2$ and any y, precisely because x^* and $x\%$ are the solutions of equations that say, among other things: Output is held to y.

Hence, we can differentiate this identity with respect to y:

$$ay \qquad dy$$

or

$$ay \qquad ay$$

This is precisely what was needed. The term in the parentheses in Eq. (8-22) equals 1, and therefore

$$A^* = \frac{dc^*}{dy} \qquad (8\text{-}23)$$

That $X^* = dC^*/dy$ is in fact a simple consequence of the envelope theorem derived in the last chapter. Recall the general maximum problem with, for simplicity here, one constraint:

maximize

$$f(xi, \ldots, x_n, a_l, \ldots, a_m) = z$$

subject to

$$g(x_u \ldots, x_n, a_u \ldots, a_m) = 0$$

The Lagrangian for this problem is

The envelope theorem says that

That is, the rate of change of the indirect objective function in which all the JC, 's can adjust to changes in a parameter is in fact equal to the rate of change of the

Lagrangian function (which numerically equals the objective function since the constraint equals 0) with respect to that parameter, *holding all the x_i 'S fixed.* Applying this theorem to the problem at hand,

minimize

$$C = w\backslash X\backslash +$$

W2X2 subject to

$$y-$$

$f(xi,x_2)=O$ we have

$$i£ = W1X1 + w_2x_2 + Hy -$$

f(xi Here, the parameter *y* enters only the

constraint; hence

$$\frac{dC^*}{dy} \_ \frac{3ig}{} \_ \_ \frac{d}{dy}$$

The Lagrange multiplier was introduced as an artifice for writing, in a convenient manner, the first- and second-order conditions for a constrained maximum problem. We see here, however, that the Lagrange multiplier can have interesting economic interpretations. This fact greatly enhances the value of Lagrangian methods. As we shall see, these multipliers more often than not provide useful formulas and insights for analyzing economic problems.

8.7 COMPARATIVE STATICS RELATIONS: THE TRADITIONAL METHODOLOGY

We now investigate the responses of cost-minimizing firms to changes in the parameters they face. In the next section, we will derive these results using the new and more powerful methodology of duality theory. However, we proceed first using the traditional procedure as outlined in Chap. 6 so that this important procedure can be illustrated and understood. As stated previously, for other than the basic models, and for nonmaximization models, this is likely to be the only available technique for investigating the responses of the decision variables to changes in the parameters.

The questions we ask are, how do cost-minimizing firms react to an increase or decrease in a factor price? Will more or less input be used when its own or some other input's price increases? How will marginal and average cost be affected? Will the firm increase or decrease its output if competitive pressures force it to remain at the minimum point on its average cost curve?

The format for investigating these questions is, again,

minimize

C = *wiX* + *W2X2*

subject to

$$= y$$

where $w\backslash$ and w_2 are the factor prices and y is a parametrically determined level of output. The Lagrangian is

$$w_2x_2 \qquad , x_2))$$

Differentiating with respect to $x\backslash, x_2,$ and A. yields the first-order conditions for constrained minimization:

$$kfi = 0 \qquad \text{(8-13a) (8-13Z?)}$$

$$A/2 = 0 \qquad \text{(8-13c)}$$

The sufficient second-order conditions are, again,

$$H = \begin{array}{ccc} & -A/12 & \\ & -/1 \quad —A/22 & \\ -h & —fi & \\ & -h \quad 0 & \end{array} < o \qquad \text{(8-14)}$$

These relations were given verbal interpretation in the previous sections.

If the production function $y = f(x\backslash, x_2)$ were actually known, then Eq. (8-13) could be used directly to characterize the least-cost solution. Everything about the firm would be completely known, including the *total* amounts of each factor that would be used at any input level and all the changes that might come about because of a change in a parameter. However, economists are not generally blessed with this kind of information. Rather, we *assert* that some sort of production relationship $y = f(xi, x_2)$ exists, with quasi-concave properties [summarized as the inequality $H < 0$, Eq. (8-14)]. We then inquire as to *changes* in response to parameter changes; i.e., we limit the analysis to *marginal* quantitites. This is accomplished by using the methodology of comparative statics outlined earlier.

Equations (8-13) represent three equations in six variables Xi, x_2, A, $w\backslash$, w_2, and y. As long as certain mathematical conditions exist, namely, that H ^= 0, these equations can be solved, in principle yielding the relations already discussed:

$$xi = x^*(w_u w_2, y) \qquad \text{(8-17a)} \quad \text{(8-17\&)}$$

$$x_2 = x^*(w_1, w_2,; y) \qquad \text{(8-17c)}$$

$$X — A^*(wi, w_2, y)$$

The new variable A. is identified as marginal cost.

The comparative statics of this model can be summarized as the determination of the signs of the nine partial derivatives:

(8-

24) • _ i 9

I — 1, /•

dy

We seek to determine, first, the extent to which the constrained minimum hypothesis generates (qualitative) information about these marginal quantities. It will also be shown that some relationships exist among these partials and that expressions can be derived which may be useful if empirical information is used in addition to the minimization hypothesis.

The first step in comparative statics analysis, of course, is to substitute the solutions (8-17) into the first-order Eq. (8-13), from which they were solved. This yields the *identities*

wi - *X*(w₁,w₂,* y)/i(ii(wi,w₂, *y), xZ(wi,w₂, y))* = 0 (8-25a)

w₂ -A*(wi,H'2j)/2(Xi(wi,W2, *y),x%(wi,w₂, y))* = 0 (8-25Z?)

y - f(x(wi,w₂, y), x₂*(w₁,w₂, y))* = 0 (8-25c)

These are identities because the solutions to Eqs. (8-13) are substituted into the equations from which they were solved. The economic significance of this step is that now it is being asserted that whatever the factor prices and output level may be, the firm will always instantaneously adjust the factor inputs (its decision variables) to those levels that will minimize the total cost of that output level. The *identities* (8-25) tell us that we have asserted that we will never observe the firm to be in any other than a cost-minimizing configuration. Having built in this strong assertion, it is then possible to alter the parameters and observe the resulting changes in the JC,- 's. These changes are observed mathematically by differentiating the identities (8-25) with respect to a parameter and solving for the relevant partial derivatives contained in the list (8-24). Let us begin the formal analysis by observing the cost-minimizing reaction to a change in *w* \, the price of factor 1. Differentiating the identities (8-25) with respect to *w*\ yields, using the product rule for A.*/i and *X*f₂* along with the chain rule,

1 - r/,, |£ _ r/12M - /, 1^- - 0 (8-260)

-^^-rfeM-A^Uo (8-266)

-f^-h^^O (8-

26c)

These relations can be more clearly summarized using matrix notation. Since (8-26) represents three *linear* equations (actually, identities) in the three unknowns *8x*/8w* i, *8x2/dwi,* and *8*/dw\\,* (8-26) becomes

/ \

-r/1 8

2 - x\

\

8

(

—

fi

8 - Cramer's rule, one gets

- 8 X * 9* \2 (8

°

8k J *

One need only solve (8-27) for the marginal quantities. Using Cramer's rule,

12

13

-H

H

-

H

H

0 -

(H (8-

H

28a)

In like fashion, replacing the second and third column of H with the right-hand column vector (—1,0, 0) for the numerator of

-28b)

(8-28c)

where H_{ij} is the (signed) cofactor of the element in row i and column j of the determinant H.

These solutions are only valid if $H \neq 0$. Otherwise, the preceding partials are undefined. The mathematics indicates that in order for these partials to exist, the solutions (8-17) must first be well defined. The implicit function theorem of Chap. 5 indicates that these solutions are valid if the determinant of first partials of (8-13) is nonzero. This determinant is exactly H, and the *sufficient* second-order condition for constrained minimization says that in fact $H < 0$. Hence, under these assumptions, the solutions (8-28) are valid expressions.

What can be said of the sign of these partials? As just noted, the denominators of these expressions, H, are all negative. The cofactor H_{11}

is a border-preserving principal minor and is negative under the minimization hypothesis. By inspection,

$$l < 0 \quad = \quad - \frac{}{fi\, f_i} - f$$

0

Hence, the qualitative result $\partial x^*/\partial w_1 < 0$, is demonstrable.

The cofactor H_{12} is *not* a border-preserving principle minor; in general, its sign will not be known, and hence the sign of $\partial x_2 / \partial w_1$ will not be determinate. However, in the *two-variable case only, assuming positive marginal products,*

,1+2
-/i

Hence, *for the two-variable case only,*

d
x >0
*

d
\
V
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The fact that it must happen that *dx*
^/
d

wi >0 for the two-factor case is easily explainable. Suppose w>i falls. The firm will then hire more JCI . If the firm also hired more x_2, then output would have to rise, given the assumption of positive marginal products. And marginal products will be positive as long as factor prices $w\backslash$ and w_2 are positive. If JCI increases, then if output is to be held constant, $X2$ must decrease. This relationship, however, need not hold for more than two factors. Some other factor (or both) must decline, but not necessarily one or the other. Finally,

#13 =

Hence, dX^*/dw_x ^0.

Similar relationships can be derived for the responses to a change in w_2. In that case, the — 1 appears in the *second* row of the right-hand side of the matrix equation, since vv_2 appears only in the second first-order relation:

dx^*

dw

dx^*

\

- 8 c

dx (

8 30a)

dx^*

dw_2

21 -

H

H

-

H

2

2

H

-

H

2

3

H

(The cofactor H_{22} is a border-preserving principal minor; it must be negative by the sufficient second-order conditions. In fact, by inspection, $H_{22} = -f_1 < 0$. Hence, the refutable hypothesis, $dx_2/dw_2 < 0$, can be asserted for these relations. The cofactor H_{12} on inspection also has a determinate sign. In fact, by the symmetry of the determinant H, H_{x2} = #21 = / 1/2 > 0. Again, that dx^*/dw_2 > 0 for this model is *not* generalizable to the n-factor case. With more than two factors present, the numerator for *dx^/dwi,* or dx^*/dw_2, will be an n x n off-diagonal cofactor. Its sign will be indeterminate. What is curious, however, is that like the profit maximization model, the reciprocity relation

dx d
*

(

dw

-,

31

is valid, since on comparing Eqs. (8-28Z?) and (8-30a) we note that $H\backslash_2 = H_{2X}$. This is a different result than that obtained for unconstrained profit maximization. In that earlier model, the JC,'s were functions of factor prices and output *price;* that is, $x_t = x^*(w\backslash, w_2, p)$. Here, the factors are functions of the output level, $y : Xi = x^*(w\backslash,$ $w_2, y)$. These are two different functions. (We have used the same notation "x*" for both in spite of this to avoid notational clutter.) The results are therefore different.

Finally, we have dk^*/dw_2 ^ 0, as we had before, since H_{23} ^ 0, being an off-diagonal (and non-border-preserving) cofactor. We shall defer explanation of this sign indeterminacy until after the following discussion with respect to parametric output level changes.

How does the firm react to an autonomous shift in output? We know, from the analysis of the previous chapters, that since 3; enters the constraint, no refutable implication can be derived for this parameter.

8-32)

Differentiating the identities (8-25) with respect to *y*, noting that *y* appears only in the third identity (8-25c),

S

H

322

-H

H

$\frac{dk^*}{}$ *H33*

31

H

\ n

-

r

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1

2

V

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–/1

Solvi

Consider this last relationship, dk^*/dy, that is, dMC/dy. This expression gives the slope of the marginal cost function. The numerator, *H^*, is the determinant

$H.7.T,$ $-x^*f_l$ =■

(noting that $/_{12}$ = / 21). Hence, H_{33} = A.^{*2}($/_{u}/_{2}$2 — $f?_{2}$)- This looks like an expression we have encountered previously: to be exact, in the profit maximization model. There, the term (/11 $/_{22}$ — fy_{2}) appeared in the denominator of the comparative statics relations. The sign of this expression was asserted to be positive by the sufficient second-order relations for profit maximization. Why, then, can we not assert from Eq. (8-33c) that dMC/dy > 0, i.e., the marginal cost curve is upward-sloping, since it appears that $//_{33}$ > 0?

In fact, in the case where these cost curves refer to a firm that is also achieving maximum profits (i.e., the firm is reaching an interior solution to the profit maximization problem), the marginal cost function is indeed upward-sloping. However, profit maximization is *not* implied by cost minimization. Cost minimization is a much weaker hypothesis, both in terms of the implied behavior of firms and, equivalently, from the mathematical conditions the cost minimiza tion hypothesis entails on the curvature properties of the production function. For profit maximization, the production function must be strictly concave (downward). Strict concavity, while sufficient for cost minimization, is not necessary. The second-order conditions for cost minimization require only *quasi-concavity,* i.e., convexity of the level curves (the isoquants, here) to the origin.

That this is a weaker condition can be readily seen. Consider the production function _y ^ JCIJC$_2$, shown in Fig. 8-9. This production function is homogeneous of degree 2. Its level curves are rectangular hyperbolas, and clearly a cost minimization solution will exist for all factor prices. But will a finite profit maximum point ever be achieved (with constant factor profits)? When both input levels are doubled,

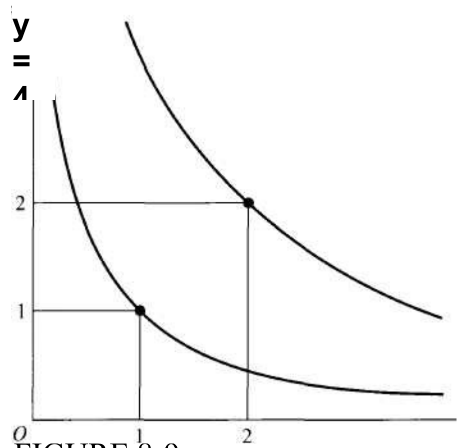


FIGURE 8-9
The Production Function y = x\Xi. The level curves of this production function are clearly convex, being rectangular hyperbolas. A cost minimization solution will necessarily exist for all factor price combinations. However, for example, when *x* = *X2* = 2, *y* — 4, whereas when *x* = *xi* = 4, *y* = 16. Revenues will always increase twice as fast as costs, and hence no profit maximum point can exist. The marginal and average cost functions are always falling here. Profit maximization is a much stronger assertion than cost minimization; i.e., the former places much stronger restrictions on the shape of the production function than does cost minimization.

output will increase by a factor of 4. Costs, C = *w\Xi* + w_2x_2, will only double, however. Hence, this firm's marginal and average cost functions must always be declining (this will be given a rigorous proof later). Therefore, a profit-maximizing point cannot ever be achieved. This firm would make ever-increasing profits, the larger the output it produced. The second-order conditions for profit maximization immediately reveal this situation:

$$/11/22-/i^2{}_2=0-0-1^2 = -1 \quad <0$$

The profit maximization hypothesis places much stronger restrictions on the shape of the production function than does cost minimization. Quasi-concavity, while implied by concavity, does not itself imply concavity.

Consider now expressions (8-33a) and *(S-33b).* These expressions say that if the output level is raised, the factor input levels can either increase or decrease. The situation is analogous to the somewhat more familiar case of inferior goods in consumer theory (to be discussed formally in Chap. 10). There, when income rises, it is commonly believed that for many individuals, the quantity of hamburger, for example, will *decrease,* not increase. Hamburger is often regarded as an "inferior good." In the same manner, factors of production can be inferior.

Consider the case, perhaps, of unskilled labor. Suppose a firm wished to dig one or two ditches. In all likelihood, it would hire a worker or two and some shovels. However, if the firm intended to dig several city blocks' worth of drainage ditches, it would undoubtedly hire some mechanical diggers (backhoes) and some skilled operators. It might reduce its demand for unskilled labor, perhaps to zero.

Thus, it is reasonable to be unable to predict the sign of *dx*/dy.* No refutable hypothesis concerning output effects emerges strictly from the minimization hypothesis. A negative or a positive sign for *dx*/dy* is consistent with the model.

It should be pointed out, however, that a factor in use cannot be inferior over the whole range of output. That is, it must have been the case that *dx*/dy* > 0 at some lower levels of output, else the factor would not ever be employed in the first place. Remember that these comparative statics relations are *local,* not global, results.

d —

H

d **H**

d -

d **H**

d -

There is one set of relationships concerning output effects that are not intuitively, or even geometrically, obvious. Note again the equations

(8-

28c)

(8-

33a)

(8-

3Oc)

(8-33*)

dy ***H***

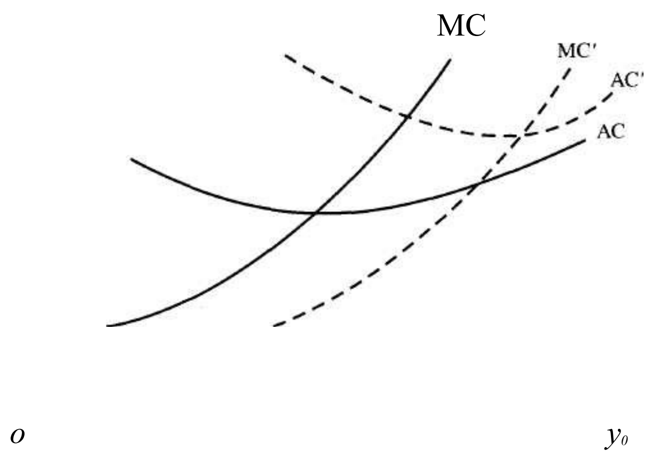


FIGURE 8-10
Changes in the Marginal and Average Cost Curves When the Price of an Inferior Factor Changes. When the price of an inferior factor increases, the marginal cost curve of the firm shifts *down,* the average cost curve shifts up. With any increase in factor costs, unit cost of production must increase. The firm will never regard an increase in factor costs as beneficial. However, if the factor is inferior, the *marginal* cost of output declines.

The symmetry of the determinant *H* immediately indicates that

$$\frac{dX^*}{dwi} = \frac{dx^*}{dy} \qquad (8\text{-}34)$$

That is, the rate of change of the marginal cost function with respect to a factor price is equal to (and therefore of the same sign as) the magnitude of the output effect for that factor. But output effects can be negative as well as positive. This leads to the strange result that the marginal cost curve, as commonly drawn (see Fig. 8-10) against output, will shift *down* when the wage of an inferior factor *increases.* If, say, $dx^*/dy < 0$, then JCI is an inferior factor (less is used as output rises). If *W* increases, the marginal cost curve will actually *fall,* i.e., shift down along the whole range where *x* is inferior.

How can we explain this result? Consider Fig. 8-11. What happens when a cost-minimizing firm experiences an increase in a factor price (holding output constant)? The firm, of course, substitutes away from that factor. However, if *x* is inferior, then for any level of JCI , increases in x_2 will result in new isoquant levels that are flatter than the previous, lower one at that level of *x*. That is, since parallel shifts in the isocost line result in tangencies to the left of the original one, then the isoquants directly vertical from the original tangency must have a lower absolute slope. This means that the distance between isoquants representing successive output levels *narrows*

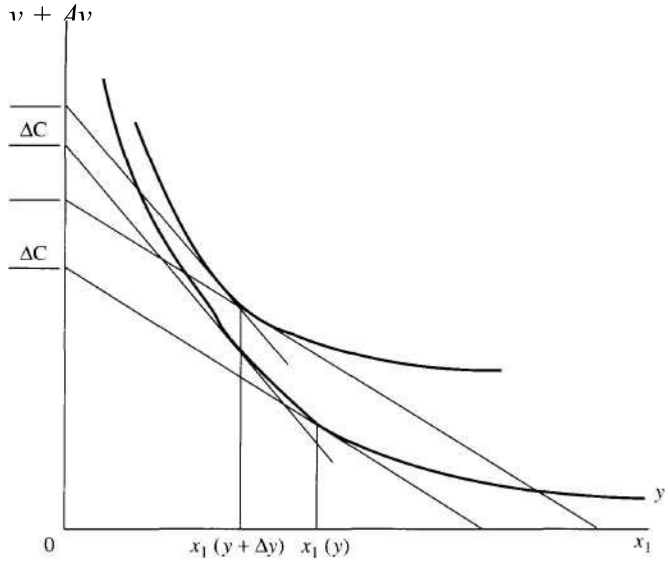


FIGURE 8-11
The Effects of an Increase in the Unit Cost of an Inferior Factor. The factor *x* is inferior, as an increase in output from *y* to *y + Ay* reduces the demand for *x*. At any level of *x*, as amounts of *X2* are increased (vertical movements), the isoquants become flatter. We can arbitrarily define the units such that *W2 — h* Then vertical movements can be identified as marginal costs, since the intercepts of the isocost lines with the vertical axis are *X2* = C/W2 = C. Since the isoquants converge as *x* is decreased, when *x* is inferior, the marginal cost of expanding output from *y* to *y + Ay* is less when less *x* is used, that is, as *w* is raised.

as *x* is reduced. Hence, an increase in *w*, which reduces *x*, *reduces* the cost of additional output.

8.8 COMPARATIVE STATICS RELATIONS USING DUALITY THEORY

We now investigate the properties of this important model using duality theory, as developed in Chap. 7.

Reciprocity Conditions

The reciprocity conditions *dx*/dwj = dx*/dwi* and 9A.*/9w,- = *dx*/dy* can be given a simpler and more powerful proof and interpretation by use of the envelope theorem. Recall again that for the general constrained maximum (or minimum) problem,

maximize

$$f(x_1,x_2,a) = y$$

subject to

$$g(xi,x_2,a) = 0$$

where *a* represents one or more parameters, that *dy*/da — d!£/da;* that is, the rate of change of the maximum value of/for any *oc,* with respect to *a,* allowing the decision variables $x\backslash$ and x_2 to "adjust" via $Xi = x^*(a)$, $x_2 = x\%(ce)$, is the same as the partial derivative of the Lagrangian $X = f(xy,x_2,a) + kg(x\backslash, x_2, a)$ with respect to *a, holding the* $x_{(}$'s *fixed.* We have already used this theorem to show that A. is interpretable as marginal cost [see Eq. (8-23)]. The cost minimization problem

minimize

$$C = W\backslash X\backslash + W_2X_2$$

subject to

$$f(x_u x_2) = y$$

has as its Lagrangian

$$i£ = wixi + w_2x_2 + X(y - f(Xi,x_2))$$

The envelope theorem thus says that

$$\frac{3C^*}{} = \frac{9i£}{} = xi = x^*(w_u w_2, y) \qquad (8\text{-}35)$$

and similarly for x_2. Also, as was shown before, Eq. (8-23), $dC^*/dy = dX/dy = k^*(wi, w_2, y)$. Equation (8-35) is often referred to as *Shephard's lemma;* it is an important part of the duality theory of cost and production functions. We also showed previously that

$$\frac{3C^*}{\_} = r(wi,w_2,y) \qquad (8\text{-}23)$$

Now $C^*(wi, w_2, y)$ is twice differentiable, assuming the production function is well behaved, i.e., that a smooth interior solution to the cost minimization problem obtains. But observe the cross-partials of $C^*(wi, w_2, y)$: Since $C^*_t = x^*(w\backslash, w_2, y)$, C^*_{JW2} is simply dx^*/dw_2; that is,

$$r^*_{WlW2} = \frac{fir^*_{l.}}{dw_2}$$

However, $C^*_{iVV2} = C^*$, since partial derivatives can be taken without regard to order. But $C^*_w = dx\%/dw\backslash$. Hence (almost) trivially

$$\frac{dxl}{W2Wl} \quad \frac{dxl}{3W,} \quad dWy$$

which was Eq. (8-31).

Likewise, since *dC*/dy* = C* = A.*(wi, *w₂, y),*

But c;wi=c;iy = ajcr/ay. Thus,

$$- \frac{dx^*}{dy}$$

which was Eq. (8-34). Similar reasoning of course shows that *dX*/dw₂ = d*

This is a very powerful, yet simple, way of regarding reciprocity conditions. (Perhaps it is powerful precisely because of its simplicity.) Reciprocity conditions are simply the statement that the cross-partials of the cost function are invariant to the order of differentiation. The reciprocity conditions appear, however, only because the first partials of C*(w>i, *w₂, y)* have the peculiarly simple forms 9C*/9w, = *x*, dC*/dy* = *X**. These simple first partials occur because the Lagrangian 56 = WIJCI + *w₂x₂* + *X(y — f{x\, x₂))* is in fact *linear* in the parameters *w\,w₂,y*. When such linearity of the Lagrangian occurs, reciprocity conditions will appear.

The modern development of the envelope theorem allows an easy derivation of the refutable implications of the cost minimization model. The comparative statics relations implied by this model are consequences of the curvature properties, in particular the concavity, of the (indirect) cost function. We proceed in an analogous manner to the analysis of the profit maximization model. For given values of factor prices and output, certain factor levels are implied:

X\ — x\ VwI'vt:2' y)

x₂ — x₂ [w₁ , w₂ , y)

In Fig. 8-12, cost is plotted vertically against Wi. Asa "reference" line, the "constrained" cost function *C* = *W{X®* + *w₂x₂* is plotted. Note that the only "variable" is *w* everything else is held fixed at the specified values. This function is clearly linear, with positive slope x°.

Consider now where C*(H>I, *w₂,* v°) has to appear in this diagram. Since by definition C* is the *minimum* cost for given output and wages, *C*(w\, w₂, y°)* cannot at any time be above the constrained cost line. However, when *w* = vt>p exactly the correct, i.e., cost-minimizing, input levels are employed; hence at wi = Wp C*(wi, *w₂, y°)* = *wix®* + ^2*2 — C. Moreover, to either side of w_p assuming unique solutions, C* < C. It is clear from the geometry that *C*(w\, w₂, y°)* must be concave in *w* (and obviously *w₂* also, by symmetry). (In fact, as the algebra of Chap. 7 shows, the Hessian matrix C* *,w.* must be negative semidefinite.) The consequences of this concavity include C^*_{JWi} < 0; however, by the envelope theorem, $C^*_{(}$ = *x**, and consequently, $C^{\wedge}_{)Wi}$ = *dx*/dwi* < 0. A similar analysis of course follows for *x%-* These results are derived algebraically, without recourse to visual geometry, in Chap. 7. The results generalize in an obvious way for ^-factor models.

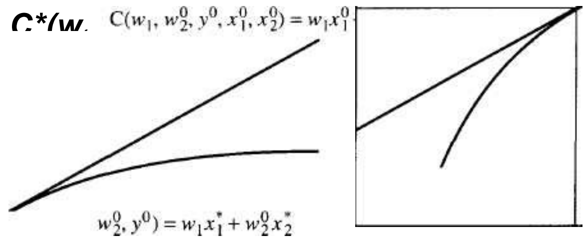


FIGURE 8-12 The cost function $C^*(w_1, w_2^0, y^0)$ for varying w_1 holding w_2 and y fixed, and the cost function $C(w_1, w_2^0, y^0, x_1^0, x_2^0)$, in which x_1 and x_2 are held fixed at $x_1^0 = x_1^*(w_1^0, w_2^0, y^0)$ and $x_2^0 = x_2^*(w_1^0, w_2^0, y^0)$. $C(w_1, w_2^0, y^0, x_1^0, x_2^0) = w_1x_1^0 + w_2^0x_2^0$ is clearly linear in w_1; that is, geome

trically it is a straight line. At $w_1 = w_1^\circ$, $C = C^*$ since x_1° and x_2° are precisely those quantities which minimize total cost subject to constraint. For $w_1 \neq w_1^\circ$, $C^* < C$, since then the "wrong" x_i's are employed for C, whereas C^* is the *minimum* cost, calculated by using whatever x_1^* and x_2^* are appropriate. But since $C^* = C$ at $w_1 = w_1^\circ$ and $C^* < C$ in the neighborhood around w_1°, C^* must be tangent to C at w_1°; also, C^* must be concave in w_1, since $C = w_1 x_1^\circ + w_2 x_2^\circ$ is linear in w_1. We therefore have, from tangency, $C^*_{w_1} = C_{w_1} = x_1^* = x_1^\circ$ and, from concavity in w_1, $C^*_{w_1 w_1} = \partial x_1^*/\partial w_1 < 0$.

Cost Curves in the Short and Long Run

In the famous Viner-Wong cost diagram (see Fig. 7-1), at any specified output level, the short- and long-run marginal cost functions are equal; however, the short-run curve either rises faster or falls slower than the long-run marginal cost function. We can show this result rigorously by the conditional demand approach.

Since in the short run a factor is held fixed, we must consider models with more than two factors. There is already one constraint, $f(x_1, x_2) = y^\circ$; adding another would completely specify the solution at some particular point on the isoquant y°, leaving no further degrees of freedom for the minimization hypothesis. Let us therefore consider the general n-factor model. Using vector notation, let $x = (x_1, \ldots, x_n)$ and $w = (w_1, \ldots, w_n)$ represent the factor and wage levels, respectively. The fundamental identity relating the short- and long-run marginal cost functions is

X k

(8-36)

where X^s is the appropriately defined short-run marginal cost function derived from cost minimization when x_n is parametric and where output y is of course a parameter.

Differentiating the fundamental identity with respect to output _y yields

dy dy

\dxiJ\dy Differentiating the identity with respect to w_n,

-----= (----- J (—) (8-38)

dw_n \dx$^{Q}_{n}$J \dw$_n$j

Applying Young's theorem and the envelope theorem to the cost function yields the reciprocity condition

dk* _ dx*

dw_n dy Using this and (8-38) to substitute for dk^s/dx^Q_n in Eq. (8-37),

ar ar (d^dy^

dy dy (dx^*/dw_n)

Equation (8-39) shows that the slope of the long-run marginal cost function differs from the slope of the short-run function by a nonpositive amount; thus

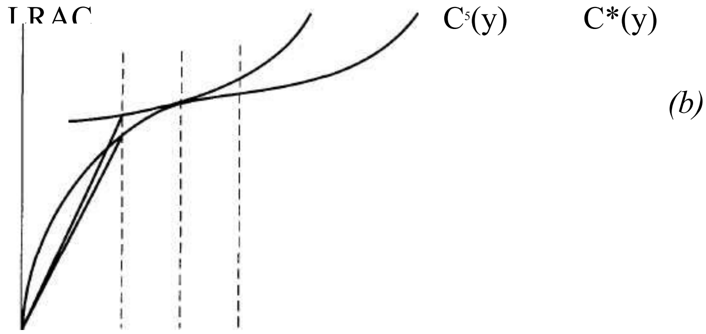
---- < ---- (8-40)

dy dy

Equation (8-39) also shows that the difference between the elasticities of the short-and long-run marginal cost functions varies directly with the size of the output effect, and inversely with the slope of the factor demand.

We can show that marginal cost rises faster or falls slower in the short run vs. the long run by considering Fig. 8-13. Panel *(a)* depicts a section of the traditional Viner-Wong diagram. The long- and short-run average cost curves are equal and tangent at some output level $y°$ and the long- and short-run marginal costs are equal there, but long-run MC is shown as falling while short-run MC is rising. Panel *(b)* depicts the corresponding total cost curves. The curve $C^*(y)$ is the long-run (total) cost function. (We momentarily suppress the other arguments of this function.) The curve C^5(_y) is the short-run cost function that results when we add any kind of additional constraint which leaves the original constrained minimum undisturbed at output level _y°. Thus C* = C^s at $y°$, but to either side of $y°$, it must be the case that C* < C^s, since C* is by definition the minimum cost at any output level, and the additional constraint can therefore only increase the cost of producing some output y^$y°$. The slope of C* is A*; the slope of $C^s(y)$ is the short-run marginal cost function $X^s(y)$. Note that C* is drawn initially concave and then convex; we mean to allow falling and then rising marginal cost. Assuming differentiability of these functions, it is clear that C^s is tangent to C* at $y°$ and is locally more convex or less concave. Algebraically, the function F = C* — C^s has an unconstrained maximum in y at $y°$. Thus $F_y = C^* - C^s_y = 0$, i.e.,

$k^* = I^s$, and $F_{yy} = C^*_{yy} - C^s_{yy} < 0$, i.e., $dk^*/dy < dX^s/dy$. Although the conditional demand relation (8-39) pertains only to the case of holding one factor fixed, the inequality (8-40), showing that the slope



(a)

FIGURE 8-13
The Viner-Wong Diagram, (a) A section of the traditional Viner-Wong diagram. At some output level y°, the short- and long-run average costs are equal and tangent, while the short-run marginal cost curve crosses the long-run marginal cost curve from below, *(b)* The total cost functions in the long run (C^*) and short run *(C^s)*. Since the additional constraint is just binding at y°, $C^* = C^s$ at y°, but to either side of y°, $C^* < C^s$, since C^* is the minimum cost. Thus C^s and C^* are tangent at y°, but C^s is more convex or less concave than C^* there. Tangency implies that the slopes of these curves, the respective short- and long-run marginal costs, are equal. The relative convexity of C^s means that SRMC has a greater slope than LRMC at y°.

of the marginal cost function increases when an additional constraint is added, holds for more general definitions of the "short run," i.e., when any kind of additional constraint is added. It is interesting that even though we cannot derive a sign for the slope of the marginal cost function, we are in fact able to derive a systematic relationship about how its slope changes, in particular, that MC rises faster or falls slower when an additional constraint is imposed on the model. Last, note that the average costs, which are the slopes of the rays from the origin to either C^* or C^s, must be equal at y°, but that short-run AC > long-run AC when $y \neq y^\circ$.

Factor Demands in the Short and Long Run

A similar procedure can be used to show the effect of holding a factor constant on the slope of the constant-output factor demands. Assuming x_n is the parametrically fixed factor, the fundamental identity is, for factors 1 through $n - 1$,

$$x_i^*(w,y) \equiv x_i^s(w_1, \ldots, w_{n-1}, x_n^*(w,y), y), \tag{8-41}$$

Differentiating with respect to w_i,

$$\frac{\partial x_i^*}{\partial w_i} \equiv \frac{\partial x_i^s}{\partial w_i} + \frac{\partial x_i^s}{\partial x_n}\frac{\partial x_n^*}{\partial w_i} \tag{8-42}$$

Differentiating the fundamental identity with respect to w_n,

(g43)

$$dw_n \qquad dx^\circ\, dw_n$$

However, $dx^*/dw_n = dx^*/dw_t$; using this along with (8-43) yields

$$\frac{dxl}{dWi} \wedge \frac{dxl}{dWi} + \frac{(dx^*Jd_{Wi})^2}{(dx^*/dw_n)} \qquad \text{(g\_44)}$$

an expression analogous to Eq. (8-39) for the marginal cost functions. Again, since the second term on the right-hand side is necessarily nonpositive, we have

$$\frac{dx^*}{d} < \frac{dx^s}{d} < 0 \qquad \text{(8-45)}$$

The larger the cross-effect between x_n and w, and the smaller the slope of x_n, dx^*/dw_n, the larger the difference in slope of the short- and long-run factor demand curves.

Similar expressions exist for the short- and long-run cross-effects; however, qualitative results depend on other assumptions about complementarity or substi-tutability of the factors. These relationships are left as exercises.

Example Consider the production function y = $X\backslash x\backslash$. This function exhibits increasing returns to scale, and thus a finite profit maximum would not be reached with constant prices. However, there is still a cost-minimizing input combination for any given output level y. Let us find the constant-output factor demand curves.

minimize

$$C = W\backslash X\backslash +$$

$W2X2$ subject to

$$X\backslash X2 - y$$

The Lagrangian is

Differentiating with respect to JCI, x_2, and X,

$$!\pounds_2 = W2 - TXx\backslash X2 = 0$$

cp ________,, $_v$ V^2 o

ot-x — y — A1A2 — w

Combining the first two expressions,

w\ _ x₂ W2 2x or

$$x_2 =$$

Now substitute this expression for x_2 into the production function:

$$\left(\frac{2w_x X]}{w_2}\right)$$

or

Solving for x_x

$$x^* = 4"^1/V^{2/3}w_2^{2/}V^{/3}$$

In like fashion one obtains, from the first-order tangency condition,

$$*i = \frac{w_2x_2}{2wi}$$

Substituting this into the production function yields

$$\frac{W_2X_2}{} \quad 2$$

or

Note that JC* and *xt,* are multiplicatively separable in the factor prices and output. Also, *x** and *x%* are homogeneous of degree 0 in $w\backslash$ and w_2. The cost function is obtained by substituting *x** and *x%* into C = $w_x x\backslash + w_2x_2$:

$$C^* = w^\wedge 'W^\wedge Ty^{1}" +$$

where k = $4^{\wedge 3}$ + $2^{1}/^{3}$ = $2^\wedge$ + $2'/^{3}$ = I'/^Z^1 + 2°) = f^1/3). Note that C* is homogeneous of degree 1 in w_x and vv$_2$, a general property of cost functions. Note also that

$$\frac{3C}{awi} = {}^{l}_{-}kw_{x\,J}^{-2/3} w\{^{2/3} y^{1/3} = x_x^*$$

and

These envelope properties are shown for the general case in Eq. (8-35).

Relation to Profit Maximization

We said earlier that the main reason to consider the cost minimization model is its relation to a firm's behavior under profit maximization. At the point of profit maximization, the firm must be minimizing the cost of that particular output level. In other words, if the parametric value _y in *x*(wi, w₂, y)* is replaced with v*, the profit-maximizing level of output, the result must be the factor demand function derived from profit maximization, *x*(wi, w₂, p).* In the ^-variable cases, the profit maximization and cost minimization models share $n - 1$ first-order conditions, *fi/fj* = *Wj/wj*. The difference between the models has to do only with the output level; in the cost minimization model, output is parametric, whereas in the profit maximization model, output is determined endogenously by the profit maximization hypothesis.

Since the symbol 'V" is being used to denote two different functions, denote the factor demands derived from profit maximization by *xf{w\, w*$_2$*, p)* and those derived from cost minimization by *xf(w\, w*$_2$*, y)*. Then the above reasoning can be summarized by the *identity*

X\(w\, w>2, *p) = x*^y_x*(w\, w*$_2$*, y*(w\, w*$_2$*, p))* (8-46)

with a similar expression for x_2. This fundamental relation can be used to derive relationships between the slopes of these two demand functions.

Suppose a profit-maximizing firm faces a decrease in some factor price, say, *w* i. Then we can imagine the response in terms of factor demand as taking place in two conceptual stages. First, a "pure substitution" effect takes place. The firm stays on the same isoquant but slides along it to a new cost-minimizing choice of *x* \ and x_2. In other words, it first responds according to the cost-minimizing demand function *x*\(wi, *w*$_2$*, y)*. Unambiguously, the firm chooses to hire more jq . Next, however, an "output effect" takes place in which the firm chooses the profit-maximizing output level. The output effect, unlike the pure substitution effect, is ambiguous. Output could either increase or decrease in response to the decrease in *w* \. Which demand function, *x*^{y}*(\V[, w*$_2$*, y)* or jcf (wi, *w*$_2$*, p),* is more elastic?

If it seems that the ability to choose some sort of "maximizing" output level would lead the firm to choose a larger absolute factor response when that option is available, that reasoning is correct in these two models. (But be careful—such intuition is not always correct, and it is not correct in an important sense in the theory of the consumer.) This result can be shown rigorously by differentiating identity (8-46) with respect towi, using the chain rule on the right-hand side:

dxf *dx*^{y} *dx\ dy**
T r ^{X} = ^ + ^ ^{1} - — (8"47)
d\V *dw* *dy dw*

Inspect the notation carefully on the right-hand side: jq is a function of the parametric output level *y;* hence the notation in the first part of the chain rule term. However, output is then chosen according to profit maximization; hence the notation *y** in the second part of the term.

Equation (8-47) shows that the difference in the slopes between the two demand functions differs by a compound term relating to an output effect, involving the rate of change of jq with respect to a change in output, *y.* Can this second term be signed? Indeed it can—recall the reciprocity condition *dx*/dp* = *—dy*/dw\,* derived by applying Young's theorem to the indirect profit function. (See Prob. 1, Chap. 4.) Substituting this in Eq. (8-47) yields

6x$_x$ *6X*$_X$ dx, *6x*$_x$

$dw\backslash \quad dw\backslash \quad dy \quad dp$

It should be clear that the last two terms on the right have the same sign, since p and y move in the same direction. This can be shown rigorously by differentiating the fundamental identity with respect to p:

$$\frac{dx^p_x}{dp} \qquad \frac{dx\{\ dy^*}{dy}$$

dp Using this and Eq. (8-47),

$$dp)\ \backslash dy \tag{8-49}$$

Since supply curves are upward-sloping, (dy^*/dp)

> 0; thus

$$\frac{dx^p}{d\backslash V\backslash} < \frac{dx\backslash}{dw\backslash} < 0 \tag{8-50}$$

A similar procedure can be used to analyze the relative magnitudes of the cross-effects dx^*/dw_2 in the two models. For these changes, a determinate sign is not available; further assumptions regarding the output effects are required. This analysis is left as an exercise.

The systematic relationships that exist between the factor demand functions derived from profit maximization versus those derived from cost minimization can be seen in terms of the general Le Chatelier relations we showed in Chap. 7. The cost minimization model is the profit maximization model with the added constraint $y = y°$. We know that $dx^*/dw_t < dx\text{-}\ /3w_{(}\text{-}$ when any constraint is added to the profit maximization model; (8-50) is just a special case of this.

8.9 ELASTICITIES; FURTHER PROPERTIES OF THE FACTOR DEMAND CURVES

The properties of the factor demand curves x, = $x^*(w\backslash,\ w_2,\ y)$ and the marginal cost curve A. = $X^*(w\backslash,w_2,y)$ are often stated in terms of dimensionless elasticity expressions instead of using the slopes (partial derivatives) directly. The elasticities of demand are defined as

$$\epsilon a = \lim \quad A\backslash Vj/Wj$$

where e,; thus represents the (limit of the) percentage change in a factor usage x_t (holding output constant) due to a given percentage change in some factor price Wj. When / = j, this is called the *own elasticity* of factor demand; when i =fc j, this is called a *cross-elasticity.*

Taking limits, and simplifying the compound fraction,

$$eu = \hat{}ip\text{-} \qquad ij, = h2 \tag{8-51}$$

? d

In like fashion, one can define the *output elasticity* of factor demand as the percentage change in the utilization of a factor per percentage change in output^ (holding factor prices constant),

$$e_{iy} = \lim Ax_i/x_i/Ay/y = \frac{y}{x_i^*}\frac{dx^*}{-\pm-} \qquad (8\text{-}52)$$

Homogeneity

The demand curves x, = x^* *(w* i, w_2, *y)* are homogeneous of degree 0 in factor prices, or, for the two-factor case, in w_l and w_2. That is, $x^*(tw\backslash, tw_2, y) = x^*(wi, w_2, y)$. Holding output y constant, a proportional change in all factor prices leaves the input combination unchanged. This is really another way of saying that only changes in *relative* prices, not absolute prices, affect behavior.

If the cost-minimizing firm faced factor prices twi, tw_2, the problem would be to

minimize

$$tW\,1X1 + tW_2X_2$$

subject to

$$f(x_u x_2) = y$$

Since $tW\backslash X\backslash + tw_2x_2 = t(w\backslash X\backslash + w_2x_2)$ is a very simple monotonic transformation of the objective function, we should expect no substantial changes in the first-order equations. Forming the Lagrangian i£ = $t(w\backslash X\backslash + w_2x_2)$ + A.(v — $f(x\backslash, x_2))$, the first-order equations for a constrained minimum are

$$5E_l = twi - kfi = 0 \qquad (8\text{-}53a)$$

$$£g_2 = tw_2 - A./_2 = 0 \qquad (8\text{-}53fc)$$

$$\%k = y - f(x_l, x_2) = 0 \qquad (8\text{-}53c)$$

Eliminating the Lagrange multiplier from (8-53a) and (8-53Z?),

$$\frac{tW\backslash}{tw_2} \quad \frac{W\backslash}{w_2} \quad \frac{f\backslash}{f_2}$$

Thus the same tangency condition emerges for factor prices *(tw* \, / w_2*)* as for (w \, w_2*)*. The isoquant must have slope $w\backslash/w_2$ for any value of *t*. And output, meanwhile,

fNote that the output elasticity is *not [(Ay/y)/(Ax_i:/*,•)], or *(xj/y)(dy/dxi)* = (1/AP,)MP,. This latter expression, though well defined, is not a measure of the responsiveness of factor demand to output changes. And it is most certainly *not* the reciprocal of e_{iy} above: e_{iy} can be positive or negative (for the case of inferior factors); (x,/j)(3y/3x$_i$) is necessarily positive as long as the marginal product of x, is positive.

is still constrained to be at level *y*. Hence the identical solution to the cost minimization problem (in terms of the JC,-'S) emerges for factor prices *(twi, tw*$_2$*)* as for (wi, *w*$_2$*)',* hence the solutions x_t = *x*(yv\, w*$_2$*, y)* are unchanged when *(w\,w*$_2$*)* are replaced by *(twi, tw*$_2$*)*. Thus *x*(w\, w*$_2$*,y)* = *x*(tw\, tw*$_2$*, y),* or the factor demand curves (holding output constant) are homogeneous of degree 0 in the factor prices. This result is perfectly general for the n-factor case; *x*(wi,..., w*$_n$*, y)* = *x*(tw\, ..., tw*$_n$*, y), i* = 1,..., *n*.

Clearly, however, *something* must be changed when factor prices are multiplied by some common scalar. What *is* changed is total cost and therefore marginal and average costs also. If factor prices are doubled, the input combination will remain the same, but the nominal cost of purchasing that input combination will clearly double. Total cost C = C*(wi, *w*$_2$*, y)* is homogeneous of degree 1 in factor prices. Total cost *C*{w\,w*$_2$*,y)* = *w\x** + *w*$_2$*x^,* a linear function of the Jt*'s. When factor prices are changed by some multiple *t,* the x*'s are unchanged, and hence

$$, tw_2, y) = tw\backslash x^*(tw\backslash, tw_2, y) + tw_2X2(tw\backslash, tw_2, y)$$

= *x* ;i, *y)* + 2(Wi, *y*
= i* *W* *, y)* +)(Wl, *y*
= (*w*$_2$*,*

Again, this result is perfectly general for the n-factor case; the cost function is homogeneous of degree 1 in factor prices.

Since total costs increase or decrease by whatever scalar multiple factor prices are changed, marginal and average costs are similarly affected. Since

$$\text{ACEE}\ \frac{C^*(\text{wi}, w_2, y)}{y}$$

$$, tw_2, y) = C^*(tw\backslash, tw_2, y) \text{ — } y \qquad (8\text{-}54)$$

$$= tC^*(w\backslash, w_2, y) \text{ — } y$$

$$= tAC(w\backslash, w_2, y)$$

Similarly, since MC = *X*(w\,w*$_2$*,* v), from the first-order equations,

$$k^*(tw_u tw_2,y) = \frac{\text{———-———}}{1,2\ fi(x^*,x_2)} \qquad \text{i} =$$

The factor inputs JC* are unchanged by the multiplication of factor prices by *t.* Hence only the numerator of the above fraction is affected in a simple linear fashion, and hence

$$k^*(twi, tw_2, y) = tk^*(wi,w_2, y) \qquad (8\text{-}55)$$

or the marginal cost function is homogeneous of degree 1 in factor prices.

It should be carefully noted that all of the preceding homogeneity results are completely independent of any homogeneity of the production function itself. These results are derivable for any cost-minimizing firm. Nowhere was any assumption about the homogeneity of the production function implied or used; therefore these results hold for *any* production function for which a cost-minimizing tangency solution is achieved.

Euler relations. Since the factor demand curve $x_t = x^* (w_1, w_2, y)$ is homogeneous of degree 0 in w_1, w_2, by Euler's theorem

$$\frac{dx^*}{\wedge} \quad \frac{dx^*}{{}^{L}{}_{2}} = O - x^* = O \qquad i = l, 2 \qquad (8\text{-}56)$$

This relation can be stated neatly in terms of elasticities. Dividing (8-56) by x^*,

$$\frac{\wedge}{x^* dw_1} + \frac{\wedge}{x^* dw_2}, 0 \qquad , = 1, 2$$

or

$$e/1 + \epsilon_{j\cdot 2} = 0 \qquad i = 1, 2 \qquad (8\text{-}57)$$

using the definitions of elasticities and cross-elasticities given in Eq. (8-51). More generally, for the n-factor case, the factor demands $x^*(w_1, \ldots, w_n, y)$ are homogeneous of degree 0 in $w_1, \ldots, w_n$. Similar reasoning yields

$$e_{,-i} + \epsilon_{i2} + , \ldots , + \epsilon_{in} = 0 \qquad / = 1, \ldots,$$

n or

$$\sum_{7=1}^{n} \epsilon_{17} = 0 \qquad i = l, \ldots, n \qquad (8\text{-}58)$$

For any factor, holding output constant, the sum of its own elasticity of demand plus its cross-elasticities with respect to all other factor prices sums identically to zero. Another relationship concerning cross-elasticities can be derived using the reciprocity relations $dx^*/d\backslash Vj$ = 3JC*/3W, . This reciprocity relation can be converted into elasticities as follows. Each side will be multiplied by 1 in a complicated way (the asterisks are omitted to save notational clutter):

$$Xj\ Wj \quad dXi\ X;\ Wi \quad dX;$$

Rearranging terms yields

$$'W;\ dxi \ \backslash \qquad Xt\ \ fwj\ dx$$

or

Dividing through by total cost C =

=

(8-59)

where $K_t = w_i x_i / C$ represents the share of total cost accounted for by factor x_t.

Never forget, incidentally, what is being held constant here. These elasticities and shares refer to constrained cost minimization, i.e., output-held constant factor curves. Slightly different relationships are derivable for, e.g., the profit-maximizing (unconstrained) firm.

The reciprocity relations in terms of elasticities, Eqs. (8-59), can be substituted into Eqs. (8-58) to yield new interdependencies of the cross-elasticities. Substituting (8-58) into (8-59),

Multiplying through by K_t yields

$$= K_1 \epsilon_{U} + K_2 \epsilon_2 \; i - \cdots {}_n\epsilon_{ni} = 0 \qquad i = 1, \ldots, n \tag{8-60}$$

The difference between (8-58) and (8-60) is that in this last relation (8-60), the elasticities being considered are those between the various factors and *on* *e particular factor price,* whereas in Eq. (8-58) the elasticities all pertain to the relationship of one particular factor JC, to all factor prices. In the former case, the shares are not involved, the relationship being derived directly from Euler's equation; in the latter case of how all factors relate to a given price change, the shares of cost allocated to those factors do play a part.

Equation (8-60) can also be derived by a different route. Consider the production function constraint /(JC*, JC|) = *y.* Differentiating with respect to some factor price *wi,*

f\ dx **=0**

From the first-order relations *Wj* =■ *kfj,* so this is equivalent to

(8-61)

Note in Eq. (8-61) that the terms refer to the change in the various factors with respect to the same factor price w,. If this expression is now manipulated in a manner similar to the derivation of

Eq. (8-59), Eq. (8-60) results.

Output Elasticities

The output elasticities are related to one another also, as can be seen by differentiating the production constraint *f(x*, x%) = y* with respect to *y:*

$$f\backslash\frac{}{dy}\;\overline{=}{+}\;{}_{1}fi\frac{}{dy}$$

Again using the first-order relations $w_t = k^*f_t$,

$$\sim k^*\sim dy\sim\ {}^{+}\ \hat{}k^*\sim dy\sim\ {}^{=}$$

To convert these terms to elasticities, multiply the first by *(y/y) (x*/x*),* that is, by unity, in that fashion. Do the same for the second term, using *x** instead of *x**. This yields

$$k^*_y \quad \text{v}\ dx^*\backslash \qquad \backslash V2x\_{}^*\ (\ y\ axl$$

or

$$K_x\epsilon_{Xy} + K'_2\epsilon_{2y} = 1 \qquad \text{(8-62)}$$

where the "weights" *K[* are the total cost of each factor divided by marginal cost times output. This result generalizes easily to the ^-factor case,

where *K[— WiXi/k*y.*

It should be noted that these weights *K[* do not themselves sum to unity, and hence Eq. (8-63) should not properly be called a *weighted average* of the output elasticities. In fact, £>/ = (£ w_t*Xi)/k*y* = *(l/k*)(C*/y)* = AC/MC. In a special case, the weights do sum to 1 —when marginal cost *k* equals average cost. This situation will occur when a firm is operating at the minimum point on its average cost curve, i.e., where marginal cost intersects average cost. Thus we could say that for a firm in long-run competitive equilibrium, the weighted average of the output elasticities of all factors sums to unity, where the weights are the share of total cost spent on that particular factor.

8.10 THE AVERAGE COST CURVE

Consider now the average cost curve (AC) of a firm employing two variable inputs JCI and x_2 at factor prices *W* and vv₂, respectively. By definition,

$$\text{AC} = \frac{C(W_1, W_2, y)}{y} = \frac{1}{y}(w_1 x_1^* + w_2 x_2^*) \qquad \text{(8-64)}$$

How is average cost affected by a change in a factor price, say *w 1* ? Can average cost ever fall in response to an increase in factor price? In this case, intuition proves correct—increased factor costs can only increase overall average cost. If this were otherwise, firms could always make larger profits by contracting for higher wage payments. This behavior is not commonly observed.

We can demonstrate the positive relationship between AC and *W* as follows. Differentiating (8-64) with respect to wi yields

3A	1	.	*dx**	*dx:*
dw₁	*y*			*dwij*
rod	rule	t	term	*c**. Using the first-order
uct	on	h	*W\i*	relations
3A	*X**	A	*(f*	
C		*	*dX**	
dw	*y*	*y*	\	

However, differentiation of the constraint identity *f(x*,* JC|) = y with respect to Wi [seeEq. (8-26c)] yields /I(3JC*/3WI) + /2(3X2/3WI) = 0. Hence, the expression in parentheses vanishes, leaving

3AC _ x*
------— — > u
dw *y*

In general, by similar reasoning

= ^- *i = l, . . . , n* (8-65)
dWi *y*

for firms with any number of factors. For positive input and output levels (the only relevant ones), therefore, average cost must move in the same direction as factor prices.

Equation (8-65) is intuitively sensible from the definition of AC directly. Average cost is a linear function of the *w[s:* AC = *(x* / y)w i + (x2/y)w₂. If W[* changes to *w* i + Aw i, at the margin the change in AC will just be the multiple of w i, *(x*/y),* that is, *(x*/y) Awi.* For finite movements, *x*/y* and *x%/y* also change, but *at the margin* the instantaneous rate of change of AC is simply *x*/y* (before *x** can change).

This is actually another simple application of the envelope theorem. Since AC = C*/y,

3AC _ 1
3C* 3wi
y

However, by the envelope theorem, recalling the Lagrangian i£ = *w\X\ + w₂x₂ + Hy - f(xi,x₂)),*

3C* 3ie _
d *d*

Hence

3AC _
JC* 3wi
y

8.11 ANALYSIS OF FIRMS IN LONG-RUN COMPETITIVE EQUILIBRIUM

The foregoing analysis can be modified and extended to analyze a well-known situation in economics. Consider a price-taking firm in a competitive industry composed of a large number of identical firms. Suppose also that entry into this industry is very easy; i.e., the costs of entry are low. What will the behavior of firms in this industry be; i.e., how will such firms respond to changes in factor prices or other parameters that might appear? (See also Chap. 6, Prob. 8.)

Under conditions of immediate entry of new firms into an industry in which positive profits appear, output price must immediately be driven down to the point of minimum average cost for all firms. Any response the firm makes to some parameter change must take into account the prospect of instantaneous adjustment of output price to minimum average cost. In this case, profit-maximizing behavior will be equivalent to each firm minimizing its average cost, since at any other point the firm would cease to exist.

Let us now investigate how the location of the minimum AC point is affected by a change in a factor price. The point of minimum average cost occurs when

$$MC(w_1, w_2, y) = AC(w_1, w_2, y) \tag{8-66}$$

The question being asked is, how does the output level y associated with minimum average cost change when a factor price changes? A functional dependence of y on factor prices w_1, w_2 is being asserted. Where does this functional relationship come from? Equation (8-66) represents an implicit function of y, w_1, and w_2. Assuming the sufficient conditions for the implicit function theorem are valid, (8-66) can be solved for one variable in terms of the remaining two; in particular

$$y = y^*(w_1, w_2) \tag{8-67}$$

We can now derive $\partial y^*/\partial w_1$ by implicit differentiation. Substituting (8-67) back into (8-66), one gets the *identity*

$$MC(w_1, w_2, y^*(w_1, w_2)) \equiv AC(w_1, w_2, y^*(w_1, w_2)) \tag{8-68}$$

This relation is an identity because output level y is posited to always adjust via Eq. (8-67) to any change in w_1 or w_2 so as to keep the firm at minimum average cost. Differentiating this identity with respect to, say, w_1,

$$\frac{\partial MC}{\partial w_1} + \frac{\partial MC}{\partial y}\frac{\partial y^*}{\partial w_1} \equiv \frac{\partial AC}{\partial w_1} + \frac{\partial AC}{\partial y}\frac{\partial y^*}{\partial w_1}$$

However, at minimum average cost, $\partial AC/\partial y = 0$ from the first-order conditions for a minimum. Hence, solving for $\partial y^*/\partial w_1$, *dy** 1 [3 AC 3MC1

r^L = HiF7H~h---------i — (8"69)

d\V 3JVlC/dy _*ow*

ow J with a similar expression holding for

dy^*/dw_2.

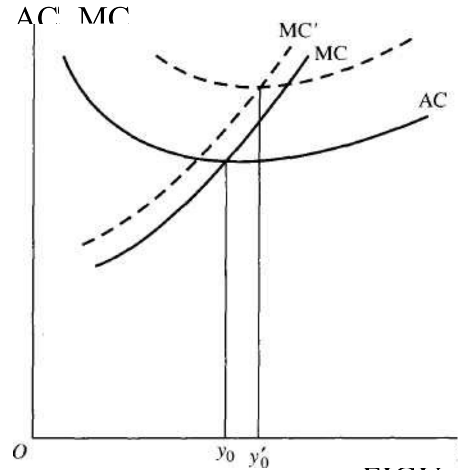


FIGURE 8-14 Shifts in the MC and AC curves when a factor price changes.

Equation (8-69) admits of

an easy interpretation. It says that if, say, *w* increases, the minimum average cost point will shift to the right (i.e., the minimum AC output level will increase) if the AC curve shifts up by more than the marginal cost curve. (Note that we know that $dMC/dy > 0$ at minimum AC.) This is geometrically obvious. Consider Fig. 8-14. The marginal cost curve always cuts through the AC curve from below, at the point of minimum average cost. When *w \,* say, increases, average cost must shift up by some amount [Eq. (8-65)]. If marginal cost shifts by less than the shift in average cost, the point of minimum average cost will clearly move to the right. And, of course, if marginal cost actually shifts down when *w* increases (indicating that *X* is an inferior factor), then the new MC curve must necessarily intersect the new (raised) AC curve to the right of, i.e., at a higher output level than, the old minimum AC point.

Equation (8-69) can be used to relate the output level changes directly to the output elasticities of factor demand. From Eq. (8-65),

9 A C *x**

and from

(8-34),

9MC *dx**

Substituting these values into Eq. (8-69),

d
x
****

dy*
1

Factoring out x^*/y,

$$dy \quad dMC/dy \quad (1 - \epsilon_{iy}) \qquad (8\text{-}70)$$

where e_{iy} — $\{y/x^idxjdy)$ is the output elasticity of factor / as defined in Eq. (8-52). The output effects of changing factor prices on firms in long-run competitive equilibrium can be read out of Eqs. (8-69) and (8-70). If a factor price, say, $w_{\{}$, rises, then if factor 1 is output-elastic $(\epsilon_{iy} > 1)$, all the firms will wind up producing less output. Minimum average costs (and thus the product price) increase, but the marginal cost curve shifts up even more. Hence, less total output is sold, since the product demand curve is downward-sloping. If factor 1 is output-inelastic (but not inferior) (0 < 6^ < 1), the marginal cost curve will shift up by less than the average curve, since by Eq. (8-70), $dy^*/dwi > 0$. Finally, if factor 1 is inferior $\{\epsilon\backslash_y < 0)$, then the marginal cost curve shifts *down* when w_x increases, average cost still shifts upward, and hence $dy^*/dw\backslash > 0$.

Analysis of Factor Demands in the Long Run

The combined effects of profit maximization and entry or exit of new firms leads firms to de facto pursue a strategy of average cost minimization. We can thus investigate the behavior of firms in the long run by explicitly considering the comparative statics implications of the model

minimize

$$AC = \frac{W_1X_1 + W_2X_2}{y} \qquad (8\text{-}71)$$

Denote the factor demands implied by this model $JC_i^L(W_1, w_2)$, $i = 1,2$. Since output price/? is endogenous, the factor demands are functions only of the factor prices. The average cost function AC*(vt>i, w_2) is the indirect objective function associated with this model. This model can be analyzed using the traditional methods of comparative statics (see Chap. 6, Prob. 8; we shall do it here using duality theory). Consider Fig. 8-15, in which AC is plotted vertically against W_1. For given values of wj! and w_2, say, wj and w®, certain factor usages are implied: JC° = $x_1^0(w_1, w_2)$, x® = x^iyvi, w_2). Also, $y^0 = f(x_1^0, x_2^0)$. Holding x_1, x_2, and w_2 constant at these values, the restricted average cost function AC = (WJJC^ + w^x^/y^0 is a straight line with positive slope x_1^0/y^0. The *minimum* AC, AC*(wi, w_2) must in general lie below this line, by definition of a minimum. However, when w_1 = wf, exactly the correct (i.e., average-cost-minimizing) levels of x_1 and x_2 are used; hence AC* = AC at that point, and AC* < AC to both sides of w°. It is clear geometrically that AC* is concave in w_1. (We leave it as an exercise to prove algebraically, using the primal-dual methods of Chap. 7, that AC* is in general concave in all factor prices.) We therefore have

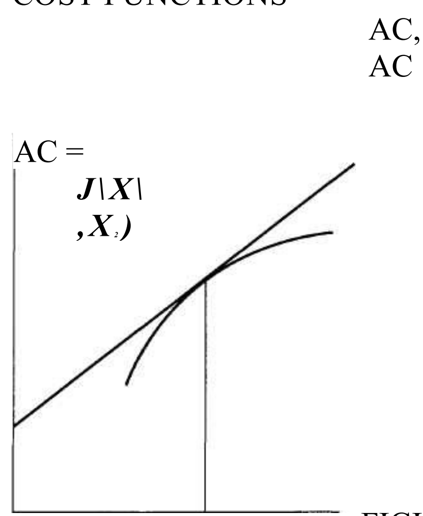


FIGURE 8-15 *Concavity of the Long-Run Average Cost Function.* The constrained AC function, in which all variables except w_1 are constant, is linear in w_1, with slope x_1^0/y^0. At w_1^0, the "correct" values of x_1 and x_2 are used; for $w_1 \neq w_1^0$, other than the averag

e-cost minimizing values are used. Thus, AC* = AC at Wp and AC* < AC to both sides of *w*®. Since AC is linear in wi, AC* must be concave in *w*\ and, by symmetry, in w>2 also. Therefore, AC*^ =

0
.

the following "envelope" results:

(8-72)

A C * < (

—

In this model, the factor demands *x[* are *not* the first partials of the indirect objective function; therefore, refutable comparative statics relations are not forthcoming with regard to these factor demands. However, the ratios of the factor inputs to output (the "relative" inputs) *are* such first partials; concavity of the indirect objective function therefore yields refutable implications for these relative input functions, as shown by Eq. (8-73). Using the quotient rule, we get

<

By

(8-74)

y

If, say, the price of factor 1 increases, the percentage change in the use of factor 1 must be less than the resulting long-run percentage change in the output of the firm. However, that output effect is not necessarily negative; thus negatively sloping long-run factor demands are not implied for competitive firms in the long run.

We can again gain greater insights into these matters by using a conditional demand procedure to analyze the relationship between the demand for factors in the short and long run. In the short run, output price is parametric; we have denoted it with the symbol *p*. In the long run, entry and exit of firms into an industry drives the price down to the minimum average cost of the marginal firm. If all firms are identical, except perhaps for scale, i.e., they all have the same minimum average cost level, competition will force the price down to this minimum average cost level. Instead of being parametric, output price will be determined by the equation $p = \min AC^*(w>i, w_2)$. The fundamental identity relating the short- and long-run demands is, therefore, for factor 1,

xfCvvi, vv$_2$) = *Xi(wi, w$_2$, p*(wi, w$_2$))* (8-75)

where*p*(w\,* w$_2$) is simply AC*. Differentiating with respect to *w\,*

(8•76)

dw\ dw\ dp dw

Using the envelope theorem, *dp*/dw* = *x\/y;* also, for the standard (short-run) profit maximization model, we have *dx\/dp = (dxj/dy) (dy*/dp).* Thus, (8-76) can be written

dx\ _ dx^p$_{\{}$ x$_x$ dxj

dy aw\ ow\ y*

ay op

We know that *dx^p /dw* < 0 and *dy*/dp* > 0; *X* and *y* are assumed positive. Therefore, if JCI is a normal factor, that is, *dx\/dy* > 0, *dx[/dw* must be less negative than 3;cf/9 w i. Moreover, it is possible that the second term on the right-hand side of (8-77) might be absolutely larger than the first term; in that case the long-run curve would have a positive slope.

Do not misinterpret this result—this is a compound effect. If, say, *w * increases, the firm will hire less JCI in the short run. However, this increase in *w* causes the minimum level of average cost, and thus output price, to rise as well. The firm may expand in response to this. If the firm becomes sufficiently larger after the factor price increase, this "expansion effect" might outweigh the short-run response to contract the use of *x*. However, *this is a description of the response of a single firm.* Since output price has increased, then assuming a downward-sloping *industry* demand curve, less total output is demanded. On the industry level, therefore, less JCI will be hired in accordance with the law of demand, but this might occur via the mechanism of many fewer firms each hiring more of that factor than prior to the factor price increase. (Curiously, the increase in *w* and thus *p* can actually lead to *entry* of firms, each much smaller than the previous ones!)

PROBLEMS

1.254 Explain why cost functions are not just technological data. Why does cost depend on the objectives of the firm and the system of laws under which the firm operates?

1.255 Are convex (to the origin) isoquants postulated because of empirical reasons or because they make the second-order conditions for constrained cost minimization valid for interior solutions?

1.256 What is the difference between the factor demand curves obtained in this chapter, i.e., from cost minimization, and those obtained earlier from the profit maximization model? What observable (in principle) differences are there between the two?

1.257 Discuss the relationships between the following definitions of complementary factors:

$(p \quad f_u > 0$ *(ii)*

$(dXi/dwj)_{WhP} < 0$ *(in)*

$(dXi/dWj)_{Why} < 0$

where $f\{x\backslash, x_2)$ is a production function for a competitive firm and where the para- meters outside the parentheses indicate that those parameters are to be held constant. 5. Consider the profit-maximizing firm with two inputs. This model can be treated as the *constrained* maximum problem,

maximize

$$py — W\backslash X\backslash —$$

w_2x_2 subject to

$$y =$$

$f(xi,x_2)$ Using the Lagrangian

$$£g = py- _{Wl}xi -w_2x_2 +X[f(x_l,x_2) - y]$$

(a) Show that if the profit maximum is conceived to be achieved in two steps: first hold
y constant and maximize over $x\backslash$ and x_2 (as functions of y) and then maximize over
the variable y, the model can be stated as

$$££ = \max_{y} (py — \min_{x\backslash,x2}\{(w,Xi + w_2x_2) + X[y - /(x,, x_2)]\})$$

1.258 Show, therefore, that profit maximization implies cost minimization at the profit-
maximizing level of output.

1.259 Derive the comparative statics of this model treating 3;, JCI, and x_2 as independent vari
ables subject to a constraint. Note that the reciprocity condition $dy^*/d\backslash Vj = —dx^*/dp$
and the supply slope $dy^*/dp > 0$ are more easily derived than in the original uncon
strained format.

6. Consider the production function $y = x''x_2^2$. Show that the constant-output factor demand
functions have the form

$$X^*=kjW; \quad Wj^{\prime} \quad yl/\backslash''\backslash ™2),-j-j \qquad * \ 1 \quad -a,7(ai+<*2) \quad <*;7(<*1+<*2) \quad 1/(ai+ai)$$

Show that the cost function has the form

$$C^* = (k_x +k_2)w_1^{a_1/(cn+a2)}w_2^{2/(ai+a2)}y^{l/(a>+ai)}$$

and that 9C73W,^JC;.

7. Suppose a production function $y = f(L, K)$ is linear homogeneous.

(a) Show that

(b) Show that

1.260 If the law of diminishing returns applies to both factors, show that the factors are
technical complements; i.e., the marginal product of either factor

rises when more of
the other factor is applied.

1.261 Show that if the marginal products are positive, the isoquants must be downward-sloping.

(e) Show that

$H = -X$ III ILK $/L$ JKL JKK JK h h 0

where H is defined as in Eq. (8-14) and A. is marginal cost.

(f) Show that $H = X(y^2/K^2)f_{LL} = X(y^2/L^2)f_{KK}$ if $f(L, K)$ is homogeneous of degree 1. Show, therefore, that there can be no "stage I" or "stage III" of the production process if the isoquants are convex to the origin.

1.262 Derive an expression analogous to Eq. (8-44) for the cross-effects $dx^*/d\backslash Vj$ and dxj/dwj, $i \wedge j$. Show that if x, and Xj are either both substitutes or both complements to x_m, then $dx^*/dwj < dx^s_t/dwj$.

1.263 Derive an expression analogous to Eq. (8-49) showing the relationship between the profit-maximizing and cost-minimizing cross-effects dxf/dwj and $dxj/d\backslash Vj$, $i =£ j$. If x, and Xj are both normal factors, which cross-effect is larger? Can these short- and long-run cross-effects have different signs?

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CHAPTER

9

COST AND PRODUCTION FUNCTIONS: SPECIAL TOPICS

9.1 HOMOGENEOUS AND HOMOTHETIC PRODUCTION FUNCTIONS*

An interesting and important class of production functions is the homothetic production functions, of which the homogeneous functions are a subset. A production function is homogeneous of degree *r* if when all inputs are increased (decreased) by the same proportion, output increases (decreases) by the rth power of that increase. Formally, if $f(x_1, \ldots, x_n)$ is homogeneous of degree r,

$$f(tx_1, \ldots, tx_n) = t^r f(x_1, \ldots, x_n)$$

Several properties of homogeneous functions in general were noted in an earlier chapter, especially Euler's theorem, already used extensively in other contexts. In addition, the geometric property that

$$\frac{f_i(tx_1, \ldots, tx_n)}{f_j(tx_1, \ldots, tx_n)} \sim f_j(x_1, \ldots, x_n)$$

i.e., that the slopes of the level curves are the same along every point of a given ray out of the origin, was proved using the homogeneity of degree $r - 1$ of the first partials f_i and f_j.

*The student may wish to review the sections in Chap. 3 on homogeneity.

However, homogeneous functions are not the only functions with this geometric property. Consider any monotonic transformation $F(z)$ of a homogeneous production function z — $f(x\backslash, \ldots, x_n)$. That is, consider _y = $H(x_{\iota}, \ldots, x_n) = F(f(x\backslash, \ldots, x_n))$, where $F'(z) > 0$. The requirement $F'\{z) > 0$ ensures that z and y move in the same direction; e.g., when z increases, y must increase. The slope of a level curve $H(x\backslash, \ldots, x_n) = y$ in the $x,$-Xj plane is

$$Hj = \underline{F'(z)fi} =$$
$$fj \quad Hj$$
$$F>\{z)fj \qquad fj$$

But we already know that fi/fj is invariant under a radial expansion. Hence the function $H(x\backslash, \ldots, x_n) = F(f\{x\backslash, \ldots, x_n))$ also exhibits this property.

The class of functions $y = H(x\backslash, \ldots, x_n) = F(f(x\backslash, \ldots, x_n))$, where F'/O and $f(x\backslash, \ldots, x_n)$ is a homogeneous function, is called the *homothetic functions*. In fact, no generality is lost if $f(x\backslash, \ldots, x_n)$ is restricted to *linear* homogeneous functions, i.e., functions homogeneous of degree 1. The reason is that if $f\{x\backslash, \ldots, x_n)$ is homogeneous of degree r, then $[f(x\backslash, \ldots, x_n)]^{1/r}$ is homogeneous of degree 1:

$$, \ldots, tx_n)]^{1/r} = [t^r f(x_u \ldots, x_n)]^{1/r} = t[f(x_1, \ldots, x_n)] \qquad {}^{1/r}$$

Taking the rth root of / can be incorporated into the monotonic transformation $y = F(z)$. That is, $F(z)$ itself can be thought of as a composite function, the first part of which is taking the rth root of $f(xi, \ldots, x_n)$ and the second part whatever transformation yields $H\{x_x, \ldots, x_n)$. Hence we can define as the class of homothetic functions all functions $H\{x_x, \ldots, x_n) = F(f(x\backslash, \ldots, x_n))$, where /(JCJ, ..., x_n) is homogeneous of degree 1 and F' ^= 0.

The statement that the slopes of the level curves are invariant under radial expansion or contraction of the original point, i.e., when $X\backslash, \ldots ,x_n$ is replaced by $tx\backslash, \ldots, tx_n$, can be expressed another way. The slope of the level curve (surface) at any point is $HJHj$. This is just another function of the x/'s; that is, define

$$\frac{H> \; - u \; (\qquad \wedge}{Hj} = tlij\{X[, \ldots, X_n)$$

The function $h_ij(xi, \ldots, x_n)$ designates the (negative) slope of the level surface of H in the x_tXj plane. This slope is unchanged under JCI , ..., $x_n \longrightarrow tx\backslash, \ldots, tx_n$. But this is simply a statement that $hij\{x\backslash, \ldots, x_n)$ is homogeneous of degree 0, that is, that $hij(tx\backslash, \ldots, tx_n) = hij(xi, \ldots, x_n)$. It can in fact be shown by more advanced methods that homotheticity can be defined in this manner also; i.e., if/z,$_7$ (JCI , ..., x_n) is homogeneous of degree 0 for all x_tXj planes, then $H\{x\backslash, \ldots, x_n)$ must have the form $H(x\backslash, \ldots, x_n) = F(f(xi, \ldots, x_n))$, where $f(x\backslash, \ldots, x_n)$ is homogeneous of degree **1** and F' =fc 0.

Example. Consider the production function $y = H\{x\backslash,x_2) = x_xx_2 + x\backslash x\backslash$. This function is not homogeneous, as can readily be verified. It is homothetic, however, since $H(x_x, x_2) = z + z^2$, where $z = x_xx_2$. That is, $H(x\backslash, x_2) = F(f(xi, x_2))$, where $F(z) = z + z^2$. Note that $F'(z)$

$= I + 2z\wedge 0$, since production is presumed to be nonnegative. The slope of a level curve of $H(x\backslash, x_2)$ is

$$H\backslash \quad X_2 \quad \backslash$$
$$H_2 \quad x\backslash$$
$$x\backslash + 2x\backslash x_2$$
$$x_2(\backslash +2x_x x_2) \qquad x_2$$

Note that $F'(z) = 1 + 2x\backslash X_2$ appears in the numerator and denominator. Hence, $H\backslash/H_2$ — $h\backslash_2(x\backslash, x_2) = x_2/X[$. The function h_u is clearly homogeneous of degree 0: $h_n\{txi, tx_2) = tx_2/tx\backslash = x_2/x_x = h_{l2}(X], x_2)$. Thus, the level curves of $X\backslash X_2 + x\backslash x\backslash$ have the same slope at all points along any given ray out of the origin.

Still another way to express homotheticity is to state that the output elasticities for all factors are equal at any given point. That is, $e_{Xy} = e_{ly} = .$ ■ ■ $= €_{ny}$. This is clear from the geometry of straight-line expansion paths. Consider Fig. 9-1. Any increase, say, in output from 3; to y' will result in a new tangency point B along a straight line through the origin and the former tangency point A. The triangles *OAx®* and *OB(tx®)* are similar; hence, $X\backslash$ increases by $OB/OA = t$. But, clearly, x_2 increases by $OB/OA = t$ also, for the same reason. Hence for homothetic production functions, output elasticities are equal in all factors.

This result can be shown algebraically by noting that a straight-line expansion path implies that the ratio *Xj* /x,, the slope of the ray out to that point in the x_t *Xj* plane, is the same for any output level as long as factor prices are held constant. That is,

dy **= *0***

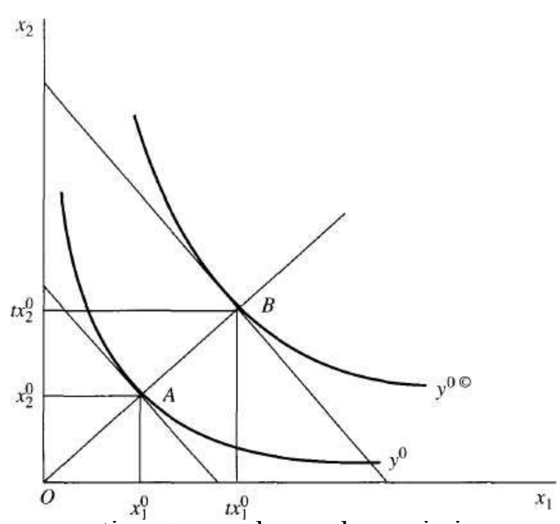

FIGURE 9-1
Homothetic Production Functions. The level curves of homothetic production functions are all radial expansions of one another; i.e., at the intersections of any ray out of the origin and the level curves, the slopes are all the same. Put another way, if output is increased autonomously, i.e., holding factor prices constant, the new tangency point will lie along the ray projecting the old tangency point from the origin. By similar triangles, it is clear that $x\backslash$ increases in the same proportion t as x_2 does when y is increased.

Using the quotient rule and multiplying through by xf yields

Jx* ***Jx* _*** ^{I} *dy* ^{j} *dy* After multiplication by y and division by x^*x^* this leads to

x) dy *x**

dy or

$$\epsilon_{jy} = \epsilon_{iy} \qquad i, j = 1, \ldots, n \tag{9-1}$$

The value of this common output elasticity can be found by applying Eq. (8-63). Since $e_{iy} = e_{jy} = e_y$, say, this constant can be removed from the summation, yielding

However,

V~^ / _ 2^ ^{w}*i*^{x}*i* _ ^<-
^K*I* ~ (MC)j ~ MC

Thus

$$\frac{\mathrm{MC}}{\mathrm{AC}} \tag{9-2}$$

The common value of output elasticity, for homothetic functions, is the ratio of marginal to average cost. Therefore, for firms with increasing average costs, the factors are all output-elastic; that is, $e_y = e_{iy} > 1$ for all factors; for firms with declining (average) cost, factors are all output-inelastic. Also, if the firm is at the minimum point of its AC curve, the output elasticities of its factors are all unity if the production function is homothetic.

9.2 THE COST FUNCTION: FURTHER PROPERTIES

We have already shown that $C^*(w_1, w_2, y)$ is homogeneous of degree 1 in W_1 and w_2, or, more generally, for the n-factor firm, $C^*(w_1, \ldots, w_n, y)$ is homogeneous of degree 1 in $W_1, \ldots, w_n$. Again, since C^* — Yl W,-JC*(WI, ..., w_n, y), and since the $x^*(w_1, \ldots, w_n, j)$'s are homogeneous of degree 0 in $w_1, \ldots, w_n$,

$$C^*(tw_1, \ldots, tw_n, y) = \sum tW_iX^*(tw_1, \ldots tw_n, y)$$

$$= tC^*(w_1, \ldots, w_n, y)$$

Suppose in addition that the production function $y = f(x_1, \ldots, x_n)$ is homogeneous of some degree $r > 0$ in $x_1, \ldots, x_n$. In this case, we shall demonstrate that the cost function can be partitioned into

(9-3)

where it is to be noted that the function $A(w\backslash,$ only. In the case where $r = 1$, that is, $f(x\backslash,$ scale,

.., $w_n)$ is a function of factor prices .., $x_n)$ exhibits constant returns to

$= yAC(wi, \ldots, w_n$ (9-4)

where $A\{w\backslash, \ldots, w_n)$ becomes the average cost function AC. But average cost AC (wi,..., w_n) is a function of factor prices only, i.e., independent of output level. This is of course as it must be; if a firm exhibits constant returns to scale, AC = MC = constant, i.e., a function of factor prices only at every level of output.

We shall prove some of these results for the case of differentiable functions. For simplicity, we shall deal with functions of only two variables, i.e., the two-factor case. The generalizations to n factors are straightforward and are left as exercises for the student. Remember, as always, that y is a parameter in the cost minimization model.

Equation (9-3) is intuitively plausible. Consider Fig. 9-2. Suppose the firm is initially at point x° utilizing inputs x° = (xj\ *x%)*. Some level of cost C(x°) would exist. Suppose now both inputs were doubled, to *(2x®,* 2^^) = x^1. Then since the production function is homothetic (indeed, homogeneous), the new cost-minimizing tangency will lie on a ray from the origin extending past the original point x° to point

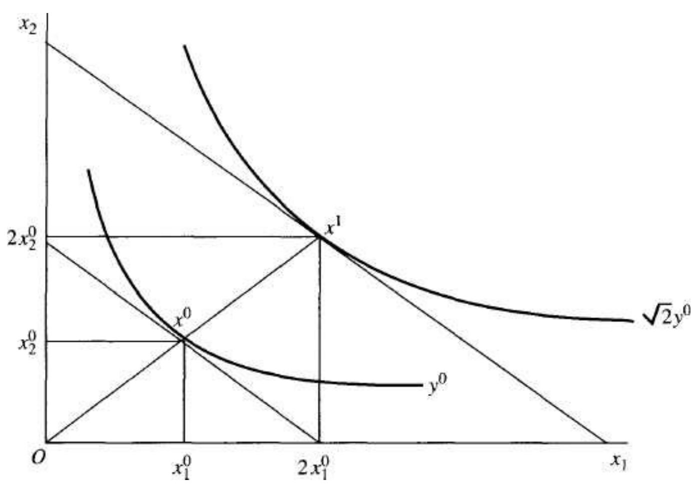

FIGURE 9-2
A Production Function Homogeneous of Degree 1/2. When input levels Jtj\ *x* are doubled, say, output increases by the factor $2^{1/2}$ = A/2. However, since C = vt>ijti + *W2X2,* cost doubles; that is, C(x°) — *\C{x')*. This means that a doubling of cost is accompanied by a A/2-fold increase in *y;* that is, cost and output are related as C = Ay^2. The constant of proportionality is constant only in that it does not involve v output. It *is* a function of factor prices; that is, *A = A(w\,* W2).

x^1 at twice the input levels. At x^1, the cost C(x^l) is clearly twice C(x°), since both inputs have exactly doubled while factor prices remain the same. Hence, C (x^1) = 2 C(x°). However, *y°,* output at x°, has grown only to $2^{1/2}y° = \backslash/2y°$, since the production function is homogeneous of degree 1/2. This means that, holding factor prices constant, cost and output are related in the proportion $C = Ay^2$, since a doubling, say, of cost is accompanied by an increase of output of the factors of V2. The proportionality constant A, in fact, must be dependent on factor prices; that is, A = *A(w\\,* w_2*).* For a different slope of the isocost line, the proportionality constant will be different; however, cost and output will still have the general relation (9-3).

The preceding reasoning cannot be applied to general nonhomogeneous functions. (It can be applied in a more complicated fashion, and we shall do so, to general homothetic functions.) If the production function is nonhomothetic, a given increase in output is not related to a simple proportionate expansion of all inputs. Instead, the ratios of one factor to another will change. Hence, the cost function will necessarily be a more complicated function than (9-3), wherein factor prices and output are all mixed together and not separable into two parts, one related to output and the other to factor prices.

In proving (9-3), we shall use the following relationship, already discussed in the first discussion of interpreting *k,* the Lagrange multiplier of the constrained cost minimization problem, as marginal cost. Since C* = *w\\x* +* w_2x*^,* then since *w* = A*/i and w_2 = ^*/$_2$,

However, for homogeneous functions, *fixi* + $f_2x_2 = ry$, where *r* is the degree of homogeneity. Hence for homogeneous functions,

* C* = *Vry* *(9-5a)*

or

si ***f)C****

— = *r* — - *(9-5b)*

y *dy*

The question now is: What general functional form *C*(w*, *w_2, y)* has the property of obeying Eqs. (9-5), which say that average cost C*/y is proportional to marginal cost, the factor of proportionality being the constant r? This question is answered by integrating the partial differential Eq. *(9-5b).* Rearranging the terms in (9-5/?) yields

C *r y*

The differential notation 3C* is used rather than *dC** to remind us that in that differentiation, *w* and w_2 were being held constant. Integrating both sides of (9-6) gives

- • • - - • - - (9_7)

C* ***r \\ J*** ***y***

As in all integrations, an arbitrary constant appears. However, since this was *a partial* differential equation with respect to y, the constant term can include any arbitrary function of the variables held constant in the original differentiation, i.e., the factor prices here. In fact, the theory of partial differential equations assures us that the inclusion of an arbitrary function in the integration constant of the variable held fixed in the partial differentiation yields the general solution to the partial differential equation.

Performing the indicated integration in Eq. (9-7) yields

$$\log C^* = \frac{1}{r}\log y + \log A(w_1, w_2) \tag{9-8}$$

Here, we have written the constant term $K(w_1, w_2)$ as $\log A(w_1, w_2)$. There is no loss of generality involved, since any real number is the logarithm of some positive number. This manipulation, however, permits us to rewrite (9-8) as

$$\log C^* = \log[y^{(1/r)} A(w_1, w_2)] \tag{9-9}$$

since the logarithm of a product is the sum of the individual logarithms, and $\log a^b = b \log a$. Since the logarithms (9-8) and (9-9) are equal (identical, in fact), their antilogarithms are equal, i.e.,

$$C^* = y^{(1/r)} A(w_1, w_2) \tag{9-10}$$

which was to be proved.

That (9-10) is a solution of the partial differential Eq. (9-5a) can be seen by substitution:

$$\frac{\partial C^*}{\partial y} = \frac{1}{r} y^{(1/r)-1} A(w_1, w_2)$$

Substituting this into the right-hand side of Eq. (9-5a) yields

But this is identically the left-hand side, C*. By definition, since the substitution of the form $C^* = y^{(1/r)} A(w_1, w_2)$ into the equation $C^* = k^* r y$ makes that equation an identity, $C^* = y^{(1/r)} A(w_1, w_2)$ is a solution of (9-5). And, it is the most general solution of (9-5) because of the inclusion of the arbitrary *function* $A(w_1, w_2)$ as the constant of integration. It is also clear that the integration constant must be positive; otherwise positive outputs would be associated with imaginary (involving $\sqrt{-1}$) costs.

To recapitulate, what has been shown is that if the production function is homogeneous of any degree r ($r > 0$), then costs, output, and factor prices are related in the multiplicatively separable fashion $C^* = y^{(1/r)} A(w_1, w_2)$. Equivalently, for homogeneous production functions, average costs are always proportional to marginal costs, the factor of proportionality being the degree of homogeneity r; that

Either Eq. (9-5) or (9-10) can be used to show the relationship of the degree of homogeneity to the slope of the marginal and average cost functions. From (9-10),

d

y and thus

3MC

dy

By inspection, if r < 1, 3MC/9_y > 0; that is, for a homogeneous production function exhibiting decreasing returns to scale, marginal costs (not surprisingly) are always increasing. Similarly, if r > 1, *dMC/dy* < 0; that is, falling marginal costs are associated with homogeneous production functions exhibiting increasing returns to scale. Lastly, if r = 1, the constant-returns-to-scale case, marginal cost is constant and equal to $A(w\backslash, w_2)$ for all levels of output.

Alternatively, from *(9-5b),* if r > 1, say, AC > MC. Since marginal cost is always below average cost, AC must always be falling, with similar reasoning holding for r < 1 and r = 1. Also, differentiating (9-5#) partially with respect to v yields

dy Solving

for *dk*/dy,* that is, 3MC/9y, gives

dy ***ry*** from

which the preceding results can be read

directly.

Homothetic Functions

Let us now consider the functional form of the cost function associated with the general class of homothetic production functions, _y = $F(f(x\backslash, x_2))$, where $f(xi, x_2)$ is homogeneous of degree 1, and $F'(z) > 0$, where $z = f(,x\backslash,x_2)$. Proceeding as before, we have

$$C^* = W\backslash X^* + w_2X2$$
$$= X\backslash F'(z)fi)x\backslash + k^*(F^f(z)f_2)xl \tag{9-11}$$

or

$$C^* = \lambda^* F'(z) z \tag{9-12}$$

using Euler's theorem. Now *y* is a monotonic transformation of z; that is, $F'(z) > 0$. This means that if *z* were plotted against *y,* the resulting curve would always be upward-sloping. Under these conditions, a unique value of *z* will be associated with any value of *y;* that is, the function $y = F(z)$ is "invertible" to $z = F\sim^{l}(y)$. The situation is the same as expressing demand curves as $p = p(x)$ (price as a function of quantity) instead of the more common $x = x\{p)$ (quantity as a function of price). Thus we can write

or, combining all the separate functions of v,

$$C^* = k^*G(y) \qquad (9\text{-}13)$$

That is, for homothetic functions, the cost function can be written as marginal cost times some function of v only, G(v). If the homothetic function were in fact homogeneous of some degree r, then $G(y)$ = rv, a particularly simple form, as indicated in Eq. *(9-5a).* As before, the question is: What general functional form of C* (w \, w_2, *y)* satisfies the partial differential Eq. (9-13)? That is, what restrictions on the form of C*(wi, w_2, *y)* are imposed by the structure (9-13)?

This question is answered as before by integrating the differential Eq. (9-13). Separating the *y* terms and remembering that A.* = *dC*/dy,* we have

The critical thing to notice about (9-14) is that the right-hand side is a function of *y* only. We shall assume that some integral function of 1/G(v) exists, and we shall designate that integral function as log *J(y).* Also, an arbitrary constant of integration must appear, and, as in the homogeneous case, this constant is not really a constant but an arbitrary *function* of the remaining variables, *w* and vv₂, which are treated as constants when the cost function is differentiated partially with respect to v. This constant function will be designated log *A(w\\,* w_2)- Thus, integrating (9-14) gives

f* 9C ***f* dy**
/ ----= /---------h log *A(W],* w_2)
G(v)

which yields

$$\log C^* = \log J(y) + \log A(vvi, w_2)$$

Using the rules of logarithms and taking antilogarithms,

we have

(9-15)

What Eq. (9-15) says is that for homothetic productions, the cost function can be written as the product of two functions: a function of output y and another function of factors prices only. C*(wi, w_2, *y)* is said to be multiplicately separable in *y* and the factor prices.

That C* should have this form is entirely reasonable. Recall that a homothetic function is simply a monotonic function of a linear homogeneous function. It is as if the isoquants of a linear homogeneous (constant-returns-to-scale) production function were relabeled through some technological transformation, represented by

$F(z)$.

But it is only a transformation of output values, not a change in the shapes of the isoquants themselves. Since the cost function for a linear homogeneous production function can be written C* = *yA(w\, w₂),* and one gets a homothetic function by operating on output y alone, not surprisingly the only change induced in the cost function is the replacement of *y* by some more complicated function of _y, designated *J(y) inEq.* (9-15).

The correctness of (9-15) as a solution to (9-13) can be checked heuristically as follows. When this form, C* = *J(y)A{w\,* vv₂), is substituted into (9-13), the right-hand side must be identically C*. Performing the indicated operations gives *X* = J'(y)A(wi, w₂),* and thus

$$C^* = J'(y)A(w\backslash, w_2) \text{ x some function}$$

of *y* and (9-15) is therefore of the requisite form.

9.3 THE DUALITY OF COST AND PRODUCTION FUNCTIONS

At this juncture let us recapitulate the analysis of production and cost functions. The starting point of the analysis was the assumption of a well-defined quasi-concave production function, i.e., one whose isoquants are convex to the origin. We asserted that the firm would always minimize the total factor cost of producing any given output level, as this was the only postulate consistent with wealth or profit maximization. The first-order conditions of the implied constrained minimization problem were then solved, in principle, for the factor demand relations *x, = x*(w\,* w₂, *y),* along with the Lagrange multiplier (identified as marginal cost) *X = X*(w\,w₂, y).* The comparative statics relations were developed yielding certain sign restrictions on some of the partial derivatives of the previous demand relations, namely, *dx*/d\Vj* < 0.

These demand relations were then substituted into the expression for total cost, C = *W\X\ + w₂x₂,* yielding the total cost function

$$C^*(\text{wi}, w_2, y) = W]X^* + \text{W2*2}$$

It was shown via the envelope theorem that 3C*/3w, = *x*, dC*/dy = X*.* Also, certain properties of the cost function regarding homogeneity and functional form were derivable from assumptions about the production function.

We now pose a new question. We have seen how it is possible to derive cost functions from production functions. Is it possible, and if so, how, to derive production functions from cost functions? That is, suppose one were given a cost function that satisfied the properties implied by the usual analysis of production functions. Is it possible to identify with that cost function some unique production function that would generate that cost function? The answer in general is yes; there is, in fact, a duality between production and cost functions: the existence of one implies, for well-behaved functions, the unique existence of the other. We shall now investigate these matters.

A critical step in the construction of the cost function was inverting the solution of the first-order relations w,- — *Xfᵢ*■ — 0, *y — f(x\, x₂)* — 0 to obtain the demand relations JC, = *x*(wi, W2,* v). The uniqueness of these solutions is guaranteed by the sufficient second-order conditions for constrained minimum, which in turn guarantees that the Jacobian matrix of the first-order equations, i.e., the cross-partials of the Lagrangian ££, has a nonzero determinant. These sufficient second-order conditions also imply that $dx^*/dw_t < 0$, $i = 1, 2$. However, $x^* = dC^*/dw_t$. Hence,

$$\frac{dx^*}{dwi} \wedge \frac{8^2C^*}{dwf} < 0 \qquad (9\text{-}16)$$

That is, the cost function has the property that the second partials with respect to the factor prices are negative. As was shown in the previous chapter, the cost functions for any well-behaved production function are weakly concave in the factor prices. Again, for the two-factor case, C* (w i, w_2, *y)* is linear homogeneous in *w\\,* W2-Thus, as shown earlier in a different manner, since *x*(w\\,* w_2*, y)* is a first partial of C* with respect to a factor price, JC* is homogeneous of degree 0 in *W\\,* w_2- Hence by Euler's theorem,

$$\frac{dx^*}{}W] \sim \frac{dx\backslash}{d}W2 = 0$$

Similarly

$$\frac{dx;}{d}$$

Eliminating *w* and w_2 (noting that $dx^*/dw_2 = 3x_2/3w^* = C^*_2$, C*. = *dx*/d\\Vj)* reveals that

The determinant of the cross-partials of C* with respect to the factor prices equals 0. In fact, C* cannot be strictly concave in *w* i and w_2 because it is linearly homogeneous in *w* and w_2; that is, radial expansions of *W* and w_2 produce *linear* expansions of C*. This result easily generalizes to the case of *n* factors using the methodology of Chap. 7.

Consider now the problem of constructing a production function from a cost function. Before proceeding, we would check to see whether in fact the given C*(wi,W2, v) exhibited "weak" concavity in w_l and w_2 and linear homogeneity in Wi and w_2. Assume that these conditions are met. Then the implied factor demands are

$$xt(w_l,w_2,y) = -\!\!-\!\!-$$

$$x_2(w_u w_2,y) = -\frac{dC^*}{dw}$$

However, x_1^* and x_2^* are homogeneous of degree 0 in w_1 and w_2; hence they can be written

$$x_1^*(w_1, w_2, y) = x_1^*\left(1, \frac{w_2}{w_1}, y\right) = g^1(w, y)$$

$$x_2^*(w_1, w_2, y) = x_2^*\left(1, \frac{w_2}{w_1}, y\right) = g^2(w, y)$$

where $w = w_2/w_1$. But (9-18) represents two equations in the four variables x_1, x_2, w, and y. Under the mathematical conditions that the Jacobian of these equations is nonzero, i.e., that

$$\begin{vmatrix} g_w^1 & g_y^1 \\ g_w^2 & g_y^2 \end{vmatrix}$$

these equations can be used to eliminate the variable w. This will leave one equation in x_1, x_2, and y, say

$$h(x_1, x_2, y) = 0$$

Solving this equation for $y = f(x_1, x_2)$ yields the production function.

How stringent is the assumption that the above Jacobian determinant be non-zero? The partials g_{1w} and g_{2w} are essentially the slopes of the factor demand relations (a reciprocal slope in the case of g_{1w}) with respect to changes in *relative* prices. In particular, using the chain rule leads to

3 J C * d w d

w d g 3 w i d w

$w_1)$ < 0

and hence $g_{1w} = -(w_1/w_2)(dx^*/dw_1) > 0$. Similarly,

dx^* 1

dw^* d w W_1

aw

$0W$

and hence

$g_{2w} =$ <0

If both factors are normal, as would be the case for homothetic production functions, then $g_{1y}, g_{2y} > 0$ and J has the sign pattern

>0

implying that $J > 0$, and thus $J \neq 0$. In the nonhomothetic case, it would be pure coincidence if $J = 0$; hence it is not implausible to assert $J \neq 0$. Hence in general we shall expect to find a unique production function associated with any well-specified cost function. This is not to say that it will be *easy* to find either the production function or the cost function from the other. In general, the equations to be solved, i.e., the first-order relations in the case of deriving the cost functions or Eq. (9-18) in the case of deriving the production function, will be complicated nonlinear functions. But we can be assured that the functions exist, in principle, and that they are unique.

Example. We previously have found the cost function associated with a Cobb-Douglas production function. It had the same multiplicatively separable form. Let us see how Eq. (9-18) can be used to reverse the process. Suppose C* is given to us or estimated econometrically as

$$C^* = y^k w^a_x w^x_2 -^a \qquad (9\text{-}19)$$

where $0 < a < 1$ (to ensure that $C^* = x^* > 0$, $C| = x_2 > 0$, $C^*_{,,}$ $C_{22} < 0$) and the exponents of $w\backslash$ and w_2 sum to unity (to ensure C* homogeneous of degree 1 in wi and vv_2). The parameter k can take on unrestricted positive values. What production function will generate this cost function?

By the envelope theorem (Shephard's lemma) $9C^*/9w_{(} = x^*$. Hence

Similarly, $x_x = y\ aw\backslash \quad w_2 \quad =aj$

$w\ \backslash$

$$x^*_2 = /(\mathrm{I} - a)w^a_x w_2{}^a = (1 - <x)y^k\ (—$$

Letting $w = w_2/w\backslash$, let us eliminate this variable. The asterisks are redundant here and will be dropped to save notational clutter. It will be easiest if we take logarithms of both sides of the equation. Then

log xi = log $a + k$ log y + (1 — a)log w log x_2 =

log(l — $a) + k$ log y — a log w Multiply the first

equation by a and the second by 1 - a and add:

a log JCI + (1 — a)log $x_2 = a$ log a + (1 — a)log(l

— $a) + k$ log y or

log x^xl''' = log $<x^a(\backslash$ -

a)¹""/ Taking antilogarithms and rearranging

slightly, we get

$$y = Kx^{a\ lk}_x x^{(}{}_2-^{a)lk} \qquad (9\text{-}20)$$

where $K = \backslash\backslash/a^a\{\backslash$ — a)¹""]¹ ^.Equation (9-20) is the production function associated with the cost function (9-19). As expected, it is of the Cobb-Douglas, or multiplicatively separable, type, and is homogeneous of degree l/k, since C* was homogeneous of degree k in y.

The Importance of Duality

The duality of cost and production functions is important for reasons other than mathematical elegance. Economists will have occasion to estimate factor demand and cost functions. There are basically two ways to approach this problem. One way is to estimate, by some procedure, the underlying production function for some activity and to then calculate, by inverting the implied first-order relations, the factor demand curves (holding output constant). The cost function can then be calculated also. This, however, is a very arduous procedure. Production functions are largely unobservable. The data points will represent a sampling of input and output levels that will have taken place at different times, as factor or output prices changed. And of what use is knowledge of the production function itself? Largely, it is to derive implications regarding factor usage and cost considerations when various parameters, e.g., factor and output prices, change.

It would seem to make more sense to start with estimating the cost functions or the factor demand curves directly; i.e., some functional form of the cost function could be asserted, say a logarithmic linear function, and costs could be estimated directly. However, this procedure would always be subject to the criticism that the estimated cost or demand functions were beasts without parents, i.e., they were derived from fictitious, or nonexistent, production processes. And that would be a serious criticism indeed.

However, the duality results of the previous sections rescue this simpler approach. We can be assured that if a cost function satisfies some elementary properties, i.e., linear homogeneity and concavity in the factor prices, then there in fact is some real, unique underlying production function. Thus, the cost function will be more plausible.

Moreover, the cost function may be easier to estimate, econometrically, than the production function. The cost function is a function of factor prices and output levels, all of which are potentially observable, possibly easily so. What is more, once estimated, the cost function can be used to derive directly the constant output factor demand curves using the relation x^* = 3C*/9w,. Thus, the simpler approach of estimating cost functions is apt to be more useful than the more complicated procedure of estimating production functions. The duality results assure us that procedure is in fact theoretically sound.

9.4 ELASTICITY OF SUBSTITUTION; THE CONSTANT-ELASTICITY-OF-SUBSTITUTION (CES) PRODUCTION FUNCTION

Neoclassical production theory recognizes the possibility of substituting one factor of production for another. The existence of more than one point on an isoquant is equivalent to such an assertion. However, we have not yet considered any quantitative measurement of the degree to which one factor can in fact be so substituted for another.

Consider a production function with L-shaped isoquants, represented in Fig. 9-3. This function can be written algebraically as *y = min[(x\/ai), (x_2/a_2)],* where *a* and a_2 are constants. This function describes an activity for which no effective substitution is possible. For any wage ratio, the cost-minimizing firm will always operate at the elbow of the isoquants. The marginal product of each factor

FIGURE 9-3 *The Fixed-Coefficient Production Function.* This production function is given by $y = \min(x_1/a_1, x_2/a_2)$, where a_1 and a_2 are parameters. No substitution among the factors is worthwhile; the marginal products of x_1 or x_2 are 0 at all points except along the corners of the production function. Although extensively used in input-output

analysis and short-term forecasting models, it is doubtful that this is a useful x_l way to look at the real world.

is 0 unless it is combined in a fixed proportion with the other input. (For this reason, this production function is described as one of *fixed coefficients.)*

How shall the degree of substitutability of one factor for another be described? Consider the Cobb-Douglas production function $y = x''x_2{\sim}^a$, where, say, $x\backslash$ is labor and x_2 is capital. A cost-minimizing firm satisfies the first-order conditions of the Lagrangian

$w_2 x_2$

or, in this case,

$- Xax^a{}_x{-}^x x\backslash{-}^a = 0$

w_2 - A(1 - a)jcf

jc$_2$"^{a} = 0 y -

$x^a{}_l x^l{}_2{\sim}^a = 0$

Upon division,

Eqs. (9-2la) and

(9-21Z?) yield

*?.
(1 —
a)\V]
aw_2

This expression can also be derived from the constant-output factor demand curves derived earlier:

$x,$ = a $\backslash$-a

1 -a

1 -a W2V

y

Equation (9-22) says that for this production function, the capital-labor ratio is (1) independent of the level of output, and (2) a function only of the *ratio* of the

wage rates (rental rate on capital to the labor wage rate). We shall shortly consider the generality of this situation.

We can therefore conceive of the capital-labor ratio as a simple function of the wage ratio. If we let $u = x_2/x_1$, $w = w_2/w_1$ for notational ease, Eq. (9-22) becomes

$$u = \frac{k}{w} \tag{9-24}$$

where $k = (1 - a)/a$. How does x_2/x_1 vary when w_2/w_1 varies? From Eq. (9-24),

$$\frac{du}{dw} = -\frac{k}{w^2} \tag{9-25}$$

where the expression is negative, as expected. Although this actual rate of change in the capital-labor ratio is a measure of substitutability, a more frequent measure is the dimensionless elasticity analog,

$$\sigma = -\frac{du/u}{dw/w} = -\frac{w}{u}\frac{du}{dw} \tag{9-26}$$

the (approximate) percentage change in the input ratio per percentage change in factor prices. A minus sign is added to make the measure positive. This measure σ is called the *elasticity of substitution.* Applying Eq. (9-26) to the Cobb-Douglas case gives

$$\sigma = \frac{w}{u}\frac{k}{w^2} = \frac{w^2}{k}\frac{k}{w^2} = 1$$

Thus, the elasticity of substitution for a Cobb-Douglas production function is constant along the whole range of any isoquant and equal to 1.

The Cobb-Douglas production function $y = x_1^a x_2^{1-a}$ is a special case of production functions that exhibit constant elasticity of substitution (CES) along any isoquant. We shall investigate these important functions, deriving their functional form and other properties. These functions have wide application in empirical work on production processes.

The concept of the elasticity of substitution is not dependent on the behavioral assertion of cost minimization. The concept can as easily be described as the percentage change in the input ratio per percentage change in the marginal rate of substitution (MRS) since the cost-minimizing firm always sets $w_1/w_2 = f_1/f_2 = \text{MRS}$. Thus we can write, as an alternative definition,

$$x_1/x_2 \quad d(f_1/f_2)$$

(Note that we are considering the inverse ratios x_1/x_2 instead of x_2/x_1, etc. As we shall shortly see, this is of no consequence.) Let us evaluate this expression. Along any isoquant, $x_2 = x_2(x_1)$. Then $dx_2/dx_1 = -f_1/f_2$, and therefore we can write

(9-27) (using the chain rule) as

$$- \frac{f\backslash x_2 d(x_1/x_2)/dx_1}{d\{f_x/f_2)/dx_{\{}}$$

Evaluating the terms in the second fraction yields

$$d(x_1/x_2) \quad (\quad dx$$

$$\{ \; X \; X$$

$$fl$$

$$(f\backslash X\backslash + f_2X_2)$$

Similarly, $d(f\backslash/f_2)/dx\backslash$ is simply $—d^2x_2/dx^2$, since $dx_2/dx\backslash = —fi/f_2$. From Chap. 3,

Combining these expressions leads to

$$o = —- \quad f\backslash x_2 \quad fl \qquad fixi + f_2x_2$$

$$(/_2{}^2/ll - 2/l/_2/l_2$$

or

$$\frac{f\backslash Mf\backslash X\backslash + flX_2)}{(/^2/2/// + /^2/)}$$

This rather cumbersome expression for *a* can be drastically simplified in the important special case of linear homogeneous production functions. First, the numerator immediately becomes $f\backslash f_2y$, upon application of Euler's theorem. For the denominator, since $f(xi,x_2)$ is homogeneous of degree 1, $f\backslash$ and f_2 are homogeneous of degree 0. Hence, applying Euler's theorem to $/\backslash$ and f_2, we have

$$0 \qquad \text{or} \qquad /_n = -f_n \frac{Xi}{x\backslash}$$

Similarly,

$$f \quad - \quad f \quad ^{xl}$$

$$hi — —Jn—$$

$$Xl$$

Making these substitutions leads to

$$(^2 \quad - 2/l/_2/l_2 + /!^2/_{22}) = \sim XiX_2f_l$$

$$= -fn(f\}xl + Ifxfixxxi + fix^2,)$$

$$+ f_2x_2)^2 = -f_ny$$

Therefore, for linear homogeneous production functions

$$\sigma = \frac{f_1 f_2}{y f_{12}}$$

a drastic simplification of Eq. (9-28) indeed.

A curiosity concerning Eqs. (9-28) and (9-29) is that they are symmetric between x_1 and x_2. That is, the identical expression results when the subscripts are interchanged. Thus we can speak of the elasticity of substitution between x_1 and x_2 rather than the elasticity of substitution of x_2 for x_1, or of x_1

for x_2. It does not matter whether x_2/x_1 is related to f_2/f_1 or x_1/x_2 is related to f_1/f_2 by derivatives in a. The formula is the same either way.

Formula (9-29) can be related to the expression for the rate of change of one factor with respect to another. Recalling Eq. (8-28Z?) on the comparative statics of cost minimization, we have

dw ***IT***

where

$H =$

-/i —

-*h* *f*

i

0

Thus

-fifi

which, for linear homogeneous production functions, becomes (as before, dropping the asterisks to remove clutter)

or

= *a*

_y^_d

*xi*_ =

j\f2

x

$x_2\ dw_2$ yf_{12}

Noting that X — w_2/f_2, we can write this as

y w_2 *"*

= *o*

$f_2 x_2\ X_1$

dw

Letting /c, = *fiXj/y (K* + K_2 = 1, by Euler's theorem) and denoting the cross-elasticity of demand by 612,

€

Thus, the elasticity of substitution is related in this simple fashion to the cross-elasticity of (constant-output) factor demand. And, of course,

$$O = \frac{1}{K_2} —6_{12} = \frac{1}{K_x} —6_{21}$$

Knowledge of *o* at any point would undoubtedly be a useful technological datum for empirical work. Beyond the strictly qualitative results of comparative statics, measurement of the degree of responsiveness to changes in parameters is an essential part of any science. Hence, it would be useful to be able to estimate a quantity like *a. A* useful first approximation in so doing is to assume that the production process is linear homogeneous and exhibits *constant* elasticity of substitution everywhere. That is, *a* is the same at all factor combinations. What would such production functions look like? We have already shown that the Cobb-Douglas function has the property *o* = 1 everywhere. What about other values of cr?

Return to Eq. (9-26), $a = —(w/u)(du/dw)$, where $u = x_2/x_x$, $w = w_2/w\backslash$. Strictly speaking, we should in general write

$$a = -\frac{w}{u}\frac{du}{dw}$$

since in general $u — x_2/x_x$ will not be a function of the wage ratio w_2/wi only but will also depend on the output level *y*. However, consider the case first of homothetic production functions. The cost function for all homothetic production functions can be written

$$C^* = J(y)A(w_l,w_2) \tag{9-31}$$

where $A(w_x, w_2)$ is linear homogeneous. (Any cost function is linear homogeneous in the factor prices.) Using the envelope theorem (Stephard's lemma), we have

$$*i = J(y)A_x(w_u w_2) \tag{9-32a}$$

$$x_2 = J(y)A_2(w_x,w_2) \tag{9-326}$$

where Ai = 9A/3w_1,etc. Since $A\backslash$ and A_2 are first partials of a linear homogeneous function, they are homogeneous of degree 0 in $w\backslash$ and w_2. But then

$$O = B_x(w)$$

and so forth, and therefore we can write

$$x, = J(y)B_l(w) \tag{9-33a}$$

$$x_2 = J(y)B_2(w) \tag{9-336}$$

(In fact, only the factor demands of homothetic production functions have this functional form.) Dividing Eq. (9-336) by *(9-33a)* gives

$$u = \frac{x_2}{x_1} = \frac{B_2(w)}{B_1(w)} = B(w)$$

That is, for all homothetic production functions, the ratio of factor inputs is a function of the ratio of wage rates only, not at all a function of output *y.* This of course is geometrically obvious, since the isoquants of homothetic production functions are merely radial blowups of each other. Hence, in formula (9-26) it is valid, for homothetic production functions, to write *o — —(w/u)(du/dw),* since *u* is indeed some well-defined function of *w* only.

Suppose now, maintaining the assumption of homotheticity, that *a* is constant everywhere. The class of homothetic functions having constant elasticity of substitution consists of those which satisfy the differential equation

$$-\frac{w}{\text{constant}\ u}\frac{du}{dw} = a$$

Let us solve this differential equation. Rearranging variables gives

$$\frac{du}{u} = -a\frac{dw}{w}$$

Integrating both sides and denoting the arbitrary constant of integration as log *c,* we have

$$\log u = -a \log w + \log c = \log cw$$

$\sim^{a}$ or

- *)* (9-34)

w J

where, of necessity, $c > 0$. Thus, all such production functions must have the property that the capital-labor ratio is proportional to the wage ratio raised to some power, that power being the negative of the elasticity of substitution. What production functions satisfy (9-34)? For cost-minimizing firms, *\/w = W\/w₂ = /1//2 = —dx₂/dx\,* the slope of an isoquant at some arbitrary output level *y.* Rewriting (9-34) in terms of the original variables yields

h)

or, taking roots $(k = c^{1/a})$,

$x^{\wedge l\prime a}$

Xi

Now *k* is any positive number. We can, for convenience, write $k = (1 - a)/a,$ where $0 < a < 1$. As a varies between 0 and 1, *k* varies from 0 to 00, so no generality is lost. Separating variables gives

dx (1 — *a)dx*$_2$

I/O (9-35)

We have to distinguish two cases now when integrating this expression. When $a = 1$, logarithms will be involved, whereas when a ^= 1, the integrals will be simple polynomials.

Case 1. Let *a* = 1. Integrating both sides of (9-35) yields

dx_{f} .. (*f* dx_2

x ***j*** ***x***

The arbitrary constant of integration can in general be any function of y, since *y* was held constant in determining the slope *8x2/dx*. Again, since Eq. (9-35) is really *a.partial* differential equation, the arbitrary constant of integration can involve any function of the variable or variables held constant, in this case output *y.* For convenience, we have denoted this constant of integration log *g(y).* Performing the indicated operations, we have

$$a \log xi = -(1 - a)\log x_2 + \log g(y)$$

or

Up to this point, the only assumption about the form of the production function we have made is that it is homothetic. Indeed, assuming *g(y)* is monotonic, we can write

$$y = F\{x?x^{*}-^{a}) \qquad (9\text{-}36)$$

where *F* is the inverse function of *g;* that is, if *z = g(y), y =* $g\sim^{l}(z)$ *= F(z)* — $F(x^{*}x\backslash\sim^{a})$. Equation (9-36) has the required form for homotheticity, being a function of a linear homogeneous function. If now we insist that}' = $f(x\backslash,x_2) = F(x''x\backslash\sim^{a})$ be homogeneous of some degree *s,* then

$$f(xux_2)=kx?x? \qquad (9\text{-}37)$$

where o;i = *as,* a_2 = (1 — *ce)s,* and thus *a.* + a_2 = *s.* If /(xi, x_2) is to be linear homogeneous, with *a* = 1, then

$$f(x_u x_2)=kx^{\circ}x^{*}-^{a} \qquad (9\text{-}38)$$

Equations (9-36) to (9-38) represent the general functional forms of production functions that exhibit constant elasticity of substitution equal to unity everywhere *(a* = 1) and, in addition, are, respectively, homothetic, homogeneous of degree *s,* and linear homogeneous. Consider now the second case, a^l.

Case 2. If *a j*= 1, integrating both sides of Eq. (9-35) yields

where again, the arbitrary constant of integration is some function of output y, designated *g(y),* since y is held constant in finding the slope 3x2/3^! of an isoquant. Performing the indicated operations and rearranging yields, incorporating the factor (-I/a)+ 1 into

$$g(y) = ax\backslash\text{-}^{l\,\prime a)+x} + (1 - a)x^{(}{}_{2}{\sim}^{l/a)+l} \qquad (9\text{-}39)$$

It will simplify matters if we let p = 1 — (I/a); that is, a = 1/(1 — p); then

$$g(y)=ax^p + \{\backslash\text{-}a)x^p \qquad (9\text{-}40)$$

Assuming again that $g(y)$ is monotonic, (9-40) can be written

$$y = F(ax^p + (\text{l-a})x_2^p) \qquad (9\text{-}41)$$

Equation (9-41) is the most general form of homothetic production functions exhibiting constant elasticity of substitution. If we wish $y = f(x\backslash, X2)$ to be homogeneous of degree 1, then

$$y = k(ax^p + (l\text{-}a)x^p)^{l/p} \qquad (9\text{-}42)$$

Equation (9-42) is what is commonly referred to as the CES production function. It assumes linear homogeneity. The elasticity of substitution, of course, varies between 0 and 00. When a —► 0, p -» —00; when a — 1, p = 0, and when a —> +00, p —> +1 . Hence the range of values for p is —00 < p < 1. When a —>■ $0(p$ -> — 00), the isoquants become L-shaped; i.e., the function becomes a fixed-proportions production function. When o —> oo(p —► +1), the isoquants become straight lines, as inspection of (9-42) reveals.

Although we have proved that when o = 1 (p = 0), the CES production function becomes Cobb-Douglas, that fact is not obvious from Eq. (9-42). In order to show this result directly, we need a mathematical theorem known as L'Hopital's rule.

L'Hopital's rule. Suppose that $f(x)$ and $g(x)$ both tend to 0 (have a limit of 0) as x —► 0. Then if the ratio $f'(x)/g'(x)$ exists,

hm -—- = km —— (9-43)

^0 $g\{x)$ x^0 $g'(x)$

The limit of the ratio of the functions, if it exists, equals the ratio of the derivatives of $f(x)$ and $g(x)$, respectively.

The formal proof of this theorem can be found in any advanced calculus text; we shall not present it here.

Consider the CES function (9-42) again, and take the logarithms of both sides:

tog $_y$ = log *$_+$'''*('''*$^{+}$ $^{(1}$-$^{g)j'}$> (9-44)

P

The right-hand side of (9-44) consists, aside from the constant, of a ratio of two functions, each of which tends to 0 as p -> 0. We find the limit as p -> 0, letting $f(p)$ = numerator, remembering that if $y = a'$, $dy/dt = a'$ log a:

$$f(P) = ax^p + (1 \text{———}]$$

$$\lim_{P \to O} f'(p) = \frac{}{1}[a \log x_1 + (1 - a) \log$$

The denominator of (9-44) is simply *p,* and thus *g'(p)* = 1; hence, *hm_p^Qg'(p)* = 1. Therefore, as *p* -> 0,

logy = log fc +log

J^"" or

$$y = kx^a_x X^{X}_2{}'^a$$

the Cobb-Douglas function, as expected.

The factor demands and the cost functions associated with the CES production function can be derived using the cost minimization hypothesis. Formally, the problem is

minimize

$$\text{WiJCi} + w_2x_2$$

= C subject to

$$ct\backslash x^p + a_2x_2 — y^p$$

where $a.\backslash + a_2 = 1$.

TheLagrangian is i£ = *W\X{* + w_2x_2 + A(_y^p — *{oi\x[* +^2^2)); differentiating with respect to x_x, x_2, and eliminating A. yields (eliminating the *'s to save notational clutter)

Multiplying through by (JCI/JC_2),

$a_2 x^p_2$

W O
i $L\backslash$
w_2 O

Now add 1 to both sides of this equation (which adds the denominator of each side to the respective numerator):

C y^p

W_2X_2 OL_2X_2

Solving for x_2,

$x_2 =$

and by symmetry,

JCI = $C^{l/il\_p)} y^{-p/(l\_p)} w;^{mi\_p)} al^{/(l\_p)}$

Therefore

$W2x_2 =$

and

Adding produces total cost; therefore

C - c^mi_p)y-p^_p)(al^ni_p)w;p/il_p) +al^/(l_p)W2^P/(l *p)) and thus

C = y(a{/(l_p)w;p/(l_p) + aU«-*w^l«-rtf-p)l_p (9-45)

We can derive the constant-output factor demands using the envelope theorem result
dC/dWi* = x*:

= *y(*

- A

or

O L .. / 1 / r i — n i _/i/M_^________1/Cl—n !
—*nl(\ — n* 1 / — n 1 / C 1 _____________^i t
_1/n_/->1

(9-46) with a similar expression for x|.

Generalizations to n Factors

Consider again the definition of elasticity of substitution given in Eq. (9-27) but now assume that the two factors in question are two of *n* factors that enter the production function:

(9-47)

This number is a measure of how fast the ratio of two inputs changes when the marginal rate of substitution between them changes. In order for this definition to make sense, the other factors must be held constant at some parametric levels *xk = x®,kj=i,j*. When more than two factors are involved, a marginal rate of substitution of one variable for another can only be defined in some two-dimensional subspace of the original space, i.e., along a plane (hyperplane) parallel to the *x(, Xj* axes, in which the other variables are held constant. Thus definitions of elasticity of substitution analogous to Eq. (9-27), for the ^-factor case, are "partial" elasticities of substitution. By holding the other factors constant, they do not represent the full degree of substitution possibilities present in the production function. These partial measures would be especially deceptive if one or more of the factors held constant were either close substitutes or highly complementary to the variable factors.

As an alternative, one could develop elasticities of substitution based on Eq. (9-26):

_v^9^ (9_4g)

where w,-;- = *Wi/\Vj, u,j = Xi/xj.* In this definition, all other *wages* are to be held constant with the other factors allowed to vary. This definition overcomes most of the objections stated above for the fixed-input definition (9-47). Clearly, *a*j* will relate to the cross-elasticities of factor demand. As such, they are less of a technological datum of the production function but most likely a more useful concept, since in reality it will be unlikely that the other factors will remain constant.

The obvious generalization of the CES functional form to many factors

*y = A(*ai*x! +* •••+«„<)^1/p (9-49)

has been shown to yield constant elasticities of the type given in (9-48); that is, the other factor *prices* are held fixed.t They are also called the *Allen elasticities}* However, all the partial elasticities are equal to each other and to 1/(1 —*p).* Also, when *p* = 0 (ai; = 1), the form reduces, as in the two-factor case, to a Cobb-Douglas or multiplicatively separable function

y — J\X ***x***

x where ^ a(- = 1 to preserve linear

homogeneity.

The Generalized Leontief Cost Function

A cost function developed by Erwin Diewert§ has been found to be useful in empirical analysis. This functional form is

C* = *yY,* E *hM'*^2 ™*)*^12 *i , j = h . . . , n* (9-50)

In order for this function to satisfy the requirements of a cost function, it must display symmetry, i.e., /3;; = *Pji-* The constant-output factor demands can be obtained by differentiation with respect to the wages:

- I = 1 , . . . , / I (9-51)

*1Wi *—' \Wi J*

Differentiating further,

dxi _ i / i y/^2

j \ i j '

Note that ^ij —*fiji* is required in order that *dx*/d\Vj = dx*/dw{.*

H. Uzawa, "Production Functions with Constant Elasticities of Substitution," *The Review of Economic Studies,* 29:291-299, October 1962.

*See R. G. D. Allen, *Mathematical Analysis for Economists,* MacMillan & Co., Ltd., London, 1938; reprinted by St. Martin's Press, New York, 1967.

§W. Erwin Diewert, "An Application of the Shephard Duality

Theorem: A Generalized Leontief Production Function," *Journal of Political Economy,* 79:481-507, June 1971.

The reason this function is called a generalized Leontief function is that input-output analysis, as developed by Wassily Leontief, utilizes "fixed coefficient" technology, i.e., L-shaped isoquants, indicating an absence of substitution possibilities among factors of production. In the special case where /3,; — *0, i ^ j, x(= yfiu.* In that case, therefore,

That is, the ratio of inputs is independent of output level and factor prices. This describes the Leontief-style technology. We shall further explore models of this nature in Chap. 17 on linear programming (General Equilibrium I).

Additional functional forms will be analyzed in the context of utility theory, though some functions have been useful in both production and consumer theory.

PROBLEMS

1.264 If a production function is homogeneous of degree $r > 1$ (r < 1), it exhibits increasing (decreasing) returns to scale. The converse, however, is false. Explain.

1.265 Suppose all firms in a competitive industry have the same production function, *y* — *f(x\,* x_2), where $f(x_u x_2)$ is homogeneous of degree $r < 1$. Show that all firms in this industry will be receiving "rents," i.e., positive accounting profits. To which factor of production do these rents accrue? In the long run, if entry is free in this industry, what will be the industry price, output, and number of firms?

1.266 Find the production function associated with each of the following cost functions:

1.267 $C = \hat{}/w\backslash\backslash v_2 e^{y/2}$

1.268 $C = w_2[1+>> + \log (w_1/w_2)]$

1.269 $C = y(w^2 + w^2)^{1/2}$

1.270 It is often said that the reason for U-shaped average cost curves is indivisibility of some factors. However, indivisibility does not necessarily lead to such properties. Suppose a firm's production function is homogeneous of some degree. Suppose the production function is also homogeneous in any *n* — \ factors when the nth factor is held fixed at some level. Show that the only function with these properties is the multiplicatively separable form *y* — *kx"'x"²* • ■ • *x""*.

1.271 What class of *homothetic* functions $y = f(x\backslash, \ldots, x_n)$ is also homothetic in any *n* — 1 factors, with the *nth* factor held fixed at some level?

1.272 Show that for homothetic production functions, the output at which average cost is a minimum is independent of factor prices.

1.273 Suppose a production function y = *f{x*, x_2) is homothetic, that is, *f(x*, x_2) = *F{h(x*$_x$,

$x_2)$), where $h(x_x,x_2)$ is linear homogeneous. Show that the elasticity of substitution is
given by $a = \{h\backslash h_2)/h_{i2}h$.

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CHAPTER

10

THE DERIVATION OF CONSUMER DEMAND FUNCTIONS

10.1 INTRODUCTORY REMARKS: THE BEHAVIORAL POSTULATES

In this chapter we shall analyze a fundamental problem in economics, that of the derivation of a consumer's demand function from the behavioral postulate of maximizing utility. The central theme of this discussion will be to study the structure of models of consumer behavior in order to discover what, if any, refutable hypotheses can be derived. Thus, our analysis is mainly methodological: We wish to find out, in particular, what it is about the postulate of utility maximization subject to constraints that either leads to or fails to generate refutable hypotheses.

The behavioral assertion we shall study is that a consumer engages in some sort of constrained maximizing behavior, the objective of which is to

maximize

$$U(x_1, x_2, \ldots, x_n) \tag{10-1}$$

where $x_1, \ldots, x_n$ represents the goods that the consumer actually consumes and $U(x_1, \ldots, x_n)$ represents the consumer's own subjective evaluation of the satisfaction, or utility, derived from consuming those commodities. However, we live in a world of scarcity, and consumers are faced with making choices concerning the levels of consumption they will undertake. The consequences of scarcity can be summarized by saying that consumers face a budget constraint, assumed to be linear:

Budget constraint

$$\sum p_i x_i = M \qquad (10\text{-}2)$$

where p_i represents the unit price of commodity x_i and M is the total budget per time period of the consumer. The classical problem in the theory of the consumer is thus stated as

$$\text{maximize} \quad U(x_1, \ldots, x_n)$$

$$\text{subject to} \quad \sum p_i x_i = M \qquad (10\text{-}3)$$

The hypothesis (10-3) is often referred to as *rational* behavior, or as what a rational consumer would do. If this were so, then another theory would have to be developed for *irrational* consumers, i.e., consumers who did not obey (10-3). (The question of how these irrational consumers might behave has never been seriously studied, probably for good reason.) Also, utility maximization has been attacked on various introspective grounds, largely having to do with whether people are capable of performing the intricate calculations necessary to achieve a maximum of utility. And, finally, it might be argued by some that since utility is largely unmeasurable, any analysis based on maximizing some unmeasurable quantity is doomed to failure.

All the above criticisms are largely irrelevant. The purpose of formulating these models is to derive refutable hypotheses. In this context, behavior indicated by (10-3) is *asserted* to be true, for all consumers. That is, (10-3) is our basic behavioral postulate. Refutation of (10-3) can come about only if the theorems derived from it are demonstrably shown to be false, on the basis of empirical evidence. This is not a postulate for rational consumers; it is for *all* consumers. If some consumers are found whose actions clearly contradict the implications of (10-3), the proper response is not to accuse them of being irrational; rather, it is our theory which must be accused of being false.[†]

This admittedly extreme view of the role of theorizing is not lightly taken. The reason is that the *stupidity* hypothesis and the *disequilibrium* or *slow adjustment* hypotheses are consistent with all observable behavior and therefore are unable to generate refutable implications. Anything in the world can be explained on the basis

† A study of chronic psychotics at a New York State mental institution, people whom society has pronounced *irrational* in some sense, showed that psychotics obey the law of demand, i.e., they too buy less when prices are raised, etc. See Battalio et al., "A Test of Consumer Demand Theory Using Observations of Individual Purchases," *Western Economic Journal,* 411-428, December 1973.

that the participants are stupid, or ill-informed, or slow to react, or are somehow in disequilibrium, without theories to describe the alleged phenomena. These terms are metaphors for a lack of useful theory or the failure to adequately specify the additional constraints on consumers' behavior. We therefore stick our necks out and assert, boldly, that all consumers maximize some utility function subject to constraints, most commonly (though not exclusively, especially if non-price or rationing conditions are imposed) a linear budget constraint of the form (10-2) above. The theory is to be rejected only on the basis of its having been falsified by facts.

We have alluded to the concept of a utility function in earlier chapters; let us now investigate such functions more closely. A utility function is a summary of some aspects of a given individual's tastes, or preferences, regarding the consumption of various bundles of goods. The early marginalists perceived this function as indicating a *cardinal* measure of satisfaction, or utility, received by a consumer upon consumption of goods and services. That is, a steak might have yielded some consumer 10 "utiles," a potato 5 utiles, and hence one steak gave twice the satisfaction of one potato. The total of utiles for all goods consumed was a measure of the overall welfare of the individual.

Toward the end of the nineteenth century, perhaps initially from introspection, the concept of utility as a *cardinal* measure of some inner level of satisfaction was discarded. More importantly, though, economists, particularly Pareto, became aware that no refutable implications of cardinality were derivable that were not also derivable from the concept of utility as a strictly *ordinal* index of preferences. As we shall see presently, all of the known implications of the utility maximization hypothesis are derivable from the assumption that consumers are merely able to *rank* all commodity bundles, without regard to the intensity of satisfaction gained by consuming a particular commodity bundle. This is by no means a trivial assumption. We assert that all consumers, when faced with a choice of consuming two or more bundles of goods, x^1 = (x|, ..., x^x_n), ..., x^* = {x\, ... ,x^), can rank all of these bundles of goods in terms of their desirability to that consumer. More specifically, for any two bundles of goods x' and x^j, we assert that any consumer can decide among the following three mutually exclusive situations:

1.274 x^* is preferred to x^j.
1.275 x^7 is preferred to x^*.
1.276 x' and x^j are equally preferred.

Only one category can apply at any one time; if that category should change, we would say that the consumer's tastes, or preferences, have changed. In the important case 3, above, we say that the consumer is indifferent between x' and x^7.

The cardinalists wanted to go much farther than this. They wanted to be able to place some psychological measure of the degree to which the consumer was better off if he or she consumed x^* rather than x^j, in situation 1, above. Such a measure might be useful to, say, psychologists studying human motivation; to economists, it turns out that no additional refutable implications are forthcoming from such knowledge. Hence, cardinality as a feature of utility has been discarded.

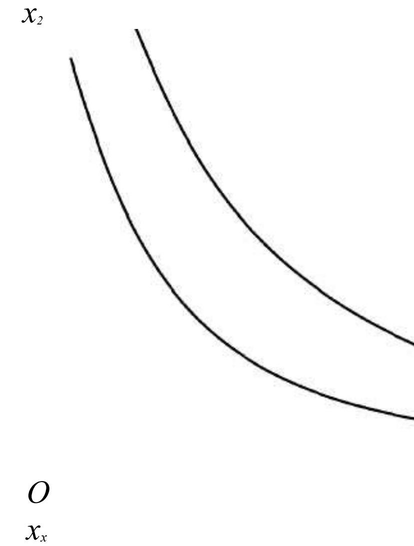


FIGURE 10-1 *Ordinal Utility Levels (Indifference Curves).* The ordinality of utility functions is expressed

by asserting that relabeling the values of the indifference contours of utility functions has no effect on the behavior of consumers.

The utility function is thus constructed simply as an index. The utility index is to become larger when a more preferred bundle of goods is consumed. Letting $U(x) = U(xi, \ldots, x_n)$ designate such an index, for cases 1, 2, and 3, above, $U(x)$ must have the properties, respectively:

1.277 $\boldsymbol{U(x^j) > U(x^l)}$.

1.278 $\boldsymbol{U(x^l) = U(x^j)}$.

How is the ordinality of $U(x\backslash, \ldots, x_n)$ expressed? Consider Fig. 10-1, in which two level curves, or indifference surfaces, are drawn for $U = U(xi,X2)$. The inner curve is defined as $U(x\backslash,X2) = 1$; the other indifference curve is the locus $U(x\backslash, x_2) = 2$. Suppose, now, instead of this U index, we decided to label these two loci by the square of U or by $V = U^2$. Then these two indifference curves, in terms of Vunits, would have utilities of 1 and 4, respectively. Or, one could consider a third index $W = \log U$, in which the "W-utiles" would be 0 and log 2, respectively. Ordinality means that any one of these utility functions is as good as the other, i.e., they all contain the same information, since they all preserve the *ranking,* though not the cardinal difference, between different indifference levels. In general, starting with any given utility function $U = U(x\backslash, \ldots, x_n)$, consider any monotonic transformation oft/, that is, let $V = F(U(x_u$

• • •, *,")) where $dV/dU = F'(U) > O$. Then Vand t/ always move in the same direction; the V index is merely a relabeli ng of the U index that preserves the rank ordering of the indifference levels. To say that utility is an ordinal concept is therefore to say that the utility function is arbitrary up to any mono-tonic (i.e., monotonically increasing) transformation. We shall check and see that all implications regarding observable phenomena which are derivable from asserting the existence of $U(x\backslash, \ldots, x_n)$ are also derivable from $V = F(JJ\{x\backslash, \ldots, x_n))$, where $F' > 0$, and vice versa. Ordinality means that $F(U(x\backslash, \ldots, x_n))$ conveys the identical information concerning a consumer's preferences as does $U(xi, \ldots, x_n)$.

The assertion that consumers possess utility functions is a statement that people do in fact have preferences.t How these preferences come to be, and why they might differ among people of different countries or ethnic groups, is a discipline outside of economics. These are certainly interesting questions. They are also exceedingly difficult to grapple with. The specialty of economics arose precisely because it was fruitful in many problems to ignore the origins of individuals' tastes and explain certain events on the basis of changes in opportunities, assuming that individuals' tastes remained constant in the interim.

Merely to assert that individuals have tastes or preferences is, however, to assert very little. In order to derive refutable implications from utility analysis, certain other restrictions must be placed on the utility function. To begin with, we shall assume that the utility function is mathematically well behaved; that is, it is sufficiently smooth to be differentiated as often as necessary. This postulate is questioned by some who note that commodity bundles invariably come in discrete packages (except perhaps for liquids, such as water or gasoline), and also, for the case of services, such as visits to the doctor, the units are often difficult to define. We note these objections and then ask, what is to be gained in our analysis by explicitly recognizing the discrete nature of many goods? In most problems, very little is gained, and it is costly in terms of complexity to fully account for discreteness. Again recall the role of assumptions in economic analysis: Assumptions are made because there is a trade-off between precision and tractability, or usefulness of theories. It is nearly always impossible to fully characterize any real-world object; simplifying assumptions are therefore a necessary ingredient in any useful theory. Hence, differentiability of utility functions is simply assumed.

In what class of problems is differentiability least likely to be a critical assumption? When consumers either singly or in groups make repeated purchases of a given item, we can convert the analysis from the discrete items to time (flow) rates of consumption. Instead of, say, noting that a consumer purchased one loaf of bread on Monday, another on Friday and another the following Tuesday (i.e., one loaf every four days), we can speak of an average rate of consumption of bread of seven-fourths loaves per week. There is no reason why the average consumption per week, or other time unit, cannot be any real number, thus allowing differentiability of the consumer's utility function. We can speak of continuous *services of goods,* even if the goods themselves are purchased in discrete units.

mere existence of preferences, however, may not be enough to guarantee the existence of utility functions. See Chap. 11.

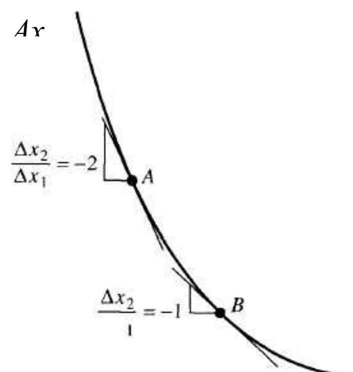
Assuming consumers possess differentiate utility functions $U = U(x_1, \ldots, x_n)$, the following properties of those functions are as serted. These are not intended to represent a minimum set of mutually exclusive properties; rather, they are the important features of utility functions which are the basis of the neoclassical paradigm of consumer-choice theory.

I. NONSATIATION, OR "MORE IS PREFERRED TO LESS." All goods that the consumer chooses to consume at positive prices have the property that, other things being equal, more of any good is preferred to less of it. The mathematical translation of this postulate is that if *x*,..., x_n are the goods consumed, the marginal utility of any good *x,* is positive, or £/, = *dU/dxi* > 0. Increasing any x_t, holding the other goods constant, always leads to a preferred position; i.e., the utility index increases.

II.SUBSTITUTION. The consumer, at any point, is willing to give up some of one good to get an additional increment of some other good. This postulate is related to postulate I. The notion of trade-offs is perhaps the most critical concept in all of economics. How do we describe the notion of trade-offs mathematically? The reasoning is analogous to that used in the definition of isoquants in the chapter on costs. Consider Fig. 10-2.

The *maximum* amount a consumer will give up of one commodity, say, x_2, to get 1 unit of *X* is that amount which will leave the consumer indifferent between the new and the old situation. Starting at point A, the consumer is willing to give up a maximum of 2 units of x_2, say, to get 1 unit of *x*. The trade-offs for any consumer are hence defined by the loci of points that are indifferent to the ini tial point. These curves are the consumer's *indifference curves;* since the consumer is indifferent to all points on the curve, *U(x\,…,* x_n*)* = *U°* — constant.

The slope of the indifference curve represents the trade-offs a consumer is willing to make. In Fig. 10-2, the slope = —2 (approximately) at point A; at point



U(x_u*x* $_2$*)* =

FIGURE 10-2
Value, in Economics, Means Exchange Value. The value of any commodity is the maximum amount of some other good that an individual is willing to part with in order to gain an extra unit of the good in question. In the limit (i.e., at the margin) the value of *x* is therefore given by the slope of the indifference curve through that point. At point *A,* the marginal value of JCI is the absolute slope of the level curve [indifference curve *U{x\,xi)* = *U°],* called the marginal rate of substitution of JCI for *X2,* and is equal to two

B the slope = — 1, indicating that the consumer will swap x_2 and *x*
one for one at that point.

For the case of two commodities, the indifference curves are the level curves of the utility function $U = U($*x*, $x_2)$, defined as $U($*x*, $x_2)$ = $U°$. Defining $x_2 = x_2($*x*, $U°)$ from this relation as before, the slope of the indifference curves at any point, using the chain rule, is found by differentiating the identity $U($*x*, $x_2($*x*, $U^0)) = U°$ with respect to JCI
:

and thus

$$dx_2$$

By postulate I, *U* and U_2 are both positive. Hence, dx_2/dx\ < 0, or the slope of the indifference curves is negative. For the n-good case,

^ < 0

A negative slope means precisely that the consumer is willing to make tradeoffs. The substitution postulate means that the indifference curves are negatively sloped, a situation implied by the postulate that "more is preferred to less." If the indifference curves were positively sloped, consumers would not be trading off one good to get some of another; rather, the situation would be better characterized by that of bribing the consumer with more of one good in order to accept more of the other. One of the goods must actually be a "bad," with negative marginal utility. Only then can *—Uj/Ui* be positive; only then would a consumer be indifferent between two consumption bundles, one of which contained more of each item than the other.

The substitution postulate is an explicit denial of the "priority of needs" fallacy. Politicians and pressure groups are forever urging that we "rearrange our priorities," i.e., devote more resources to the goods they value more highly than others. While it is useful for such groups to talk of "needs" and "priorities," it is fallacious for economists to do so. The notion of a trade-off is inconsistent with one good being "prior" to another in consumption.

The ultimate reason for rejecting the notion of priority of some goods over others is by appeal to the empirical facts, however, and not from logic. "Nonpriority" is an empirical assertion. How could one test for it? Consider a consumer who, by all reasonable measures, is considered to be rather poor. Suppose he or she is made even poorer by taxation or appropriation of some of his or her income. As income is lowered, if this consumer held the consumption of all goods except one constant and reduced some other good to zero, and then repeated the process for the other goods, we would have to conclude that such behavior indicated that some goods were prior to others in fulfilling the person's desires. However, it is unlikely that we should find such individuals. In all likelihood, all people, even very poor people, when faced with a reduction of income will tend to spread out the reduction among several goods, rather than merely cons uming only less clothing, say, or only less shelter. Real-world behavior is consistent with $U_i > 0$, $i = 1, \ldots, n$, for the goods actually consumed by a given individual.

The notion of substitution and trade-offs provides the critical underpinnings of the concept of *value* in economics. It is only by what people are willing to give up in order to get more of some other good that value can be meaningfully measured. In Fig. 10-2, the consumer at A is willing to give up 2 units of x_2 to get 1 unit of x_1; we conclude from this that the consumer *values* x_1 at 2 units of x_2, or that he or she values x_2 at 1 unit of x_1. This value, indicated by the slope of the indifference curve at some point, is called the marginal rate of substitution (MRS) of x_1 for x_2; it is the marginal value of x_1 in terms of x_2.

The last postulate economists make regarding utility functions is a restriction on the behavior of these marginal values. Specifically, it is asserted that:

III. ALONG ANY INDIFFERENCE SURFACE, THE MARGINAL VALUE OF ANY GOOD DECREASES AS MORE OF THAT GOOD IS CONSUMED. This says that

d **>0** $j = 1, \ldots, n$, $ij=j$

We shall show, however, that this generalization of diminishing marginal rate of substitution, while implied by the second-order conditions for maximization of utility subject to a budget constraint, is insufficient in itself to guarantee an interior constrained maximum. The condition required is that the indifference surfaces (actually, "hypersurfaces" in *n* dimensions) be convex to the origin, analogous to the convexity of the two-dimensional indifference curves. Mathematically, this is the condition of "quasi-concavity" of the utility function explored in Chap. 6. Its algebraic formulation, none too intuitive, is that the border-preserving principal minors of the following bordered Hessian alternate in sign:

$$H = \begin{matrix} U_{n} & \cdots & U \\ U_{2i} & u_{2n} & u \\ U_{nl} & U_{nn} & U_{n} \\ U_{1} & U_{n} & 0 \end{matrix} \qquad (10\text{-}3)$$

The border-preserving principal minors of order 2 in *H* above have the form

U_{n}

U_j +

In Chaps. 3 and 6 we found

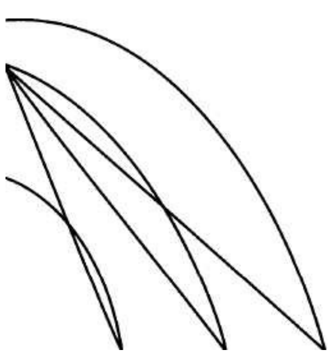
$$\frac{d^2x_2}{dx_1^2} = \frac{d(-U_1/U_2)}{dx_1}$$

$$U_{22}U_1^2)$$

The bordered Hessian H_2 is precisely a generalization of this to the case of n goods, wherein all goods except x_i and x_j are held constant. $H_2^{ij} > 0$ then says that in the $x_i x_j$ (hyper)plane, at stipulated values of the x_k's, $k \neq i, j$, the MRS of x_i for x_j decreases, or $d^2x_j/dx_i^2 > 0$. If this diminishing MRS holds for *every* pair of goods x_i and x_j, $i, j = 1, \ldots, n$, this says only that all the border-preserving principal minors of order 2 are positive; this is insufficient information from which to infer anything about the higher-order minors or H itself. Hence the notion that the indifference hypersurface is convex to the origin is a much stronger assumption, in an n-good world, than simply diminishing MRS between any pair of goods, other goods held constant. Only in the case of only two goods, where there are no other goods to be held constant, is quasi-concavity equivalent to diminishing MRS.

All the preceding postulates can be summarized as saying that we assert that all consumers possess utility functions $U = U(x_1, \ldots, x_n)$ that are differentiate everywhere and that are strictly increasing $(U_i > 0, i = 1, \ldots, n)$ and strictly quasi-concave. The adjective *strictly* is used to denote that there are no flat portions of the indifference curves anywhere; this guarantees uniqueness of all our solutions.

These mathematical restrictions are asserted not merely because they guarantee an interior solution to the constrained utility maximization problem, which they do, but more fundamentally, because such restrictions are believed to be confirmed by data involving real people. To deny these postulates is to assert strange behavior. As in the case of factor demands discussed in an earlier chapter, the assumption that, for example, indifference curves are concave to the origin implies that consumers will spend all of their budget on one good. A corner solution is achieved, point B in Fig. 10-3. At certain prices, only JCI will be consumed. Then, as p_1 is increased past



A'

FIGURE 10-3
Nonquasi-Concave Utility Functions. As p_1 increases, the budget line shifts from $A'B$ to $A'A$ to $A'C$. The maximum utility point will change suddenly from lying on the x_1 axis, to lying on the x_2 axis at A'. This behavior is not observed; for that reason it is asserted that indifference surfaces are convex to the

origin; i.e., the utility function is quasi-concave.

a certain level, the consumer suddenly switches over entirely to x_2. This inflexible and then erratic behavior is hard (impossible?) to find in the real world; it is for that reason and that reason only that the assumption of quasi-concavity is made.

10.2 UTILITY MAXIMIZATION

Let us now begin our analysis of the problem at hand, stated in relations (10-3), maximize

$$U(x_x, \ldots, x_n)$$

subject to

$$\text{^}T\ PiXi = M$$

We will, for simplicity, consider the two-variable case only, in the formal analysis, and briefly sketch the generalizations to *n* variables.

Suppose, then, the consumer consumes two goods x_x and x_2 in positive amounts. These goods are purchased in a competitive market at constant unit prices p_x and p_2, respectively. The consumer comes to the market with an amount of money income *M*. Under the assumption of nonsatiation, the consumer will spend all of his or her income *Monxi* and x_2, since *M* itself does not appear in the utility function. Income *M* is useful only for the purchase of JCI and x_2, as expressed by writing the utility function as $U = U(x_x,x_2)$.

We assert that the consumer (i.e., all consumers) act to

maximize

$$U = U(x_x,x_2)$$

subject to

$$Pix_1+p_2x_2 = M \tag{10-4}$$

A necessary consequence of this behavior is that the first partials of the following Lagrangian equal zero:

$$u\ x_2) + k(M - p_xx_x - p_2x_2) \tag{10-5}$$

where *X* is the Lagrange multiplier. Hence

$$Xi = Ui - Xp_x = 0 \tag{10-7a}$$

$$\$_2 = U_2 - kp_2 = 0 \tag{10-76}$$

$$£_k = M - p_1x_x - p_2x_2 = 0 \tag{10-7c}$$

The sufficient second-order condition for this constrained maximum is that the bordered Hessian determinant of the second partials of ££ be positive:

$$D = \quad 22 \quad \begin{matrix} U_n & U_{i2} & \sim P\backslash & U_{2}\backslash \\ U_{22} & -pi & -P\backslash & \\ -Pi & 0 & & \end{matrix} > 0 \qquad (10\text{-}8)$$

We will, of course, assume that D is strictly greater than zero; only $D >$ 0 is implied by the maximization hypothesis.

Thus far we have accomplished little. Most of the terms in Eqs. (10-7) and (10-8) are unobservable, containing the derivatives of an ordinal utility function. As we have repeatedly emphasized, the only propositions of interest are those which may lead to refutable hypotheses; in order to do so, all terms must be capable of being observed. Thus the objects of our inquiry are the demand functions implied by the system of Eqs. (10-7). These three equations contain six separate terms: JCI, x_2, A, $p\backslash$, $P2$, and M. Under the conditions specified by the implicit function theorem that the Jacobian determinant formed by the first partials of these equations *{X* — 0, $££_2$ = 0 and X_k = 0) is not equal to zero, this system can be solved, in principal, for the variables JCI , x_2, and A. in terms of the remaining three, $p\backslash$, p_2, and M. In fact, this Jacobian is simply the determinant D in Eq. (10-8). Each row of D consists of the first partials of the corresponding first-order equation in (10-7). Since the system of Eqs. (10-7) is itself the first partials of *!£,* the Jacobian determinant consists of the second partials of *$£* with respect to $x\backslash$, x_2, and A.. The *sufficient* second-order conditions guarantee that D ^ 0 (in fact, D > 0); hence in this case we can write

$$x_{\{}=xf(_{Pl},p_2,M) \qquad (\backslash 0\text{-}9a) \quad (10\text{-}\%)$$

$$x_2 = x?(pi,p_2,M) \qquad (10\text{-}9c)$$

$$X = k^M(p_{uP2},M)$$

Equations (10-9) are the *simultaneous* solution of Eqs. (10-7). Note the parameters involved: prices and money income. Equations (10-9a) and *(\0-9b)* indicate the chosen levels of consumption for any given set of prices and money income. Hence, these equations represent what are commonly referred to as the *money-income-held-constant* demand curves. These functions are also commonly referred to as the *Marshallian* demands, after the great English economist Alfred Marshall.* The superscript M in these functions is a mnemonic for either "money" or "Marshall."

The phrase "money income held constant" is somewhat of a misnomer. Money income M is simply one of the three parameters upon which demand depends. The phrase arose from the usual graphical treatment of these demand curves in which p_x,

tMarshall's *Principles,* first published in 1890, is the seminal synthesis of the neoclassical paradigm of economics. In it, Marshall recognized demand as a schedule of prices and quantities.

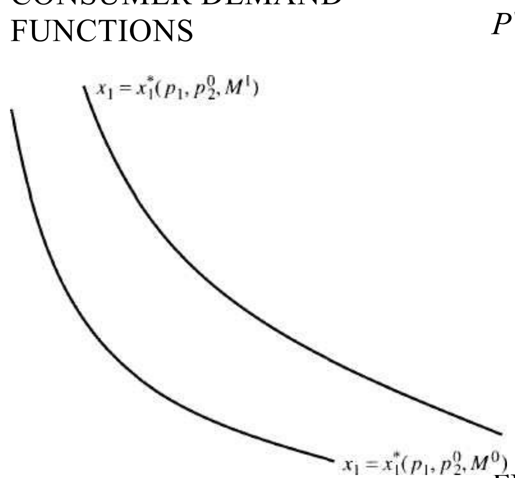


FIGURE 10-4 *The Money-Income-Held-Constant Demand Curve.* The notion that "money income is held constant" is simply a way of stating that in fact x_1 is a function not only of p_1, against which it is plotted, but also of p_2 and M. What is being held consta

nt is merely a convention as to which variables are chosen to be plotted. In the usual case, depicted here, where $x\backslash$ is plotted against its price $p\backslash$, changes in M result in a different projection of the demand curve $x *\{p\backslash, pi, M)$, and hence the drawn demand curve in the figure shifts.

say, is plotted vertically and JCI is plotted on the horizontal axis, as in Fig. 10-4. In this usual graph, since only two dimensions are available, only the parameter p_x is varied, and p_2 and M are held fixed at some levels $p\backslash$ and M°. Thus this graph really represents a projection of the function JCI = $x\^\{p\backslash, Pi, M)$ onto a plane parallel to the $X\backslash, p\backslash$ axes, at some fixed levels of p_2 and M. Because these two-dimensional graphs obscure the other variables in the demand curve, one has to specify what they are; e.g., in this case they are p_2 and M. These ceteris paribus (other things held fixed) conditions are simply another way of indicating exactly what variables are present in the demand function. "Movements along" the demand curve $x\backslash = x^{\wedge l} (p\backslash, p_2, M)$ simply refer to the response of quantity $x\backslash$ to changes in its own price $p\backslash$, where "shifts in the demand curve" represent responses to either p_2 or M. But it all depends on which variables are chosen to be graphed.

Although the marginal relations from which they are solved are not observable, the demand relations (10-9a) and (10-%) relate to observable variables and hence are potentially interesting.

If the demand functions (10-9a) and (10-%) are substituted into $U(x_i, x_2)$, one obtains the *indirect utility function* $U*(p_u p_2, M) = U(x_f*(p_l, p_2, M),$

x

(10-10)

Note that U^* is a function only of the parameters: prices and money income. The function $U^*(p\backslash, p_2, M)$ gives the maximum value of utility for any given prices and money income $p\backslash, p_2, M$, since it is precisely those quantities $x_{(}$ and x_2 that maximize utility subject to the budget constraint that are substituted into $U(x\backslash, x_2)$. Let us now investigate the first-order marginal relations (10-7). In so doing, we can discover some aspects of the nature of maximizing behavior and some of the properties of the demand relations (10-9a) and (10-9/?). The first proposition is one alluded to earlier, that no assumption of cardinality is necessary for the derivation of the demand curves *xf'(p\, P2,* Af); the same demand curves will occur if the indifference levels are relabeled by some monotonic transformation of *U(x\, xj)-*

Proposition 1. The demand curves implied by the assertion maximize

$$U(x_u x_2)$$

subject to

$$+ p_2 x_2 = M$$

are identical to those derived when $U(x\backslash, x_2)$ is replaced by $V(x\backslash, x_2) = F\{U(x_x, x_2)),$ where $F'(U) > 0$.

Proof. Consider how the demand curves are in fact derived. The two demand curves *xf (pi,* p_2*, M)* and;c^(/?i, p_2*, M)* are derived from a tangency condition and the budget constraint. The tangency condition is obtained by eliminating the Lagrange multiplier A. from Eqs. (10-7a) and *(10-1 b),* or

$$\wedge = \wedge \qquad (10\text{-}11)$$
$$U_2 \quad p_2$$

(This is the condition that would be obtained without the use of Lagrangian methods.) This equation, and the budget constraint

$$M - p_{\{} x_{\{} - p_2 x_2 = 0$$

are the two equations whose solutions are the demand curves above. How are these equations affected by replacing $U(x\backslash,x_2)$ by $V(x_u x_2) = F\{U(x\backslash,x_2)),$ that is, by relabeling the indifference map, but preserving the rank ordering? Instead of (10-11) we get

$$\text{£} = \wedge\text{i} \qquad \textbf{(10-12)}$$
$$V_2 \quad Pi$$

However, $V_x = F'(U)U_U$ $V_2 = F'(U)U_2,$ and therefore

$$V, = F'U_X \wedge U_x = Pi$$
$$V_2 \quad F'U_2 \quad U_2$$
$$p_2$$

Since V_x/V_2 is identically $U\backslash U_2$ everywhere, the equations used to solve for the demand curves are unchanged by such a transformation of *U.* That is, the solutions of (10-11) and the budget constraint are identical to the solutions of (10-12) and the budget constraint.

We must of course show that $V\backslash V_2 = p_x/p_2$ is indeed a point of *maximum* rather than minimum utility subject to constraint. That is, one must check that the consumer will actually set $V_x/V_2 = p\backslash/p_2$. If $F' < 0$, then V_x/V_2 would still equal $U\backslash U_2$, but $V\backslash V_2 = p\backslash\ I\ pi$ would not be a tangency relating to *maximum* utility, since with $F' < 0$, increases in both $X\backslash$ and x_2, which would increase *U,* will *decrease V.* Since $V = F(U)$ and $F'(U) > 0$, *V* and *U* necessarily move in the same direction; thus *U* achieves a maximum if and only if *V* does likewise.

The demand curves are independent of any monotonic transformation of the utility function; i.e., they are independent of any relabeling of the indifference map. This proposition simply reinforces the notion that it is only *exchange values* that matter. Along any indifference curve, the slope measures the trade-offs a consumer is willing to make with regard to giving up one commodity to get more of another. These marginal evaluations of goods are the only operational measures of value; it matters not one whit whether that indifference curve is labeled as 10 utiles or 10,000 or 10^{10} utiles. It is the *slope,* and only the slope, of that level curve which matters for value and exchange, not some index of "satisfaction" associated with any given consumption bundle. In fact, it is impossible to tell whether a consumer is pleased or displeased to consume a given commodity bundle. If those are the only goods over which he or she has to make decisions, the *exchange* values do not in any way reflect whether the consumer is ecstatic or miserable with his or her lot.

The preceding derivation also makes clear why the concept of *diminishing marginal utility* is irrelevant in modern economics. With strictly ordinal utility, the rate at which marginal utility changes with respect to commodity changes depends on the particular index ranking used. Since *V\ = F'(JJ)U\,* using the product and chain rules, *Vn = F'U\\ + U\F"U\,* and in general

Vtj = F'Utj + F'ViUj (10-13)

Now *F'* > 0 is assumed, and U_t and *Uj* are positive by nonsatiation. However, *F"* can be positive or negative; for example, if *F(U)* =log *U, F'* > 0 and *F"* < 0; if *F(U)* = e^u*,F'>0, F"* > 0. Suppose t/,$_7$ < 0. Then if *V* is chosen so that *F"* > 0, it is possible that *V,j* > 0. Similarly, if t/$_{(J}$ > 0, there is some monotonic transformation that would make *V,j* < 0 by having *F"* sufficiently negative. Hence *U,j* and *V,j* (which include the case £/,-,- and *Va)* need not have the same sign, and yet the identical demand curves are implied for each utility function. Thus a given set of observable demand relations is consistent with a utility function exhibiting diminishing marginal utility and some monotonic transformation of it exhibiting increasing marginal utility. Hence, the rate of increase or decrease of marginal utility carries no observable implications.

In a similar way, economists once defined complementary or substitute goods in terms of marginal utilities as follows: Two goods were called complements if consuming more of one raised the marginal utility of the other, and vice versa for substitutes. For example, it was argued that increasing one's consumption of pretzels raised the marginal utility of beer; hence beer and pretzels were complements. The algebra above shows why this reasoning is fallacious. The term being considered in this definition is *dUi/dXj* = *U,j* = *Uj$_t$*. But if £/$_{;;}$ > 0, say, some monotonic transformation of *U, F(U),* with *F"* < 0 can produce a new utility function with *dVj/dXj* = *Vjj* < 0, opposite to £/,-;, and yet imply the same *observable* behavior, summarized in the demand relations. Hence this definition is incapable of categorizing observable behavior and is thus useless.

We now come to the second proposition concerning the demand curves that can be inferred directly from the first-order relations (10-

7):

Proposition 2. The demand curves $x_t = x^{\wedge}\{p_x, p_2, M)$ are homogeneous of degree 0 in $pi,p_2,$ andM. That is $xf^{l}(tp_{l},tp_2, tM) = x^{\wedge}(p_u\, p_2, M)$.

Proof. Suppose all prices and money income are multiplied by some factor /. Then the utility maximum problem becomes

maximize

$$U(x_u x_2)$$

subject to

$$tp_2x_2 — tM$$

But this "new" budget constraint is clearly equivalent to the old one, $p\backslash X\backslash + p_2x_2 = M$. Hence the first- and second-order equations are identical for these two problems, and thus the demand curves derived from this one, being solutions of those same first-order equations, are unchanged.

The meaning of this proposition is that it is only *relative* prices that matter to consumers, not absolute prices, or absolute money-income levels. This simply reinforces the tangency condition $U\backslash/U2 = P\backslash lPi$- It is the price *ratios* and the ratios of income to prices that determine marginal values and exchanges. Again, as mentioned earlier, some economists in the 1930s argued that consumers and producers would react to changes in nominal price levels even if real (relative) price levels remained unchanged. This concept, called money illusion, has been largely discarded. It was a denial of the homogeneity of demand curves.

Interpretation of the Lagrange Multiplier

Let us now consider the meaning of the Lagrange multiplier A. From the first-order relations

Pi *Pi* Also, by

multiplying *(10-7a)* by xf^{l} and (10-7Z?) by x™ and adding,

$$UiX? + U_2x? = X^M(_{Pl}x™ + nx?) = k^M M$$

Hence

x

(1014)

P *Pi* *M*

These relations provide an important clue to the interpretation of X^M. At any given consumption point, a certain amount of additional utility $U\backslash$ can be gained by consuming an additional increment of $x\backslash$. However, the marginal cost of this extra JCI is $p\backslash$. Hence the marginal utility per dollar expenditure on $x\backslash$ is $U\backslash/p\backslash$. Similarly, the marginal utility per dollar expenditure on JC_2 is $U_2/'Pi$- What the first equalities in (10-14) therefore say is that at a constrained maximum the marginal utility per dollar must be the same at "both margins," i.e., for JCI and x_2. If $U\backslash/p\backslash > U_2/p2,$

say, the consumer could increase his or her utility with the same budget expenditure simply by reallocating expenditures from x_2 to $x_{\prime}$.

What of the third equality in (10-14)? This relation says that the same marginal utility per dollar must occur when the incremental expenditure is spread out over both commodities, as when it is spent at either margin. It is an envelope-related phenomenon, exhibiting the property that the rate of change of the objective function with respect to a parameter is the same whether or not the decision variables adjust to that change. The rate of change of utility with respect to income is the same at each margin and at all margins simultaneously.

Thus far, however, we have not shown mathematically what has been inferred on the basis of intuition. To say that X^M is the marginal utility of money income is to say that $X^M = dU^*/dM$, where again

$$U\backslash p_u\, p_2, M) = U(xf(_{Pl}, p_2, M), x?(_{Pl}, Pi, M)) \qquad (10\text{-}10)$$

This can be shown directly. Differentiating (10-10),

8U* dU dx™ dU 9x$_2$^{M}
dM a^3M^r + dx$_2$dM
dxf^1 dx™
+ U^L (10-15)
U$_\prime$ + U$_2$
dM dM

Using the first-order relations (10-7), $U_x = k^M p_u$ $U_2 = X^M p_2$,

dU M
M(3JC, 3JC.\
=X^M P1_L + p _J_ (10-16)
y dM y dM J

Now consider the budget constraint $p_x xi + p_2x_2 = M$. When the demand curves are substituted back into this equation, one gets the identity

$$\text{pixf} + p_2x_2{}^M = M$$

Differentiating with respect to M yields

dxf^1 dx™

$$Pi \quad + p_2 \quad = 1$$

But this is precisely the expression in parentheses in Eq. (10-16). Hence, substituting (10-17) into (10-16) yields

$$y \; dM \quad dM$$
$$\wedge \quad {}^{A} \quad 1 \qquad (10\text{-}17)$$

What we have just done is in fact merely a rederivation of the envelope theorem for the utility maximization problem. Using the envelope theorem, recalling that the

Lagrangian i£ = $U(x\backslash, x_2)$ + A(M — $p\pm x\backslash$ — p_2x_2),

du*

M

= A

Nonsatiation implies that A^M = dU^*/dM >

0.

Roy's Identity

A similar procedure yields an important relation regarding the rate of change of maximum utility with respect to a price, that is, 3 *U*/dpi*. Using the envelope theorem,

x x (1019)

dpi *dpi*

This result is known as *Roy's identity,* after the French economist Rene Roy, who first published it in 1931.^ Moreover, solving for *xf* and using (10-18),

M = _ *I* ^ *Z I* _ *L* (10-20)

dU/dM*

Note that in the case of the demands derived from profit maximization and those derived from constrained cost minimization, the choice functions are the partial derivatives of the indirect objective function with respect to the prices. In the utility maximization model the implied choice functions, i.e., the Marshallian demand functions, do *not* have this simple property; rather, they are the (negative) partials with respect to prices divided by the partial derivative with respect to money income. By applying Young's theorem to the indirect utility function, we can derive reciprocity relations for the utility maximization model. From Roy's identity,

77* ~ ~k x|M I Y

Therefore,

PIP2 d_{P2} d_{P1} P2P1

Applying the product rule yields

A *Mdxf* ---h X_x^M dX^M — — A^M *dx*™ ---h X_2 $^M dX^M$ — (1U-Z1)

dp_2 dp_2 *dpi* *dpi*

In the profit maximization and constrained cost minimization models, the simple reciprocity results *dx*/dpj* = *dx*/dp*(were derived. Since in the utility maximization model the explicit choice functions are not the first partial derivatives of the

^Rene Roy, *De L'Utilite, Contribution a la Theorie de Choix,* Dunod, 1931. More accessible is "La Distribution du Revenu Entre Les Divers Biens," *Econometrica,* 15:205-225, 1947.

indirect objective function, the reciprocity conditions take on a more complicated form. It can be seen from (10-21), however, that if $dX^M/dpi = 0$ for all prices, then $dxf/dpj = dx^M/dpj$. This condition implies that the utility function is homothetic, or, equivalently, the income elasticities are all unity. We will return to this at the end of the chapter.

Another reciprocity relation is available regarding responses to changes in income. Since $U^*_M = X^M$, $U^*_{Mpi} = dX^M/dpi$. However, $U^*._M = 8\{-X^M xf^l)/dM = -[x$™$(dX^M/dM) + X^M\{dxf^l/dM)l$ From Young's theorem, therefore,

$$\frac{dX^M}{dpi} = -x^M \frac{dX^M}{8M} - X^M \frac{dxf^L}{8M} \qquad (10\text{-}22)$$

This expression is used, among other places, in the analysis of consumer's surplus.

Example. Consider the utility function $U = x_x x_2$. The level curves of this utility function are the rectangular hyperbolas $x_x x_2 = U°$ — constant. What are the money income demand curves associated with this utility function? The Lagrangian for this problem is

=X — XxX_2 ~~t~ A^iKi — $P\backslash Xx$ — $P2$-%$_2$)

The first-order equations are thus

i£i — x_2 — $Xpx = 0$

$tc_2 = Xx — Xp_2 = 0$!£i

$= M — p\backslash Xx — p_2x_2$

$= 0$

The demand curves are the simultaneous solutions of these equations. Before proceeding, let us check the second-order determinant. Noting that $U_u — U_{22} = 0$, $U\backslash_2 = U_n = 1$,

$D = p_x p_2 > 0$

1.279 1 $-px$
1.280 0 $-p_2$
$-px$ $-p_2$ 0

The second-order condition is satisfied since both prices are assumed to be positive.

Returning to the first-order equations, eliminate X from the first two equations:

$x_2 = Xpx$ Xx — Xp_2 Dividing,

fi - *El* x_x p_2 or

This equation says, incidentally, that the total amount spent on x_x, $pxXx$, always equals the amount spent on x_2, p_2x_2, at any set of prices. We should thus expect the demand curves to be unitary elastic. (Why?)

The relation $p_2x_2 = p_xx_x$ is derived solely from the tangency condition $U\backslash / U_2 = Px/p_2$, not at all from the budget constraint. This equation therefore holds for all

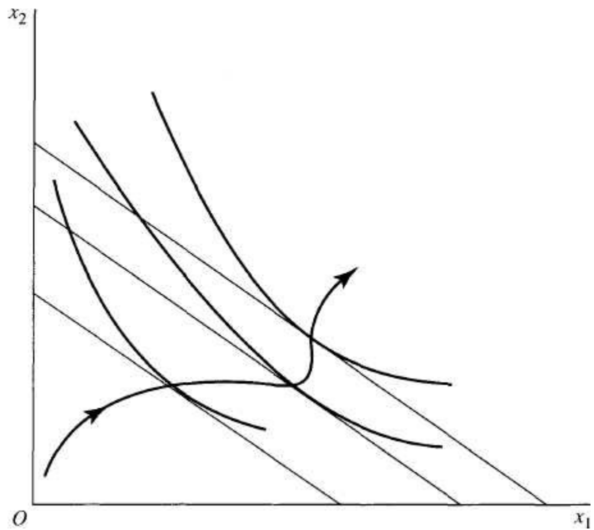


FIGURE 10-5
The Income-Consumption Path. The income-consumption path is the locus of all tangencies of the indifference curves to various budget constraints. That is, it is the locus of points *(x\,x_2)* such that *U1/U2 = P\/P2,* where the slope of the indifference curve equals the slope of the budget constraint. Thi s equation is independent of money income M; hence it represents the solutions of the first-order equations that correspond to all values of M. As M is increased, the implied consumption bundle moves in the direction of the arrow along the curve, reaching higher indifference levels for higher *M.*

possible income levels. It is the locus of all points *{x_x, x_2),* where the slopes of the level curves are equal to *—p\/pi-* Hence, *p_2x_2 = P\Xi* represents what is called the *income-consumption* path, shown in Fig. 10-5. The income-consumption path, one of the so-called Engel curves, illustrates how a consumer would respond to changes in income, holding prices constant. Rewriting the present equation slightly as *x_2 = (p\/pi)x\,* we see that the income-consumption path is a straight line, or ray, emanating from the origin. [The point (0, 0) obviously satisfies the equation, and *x_2* is a linear function of X].] The slope of this line is *p\ / p_2.* Since the income-consumption path is a straight line, by an easy exercise in similar triangles, a given percentage increase in money income *M* leads to that same percentage increase in the consumption of both commodities (see Fig. 10-6). We therefore expect to find that the demand curves derived from this utility function *(U = X\X_2)* possess unitary income elasticity as well as unitary price elasticity. In order to derive the demand curves, the budget constraint must be brought in. The demand curves, we recall, are the simultaneous solutions of the tangency condition *U\/U_2 = P\/pi* and the budget constraint *p_xx_x + p_2x_2 = M.* Since the former gives *p_2x_2 = p\X\,* substitute this into the budget equation, yielding

P1X1 + *= M*

or

2p\X\ = M

Therefore

is the implied demand curve for *x\.* In similar fashion,

Xl ~ 2_{P2}

is the implied demand curve for *x_2.*

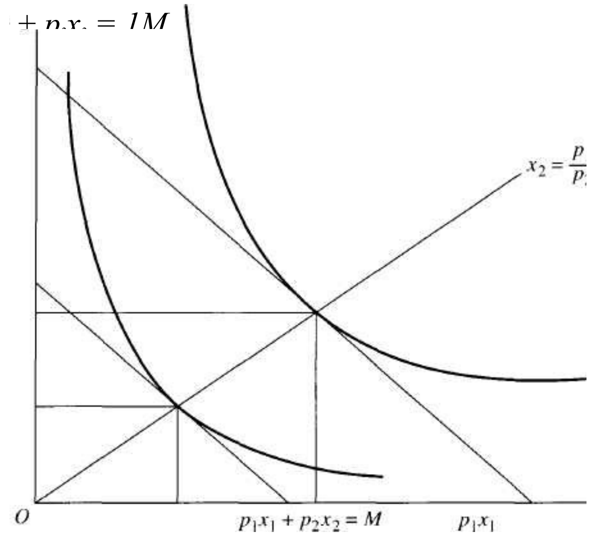


FIGURE 10-6 *The Income-Consumption Path for the Utility Function $U = x_1x_2$.* The income-consumption

path is the solution of the tangency condition $U_1/U_2 = p_1/p_2$. For the utility function $U = x_1x_2$, $U_1 = x_2$, $U_2 = x_1$, thus the income-consumption path is $x_2/x_1 = p_1/p_2$, or $x_2 = (p_1/p_2)x_1$. This is represented geometrically as a straight line emanating from the origin. Doubling M will move the budget constraint twice as far from the origin as previously; when this is done for this utility function, clearly the consumption of x_1 and x_2 will exactly double. Hence, we expect to find demand curves with unitary income elasticities for this utility function.

Let us check the envelope result for X^M and Roy's equality. The indirect utility function is $U^*(p_1,p_2, M) = x_1^*x_2^* = (M/2p_1)(M/2p_2) = M^2/4p_1p_2$. Differentiating with respect to M, we find, as expected, $X^M = dU^*/dM = M/2p_1p_2$. Also

$$dU^*/dp_1 = -M^2/(4p_1^2p_2) = z M$$

$$\frac{\partial U^*/\partial p_1}{\partial U^*/\partial M} \sim \frac{M/(2p_1p_2)}{2p_1} \; x_1^M$$

Let us check the properties of these demand curves. We n ote, first, $dx_1^M/dp\ = —M/2p^2 < 0$, $dx\%'/dp_2 = —M/2p\ < 0$; the demand curves are downward-sloping. The cross-effects are both 0; since xf^l is not a function of p_2, and x™ is not a function of $p\$, $dx^/dp_2 — dx^/dpi — 0$. This is a very unusual property for the money income demand curves. In general, $dx^/dpj =fc dx^/dpi =fc 0$, $i =fc j$.

The price elasticity of each demand curve is given by

Pi

dx

™

Thus, forjcf = $M/2p_u$ $dx^/dp_x = -M/2p^2$. Hence,

$$P\ \quad dp\ i \quad \frac{2p\ -M}{M \quad 2p\} = -1$$

with a similar result for $e_2$2- As indicated earlier, the price elasticities of demand are indeed equal to —1, as expected, since total expenditures $p_ix\backslash$ and *P2X2* are the same for all prices.

Regarding the income elasticities,

$$M\ 3xf$$

Here, $dxf/dM = 1/lpuM/xf^1 = M/(M/2p_x) = 2p_i$. Hence, $\epsilon_{lM} = 1$ as expected from the linearity of the income-consumption path. Similar algebra shows that $\epsilon_2 M = 1$ also.

10.3 THE RELATIONSHIP BETWEEN THE UTILITY MAXIMIZATION MODEL AND THE COST MINIMIZATION MODEL

In Chap. 8 we studied the

problem minimize

$$C = W1JC1 +$$

W2X2 subject to

$$f(.x_l,x_2) = y°$$

where $y = f(x\backslash,x_2)$ was a production function and $W\backslash$ and w_2 were the factor prices. Consider now a problem mathematically identical to this, that of minimizing the cost, or expenditure, of achieving a given utility level *U°,* or

minimize

$$M = p_xx\backslash + p_2x_2$$

subject to

$$2) = U^0 \qquad (10\text{-}23)$$

where $p\backslash$ and p_2 are the prices of the two consumer goods JCI and x_2, respectively, and $U = U(x_u x_2)$ is a utility function. The entire analysis of Chap. 8 applies to this cost minimization problem. The only changes are in the interpretation of the variables; the mathematical structure is the same.

The first-order conditions for this problem are given by setting the partials of the appropriate Lagrangian equal to 0:

$$X = {}_{Pl}xi + p_2x_2 + \mathrm{ME}/^0 - U(x_u\ x_2))$$

$$\pounds i = pi - \mathrm{A.E}/1 = 0 \qquad (10\text{-}24a)$$

$$\pounds g_2 = {}_{P2} - XU_2 = 0 \qquad (10\text{-}24\hat{})$$

$$X_x = U° - U(xi,x_2) = 0 \qquad (10\text{-}24c)$$

The sufficient second-order condition for a constrained minimum is

$$-XU_U \quad -MJ_n \quad \sim U$$

$$-XU_2\backslash \quad -XU_{22} \quad -U_2 \quad < 0 \qquad (10\text{-}25)$$

$$-Ui \quad -U_2 \quad 0$$

Assuming (10-24) and (10-25) hold, choice functions of the following type are implied, as simultaneous solutions to the first-order relations (10-24):

$$x_t = x\backslash^r(pi, p_2, U°) \qquad (10\text{-}26a)$$

$$x_2 = x\%(p_u p2, U^0) \qquad (10\text{-}266)$$

$$k = X^u(p_u p_2, U°) \qquad (10\text{-}26c)$$

Whereas the demand curves (10-9), $x_t = x\hat{}(p\backslash, p_2, M)$, are called the *"money income* held constant" demand curves, the demand curves (10-26), x_t — $xV(p\backslash, p_2, U°)$, are called the *"real income* held constant," or "income-compensated" demand curves. These latter curves hold utility, or "real" income, constant; they are mathematically equivalent to the "output held constant" factor demands of the previous chapter. The functions $xf\ (p\backslash, p_2, U°)$ are also commonly referred to as *Hicksian* demands, after Sir John R. Hicks, the British Nobel Laureate in Economics. The partial derivatives of the Hicksian demand functions with respect to the prices represent *pure substitution effects,* since, utility being held constant, the consumer remains on the same indifference level. Substituting the Hicksian demands into the objective function yields the *expenditure function* $M^*(p\backslash, p_2, U°) = Pix\hat{}\ (pi, p_2, U°) + p_2x\backslash\ (pi, p_2, U°)$, indicating the minimum expenditure needed to achieve utility $U°$ at prices p_x and p_2.

What is the relation between the demand curves (10-26), derived from cost minimization $x_t = xf\ (p\backslash, p_2, U°)$, $i = 1, 2$, and the demand curves (10-9), derived from utility maximization x_t — $xf\ (p\backslash, p_2, M)$, $i = 1,2$? Consider the first-order relations (10-24a) and (10-246). Eliminating X yields

$$PI \quad U_2$$

This is the same tangency condition as that derived in the utility maximization problem. In both cases, the budget line must be tangent to the indifference curve. In fact, consider Fig. 10-7. In the utility maximization problem, given parametric prices and money income M, some maximum level of utility U^* will be achieved, at, say, point A, where the consumer will consume x^* and $x\hat{}$ amounts of xi and x_2, respectively.

Suppose now the indifference level U^* were specified in advance; that is, $U^* = U°$, and the consumer minimized the cost of achieving $U^* = U°$ with the same prices. Then, clearly, the consumer would wind up at the same A, consuming the package $(x^*, x\%)$. *But the comparative statics of the two problems are not the same!* The adjustments to price changes are different because different things are being held constant. Consider Fig. 10-8. In the case x, = $xf(p\backslash, p_2, M)$, as $p\backslash$, say, is lowered, the budget line MM' swings out along the JCI axis to MM'' to a new,

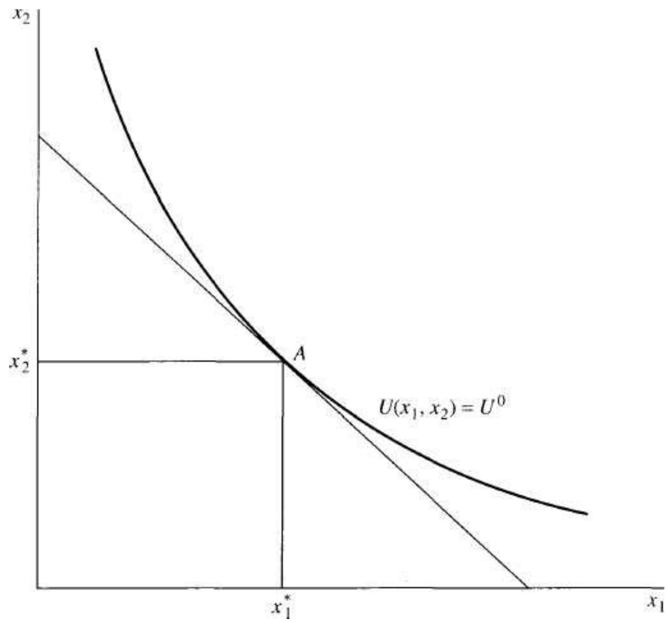


FIGURE 10-7
The Tangency Solution to the Cost-Minimization and Utility-Maximization Problems. If $U(x_1, x_2)$ is maximized subject to $p_1x_1 + p_2x_2 = M$, some level of utility U^* will be achieved at point A. If now U^* is set equal to U^0, and the consumer minimizes the cost of achieving $U^* = U^0$, the point A. will again be achieved. However, the comparative statics of the two problems differs, because the parameters of the problems are not identical (see Fig. *10-8a* and *b*).

higher intercept, as depicted in panel *(a)*. This increases the achieved utility level to U^{**}. However, in the cost minimization problem [panel *(b)*], if p_1 is lowered, the level of U is parametric: it is held fixed at U^0. It is the achieved minimum budget M^* that decreases, as the new tangency at A'' is reached, at the new expenditure level M^{**}.

Finally, we note from (10-7*a*) and *(10-7b)*,

$$M$$

$$P_1 \quad P_2$$

However, from (10-24a) and (10-24b),

$$X^u = — = —$$

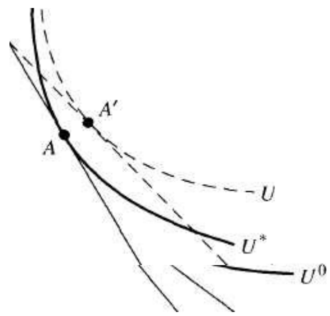
Hence, at any tangency point, for the proper U^*

and M^*,

(10-27)

Y Y

M M M
(



M *(b)*

FIGURE 10-8

Utility Maximization; Cost Minimization. The comparative statics of the cost minimization problem differs from the statics of the utility maximization problem in that different parameters are held constant when a price changes. When $p\backslash$ changes, say, decreases, in the utility maximization problem (Fig. 10-8a), the horizontal intercept, which equals *M/pi,* shifts to the right, to keep *M* constant. A new tangency, *A',* on a higher utility level, is implied. In the cost minimization problem (Fig. 10-8£), as $p\backslash$ decreases, the utility level is held constant at *U°,* and hence, the tangency point slides along *U{x\, xi) =* U^{o} to point *A",* where a new, lower expenditure level *M**** is achieved.

In the production scenario, X^{u} is the marginal cost of *output.* Here, in utility analysis, it is the (unobservable) marginal cost of *utility.* It is the reciprocal of k^{M}, the marginal utility of money income, as the units of each term would indicate.

The student is warned, however, not to simply regard 3 *M**/ 3 *U°* = l/(3£/*/3M) as trivial. These partials cannot simply be inverted; *dM*/dU°* and *dU*/dM* refer to two separate problems. It is a matter of some curiosity that the simple relation (10-27) holds.

The fundamental contribution to the theory of the consumer, known as the Slutsky equation (developed by E. Slutsky in 1915), relates the rates of change of consumption with respect to price changes when money income is held constant to the corresponding change when real income, or utility, is held constant. That is, a relationship is given between *dx^/dpj* and *dx^/dpj.* This relationship will be derived in the next two sections.

We must check that the second-order conditions for the uti lity maximization and cost minimization problems are identical. This fact is visually obvious from Fig. 10-7. Clearly, interior solutions to both problems require that the indifference curves be convex to the origin, at all levels. Therefore, we should be able to show that the determinant D given in (10-8) is positive if and only if H, given in (10-25), is negative. We leave it as an exercise in determinants that in fact $H = —X^M D$. By nonsatiation, $X^M > 0$, and thus $H < 0$ if and only if Z) > 0. For either cost minimization or utility maximization, the utility functions must be quasi-concave.

Let us recall the comparative statics of the cost minimization problem. As was shown in Chap. 8, differentiating the first-order conditions (10-24) with respect to

Pi yields the comparative statics equations

$-kU_n$ ~ *dpi*

$-kU_2\backslash$ $-kU_{22}$ $-U_2$ ^-Ui *dp i*

$-U_2$ 0 \

Differentiating with respect to p_2 would place the — 1 in row 2 on the right-hand side. In general, we find, again,

dpj *H*

Inspection of *H* and *D* quickly reveals that H_{ij} = (A^M)²Z)$_{ij}$; thus

(10-28)

O U T_J \ M J~~
OX; — n a A
LJ I*;

H *D*

Last, *dx^/dpi* < 0; when *i* ^= *j,* however, *dxf/dpj* ^ 0 (except in the two-variable case).

10.4 THE COMPARATIVE STATICS OF THE UTILITY MAXIMIZATION MODEL; THE TRADITIONAL DERIVATION OF THE SLUTSKY EQUATION

It is apparent from the structure of the utility maximization model that no refutable hypotheses are strictly implied on the basis of the maximization hypothesis alone. All of the parameters appear in the constraint. As the general analyses of Chaps. 6 and 7 show, no testable implications appear in any model for any parameter appearing in the constraint function.

The interest in this model stems from the analysis of E. Slutsky in 1915, and expanded by John R. Hicks in 1937, in which the response to a change in price was conceptually partitioned into two separate effects; a pure substitution effect, in which "real" income (utility, in Hicks' formulation) is held constant, and a pure income effect, in which prices are held fixed, and the budget line shifts parallel to itself to the final maximum utility level. As we shall presently show, whereas the income effect is indeterminate in sign, the pure substitution effect, which is precisely the response derived from the minimum expenditure model, is always negative.

We can illustrate this analysis graphically as follows. Suppose a consumer with preferences given by the indifference curves shown in Fig. 10-9 initially faces the budget constraint *MM* and achieves maximum utility at point *A,* consuming JC° amount of JCI. Suppose $p\backslash$ is lowered. The budget line will pivot to the right, producing a new utility maximum at point *B.* The total change in consumption of $x\backslash$

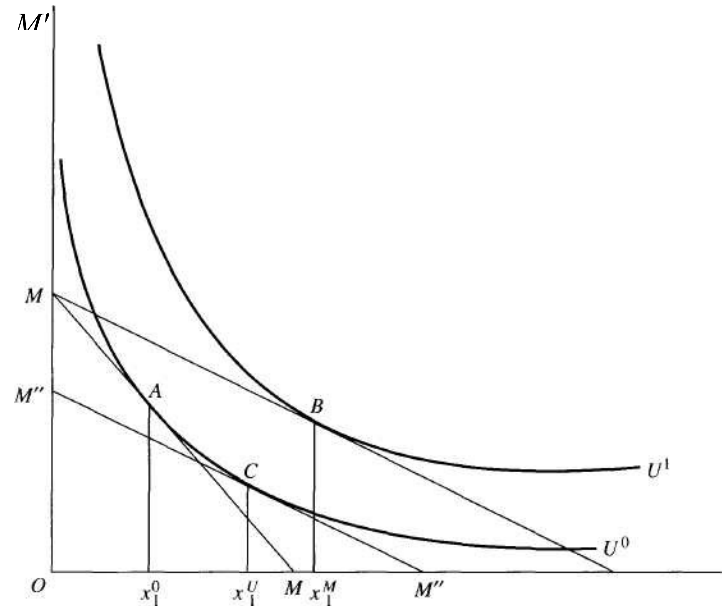


FIGURE 10-9
The Substitution and Income Effects of a Price Change. This diagram relates to finite movements in the consumption of *x* due to a finite change in *p*. It is therefore not directly comparable with the Slutsky equation, which deals with instantaneous rates of change. However, the income and substitution effects of a price change are easily seen in the above well-known diagram. The original tangency is at A, on budget line *MM*. When *p* is lowered, the horizontal intercept increases, and the budget line pivots to *MM',* yielding a new tangency at *B*. The total change in consumption of *x* is *xf* — *x*®. This amount can be partly attributed to *x** — *x*®, a pure substitution effect obtained by sliding the budget line around the indifference curve *U*° until it is parallel to the budget line *MM',* reflecting the new prices. Since utility is held constant, this is indeed a pure substitution effect. The remaining part of the total change in *x\, xf* — jcf, is attributable to a parallel shift in the budget line from *M"M"* to *MM'*. This is a pure income effect since prices are held constant.

is JCJ — *x*. This amount, however, is partitionable into

***«-*?= (*?-,») + (*?-,?)**

The first term, jtf — *x*®, is a change in *x\,* holding utility constant. The tangency point *C* occurs at the new, lower, *p\,* but at a reduced budget level represented by the budget constraint *M"M"*. Point *C* is the combination of x_x and x_2 that minimizes the cost of achieving the old utility level at the new prices (i.e., new price p_x *oix\)*. Hence, the change Jtj — *x*® is a pure substitution effect, and would be generated by the cost minimization problem.

The remaining part of the total change, *xf* — JCJ, is generated by a parallel shift of the budget equation from *M"M"* to *MM'.* Since prices are held constant, this is a pure income effect.

The preceding graphical analysis, while a useful aid to understanding this model, does not correspond exactly to the comparative statics analysis. Comparative statics relations are the instantaneous rates of change of choice variables with respect to parameter changes; they are partial derivatives evaluated at a particular point. Let us now proceed with the traditional analysis of the utility maximization model, even though, as we shall see, a more powerful technique, using modern duality theory, is available for deriving the main result. However, the traditional technique is still important for nonstandard models, and so we apply it here to illustrate its use.

The first-order equations of the utility maximization problem, in identity form, are, again,

=0

(10-7)

? $M - p_x x_x^M - p_2 x? = 0$

How will the consumer react, first, to a change in his or her money income M, prices being held constant? Differentiating these identities with respect to M, noting that M itself appears only in the third equation, the following system of equations is found:

" dx

dM dx*

dM dM =0 (10-29a)

dk^M JM

dM (10-2%)

1 — p_x dM dx ? = 0 (10-29c)

In matrix form, this system of equations is

U_n $-p$ dx_x^M

$U_2 2$ —pi dx™

(10-30)

~P\ —Pi 0 M dx dM V"1/

The coefficient matrix is, again, the second partials of the Lagrangian function $£ = U + X(M — p_x xi — p_2 x_2)$, and the right-hand coefficients are the negative first partials of the first-order equations with respect to the parameter in question, here M, as the general methodology indicates. Solving this system by Cramer's rule yields

dM 0 ~P | | 0 ~P | -1 -Pi 0 |

$\boldsymbol{D}$

$\boldsymbol{D}$ (10-31a)

and similarly

(10-31Z?)

3*f ***-D*₃₂**

dM *D*

dX *—D33*

dM *D*

In none of these instances can a definitive sign be given. The denominators *D* are positive by the sufficient second-order conditions. However, inspection reveals

$$+ P\backslash U_{22} \wedge 0$$

and likewise

$$D_{32} = piU_{u} - P\backslash U_{2[} \wedge 0$$

Also,

because D33 is not a *border-preserving* principal minor.

What Eqs. (10-3la) and (10-31Z?) say, not surprisingly, is that convexity of the indifference curves is insufficiently strong to rule out the possibility of inferior goods. That is, it is entirely possible to have *dx^/dM* < 0 or *dx^/dM* < 0, as Fig. 10-10 shows. It is *not* possible, however, for both $x\backslash$ and x_2 to be inferior. If that were so, more income would result in reduced purchases of both $x\backslash$ and x_2, violating the postulate that more is preferred to less. On a more formal level, the third equation in the comparative statics system, Eq. (10-29c), the differentiated budget constraint, says that *pidx^/dM* + p_2*dx^/dM* = 1 > 0. Since the prices $p\backslash$ and p_2 are both positive, it cannot be that *dxf/dM* < 0 *and* *dx^/dM* < 0. Also, inferiority is of necessity a *local* concept. Goods cannot be inferior over the whole range of consumption, or else they would never be consumed in positive amounts in the first place!

Let us now differentiate the first-order Eqs. (10-7) with respect to the prices, in particular, $p\backslash$. This operation will yield the rates of change of consumption of any good with respect to a change in one price, holding all other prices and money income constant. Performing the indicated operation,

3pi *dpi* *dpi*

U_{2l}^- + U_{22}---p_2----------- = 0 (10-32Z?)

dpi *dpi* *dpi*

-pi^- - xf - p_2-^- = 0 (10-32c)

dp dp_x

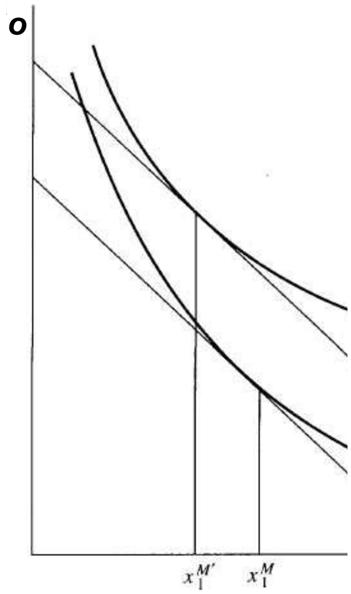


FIGURE 10-10 *Convexity of the Indifference Curves Allows Inferior Goods.* If money income is raised from M to M', the consumption of one good, say x_1, can decrease. A common example is the case of hamburger. As incomes rise, say, as students leave college and acquire jobs, hamburger is often replaced by steak. A word of warning: inferiority is a "local" concept. A good cannot be inferior over the whole range of consumption, or else it would never have been consumed in positive amounts in the first place!

where the product rule has been used to differentiate —

$X^M p\backslash$ and —$p\backslash x$^ (partially) with respect to p_x. In matrix form, this system of equations is

P \ I ,1 \ d

U 2 1 - p i . \

I

It is apparent right here that no comparative statics results will be forthcoming from this model; i.e., no definitive sign for dx^$/dpi$ or $dX^M/dp\backslash$ is implied by utility maximization. The reason is that there are two nonzero entries in the right-hand column. This means that knowledge of the signs of *two* cofactors in a given column of D will have to be determined; since only one can be a border-preserving principal minor (whose sign *is* known), at least one must be an off-diagonal co-factor whose sign and size is indeterminate from the maximization hypothesis alone.

3 / 7 1

Solving via Cramer's rule,

		-Pl
0		*-Pl*
x	*-Pl*	0

dx^M

dpi *D* *D* *D*

Likewise, putting (A^M, 0, x_t^M) into the second and third columns, respectively, in the numerator,

+ -1—* (10-33*)

3~M iM Di? x«n

+

dpi *D* *D*

IT" = —F^ + ~4T^ (10,,33c)

dpi *D* *D*

The determinant D_u is a border-preserving principal minor and is negative by the second-order conditions. Actually, by inspection, Du = — *p* < 0, quite apart from the second-order conditions. The determinant D_{33} is on-diagonal; however, it is not border-preserving; hence its sign is unknown. All the other cofactors are off-diagonal and are thus of indeterminate sign.

As expected, no sign is implied for either *dxf/dpj* or *dx^/dpj (i* =/=./). We define consumer goods as *substitutes* if an increase in the price of one good *increases* the demand for the other, and as *complements* if an increase in the price of one good *decreases* the demand for the other good. For example, an increase in the price of gasoline would likely decrease the demand for cars (a complement) and increase the demand for coal (a substitute). Substitutes and complements can be defined to either include or exclude the income effects, i.e., by using either the Marshallian or Hicksian demand functions. If the income effects are included, then the goods are called *gross* substitutes or complements; otherwise they are termed *net* substitutes or complements. Thus, *dx^/dpj* > 0 means that x_t and *Xj* are gross substitutes; *dxf/dpj* < 0 means x_t and *Xj* are net complements. Convex indifference curves (i.e., strictly increasing, quasi-concave utility functions) allow both substitutes and complements (by either definition), except in the two-good case, where the goods must be net substitutes (why?).

The interest in Eqs. (10-33a) and (10-33Z?) stems from the interpretation of the individual terms in the expression. Recall Eq. (10-28). In fact, the first terms on the right-hand side of Eqs. (10-33a) and (10-33Z?) are the pure substitution effects of a change in price as derived from the cost minimization model. Consider also Eqs. (10-31a) and (10-31*) relating to the income effects. These expressions are, respectively, precisely the second terms of the preceding equations when multiplied by the term —JCJ^M. Hence, Eqs. (10-33a) and (10-33*) can be written

^ i - = ^ - - x f ^ - (10-34a)

$$\frac{dpi}{dx^{\text{™}}}\hat{\ } - \frac{dpi}{dpi} = \frac{dpi}{dx''}\hat{\ } - - x f^{M}_{l} \frac{dM}{dx^{\text{™}}} ?- \frac{}{dM} \qquad (10\text{-}34^{*})$$

The equations for the response of the money-income-held-constant demand curves to price changes, when written in this form, are known as the *Slutsky equations.* Similar expressions can be written with respect to changes in p_2 and are left as an exercise for the student:

M

2

dp_2 dp_2 2 dM

dx™

(10-34J)

JC$_9$

dp_2 dp_2

In general (and this result is, in fact, a general result for the case of n goods),

dM (10-34e)

The Slutsky equation shows that the response of a utility-maximizing consumer to a change in price can be split up, conceptually, into two parts: first, a pure substitution effect, or a response to a price change holding the consumer on the original indifference surface, and second, a pure income effect, wherein income is changed, holding prices constant, to reach a tangency on the new indifference curve.

10.5 THE MODERN DERIVATION OF THE SLUTSKY EQUATION

In the previous section, the Slutsky equation was derived via the traditional methods of comparative statics. The procedure is somewhat tedious and long, an unfortunate requisite for doing that derivation correctly. However, a much shorter route is available by way of the more modern duality analysis. The new method is much more revealing than the old.

We start off with a money income demand curve, JCI = *xf*(p\\, p₂, M).* When *Px* changes, p_2 and M are held constant, producing a change in utility, since, by Roy's identity, *dU*/dpi = —X^Mxf^1* < 0. (When *p\\,* for example, is lowered, the opportunity set of the consumer expands, hence the attained utility increases.) Suppose, now, when p_x changes, M is also changed to the minimum amount necessary to keep utility constant. That is, define the function *M — M*(p₍, p₂, U°)* such thatM* is exactly that minimum money income level that keeps $U = U°$ when *p* (or any other price) changes. Then, by definition, if *x^ipi, p₂, U°)* is the utility-held-constant demand curve,

*r(Pi, *P^ ᵘ°) =* *?(Pi. P2» *M\\p_{uP2}, U⁰))* (10-35)

This is an identity—it defines *x^(pi, p₂, U°).* Differentiate both sides with respect to *pi,* say, using the chain rule on the right-hand side:

$dpi \quad dpi \quad dM \quad dpi$

What is *dM*/dpi?* The function $M^*(p_t, p_2, U°)$ is the minimum cost, or expenditure, of achieving utility level $U°$ (at given prices). M^* is therefore simply the (indirect) cost or expenditure function from the cost minimization problem

minimize

$$M = p_x x_x + p_2 x_2$$

subject to

$$U° - U(xi, x_2) = 0$$

the Lagrangian of which is $X = p\^xi + p_2x_2 +$ A.(t/° — $U(x\backslash, x_2))$. By the envelope theorem,

$$\frac{dM^*}{dpi} \_ \frac{dX}{dp_x} \_ \quad \_ \quad {}_l \quad {}_l$$

at any given point. Substituting this into the preceding equation yields

$$\frac{ox\backslash}{dp_x} \sim \frac{dpi}{3M} \quad \overline{M}^{ax}\backslash \quad {}^{ax}i \quad {}_l$$

This is precisely the Slutsky equation (10-34a)! (Note that here *dx^/dpi* appears alone on the left-hand side; we have merely rearranged the terms.) This proof is perfectly general. For *n* goods,

$$X?(p_l, \ldots, p_n, U°) = Xfrpu \ldots, p_n, M\backslash_{Pu} \ldots, p_n, U°))$$

where $M^*(p\backslash, \ldots, p_n, U°)$ is the minimum cost of achieving utility level $U°$ at given prices. By the envelope theorem from the cost minimization problem, $dM^*/dpj =$ x^ $= xf$ at a given point. Thus

dpj dM *pj* *8M*

()x^M *f)* $__$I ${}_x^M$ The Slutsky equation can be derived in this fashion by starting with the compensated demand curve $xf\ (p\backslash, \ldots, p_n, U°)$ and using it to derive the uncompensated demand curve JC$_{(}^{M}$ $\{p\backslash, \ldots, p_n, M)$. Specifically, if some *pj* changes, change $U°$ also by that *maximum* amount consistent with holding money income M constant. That is, define $U° = U^*(pu \ldots, p_n, M)$ to be the maximum achievable utility level for a given budget M at given prices. Then U^* is simply the indirect utility function of the utility maximization problem: max $U(x\backslash, \ldots, x_n)$ subject to $J2\ Pi^x i$ — M. The associated Lagrangian is $X = U(xi, \ldots, x_n) + X(M — J2\ Pi^x i)$- By me envelope theorem, $dU^*/d_{Pj} = d\^/dpj = -X^M xf$.

By definition, then,

$$Xfipu \ldots, p_n, M) = {}_X f\{_{Pu} - \ldots, Pn, U^*(p_l, \ldots, p_n, M))$$

A similar procedure to the preceding, together with an extra step, yields the Slutsky equation. This derivation is left to the student as an exercise. The Slutsky equations are sometimes written

$$— Xj$$

where the parameters outside the parentheses indicate the ceteris paribus conditions, i.e., what is being held constant. This representation is satisfactory, but it obscures the source of these partial derivatives. As has been constantly stressed, the notation *dy/dx, df/dxi,* etc., makes sense only if well-specified functions _y = *f(x), y = f(xi,x₂,* ...), etc., exist (and are differentiable). It is nonsense to write derivative-type expressions when the implied functional dependence is lacking. The Slutsky equation should be regarded as a relationship between two different conceptions of a demand function:

$$x_i=xf^l\{p_u p_2,M) \qquad (10\text{-}9)$$

and

$$Xi=xj'(p_l,p_2,U) \qquad (10\text{-}26)$$

Each equation is a solution of a well-defined system of equations stemming from an optimization hypothesis; in the case of Eq. (10-9), from utility maximization, and in the case of Eq. (10-26), from cost minimization. The Slutsky equation shows that these two equations are related in an interesting manner.

Let us examine the Slutsky equation again and see why it makes sense. We have

$$-x^M \quad l$$
$$pj \quad pj \quad dM$$

When a price changes, the consumer begins to substitute away from the good becoming relatively higher priced. However, the price change also changes the opportunity set of the consumer. If the price *pj* falls, the consumer can achieve certain consumption levels previously outside his or her former budget constraint. This is like a gain in income. However, what determines the size and sign of this income effect? If *pj decreases,* an effect similar to an *increase* in income is produced. Both produce larger opportunities. Price increases and income decreases are similarly related. Hence, it is plausible that the income term in the Slutsky equation be entered with a negative sign. The negative sign indicates that the implied change in income is in the opposite direction to the price change.

What about the multiplier *x*™ in the income term? What is its meaning and/or function? Suppose the commodity whose price has changed is salt. Salt is a very minor part of most people's budget. Hence, the income effect of a price change in salt should be small, even for large price changes. Suppose, however, the price of petroleum changes. Petroleum products may occupy a large part of our budgets, especially of those people who commute by car or heat their homes with oil. These income effects can be expected to be large. It is plausible, therefore, to "weight" the income effect *dxf/dM* by the amount of the good *Xj* whose price has changed. If the price of Rolls Royces increases, the effect on my consumption of that and other goods is negligible. Change the price of something I consume intensively and my *real* income, or utility, is apt to change considerably.

In the case where / = *j,* the Slutsky equation takes the form

(10-36)

dpi *dpi* ' *dM*

The important question, again, is, what refutable hypothesis emerges from this analysis? Can anything be said of the sign of *dx^/dpi?* Strictly speaking, no. However, we know that *dx^/dpi* < 0. If x_t is not an inferior good, that is, if *dx^/dM* > 0, then *dx^/dpi* < 0 necessarily. This proposition is nontautological only if an independent measure of inferiority (i.e., not based on the Slutsky equation) is available.

It is conceivable, though not likely, that *dxf/dpi* > 0, the so-called Giffen good case. Do not make the mistaken assumption that because something is mathematically possible, it is therefore likely to be observed in the real world. The refutable proposition *dx^/dpi* < 0 cannot be inferred from utility maximization alone; it is not on that account less usable. Utility maximization is a hypothesis concerning individual preferences for more rather than less, and provides probably the most successful framework for analyzing economic problems.*

A similar analysis can be applied to the Lagrange multiplier k^M, the marginal utility of income. A "compensated" or "Hicksian" marginal utility of income, k^H, would show responses in this value as one moved along a single indifference curve. Proceeding in exactly the same manner,

$$k^H(p_1, p_2, U^\circ) = k^M(p_u\, p_2, M^*(p_1, p_2, U^0)) \qquad (10\text{-}37)$$

Differentiating with respect to, say, *p\,*

dk^H *dk^M* *fdk^M\ fdM**

T- = T- + hn7 hr- (10-38)

dpi *dpi* *\dM J \dpi J*

Substituting *xf' = x^ = dM*/dpi,* we obtain a "Slutsky" equation for the marginal utility of income:

M

(10-39)

dpi *dpi* *' dM* The result is of course valid for any price /?,, and for models involving *n* goods.

^There are "general equilibrium" reasons for not believing that *dx^/dpt* > 0. If p_t falls, the consumers of x, experience a gain in wealth; however, the current owners and sellers of x, experience a wealth loss. Since at any time the quantity bought equals the quantity sold, the overall income effects of price changes are apt to be small.

Applying Eq. (10-22) to the right-hand side of (10-39), for any good *i*,

^ _ ^ _ A ^ ^ (10-40)

dpi *dM*

This equation says that along a given indifference curve, as some price changes, the change in the marginal utility of income is related in a very simple manner to the income effect of the good whose price has changed: it always has the opposite sign. We would in general expect that a lower price increases the purchasing power (measured in utility received) of additional income. If a good is inferior, however, increases in income lead to decreases in consumption of the good. This must mean that the marginal utility of income is relatively greater, if less rather than more of the good is consumed. Therefore, if the good whose price decreases is inferior, the greater consumption (along an indifference curve) leads to a fall in the marginal utility of income.

Conditional Demands*

In Chapter 8, we investigated the effect on the constant-output factor demands when one input was held constant (see Sec. 8-8). As we have already noted, the algebra of the cost minimization model is identical to the model in which expenditure is minimized subject to a utility-held-constant constraint. Restating this analysis in the context of consumer theory, the fundamental identity relating the Hicksian demand for *Xi* with a "short-run" Hicksian demand when, say, x_n is held constant at its expenditure-minimizing value, is

xf{pu . . . , Pn, U°) = x\(pu . . -, Pn-x,xun, U°) (10-41)

Differentiating both sides of this identity first with respect to p_t and then with respect to p_n, we obtain [see the derivation of Eq. (8-44)]

f *dxf* *(dx^/dpn)*

__!-------L = ——— < 0 (10-42)

dpi *dpi* *dx^/dpn*

A similar expression is obtained for the differences in the Hicksian responses of *Xi* with respect to *pj,* except that the numerator on the right-hand side becomes *(dx^/dp^idx^/dpn)* and is unsigned. (The derivation of this expression is left as an exercise.)

The analysis of the conditional Marshallian demands is somewhat more complicated, because the reciprocity results used in the derivation of (10-42) are not available for the uncompensated demand functions and because p_n, the price of the

t The short- and long-run results for the Hicksian and Marshallian demands were first developed by Robert Pollak, "Conditional Demand

Functions and Consumption Theory," *Quarterly Journal of Economics,* 83:60-78, February 1969.

good held constant, appears explicitly in the short-run demand. If x_n is held constant at its utility-maximizing level, the fundamental identity relating the Marshallian demand for some *X{ (i =fcn)* is

xfip, M) = x^n$_{\{}$ *(p, x^(p,* M), *M)* (10-43)

where *p = (pi, …,* $p_{n,}$), the price vector consisting of *all n* prices. In the case of the Hicksian demands, when x_n is held constant, the price p_n drops out of the first-order equations so that *x** is not a function of p_n. However, in the derivation of the Marshallian demands, p_n is in the budget constraint and does not drop out of the first-order equations. Thus the "short-run" Marshallian demand *x"* is a function of all *n* prices, *p\, …,* p_n.

Differentiating the fundamental identity (10-43) with respect to /?,- and then with respect to *M,*

SSHSHS)

£■£♦(£)(£)

If we now multiply (10-45) by x_t and add the two identities together, we create pure substitution terms, for example, *dx^/dpi* +Jt, *(dx^/dM)* on the left and similar terms on the right:

$$\frac{\partial x_i}{\partial p_i} + \left(\frac{\partial x_i}{\partial x_n^0}\right)\left(\frac{\partial x_n}{\partial p_i}\right)$$

3$_{Pi}$

The function JC* is the Hicksian demand for JC, holding x_n constant as defined in Eq. (10-41). Combining this with Eq. (10-46) yields

$$\left(\frac{\partial x_i^n}{\partial x_n^0}\right)\left(\frac{\partial x_n^U}{\partial p_i}\right)$$

dx[= *(dx"/dp*$_n$*)*

d$_{Pl}$ *d*$_{Pi}$

Therefore,

dx?

$$\left(\frac{\partial x_i^U}{\partial p_n}\right)\Big/\left(\frac{\partial x_n^U}{\partial p_n}\right)$$

Using this to eliminate *dx?/dx%* in Eq. (10-44),

dx? dx? (dx^u$_n$*/d*$_{Pi}$*){dx^d*$_{Pi}$*)*

dpi dpi dxV/dp$_n$ (10-47)

The denominator in (10-47) is negative; however, the numerator could possibly be negative, if the income effect on x_n is large enough relative to the cross-effect. Therefore, except in this unusual case, the Marshallian demand curve is more elastic than its associated "short-run" curve.

The Addition of a New Commodity

As a last example of this technique, let us examine the situation in which a consumer, as a result of an increase in income, decides to consume some new good x_{n+1} offered at price p_{n+1}. In most of traditional consumer theory, the problem of choosing the bundle of goods to be consumed at positive levels is not analyzed. It is in fact a very complicated problem of the type known as *nonlinear programming,* and would require detailed knowledge of the utility function for its execution. Comparative statics techniques can only analyze the signs of partial derivatives in the neighborhood of some point. However, we can gain some insight by considering the conditional demand for some good already consumed at a positive level x_i, $i = 1,\ldots, n,$ in terms of the new good x_{n+1}, fixed initially at $x^0_{n+1} = 0$. The Hicksian demand for any of the previously consumed goods is

$$x_i^U(p_1, \ldots, p_n, p_{n+1}, U^0) \equiv x_i^s(p_1, \ldots, p_n, U^0, x^U_{n+1}(p_1, \ldots, p_{n+1}, U^0)) \qquad (10\text{-}48)$$

This fundamental identity establishes the relationship between the demand for x_i when x_{n+1} is absent (fixed initially at zero) and when x_{n+1} is present. Suppose now that the consumer's income increases. We can analyze the effect of this on the Hicksian demands by increasing the parametric indifference level U^0. Differentiating the fundamental identity (10-48) with respect to U^0,

Differentiating the fundamental identity with respect to p_{n+1},

$$\left(\frac{\partial x_i^s}{\partial x_{n+1}^0}\right)\left(\frac{\partial x_{n+1}^U}{\partial p_{n+1}}\right) \qquad (10\text{-}50)$$

Using this in (10-49) and reciprocity yields

$$dx'' \qquad dxf \qquad \underline{(dx^\wedge/dp_i)(dx^\wedge/dU)}$$

The denominator of the latter term in (10-51) is negative, and since we are assuming that x_{n+1} is initially zero and becomes positive due to the increase in income, $\partial x^U_{n+1}/\partial U^0 > 0$ at that margin. Thus, the latter term has sign opposite that of $\partial x^U_{n+1}/\partial p_i$. We therefore see that if the new good is a substitute for x_i, so that $\partial x^U_{n+1}/\partial p_i > 0$, an increase in income will produce a smaller income effect on x_i than if x_{n+1} were not present, and vice versa for complements. Consider, for example, a person who experiences an increase in current income, due, say, to completing an advanced degree and commencing gainful employment. Having been a student for most of his or her life, the individual will likely have good information on the goods available for those on low incomes and probably less information regarding more momentous purchases. As the information on these new items is acquired, the income elasticities of the goods the items tend to replace (substitute) will fall, and the opposite holds true for complements. Conversely, people who experience a fall in income may have a more difficult time of it than people who are always at that lower income level, since acquiring the knowledge of how to be poor, i.e., which goods to give up, may take some time.

Example. We previously showed that the money income demand curves implied by the utility function $U = x_x x_2$ were $x_i^M = M/2p_x$, $x^\wedge = M/2p_2$. Let us find the compensated demand curves $x_l^u(p_x, p_2, U°)$, $x^\wedge\{p_x, p_2, U°)$ and show that the relationship between $x^\wedge$ and $x\backslash *$, etc., is as given by the Slutsky equation.

The compensated demand curves are solutions to the model

minimize

$$M = pixi +$$

p_2x_2 subject to

$$U\{x_x,x_2) = x_x x_2 = U°$$

The Lagrangian is

$$X = p_x x_x + p_2 x_2 + X(U° —$$

$x_x x_2)$ producing the first-order equations

$$JL_x = p_x — hx_2 =$$
$$0\ i£_2 = p_2 - Xx_x =$$
$$0\ £_k = U°- x_x x_2$$
$$= 0$$

The conditions $X_x = X_2 = 0$ yield the tangency condition $U_x/U_2 = Xx_2/Xx_x = P\backslash/P2$, or

$$p_x x_x = p_2 x_2$$

The same tangency condition is obtained for the cost minimization model as for utility maximization. The constraint, however, is not the budget equation but rather the constant utility equation $x_x x_2 = U°$. Combining the tangency and constant utility conditions yields

$$x^u \quad \frac{P\ ;\ (\ P}{} = 1/\ \nu\ \backslash$$

Similarly

We note in passing that, as required by the second-order conditions,

Also, the reciprocity condition $dx^\wedge /dp_2 = dx^\wedge/dp_x$ holds:

The cost, or expenditure function, is given by $M^* = p_x x_x^u + p_2 x^\wedge — 2(p_l p_2 U°)^{1/2}$. The Slutsky equation is a relationship that holds at any particular point of tangency with a budget line and an indifference curve. Thus, the terms $dx^\wedge/dpj$ and $dx^\wedge/dpj$ must be evaluated at the same point. Since xf^* and $x^\wedge$ are functions of different variables, M and U, respectively, we have to make sure the same point is being considered, such that $xf(puPi, M) = x_l^u(p_l, p_2, U°)$ and $x?(p_u p_2, M) = x\%(p_u p_2, U°)$. This is most easily handled via the indirect utility function $U^* = x^\wedge x^\wedge = M^2/4p_x p_2$. This can be viewed as relating, at given prices, the utility levels to money income levels. If utility is maximized subject to a given budget M, then minimizing cost subject to being on the indifference level $U° = M^2/4p_l p_2$ will lead to the same consumption bundle for this utility function. We also note that this is the same relationship as was given by the expenditure function $M^* = 2(p]p_2 U°)^{1/2}$.

Let us now evaluate the partial derivatives in a Slutsky equation:

dx dx dx

dp\ dp\ dM

Here, $dx?/dp_x = -M/2p^2$,

dM 2/7,

M 1 M

dM 2/7, 2/?, $4p^2$

and $dx^\wedge/dpi = —^\wedge(p_2 U°)^{1/2} pl^{3/2}$. The money income level M corresponding to $U°$ is given by either the expenditure or indirect utility function as $U° — M^2/4p_x\, p_2$. Thus, at any given point,

d 1

9 ~ '2

~ \

-M -

2p\ ~

M **-1/2 -1/2**

l

Hence

$$M$$

$$2pJ$$ as required.

10.6 ELASTICITY FORMULAS FOR MONEY-INCOME-HELD-CONSTANT AND REAL-INCOME-HELD-CONSTANT DEMAND CURVES

The Slutsky Equation in Elasticity Form

The Slutsky equation can be written in terms of dimensionless elasticity coefficients.^ First, multiply the entire.equation through by *Pj/x;.* Then we have

f1

Pj dxf' = Pj dx" pjxj dx
x_t dpj Xt dpj Xi dM

The first two expressions are already elasticities; the income term can be made one by multiplying it by *M/M,* that is, by 1, yielding

€%=€%- Kj€$_{iM}$ (10-52)

where *ef-* = elasticity of response of x, to change in *pj,* holding money income constant

€^ = elasticity of response of x_t to change in *pj*, holding utility constant *Kj = pjXj/M,* the share of the consumer's budget spent on good *j* e_{iM} = income elasticity of good *i*

The difference between the (cross) elasticities of the uncompensated and compensated demand curves depends on the size of the income elasticity of the good and the importance of the good whose price has changed, measured by the share of the consumer's budget spent on the good whose price has changed.

Certain useful relations concerning the various elasticities of demand are derivable from the utility maximization model. In general, they stem from either of two sources:

1.281 The homogeneity of the demand curves in prices and money income

1.282 The budget constraint

Homogeneity. We know that *xf*(pi, p_2, M)* and *x^ipx, p_2, M)* are homogeneous of degree 0 in prices and money income. Thus, by Euler's theorem, for *x^,*

*dx*l dxf* dxf'*
^PI + -^P2 + ^VTM =
0
dp i *dp_2*

Dividing this expression by *xf'* yields

^To reduce notational clutter, we will leave off the superscripts for x, when they is not needed.

Similarly

21 + 622 + €2M = 0 In general, for the case of n goods, with x, = $x^\wedge(pi, \ldots, p_n, M)$,

$$€n + *,? + \bullet\bullet\bullet + €\% + €_{iM} = 0 \qquad (10\text{-}53)$$

The budget constraint

(a) Income elasticities Differentiate the budget constraint with respect to M:

$$\frac{dx?*}{y\ dM} + Pi\frac{dx™ Pi}{F\ dM} = 1$$

This expression is equivalent to

p2^2 (M_dxi\ = M yxi dM J M \x₂ dM or

In general, for the case of n goods,

$$\text{H----h } K_n€_{nM} = 1 \qquad (10\text{-}54)$$

The weighted sum of the income elasticities of all goods equals 1. The weights are the shares of income spent on each good; the shares themselves sum to 1.

If income, say, increases by a certain percentage and consumption of some good Xi increases by some greater percentage, we say the good is *income elastic;* if consumption of that good increases by a smaller percentage, it is *income inelastic.* If a good is income elastic, then obviously the share of income spent on that good must rise as income rises. Algebraically, letting $r]_t$ = $e_{,M}$ to reduce clutter, we leave it as an exercise to show that

(<f) = (l 1) (10 55)

Clearly, 3/c,/3M ^ 0 as $r)_l$ ^ 1. Also note that homotheticity of the utility function, meaning unitary income elasticities of all goods, can also be described by the condition of unchanging shares of income spent on each good as income changes.

It also follows that as income continues to increase, goods that remain income elastic will eventually take over the entire budget (and would, in fact, eventually exceed it). It must be the case, therefore, that the income elasticities of income elastic goods must eventually fall. This might take the form of substitution toward higher-quality goods.[t]

There is an algebraic statement of this reasoning. Differentiating the identity *J2 Ki rji* = 1 with respect to income,

= 0 (10-56)

*-^ \dMj ^

—' \ Substituting (10-55) into this expression yields

or

Since the last term on the right-hand side is unity,

\dMJ \M

If the utility function is homothetic so that all income elasticities are unity, then the right-hand side of (10-57) vanishes, since $Yl^K i = 1$. In that case, the sum of the weighted rates of change of the income elasticities with respect to income changes, is 0, i.e., this weighted sum does not change as income changes. For all nonhomothetic utility functions, however (and this is the empirically important case), X) *Ki r]j* > 1. This relation follows from the fact that *tf* is a strictly convex function (think of the shape of $y = x^2$). Therefore, any convex combination of these squared income elasticities, ^ K_t $r\|^2$, must be greater than ^ K_t $r)_x$: = 1. For this general case, therefore,

5>(&)° (10-58)

In other words, for general (nonhomothetic) utility functions, the sum of the weighted rates of change of the income elasticities, with respect to income, falls as income rises. Could it be, as a general proposition, that all income elasticities must eventually fall?

(b) Price elasticities Now differentiate the budget identity *p\X^ + P2X2* = *M* with respect to *p*

^Levis Kochin and Yoram Barzel enlightened us about this intriguing argument.

This is equivalent to

(pi_dx?\ p^f p^

M [xf1 dpi J M Uf dpi I M

or

In general, for *n* goods, using the same technique,

K^ + .-. +K^^-KJ (10-59)

The weighted sum of the elasticities for all goods with respect to the price of a certain good sums to the negative of the share of the budget spent on the good in question. The weights are again the shares of the budget spent on each good. Note a difference between (10-53) and (10-59): Eq. (10-53) relates to *one* good and *all* prices (and income), whereas Eq. (10-59) relates to *all* goods and *one* price change.

Compensated Demand Curves

The compensated demand curves *x^ipi, p2, U°)* have slightly different elasticity properties. These properties are again derived from two sources: homogeneity and the constraint equation, in this case *U(x\, x2) = U°*.

Homogeneity. The demand curves *x^(pi, p2, U°)* and *x2(p\, p2, U°)* are homogeneous of degree 0 in the prices only. If prices are both doubled, say, since relative prices are unaffected, the tangency point remains the same. Hence, forx^, by Euler's theorem,

dxf7 dx" T-^I

+ ~Pi = o

dpi dp2

Dividing by jcf7 yields

where

Pi dx?

lJ x» dp

is the (cross-) elasticity of compensated demand of good / with respect to the price *Pj* of goody. For the case of *n* goods, using the same technique, one finds

en+4 + - - - + <£5E0 (10-60)

The constraintf/Qcf', x^) = U°. Differentiating this identity with respect to p\, say,

1/A + 1 / 2 E 0
dpi *dp*x

From the first-order conditions for cost minimization, U\ = *pi/k*, U_2 = hence (after multiplying by A.),

dpi *dpi*

This is almost the same relation as derived previously in Eq. (10-60); it in fact is derivable from that equation by noting that for compensated demand curves, *j* = *dx^/dpi*. Converting this expression to elasticities gives

$$\frac{p_1 x_1^U}{}\left(\frac{p_1}{x_1^U}\frac{\partial x_1^U}{\partial p_1}\right) + \frac{p_2 x_2^U}{M}\left(\frac{p_1}{x_2^U}\frac{\partial x_2^U}{\partial p_1}\right)$$

M or

*K*f*e*u u +*K*2€^ =0 In

general, for the n-good case *x^(pi, p*2*, …, p*n*, U°),*

U + *K*2€%*j* + • • • + *K*n£U nj = 0 (10-61)

Note the difference between (10-60) and (10-61): In Eq. (10-60), only one demand relationship *xf (p\, …, p*n*, U°)* is being considered, and the cross-effects of that good and all other prices are related. In (10-61), the responses of all goods to a given price change are related. These identities are the elasticity formulas commonly encountered in the theory of the consumer.

Return a moment to the derivation of Eq. (10-60) or (10-61). From the homogeneity of degree 0 of the compensated demand curves JC, = *xV{p\,…, p*n*, U°)* with respect to prices, from Euler's theorem,

dx" 1
P i r + + />„r s O
op I *dp*n

Letting *s*i*j* = *dx^/dpj,* the pure substitution effect on x, of a change in *p*j, we have

P\Sn + *PiSn* -\---1- *p*n*Sin* = 0

However, for compensated changes *s*i*j* — *Sj*i. Hence,

u + *p*2*S*2*i* H-----\~ *PnSni* = 0

These results are known as Hicks' *third law.* (The first two are, respectively, *s*ij *Sji*, *sa* < 0.) The law can be stated succinctly as

We have shown how the behavioral assertion of utility maximization leads to certain propositions that are, at least in principle, refutable. Specifically, the proposition that if a good is noninferior and if its price is lowered more will be consumed is implied by utility maximization. In addition, the income-compensated demands *xf(p\, ..., p_n, U°)* have the property of negative slope in their own price and also possess the reciprocity properties *dxf/dpj* = *dx^/dpi*. In addition, the preceding elasticity properties are implied.

In the example worked out earlier for the utility function U = *x\\X_2,* certain special results were obtained, in particular, *dx^/dpj* — *dx^/dp_t* = 0, together with unitary price and income elasticities. In general, what types of utility functions yield these properties?

Consider the unusual case *dx^/dpj* = *dx^/dpi.* This condition always holds for the Hicksian demands but not generally for the Marshallian demands $x_;^M$. If this does hold, then using the Slutsky equation,

ft U o $_y$M g_xU Q_xM x M
Q pj pj
dM
dpt ' dM

However, *dx^/dpj* = *dx^/dpi* always. Hence, we are left with

x
^ dM ' dM

Multiplying this equation through by *M/(x^x^)*

yields

$$\frac{M}{Xf}\frac{dxf^1}{dM} = \frac{M}{Xj}\frac{dxf}{dM}$$

Thus, all pairs of goods for which *dxf¹/dp j* = *dx^/dpi* have equal income elasticities. Suppose this is true of all commodities consumed. Then, using Eq. (10-54) that the weighted sum of income elasticities sums to unity, denoting the common value of the income elasticities as e_M,

-------h $K_n€_M$ = $€_M${$K_ℓ$ -\----\-$_{K_n}$) = \

Thus, $€\_M$ = ■ ■ ■ = $€_{nM}$ = $€_M$ = 1, since the shares $K_ℓ$ = *piXi/M* sum to unity.

What types of utility functions possess unitary income elasticities for all goods? Recall Fig. 10-6 for the specific case of U = *Xix_2*- The income elasticities were unity because the income consumption paths, the locus of all possible tangency points, was a straight line out of the origin. This is the property of homotheticity, of which the homogeneous function U = *Xix_2* is a particular case.

Mathematically, consider Eq. (10-63) once more. This equation is equivalent to

$$K + \frac{d(x f /_{X} M)}{dM} = 0 \qquad (10\text{-}64)$$

For using the quotient rule

Since xf^{1} is presumed positive, Eq. (10-63) results. Equation (10-64) says that the ratio of consumption of Xj to $x_{(}$- is the same at all income levels. This ratio, Xj/x_{t}, is simply the slope of the ray from the origin through (x,- $,Xj)$. To say that this ray has constant slope in the $x,x_{;}$-plane, for all pairs of goods, is to say that the utility function is homothetic. The reasoning can be reversed, using Eq. (10-64) as a definition of homotheticity to show that homothetic utility functions imply demand curves which have unitary income elasticities and exhibit the property that $dx^/dpj = dxf/dpi$. Any one of these three statements implies the other two; they are all equivalent.

10.7 SPECIAL TOPICS

Separable Utility Functions

In the early development of utility theory, utility was conceived as an *additive* function of utilities received from the consumption of separate goods, i.e.,

$$U(x_{lf}, \ldots, x_n) = U_x(xi) + U_2(x_2) + \cdots + U_n(x_n)$$

Such a function is called *additively* or *strongly separable.*^ [If a utility function were multiplicatively separable, i.e., $V = U\backslash(x\backslash) \cdot$ L^fe) $\cdots$ $U_n(x_n)$, the same implications for the demand system would occur, since taking the logarithm of V (a monotonic transformation) would produce an additively separable form and leave the demand functions unchanged.] In the additive case, the marginal utility derived from consuming some good x,- is a function of x, only, $t//(x_{;})$. The marginal utility would be unaffected by changes in consumption of some other good x_7, since $dU[\{Xi)/dXj = 0$. It is tempting to conclude from this that separability of the utility function implies independence of the demand for x, on the prices of other goods, that is, $dxf/dpj = 0$, $j \wedge i$. Such a conclusion is false. We leave it as an exercise to show that in the two-variable case, neither $dxf/dpj = 0$ nor $dx^/dpj = 0$ is implied. It can, however, be shown that $dx^/dpj = 0, j \wedge i$ implies that the utility function is Cobb-Douglas (or a monotonic transformation thereof). This is a more advanced exercise, involving solutions to partial differential equations. Furthermore, it follows immediately from Hicks' third law, Yl $Pi^{s}ij$ — 0 [Eq. (10-62)], that it can never be the case that $s_{i}j = dx^/dpj =$ 0 for all $j \wedge i$, since $s_{ti} < 0$.

Strong separability does, not surprisingly, place restrictions on observable behavior. For example, either all goods are noninferior and net substitutes for each other

^A slightly more general formulation would specify utility as a sum of functions of *groups* of commodities.

> 0, j =fc i), or all goods but one are inferior, and the noninferior good is a net substitute for the other goods, while the others are all net complements to each other. Additional restrictions on the demand functions, due mainly to Samuelson and Houthakker, are left as exercises at the end of Chap. 11.

Weak separability specifies utility as a function of categories of goods, e.g., food, clothing, etc., each of which in turn contain one or more individual goods. For example, in a four-good case, we might write $V(x\backslash, x_2, x_3,$ x$_4$) = $U(f(x\backslash, x_2), g(x_3,$ JC4)) = $U(z\backslash, z_2)$. Strong separability is a special case of this specification. Of course, as with strong separability, it is not necessarily the case that *dx'^/dpj* = 0, for goods / and j in different categories. However, consider the marginal rate of substitution between any two goods in the same category, say $X\backslash$ and x_2:

$$\frac{dV/dx,}{dV/dx_2} = \frac{UJ_1}{Uxh} = \frac{Mx_{1,}x_2)}{f_2(x_1,x_2)}$$

It is apparent that the marginal rate of substitution between any two goods in the same category is a function only of the goods in that category. If the total expenditure on *xi* and x_2 were known, this tangency condition plus the implied budget constraint for JCI and x_2 could be solved for the Marshallian demand functions, which would then be a function only of the prices of JCI and x_2 (i.e., the goods in that category), and the total expenditure on the goods in that category. In that case we could imagine a two-stage budgeting process, whereby the consumer first decides the expenditures on the categories food, clothing, shelter, etc., and then allocates his or her budgets within each of those groups of goods on the basis of only the prices of goods in that group. However, such a two-stage budgeting process is not implied by weak separability. The total expenditure on a given category, say food, in fact depends inexorably on the prices of all goods, not just the prices of the food items. Only with further very stringent conditions is such two-stage budgeting possible.^ It is *not* the case, for example, that if the "subutility" functions / and g above are homothetic or homogeneous, that two-stage budgeting is possible. For example, consider the utility function

$$U = f(x_U X_2) + g(x_3, X_4) = \text{(y +XJX2J + (y +} X_3 X_4\backslash$$

Clearly,/and g are both homogeneous of degree 2. We leave it to the reader to confirm that for $p\backslash > p_2$ this function achieves a positive interior constrained maximum subject to a linear budget constraint, and that, for example, *dx^/dpj* ^ 0, / = 1,2, j = 3, 4. Moreover, letting /* be the utility-maximizing value of /, *df*/dpj*

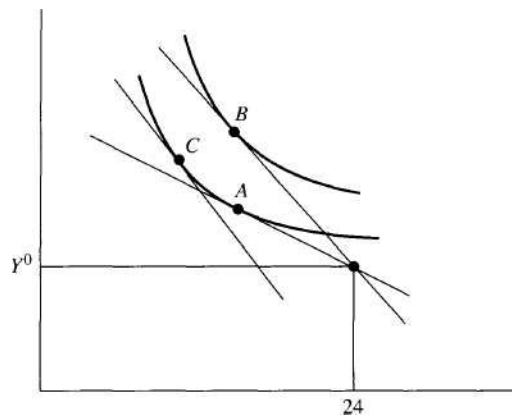
a more formal development of the implications of separability, see Robert Pollak, "Conditional Demand Functions and the Implications of Separability," *The Southern Economic Journal,* 37:423^-33, April 1971. The most complete analysis of separability is C. Blackorby, D. Primont, and R.R. Russell, *Duality, Separability and Functional Structure: Theory and Applications,* Elsevier, New

York, 1978.

$j = 3, 4$. Thus even with these restrictions on the utility function, the amount of "food" a consumer will purchase will depend on the individual prices of "clothing" items.

The Labor-Leisure Choice

The decision as to how many hours out of the day to devote to work is an important choice made by individuals. We model this choice by assuming that consumers desire leisure L as well as the consumption of goods. Rather than listing out the goods individually, we simplify the model by asserting that utility is a function of income *Y* and leisure: *U* = *U(Y, L)*. Income is produced by working (24 — L) hours at wage *w* per hour. In addition, nonwage income



T

Y = w(24- L)

This situation is pictured in Fig. 10-11. The individual is endowed with 24 hours of leisure and a nonwage income, assumed positive, of *Y*°. The budget line passes through the point (24, *Y*°) and has slope —*w*. The consumer maximizes utility at some point A, where the indifference curves are tangent to the budget line. An increase in *w* is represented by rotating the budget line clockwise through the endow-ment point, resulting in a new maximum position Bona higher indifference curve.

The Lagrangian for this model is

££ = *U* *(Y*, *L)* + *k(* *Y*° -*Y* + w(24 - *Q* *)* *he Labor-Leisure Choice.* A consumer is endowed with 24 hours of leisure and nonwage income Y°. At some wage rate w, the utility maximum occurs at point A. An increase in w produces a pure substitution effect from A to C and an income effect from C to B. Assuming leisure is a normal good, the income effect acts in the *opposite* direction of the substitution effect, since the consumer *sells* leisure.

The first-order conditions are

$$U_Y - l = 0 \qquad (10\text{-}65a)$$

$$U_L - kw = 0 \qquad (10\text{-}656)$$

and the constraint

$$Y^\circ - Y + w(24 - L) = 0 \qquad (10\text{-}65c)$$

From (10-65a) and (10-656), $U_L/U_Y = w$. This says that the marginal value of leisure, in terms of income forgone, is the wage rate. If a person can choose how many hours to work, then the decision not to work an additional hour entails giving up an hour's income, *w)*

Assuming the sufficient second-order conditions hold, the Marshallian demand functions

$$L = L^M(w, Y^\circ) \qquad (\backslash 0\text{-}66a)$$

$$Y = Y^M(w, Y^\circ) \qquad (10\text{-}666)$$

and an expression for the Lagrange multiplier

$$X. = k^M(w, Y^\circ) \qquad (10\text{-}66c)$$

are implied. We can interpret X^M as the marginal utility of nonwage income.

What is the effect on L and *Y* of an increase in the wage rate w? We already know that mathematically no refutable implication is available. An increase in the wage rate raises the opportunity cost of leisure; we should expect on this account the individual to substitute away from leisure, i.e., toward more work. However, this is just the pure substitution effect. As the wage rate increases, income also increases. If leisure is a normal good, we should expect the person to consume more leisure, i.e., to work less. Let us derive the associated Slutsky equation.

The Hicksian, or utility-held-constant demand, functions for this model are derived from the expenditure minimization problem,

minimize

$$Y^\circ = Y - w(24 - L)$$

subject to

$$U(Y, L) = U^\circ$$

In this model, Y° is no longer a parameter; it is the value of the objective function. The utility level is now a parameter. The Lagrangian for this model is

$$SE = Y - \mathrm{w}(24 - L) + X(U^\circ - U(Y, L))$$

^Even though in the short run hours per week may be fixed, in the long run individuals make choices in jobs and careers for which that and other job characteristics are presumably variable.

Assuming the first and second-order conditions hold, the Hicksian demand functions

$$Y = Y^u(w, U°) \tag{10-67a}$$

$$L = L^u(w, U°) \tag{10-67Z?}$$

are implied. The associated expenditure function is derived by substituting these solutions into the objective function:

$$Y^*(w, U°) = Y^u(w, U°) - w[24 - L^u(w, U^0)] \tag{10-68}$$

The Hicksian and Marshallian demand functions for leisure are related to each other through the fundamental identity

$$L^u(w,U°) = L^M(w,Y^*(w,U^0)) \tag{10-69}$$

Differentiating both sides with respect to *w,*

$dL^u \quad dL^M \quad (dL^M \backslash fdY$

$\overline{\partial Y^0} \quad \Big\backslash \overline{\partial w} \quad \Big)$

Applying the envelope theorem to Eq. (10-68), $dY^*/dw = -(24 - L^u)$. Thus, rearranging (10-70) slightly,

$$L^U)\left(\frac{\partial L^M}{\partial Y^0}\right) \tag{10-71}$$

an equation analogous to the traditional Slutsky equation (10-34e).

Notice in this case, however, the term multiplying the income effect is the amount of leisure "sold," 24 — L^u, not the amount of some good purchased. When the consumer comes to the market with money income, which does not enter the utility function directly, and uses it to purchase goods that do enter the utility function, the income effect for normal (noninferior) goods reinforces the substitution effect. In this case, since the consumer is *selling* leisure, not buying it, the income effect acts in the opposite direction of the substitution effect for normal goods. There is ample evidence that leisure is a normal good. (How does winning one of the various state lotteries now in existence affect the winner's time spent working?) Since (24 — L^u) is positive, the income effect is positive, while the pure substitution effect $3L^u/dw$ is necessarily negative. Because of this, the slope of the Marshallian (uncompensated) demand for leisure, dL^M/dw is less predictable than the slope of the Marshallian demands for ordinary goods and services.

A recurring public policy question concerns the effects of tax rates on work effort. The 1986 U.S. tax changes lowered the marginal rates on federal income taxation to 28 to 33 percent, from 50 percent. Some countries have tax rates in excess of 90 percent. It can be seen from the above analysis that lowering tax rates, which effectively raises the after-tax wage rate, does not have an implied effect on hours worked. Since the opportunity cost of leisure is now higher, the substitution effect produces less leisure. However, the individual is also wealthier; the income effect leads therefore to *more* leisure. The net effect is an empirical matter. [Of course, at a tax rate of 100 percent, no effort will be forthcoming (legally); the income effect of lowering taxes at that margin will certainly dominate, and induce greater effort.]

The preceding model of labor-leisure choice is a special case of a model that appears in the literature on general equilibrium. Assume that, instead of the consumer bringing an amount of money income *M* to the market to purchase goods and services, the consumer comes to the market with initial endowments of *n* + 1 goods *XQ* , x}¹, ..., *x*®. The market sets prices of *po, p*\, ..., p_n for these goods, and the consumer maximizes utility subject to the constraint that the value of the goods purchased equal the value of the initial endowment, i.e.,

maximize

$$U(x_o, xi, \ldots, x_n)$$

subject to

$$-\backslash- h\ p_n x^o_o = PQXQ -\backslash h\ p_n x_n$$

that is, subject to

$$n$$

$$i=0 \qquad (=0$$

The first-order conditions are obtained by setting the partials of the Lagrangian equal toO:

$$££ =$$

$$U(x_0, \ldots, x_n) +$$

$$2o = Uo - Xp_Q = 0$$

$$Xi = Ui - X_{P1} = 0$$

$$n = U_n - kp_n = 0$$

The first-order equations are solved for the demand functions:

$$x_t = x\hat{}(p_0, \ldots, p_n, x^0{}_0, \ldots, x^\circ{}_n) \qquad i = 0, \ldots, n \qquad (10\text{-}72)$$

It is apparent, using reasoning similar to that used before, that these demand functions are homogeneous of degree 0 in the *n* +1 prices *po,* ..., p_n. It is customary to choose one commodity and set its price equal to 1. This commodity, say x_0, is called the *numeraire;* it is the commodity in terms of which all prices are quoted. The situation being described is one of barter. If one of the goods is, say, gold, it may turn out that in addition to its amenity values (for which it enters the utility function, being useful in jewelry, dentistry, etc.), this commodity will also serve as a medium of exchange, being the commodity for which transactions costs are least. This model is incapable of predicting which commodity, if any, will be so chosen, but we can designate *XQ* as that commodity which is the numeraire and set *po* — 1. The remaining prices *P*\, ..., p_n then become *relative* prices.

Similar results are obtained in this model as in the standard utility maximization problem. The endowment of the numeraire JCQ serves the same function as *M,* the money income of the consumer. The compensated demand curves $x_t = x\hat{}(p\backslash, \ldots, p_n, U°)$ are derivable from

minimize

i=0 1=1

subject to

$$U(x_0, \ldots, x_n) = U°$$

Note that the implied compensated demand curves are *not* functions of the initial endowments, which enter the objective functions as constants and drop out upon differentiation. Once the utility level $U°$ is specified, the original endowment is irrelevant—the demands are determined by tangency and the utility level $U°$.

The indirect "endowment function" (formerly cost, or expenditure function) is given by

$$x_0 = x^*_0(p_u \ldots, p_n, x l \ldots, x°_n, U°) = Y.P^{rx} i \quad " Z>*,^? \qquad \text{t}^{10"73})$$

j=0 /=1

Thus, by the envelope theorem,

$$\hat{} = x ? - x° \qquad (10\text{-}74)$$

dpi

We can use these results to derive the implied Slutsky equation for this general equilibrium system. Proceeding as before, starting with the ordinary demand curves

$$x?(p_l, \ldots, p_n, x l \ldots, x°_n) \qquad i = 0, \ldots, n$$

define *XQ* to be the minimum *XQ* to keep $U(x_0, \ldots, x_n) = U°$. Then *XQ* is just the indirect function (10-73). Thus, by definition,

$$x_i(pi, \ldots, p_n, u) = x_t \ [pi, \ldots, p_n, x_0, x_t, \ldots$$

$, x_n)$ Differentiating with respect to some *pj,*

$$d_Pj \quad dpj \quad dxg$$

Using Eq. (10-74) and rearranging,

$$\frac{T}{dpj} \quad \frac{IT}{dpj} + (^x \sim {}^x f) \frac{iro}{dx\%} \qquad (^{10}\text{-}^{75})$$

Thus, the Slutsky equation has the same form as previously, with the important exception that the *income* effect $dx\hat{}/dx\hat{}$ is weighted by the *change* in the consumption of *Xj,* x° — $x\hat{}$. If the amount of *Xj* was unchanged after going to the market, that is,

$x^{™} = x^{®}$, there would be no income effect at all. Also, if, say, some price p_j goes up, then while formerly this acted as a decrease in real income, if the consumer is a net *seller* of X_j, this income effect is positive, i.e., it raises his or her real income.

Slutsky Versus Hicks Compensations

Although we have been referring to Eq. (10-34e) as the Slutsky equation, this version was in fact first introduced by J. R. Hicks in *Value and Capital* (1937), based on Pareto's discussion of the phenomenon. Slutsky compensated the consumer in a slightly different form: After a price change, instead of adjusting M to return the consumer to the original indifference curve, Slutsky gave the consumer enough income to purchase the original bundle of goods. This is in fact more than $M^*(p_1, \ldots, p_n, U^\circ)$, the minimum M to return the consumer to the original utility level. How does this affect the Slutsky equation? Surprisingly, not at all. In the limit (at the margin, that is), the Hicks and Slutsky compensations are identical.

Consider Fig. 10-12. The original tangency is at (x_1^0, x_2^0). Suppose p_1 is lowered. Then compensating a la Hicks leads to a new level of x_1^h, x_2^h, at a new tangency of the same indifference curve U^0, and a new budget line. Compensation according to Slutsky, however, places the new budget line through (x_1^0, x_2^0) at the new prices. Whether the prices are raised or lowered (the diagram is for p_1 lowered relative to p_2), the consumer can achieve a higher level of utility, say U^s (for Slutsky). If x_1 is a

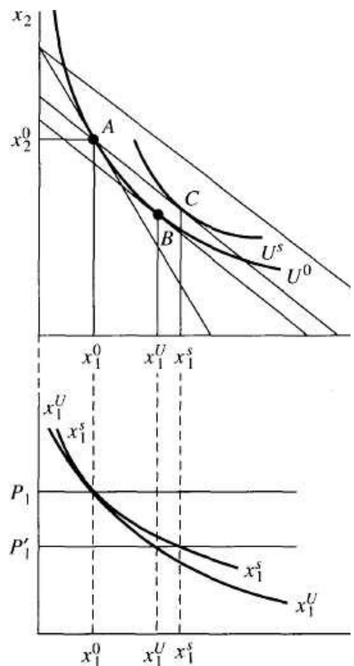


FIGURE 10-12 *The Hicks and Slutsky Compensations.* The consumer starts at point (xj', *x^*). When *p* is lowered, compensating according to Hicks leads to the new tangency *B,* where Xi — *x*. If a Slutsky compensation is made through the original point *A,* a new tangency on a higher indifference level, at point C, is attained. If xi is a normal good, the Slutsky demand *x* is greater than *x*. The same situation (xj > xf') occurs if the price change is in the other direction. The consumer can always achieve a higher indifference level by moving away, i.e., adjusting to the price change. Hence, *x^* = x[at x", but *x* > xj' everywhere else. Hence, x[and *x* are tangent at if, i.e., they have the same slope there, or *dx\/dpi* = *dx^/dpi.*

normal good, this will raise the consumption of *x*. Hence, the Slutsky demand curve jCp while equal to the Hicks curve at JC°, JC°, lies to the right of xf[7] for p_x not equal to the original price. Assuming that *x\ (p\, P2, U°)* and *x\ (p\, p₂, x®, x%)* are both differentiable, the diagram clearly indicates that the two have a point of tangency at (jcf, *x®)*. But this says that the slopes of the two demand curves are equal there, or

	dx?	dx[
	dpi	dpi
This result is perfectly		similar
	dxf	dx^s_t

If JC, is inferior rather than normal, a tangency still occurs, but with the Slutsky demand curve to the left of the Hicksian curve.

An algebraic proof follows trivially from the general equilibrium variant of the Slutsky Eq. (10-75),

dx "

A Slutsky compensation is equivalent to starting the consumer off at x, = *x®, i* = 1, ..., *n*. In this case, there is no income effect, and *dx^/dpj = dx-/dpj* by definition, and, hence, *dx^/dpj = dxf/dpj*. Note, however, in the figure, that the Slutsky demand curve is more convex than the Hicks curve. The second derivatives are not equal, and, in fact, for normal goods, *d²x-/dpf > d²xjʲ/dpf.* This was first brought out by A. Wald and J. Mosak, who resolved the conflict between the Hicks and Slutsky variants of compensation.^ What Mosak showed was that if *pj* changed by an amount *Apj*, the difference between the Hicksian demand and the Slutsky demand was of second-order smallness; i.e., it involved powers of *Ap* of order 2 and higher.

The importance of this result is that in general it will not matter much which type of compensation is used if the price change is not too large. Although the Hicks compensation is probably neater from the standpoint of the mathematical theory, this compensation will not be easy to observe. The Slutsky compensation, on the other hand, is calculable on the basis of simple arithmetic. Using the Wald-Mosak result, we can be assured that the compensations will not be very different, and that the easily observed Slutsky compensation is a good approximation to the "ideal" compensation a la Hicks.

This issue comes into play in the definition of index numbers. The Laspeyres index, used by the United States and other countries to define the consumer price

Mosak, "On the Interpretation of the Fundamental Equation of Value Theory," in O. Lange et al. (eds.), *Studies in Mathematical Economics and Econometrics,* University of Chicago Press, Chicago, 1942.

index (CPI), is essentially a Slutsky compensation. The price index indicates the amount of dollars needed in the current year to purchase *the original consumption bundle* in the base year. Substitution away from that original basket of goods is not considered (a feature that biases the CPI upward, i.e., it exaggerates the impact of price changes by not allowing the consumer to adjust to the change). However, for small relative price changes, the bias should not be much worse, since the Slutsky compensation is a good approximation to the Hicksian compensation, which a "true" price index would try to calculate.

The Division of Labor Is Limited by the Extent of the Market

We have thus far considered utility maximization subject to only a linear budget constraint. This specification of the constraint expresses a consumer's inability to affect prices by his or her consumption decisions. Suppose, however, that an individual engages in actual production of the goods consumed. In what ways would a consumer's choice of goods to consume be affected by the opportunity for exchange after production?

It is a familiar exercise in the theory of comparative advantage (demonstrated first, in virtually its current textbook form, by David Ricardo in his *Principles)*^ to show that if, say, Robinson Crusoe can either gather three coconuts or catch three fish (or any convex combination thereof) in a day, and Friday can either gather eight coconuts or catch four fish in a day, then mutual gains are possible if they specialize in their *comparative* advantages. In this case, Crusoe's marginal cost of producing fish is one coconut, whereas for Friday it is two coconuts; likewise, Friday's marginal cost of producing fish is half a coconut, whereas for Crusoe it is one coconut. Minimization of costs would therefore lead Crusoe to specialize in the production of fish, and Friday in coconuts. In that manner, they could share an output of eight coconuts and three fish, a consumption opportunity beyond their capabilities if specialization were not pursued. Since more is preferred to less, utility maximization would therefore tend to lead to such behavior.

Earlier, in an otherwise famous year, 1776, Adam Smith had outlined the benefits of specialization with a striking example of pin manufacturing:*

> A workman, not educated to this business (which the division of labor has rendered a distinct trade), nor acquainted with the use of the machinery employed in it (to the invention of which the same division of labor has probably given occasion), could scarce, perhaps, with his utmost industry, make one pin in a day, and could certainly not make twenty. But in the way in which this trade is now carried on, not only the whole work

^David Ricardo, *The Principles of Political Economy and Taxation, Chapter VII,* 1817. The accessible publication is *The Works and Correspondence of David Ricardo,* P. Straffa (ed.), Cambridge University Press, Cambridge, 1966.

*Adam Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations.* Reprinted by Modern Library, New York, 1776.

> is a peculiar trade, but it is divided into a number of branches, of which the greater part are likewise peculiar trades. One man draws out the wire, another straightens it, a third cuts it, a fourth points it, a fifth grinds it at the top for receiving the head ... [tjhose ten persons, therefore, could make among them upward of forty-eight thousand pins in a day.... This great increase in the quantity of work … is owing to three circumstances; first to the increase in dexterity in every particular workman; secondly, to the saving of time which is commonly lost in passing from one species of work to another; and, lastly, to the invention of a great number of machines which facilitate labor and enable one man to do the work of many.… It is naturally to be expected ... that some one or other of those who are employed in each particular branch of labor should soon find out easier and readier methods of performing their particular work.

Since the incentives to specialize are derived from exchange, the extent to which exchange is available sets limits on specialization:

> But man has almost constant occasion for the help of his brethren.… As it is by treaty, by barter, and by purchase that we obtain the greater part of those mutual good offices which we stand in need of, so it is this same trucking disposition which originally gives occasion to the division of labor … so the extent of this division must always be limited by the extent of that power or, in other words, by the extent of the market. When the market is small, no person can have any encouragement to dedicate himself entirely to one employment, for want of the power to exchange all that surplus part of the produce of his own labor, which is over and above his own consumption, for such parts of the produce of other men's labor as he has the occasion for.

Though it hardly does justice to Smith's and Ricardo's masterful analyses, we can depict this discussion mathematically by postulating a production frontier $g\{x\backslash,x_2) — k$, representing the amounts of two goods an individual could produce with his or her own labor, and possibly other inputs. If the individual is unable to engage in trade, he or she will produce that bundle of goods that maximizes utility subject to that production constraint, i.e.,

maximize

$$U(x_u x_2) = U$$

subject to

$$g(x_u x_2)=k \qquad (10\text{-}76)$$

The Lagrangian for this problem is

producing the first-order conditions

$$£, = U_l(x_l,x_2)-kg_l(x_l,x_2) = 0$$

$$\$2 = U_2(x_u x_2) - Xg_2(x_u x_2) = 0$$

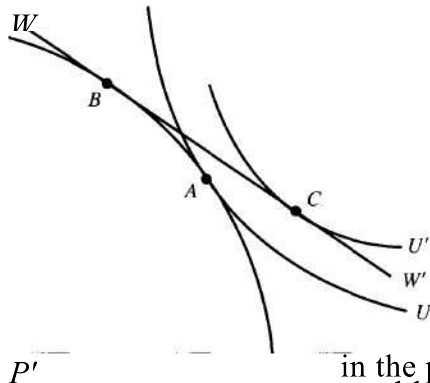


FIGURE 10-13
The Division of Labor Is Limited by the Extent of the Market. If trade is highly restricted, a consumer endowed with some production frontier *PP′* must consume largely what he or she produces. This utility maximum occurs at *A*. With efficient markets, the consumer can specialize in the production mix generating the highest wealth, *B,* and trade at market prices to achieve the higher utility level at *C*.

The first two conditions imply $U1/U2 = 81/82'$, this plus the last condition (the constraint) indicates that the indifference curve must be tangent to the production frontier. This solution is shown as point A on Fig. 10-13.

In the preceding situation, the consumer must consume the identical bundle of goods he or she produces. This situation might have been approximated on the North American frontier in the nineteenth century, or perhaps in remote villages today. (The existence of itinerant traders in those locales is testimony to the advantages of specialization.) Suppose, however, there is a market for these goods, so that once produced, the individual can trade these goods for some other, more preferred bundle. In this case, the consumer will produce that bundle of goods *with the hishest market value,* which will not in general be the mix of goods desired in consumption, and will then trade these goods for the bundle that maximizes utility. With "extensive" markets, the consumer achieves point *C* in Fig. 10-13, on a utility level higher than when markets are so limited that no trade can take place. (Point *C* cannot be less preferred; the individual can always choose not to trade and remain on the production frontier.) By separating the problem of consumption from production, the consumer is able to exploit his or her comparative advantage, without having to worry whether he or she would like to consume only that bundle of goods produced.^

This model is formulated mathematically as follows. There are in fact 4 (i.e., *2ri)* decision variables: the bundle produced, $(y\backslash, y_2)$, and the bundle consumed, $(x\backslash, x_2)$. The individual's problem is to

^This same idea was exploited by Irving Fisher, who explained that the existence of capital markets, in which individuals borrow and lend, allows individuals to first maximize wealth (the present value of all future income) and then rearrange consumption so as to maximize utility over time. This result is known as the *Fisher separation theorem.* See Irving Fisher, *The Theory of Interest,* The Macmillan Company, New York, 1930. Reprinted by Augustus Kelley, New York, 1970.

maximize

$$U(x_1, x_2) = U$$

subject to

where p_1 and p_2 are the market prices of the two goods. It is easier to analyze the problem by introducing a fifth variable *W,* the total value of the individual's output (wealth). We can then state the model as

maximize

$$U(x_1, x_2) = U$$

subject to

$$p_1 x_1 + p_2 x_2 = W$$

$$w \qquad (10\text{-}77)$$

It is clear from the last two constraints that for any y_1 and y_2 satisfying the production constraint, wealth W is determined. The problem then reduces to maximizing utility subject to the ordinary budget constraint $p_1 x_1 + p_2 x_2 = W$, where W is "conditional" on y_1 and y_2. However, assuming nonsatiation, increases in W will necessarily increase utility. It thus follows that in order to maximize utility, the consumer must first choose the output mix (y_1, y_2) that maximizes wealth; this occurs at point *B* on Fig. 10-13. The consumer then maximizes utility subject to the budget line *WW* tangent to the production frontier at *B,* achieving consumption at point *C.* We leave it as an exercise to set this model up formally and derive the first- and second-order conditions. We note in passing, that as in all such utility maximization models, the prices appear in the constraints, making refutable comparative statics implications dependent upon further assumptions in the model.

Modern societies are characterized by a high degree of specialization. No one worries that they will have to consume what they produce; individual production is directed toward maximization of that individual's value of output. Adam Smith went on to say that

> As every individual direct[s] [his] industry that its produce may be of the greatest value, every individual necessarily labors to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest nor knows how much he is promoting it,..., and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention.

The maximization of *society's* total value of output depends upon further assumptions about property rights and one individual's effects on others.^ We shall return to this issue in Chap. 19 on welfare economics.

PROBLEMS

1.283 What is the difference, in a many-commodity model, between diminishing marginal
rate of substitution between any pair of commodities, and quasi-concavity of the utility
function? Which is the more restrictive concept?

1.284 Why does the proposition "More is preferred to less" imply downward-sloping indiffer
ence curves?

1.285 What dependence, if any, does the homogeneity of degree 0 of the money-income-held-
constant demand curves have on the homogeneity of the consumer's utility function?

1.286 Show that the marginal utility of money income, A^M, is homogeneous of degree — 1.

1.287 Consider the utility functions of the form $U = x''_1 x''_2$. Show that the implied demand
curves are

M ori *M*

x_2 a, + a_2
p_2

Find X^M and $U^*(x^\wedge, xf)$, and verify that $X^M = dU^*/8M$.

1.288 Prove the elasticity formulas (10-53), (10-54), (10-59), (10-60), and (10-61) for the
^-commodity case.

1.289 Is it possible to define complements in consumer theory by saying that the marginal
utility of *Xi* increases when more *Xj* is consumed? *(Hint:* What mathematical term is
being defined, and is it invariant to a monotonic transformation?)

1.290 Substitutes can be defined by the sign of the *gross* (including income effects) cross-
effects of prices on quantities, or the *net* effect (i.e., *not* including income effects). That
is, one may define "x,- is a substitute for x/' if:

dx^M

<) 0

(with the reverse sign on the inequality for "complements").

1.291 Which term is likely to be the more observable (empirically)?

1.292 Are these terms invariant to a monotonic transformation of the utility function?

1.293 According to the preceding definitions, if x, is a substitute for *Xj,* is *Xj* necessarily
a substitute for x, ?

9. Considering Hicks' "third law" and the preceding definition *(ii)* of substitutes and complements, show that there is a tendency toward substitution of commodities in the sense

t Smith's famous passages, quoted above, in fact appear in a section of the book entitled *Economic Liberalism,* **The Ideal** (emphasis added); Smith went on to consider *The Reality,* dwelling on such problems as monopoly, tariffs, and the like.

that

10. Describe the effects of a monotonic transformation of the utility function on:
1.294 The rate of change of the marginal utility of one good with respect to a change in another good.
1.295 The law of diminishing marginal utility.
1.296 The slopes of demand curves.
1.297 The values of income elasticities.
1.298 The homogeneity of the demand functions.
if) The size and sign of the marginal utility of income.

11. For the utility maximization model, show that

$$^{M} \quad {}_{M}dX^{M}$$

where X^M is the marginal utility of money income.

12. Suppose a consumer will have income x° this year and x° next year. He or she consumes $X\backslash$ this year and x_2 next year, being able to borrow and lend at interest rate *r.* Assume the consumer maximizes the utility of consumption over these two years.
1.299 Derive the comparative statics for this problem. Will an increase in this year's income necessarily lead to an increase in consumption this year?
1.300 Prove that the consumer will be better off (worse off) if the interest rate rises if he or she was a net saver (dissaver) this year.

13. Consider the utility maximization problem, max $U(x_x,x_2)$ subject to $p_x x_x + p_2 x_2 = 1$, where prices have been "normalized" by setting $M = 1$. Let $U\%p_x$, $p_2)$ be the indirect utility function, and X be the Lagrange multiplier.
1.301 Show that $X^M = \{dU/dx_x)x\backslash + (dU/dx_2)x^*$.
1.302 *ShowthatdU*/d*$_{Pl}$$=-k^M x;,\quad dU^*/dp_2 = -X^M x^*$.
1.303 Show that $X^M = -[(dU^*/dp_x)p_x + (dU^*/dp_2)p_2]$.
1.304 Prove that if $U\{x_{\backslash}, x_2)$ is homogeneous of degree r in (x_x, x_2), then $U^*(p_x, p_2)$ is homogeneous of degree — r in (p_x, p_2).

14. Consider the class of utility functions that are "additively separable," i.e.,

1.305 Find the first- and second-order conditions for utility maximization for these utility functions. Show that diminishing marginal utility in at least one good is implied.
1.306 Show that if there is diminishing marginal utility in *each* good, then both goods are "normal," i.e., not inferior.
1.307 Show that this specification does not imply *dxf /dpj* — 0, / =/= *j.*
1.308 Show, however, that if *dx^/dpj ~ dxf/dpt* = 0, then $U\{x_x,x_2) = a_x$ log*! +

a_2 $\log x_2$.

1.309 Assume now that x_x is a Giffen good, i.e., $dxf/dp_x > 0$. Prove that $dX^M/dM > 0$.

1.310 Consider the two-good utility maximization model and assume JCI is a Giffen good, i.e.,
$dxf^1/dp_x > 0$. Prove that dx_2^M/dp_x and dx^/dpx must be of opposite signs.

1.311 Derive an expression analogous to Eq. (10-42) for the difference between dx^/dpj and
$dx'J/dpj$, i ^ j. Show that if JC, and Xj are either both net substitutes or both net com
plements of x_n, the Hicksian cross-elasticities of demand are numerically smaller in the
long run than in the short run.

1.312 Let $U = f(x_1, x_2) + g(x_3, x_4) = (x_1^2/2 + x_1x_2) + (x_3^2/2 + x_3x_4)$. Show that if $p_1 > p_2$, this utility function achieves an interior constrained maximum subject to a linear budget constraint and that $\partial x_i^* / \partial p_j \neq 0$, $i = 1, 2$, $j = 3, 4$. Show also that if f^* is the utility-maximizing value of f, $\partial f^*/\partial p_j \neq 0$, $j = 3, 4$. That is, strongly separable utility functions do not imply the possibility of "two-stage" budgeting.

1.313 The Hicksian "real income," or utility-held-constant demand curves are written

Suppose now, when p_1 changes, U^0 is also adjusted to that maximum amount achievable so as to keep money income M constant, i.e.,

$$U^0 = U^*(p_1, p_2, M)$$

is that functional relationship which keeps M constant by adjusting utility, when p_1 or p_2 changes. Thus, the money-income-held-constant demand curves can be written

$$x_1^M(p_1, p_2, M) = x_1^U(p_1, p_2, U^*(p_1, p_2, M))$$

1.314 Show that the income effect on x_1 is proportional to the "utility effect" on x_1, i.e., the change in x_1^U when U is changed, the factor of proportionality being the marginal utility of money income.

1.315 Show that

$$\frac{\partial x_1^*}{\partial p_2} = \frac{\partial x_1^U}{\partial p_2} - x_2 \frac{\partial x_1^*}{\partial M}$$

(This is an alternative derivation of the Slutsky equation to that given in the text.)

19. In a leading economics text, the following form of the "law of diminishing marginal rate of substitution" is given: The more of one good a consumer has, holding the *quantities* of all other goods constant, the smaller the marginal evaluation of that good becomes in terms of all other goods, i.e., the indifference curves become less steep. (Sketch this condition graphically.)

1.316 This is a postulate about the slopes of indifference curves, i.e., about the term $(-U_1/U_2)$. What is the sign, according to this postulate, of $\partial(-U_1/U_2)/\partial x_1$, $\partial(-U_1/U_2)/\partial x_2$?

1.317 Show that this postulate implies that the indifference curves are convex to the origin.

1.318 Suppose this postulate is violated for good 2. Show that x_1 is an inferior good. Show that if the postulate is violated for good 1 also, then the indifference curves are *concave* to the origin.

1.319 Show that the preceding postulate rules out inferior goods (for the two-good case).

1.320 Show that in part (c), in which the indifference curves are still assumed to be convex to the origin, the marginal evaluation of x_2 *increases* the more it is consumed relative to JCI . Explain intuitively.

1.321 Show that in a three-good world, the preceding postulate is insufficiently strong to imply indifference curves which are convex to the origin.

20. An historically important class of utility functions includes those functions which exhibit vertically parallel indifference curves; i.e., with $X\backslash$ on the horizontal axis and x_2 on the vertical axis, the slopes of *all* indifference curves are the same at any given level of $X\backslash$.

For these utility functions:

1.322 Prove graphically and algebraically that the income effect on x_x equals 0.

1.323 Show that the "ordinary" demand curve for $x\backslash$, $x''(p\backslash, p_2, M)$ and the compensated demand curve for x\, x"(p!, p2,U) are identical by showing that at any point, the slopes of x™ and x\ are the same, and that the shifts in x™ and x\ are the same with respect to a change in p_2, the price of the second good.

1.324 Consider the utility function $U = x_2 + \log x_1$. Show that this function has vertically parallel indifference curves.

1.325 For $U = x_2 + \log X_1$, show also that the price consumption paths with respect to changes in p_1 are horizontal, i.e., that the amount of x_2 consumed is independent of the price of good 1.

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CHAPTER

11

SPECIAL TOPICS IN CONSUMER THEORY

11.1 REVEALED PREFERENCE AND EXCHANGE

Any economic system solves, in some way, the problems of production and allocation of goods and resources. Starting with various factor endowments, resources are somehow organized and combined, and a certain set of finished goods emerges. All along the way, decisions are made concerning two fundamental problems:

1.326 What final set of goods shall be produced?

1.327 How shall factors of production be combined to produce those goods?

These problems are not independent. The choice of factors and their least-cost combinations vary depending on the level of demand for the goods. A person building a car in the backyard will use inputs different from those used by General Motors. These matters aside, how does it come to pass that producers of goods have any idea at all what to produce? What is it that guides these decision makers in selecting a certain, usually small, set of goods to produce, out of the vast array of conceivable alternative goods and services?

The problem is by no means trivial. Imagine yourself as the chief economic planner of a society in which it has been mandated by the ruling political party that all goods are to be handed out free of charge. To make life easy for you, the government has provided you with a complete set of costs of producing all existing and potential goods. How much of each should you produce, assuming you had the best interests of the consumers in mind? To achieve your goal, you would need to know how much consumers valued the alternative goods. Without this information, a planner might decide to produce meat for a nation of vegetarians, or, on a less grandiose scale, too much wheat for people who would rather consume more rice or corn, or trains and buses for people who would rather drive their own cars. What mix of these goods and services should be produced?

The solution to this allocation problem in any economy depends upon the production of information concerning the valuation of goods by consumers and the ability of individuals to utilize that information. The latter problem has to do with the system of property rights developed in the nation in question. We shall not inquire into these matters here. Suffice it to say that a system that allows private ownership and free contracting between individuals will in all likelihood produce a different set of goods than a society where these rights are attenuated.

The former problem, how information is produced regarding consumers' valuations of goods, is the topic at hand here. Recall the definition of *value.* The value of goods (at the margin) is the amount of other goods consumers are willing to give up in order to consume an additional increment of the good in question. In most private exchanges, information about these marginal values is produced automatically by the willingness or reluctance of the participants to engage in trade. When a trade takes place, the value of the goods traded is revealed to the traders and other observers. Since, under the usual behavioral postulates of Chap. 10, individuals will purchase goods until the marginal value of those goods falls to the value of the next best alternative, *prices,* in a voluntary exchange economy, provide the information of consumers' *marginal* (though not total) value of each traded good. Any producer whose marginal costs of production are less than that price can benefit by producing more of that good and in so doing will be directing resources from low-valued to higher-valued uses. In this way the gains from trade will be further exhausted.

The value of goods will also be revealed, though not as precisely, when other means of allocation are used. When goods are price-controlled, e.g., gasoline in the winter of 1973-1974, waiting lines and other nonprice discrimination appeared. These phenomena provided evidence that the good was valued higher, at the margin, than the official controlled price. But exactly how much higher (a subject of intense debate at the time) was not known. The information on the precise marginal evaluation of gasoline during that time was never allowed to be produced. And, in the extreme case, where goods are handed out "free," very little information is produced concerning consumers' valuations of those goods.

In the usual case of so-called private goods in which congestion is so extreme that only one person can consume the item, preferences are revealed automatically through the act of exchange. Intensity of preference will be revealed through the level of purchase of goods and services. An important class of goods for which this does not easily occur is made up of the so-called public goods, in which congestion is absent, so that adding an additional consumer to the consumption of that service in no way diminishes the level of service provided the other consumers. The services national defense, lighthouses, or uncrowded freeways are classic examples of such goods. In some cases, the ability to *exclude* nonpayers from the benefits of these services would be difficult to arrange. (The right of exclusion, a fundamental part of property rights, is not peculiar to public goods, nor are all public goods incapable of having rights of exclusion cheaply enforced.) In the case of nonexclusive public goods, particularly, information concerning consumers' valuations of the good will be difficult to observe. Consumers will often have an incentive to understate the intensity of their preferences, and to "free-ride." Imagine how the production of such goods might be attempted: If the costs of production are to be assessed on the basis of the value of the service to the consumers, the consumers will tend to indicate how little they value the service (if at all), each hoping that enough others will indicate a high enough level of willingness to pay to make the project viable. The end result may be that the service is not produced at all, or that "too little" is produced. In these situations, coercive schemes such as government provision of the good through mandatory taxation or the formation of private clubs with assessment of dues are often resorted to as a means of lowering the contracting costs between consumers eager to exhaust the gains from exchange. But the preferences of individuals for these types of services will not be completely revealed, since individuals in the group will still, in all likelihood, have different marginal evaluations of the final level of public good produced.

Is it possible, given the nature of exchange explored above, to replace the utility maximization hypothesis with one based entirely on observable quantities? That is, can a behavioral postulate yielding refutable hypotheses be formulated in terms of exchanges? This question was initiated by Samuelson, Houthakker, and others in the 1930s and 1940s, resulting in what is known as the *theory of revealed preference.* It is intimately tied in with another classical question of the theory of the consumer, viz., whether the Slutsky relations of Chap. 10 constitute the entire range of implications of the utility maximization hypothesis. That is, is it possible, starting with a set of demand relations which obey symmetry and negative semidefiniteness of the pure substitution terms, to infer that there exists some utility function (together with all its monotonic transformations) from which those demand functions are derivable? This issue is known as the problem of *integrability.* A complete discussion of these issues is beyond the scope of this book, the integrability issue in particular being dependent upon subtle mathematical details. We shall, however, indicate the general nature of the problems.

Let us suppose that a consumer possesses a well-defined set of demand relations,

$$x_i = x_i^*(p_1, \ldots, p_n, M) \qquad i = 1, \ldots, n \qquad (11\text{-}1)$$

At this point we need not even assume that these relations are single-valued; i.e., we allow, for the moment, that confronted with a set of prices $p_1, \ldots, p_n$ and a given money income M, the consumer might be willing to choose from more than one consumption bundle. Strictly speaking, then, the relations (11-1) are not functions, since single-valuedness of the dependent variable is part of the definition of a function; instead system (11-1) represents what are sometimes called *correspondences* or just simply *relations.* What is being insisted on here is that a consumer *will choose* some consumption bundle $x^0 = (x_1^0, \ldots, x_n^0)$ when confronted with a price-income vector

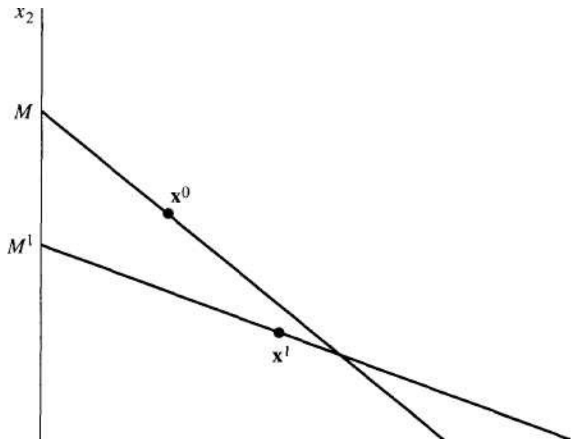

FIGURE 11-1
The Weak Axiom of Revealed Preference. At prices p°, the consumption bundle x° is chosen, implying a budget line *MM.* The consumption bundle x^1, since it lies interior to *MM,* could have been chosen but wasn't. Hence, x° is said to be revealed preferred to x^1. This does not mean that x^1 will never be chosen. What it does mean is that when x^1 is chosen, at some price vector p^1, implying a budget line $M^x M^l$, x° will be more expensive than x^1 at those new prices. In other words, if p°x° > p°x^1, when x^1 is chosen at p^1, necessarily p^1x' < p^°. This is illustrated in this diagram, since x° lies outside the budget line

(p°, *M°)* = (/?p ..., /?°, M°). Let us also assert that the consumer, in so choosing, will spend his or her entire budget; i.e., the choice x° will satisfy the budget relation

It will be much easier going if some elementary matrix and vector notation is used in the following discussion. Recall the definitions of vectors and matrix multiplication in Chap. 5. The scalar (or inner) product of two vectors x = (*!,..., jc_n)and y = (ji, ..., y_n) is defined as xy = *Y^=i* ^{x}*iyi-* With this notation the budget equation *Yl* $Pi^x i$ *= M is* simply written px = *M.* The set of differentials dx_l, ..., dx_n is written simply dx. The expression p dx means *YM=I Pidxi,* etc. The entire set of demand relations (11-1) is written simply as x = x^M(p, *M).*

In Fig. 11-1, a consumer is faced with a price-income vector (p°, M°) and chooses the consumption bundle x°, where p°x° = M°; that is, the budget equation is satisfied. In so doing, we shall say that the consumer reveals a preference for bundle x° over some other bundle, say x^1, which was not chosen. We say x° is *revealed preferred* to x^1. We cannot yet speak of the consumer being indifferent between x° and x^1, since indifference is a utility-related concept, which is not yet defined. The phrase "x° revealed preferred to x^1" simply means that where the consumer was confronted with two *affordable* consumption bundles x° and x^1, x° was chosen and x^1 not, although x^1 was no more expensive than x°. It is not likely that we would be able to formulate a hypothesis about choices if the chosen bundle were less expensive than the nonchosen one; people choose Chevrolets instead of Cadillacs not necessarily because they prefer Chevrolets to Cadillacs but because the latter cost more. The statement that x^1 is no more expensive than x° is written **pV^pV.**

Having so *defined* revealed preference, let us now assert something about behavior in terms of it.

> **The weak axiom of revealed preference.** Assume that x° is revealed preferred to x^1, that is, at some price vector p°, x° is chosen and $p^\circ x^\circ > p^\circ x'$, so that x^1 could have been chosen but was not. *Then* x^1 *will never be revealed preferred to* x°.

The weak axiom (we shall presently explain the reason for the adjective *weak)* does not say that x^1 will never be chosen under any circumstances. The bundle x^1 may very well be chosen at some price vector p^1. What the weak axiom indicates is that if x^1 is chosen at some price p^1, then x° will be more expensive than x^1 at prices p^1. Consider Fig. 11-1 again. At prices p°, the consumer chooses x° even though x^1 could have been chosen, since x^1 lies below the implied budget line *MM* defined as $p^\circ x^\circ = M^\circ$. At some other set of prices p^1, x^1 might be the chosen bundle, forming a new budget equation $p'x^1 = M^1$. But note that at prices p^1, x° is more expensive than x^1; that is, $p'x^0 > p^{\wedge 1}$. Hence, x^1 is *not* revealed preferred to x° merely because it was chosen, for the same reason that one would not want to infer that Chevrolets are preferred to Cadillacs. The bundle x^1 is simply cheaper than x° at prices p^1; nothing can be inferred about the desirability of x° and x^1 from $p^{\wedge 0} > p'x^1$ alone.

Algebraically, then, the weak axiom of revealed preference says

if

$$p^\circ x^\circ > pV$$

then

$$p'x^0 > p'x^1 \qquad (11\text{-}2)$$

where the consumption bundle chosen is the one whose superscript is the same as that on the price vector. Figure 11-2 shows a price consumption situation that would contradict the weak axiom. There, x^1 is chosen at p^1 when x° could have been chosen; we have both $p^\circ x^\circ > p^\circ x^1$ and $p^1 x' > p'x^0$. The weak axiom therefore does imply some restrictions in the range of observable behavior. What are they?

> **Proposition 1.** The demand relations (11 -1) are homogeneous of degree 0 in all prices and money income; that is, *xf{tp\,... ,tpₙ, tM) = xf{p, …,/?„, M).*

> ***Proof.*** Let the consumption bundle x° = *{x®,..., x°ₙ)* be chosen by the consumer when prices and income are (p°, M°) = (p°,..., *p°ₙ*, M°) and let x^1 = *{x\,... ,x\)* be chosen at prices and income (p^1, M^1) = *(p\, ..., pˣₙ*, M^1). By hypothesis, $p^1 = fp^\circ$, $M^1 = tM^\circ$. Assume now that x^1 ^x°, that is, that two distinct points are chosen in these situations. We shall show that a contradiction arises. Since $tM^\circ = M^1$ and the

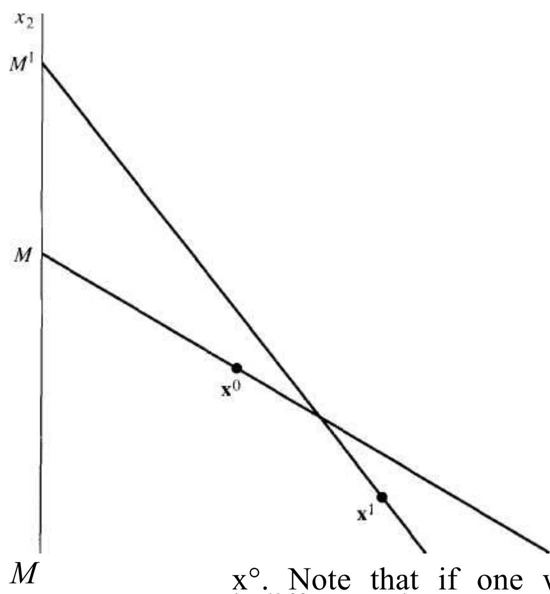

FIGURE 11-2
Violation of the Weak Axiom of Revealed Preference. In the initial situation at prices p°, x° is chosen even though x^1 could have been chosen. Hence, x° is revealed preferred to x^1. When x^1 is chosen at prices p^1, implying a budget line *M*^{x}*M*^{x}, x° could still have been chosen, and thus x^1 would be revealed preferred to x°. This contradicts the weak axiom, which says that if x° is revealed preferred to x^1, then x^1 will never be preferred to x°. Note that if one were to try to draw an indifference locus tangent to *MM* and *M*^{l} *M*^{l} at x° and x^1, respectively, the locus would be concave to the origin. This behavior is ruled out by the weak axiom.

consumer spends the entire budget, However, p^1

?p°x° = pV

= *tp*°. Hence,

= *tp°x*^{l}

or

p°x° = pV

(11-3)

Equation (11-3) says that x° is revealed preferred to x^1, since x^1 could have been chosen and was not. Therefore, when x^1 *is* chosen, x° must be more expensive, i.e.,

pV < px°

(11-4)

by the weak axiom of revealed preference. However, p^1 = *tp*°. Substituting this into (11-4) yields

fpV < *tp°x°*

or

pV

p°x° (11-5)

However, (11-5) and (11-3) are contradictory; hence, the assumption that x^1 ^ x° must be false, and the weak axiom of revealed preference implies that the demand relations (11-1) are homogeneous of degree 0.

Proposition 2. The weak axiom implies that the demand relations (11-1) are single-valued; i.e., for any price income vector (p, *M)* the consumer chooses a single point of consumption.

Proof. This proposition is actually a special case of proposition 1; simply let $t = 1$ in the above proof. Proposition 1 includes the case where t — 1 (since it holds for all $t > 0$), so when $p^1 = p°$, $M^1 = M°$, one and only one consumption bundle is chosen. If two points were chosen, each would be revealed preferred to the other, an obvious contradiction.

Thus, two properties of demand functions implied by utility analysis, single-valuedness and homogeneity of degree 0, are also implied by the weak axiom of revealed preference. Most important, however, the axiom also implies the negativity of the Hicks-Slutsky-type substitution terms *dx^/dpi + Xjdx^/dM.* Let us define

dxf¹

We are not yet entitled to call these terms *pure substitution effects,* or *compensated changes,* because we have not yet shown (the weak axiom is insufficient for that purpose) that a utility function exists for this consumer. With utility as yet undefined, the concept of indifference or utility held constant has no meaning. However, we can show the following.

Proposition 3. The matrix of 5,/s is negative semidefinite, under the assumption of the weak axiom of revealed preference.

Proof. Let us assume also that the demand functions (11-1), x = x^M(p, M), are differentiable. Let $p^1 = p° + dp$, $x^1 = x° + dx$, where the differentials indicate movements along the tangent planes. Then from the weak axiom,

p°x° = p°x' implies p'x¹ <

p'x° With p¹ and x¹ defined as stated this

becomes

$$p°x° = p°(x° + dx) \qquad (11\text{-}7)$$

which implies

$$(p° + dp)\,(x° + dx) < (p° + dp)x° \qquad (11\text{-}8)$$

Equation (11-7) simplifies to p° dx = 0, and (11-8) reduces to

(p° + dp)x° + (p° + dp)dx < (p° +

dp)x° or

(p° + dp)dx < 0 Hence, for

differentiable demand functions, the weak axiom can be

stated as

$$dp\,dx<0 \qquad (11\text{-}9)$$

whenever

$$pdx = 0 \qquad (11\text{-}10)$$

That is, *^dpjdxi* < 0 whenever ^ *ptdx*$_t$ = 0, where the equality holds in (11-9) only when all prices change in the same proportion; otherwise dp dx < 0. Now relate Eqs. (11-9) and (11-10) to Hicks-Slutsky terms, the s,/s defined in (11-6). For each demand function JC, = *xf{p\,..., p_n, M),*

——*dpj*\-\- -ⁱ—*dM* (11-11)

However, when XT=i *Pjdxj* = 0,

n n n

dM — y

Xidp: + y pjdxi = y x:dpj (11-12)

Substituting (11-12) into (11-11) gives

7=1 **7=1**

or

:ᴹ *dxj*

Applying Eq. (11-9) to (11-13) gives

n n

" ^ ^ " <0 (H-14)

,=i i=i *j*=

where the equality holds when all prices change in the same proportion. Equation (11-14) says, by definition, that the matrix of Slutsky terms is negative semidefinite. As such, with the methods employed in deriving the conditions for maximization, ■Si/ *S* 0 (usually s_{ti} < 0); that is, the pure substitution own effects are negative.

What is the meaning of (11-9) and (11-10)? The condition ^ *Pidxi* = 0 is precisely what is implied when, starting from the utility framework, utility is held constant. When *U(x\, ... ,x_n) = U°,* a constant,

dU =

using the first-order equations for utility maximization subject to a budget constraint. Hence, in that case, assuming nonsatiation (A. ^0), *J2 Pi*ᵈˣ*i* — 0- Thus, the JJC,'S would be interpretable as pure substitution movements. If only one price *pj* is changed, that is, *dp_t = 0, i ^ j,* then Eq. (11-9) says that *dpjdxj* < 0, or that the own substitution effect is negative, as implied by utility analysis. But again, these are mere analogies at this point, since the existence of a utility function has not yet been shown.

The revealed preference approach to consumer theory was originally offered as an operational alternative to the sometimes vague and mysterious utility analysis. We see that, in fact, the weak axiom of revealed preference implies almost as much as utility analysis itself and hence is practically equivalent to it. The only result not implied by the weak axiom is the symmetry of the Slutsky terms; that is, s_{ij} —S_{ji}. Without this, a utility function cannot exist, since $s_{ij} = S_{ji}$ is a necessary consequence of utility theory. The question thus remains: Can the weak axiom of revealed preference be strengthened so that it implies symmetry and hence the possible equivalence of revealed preference theory and utility analysis? The answer was provided by Houthakker in 1950, with results discussed in the next section.

11.2 THE STRONG AXIOM OF REVEALED PREFERENCE AND INTEGRABILITY

The inability to deduce the symmetry of Slutsky-type substitution terms from the weak axiom of revealed preference seems at first to be mainly an annoying detail. However, if a utility function does not exist for a given consumer, we should expect occasionally to observe behavior that most of us would regard as strange and not in conformity with the usual observations on consumer behavior. Let us see what type of behavior is *not* ruled out by the weak axiom.

Consider three consumption bundles, x°, x^1, and x^2, which the consumer purchases at price vectors p°, p^1, and p^2, respectively. Each bundle represents consumption levels of three separate goods. Let $x^j = (x\{, x\{, x^j_3), p^j = (p\{, p\{, p^j_3)$, 7 = 0, 1, 2.

$$x^\circ = (2, 2, 2) \qquad p^\circ = (2, 2, 2)$$
$$x^1 = (3, 1, 2) \qquad p^1 = (1, 3, 2)$$
$$x^2 = (4, 1, 1i) \qquad p^2 = (2, 1i, 5)$$

In the initial situation, when each good is priced at \$2, 2 units each are bought, for a total expenditure of \$12. When $p\backslash$ is lowered from \$2 to \$1 and p_2 raised to \$3 from \$2, to produce $p^1 = (1,3,2)$, this consumer evidently increases consumption of the first good JCI and lowers that of x_2. This is in accordance with substitution toward the lower-priced good. Similarly, when p_3 is raised from \$2 to \$5, among other changes, the consumer decreases consumption of *X3,* from 2 units to 1| units. Although p_x increases absolutely from \$1 to \$2, relative to the change in p_3, $x\backslash$ becomes relatively cheaper and consumption of JCI increases. Hence, these consumption bundles and prices seem plausible enough.

They are even more plausible in that the weak axiom of revealed preference is satisfied for these points. In particular, we note

$$pV = pV = 12$$

and thus x° is revealed preferred to x^1. When x^1 is in fact purchased, x° is more expensive than x^1:

$$px^1 = 10 < p'x^0 = 12$$

What is more, x^1 is revealed preferred to x^2:

$$pV = p^J x^2 = 10 \text{ and}$$

when x^2 is purchased, x^1 is more expensive:

$$p^2x^2 = 17 < pV = 17\pm$$

Now, however, something utterly revolting occurs: x^2 is revealed preferred to x°.

$$p^2x^2 = p^2x° =$$

17 and, when x° is purchased, x^2 is more expensive:

$$p°x^0 = 12 < p°x^2 = 13$$

We see from this example that the weak axiom of revealed preference allows intransitivity of preferences to occur. If revealed preference is to be associated with the usual notions of consumers' preferences, we cannot allow the situation where x° is preferred to x^1 and x^1 is preferred to x^2 and then have x^2 preferred to the original bundle x°. Such intransitivity could not occur under the usual assumptions of utility analysis—in particular, the assumption that indifference curves are nonintersecting. Yet this situation is precisely what occurs in the preceding example, an example in complete conformity with the weak axiom of revealed preference.

It is therefore not surprising that something less than what is implied by utility maximization is implied by the weak axiom. This took the form of allowing *s,j ^= Sj,*. It is not obvious or easy to explain but nonetheless true that this asymmetry and the occurrence of nontransitive revealed preferences are equivalent in the sense that, together with the weak axiom, eliminating either one rules out the other also. In other words, if the weak axiom of revealed preference is strengthened to include the additional assertion that revealed preferences will not be nontransitive, i.e., that nontransitivity will not occur, then in fact it can be shown that a utility function exists for that consumer with the usual properties. These properties include the condition that *Sjj = Sji, i, j* = 1, ..., *n*. Conversely, if, in addition to the weak axiom, it is also assumed that s(y = *Sji, i* = 1, ..., *n,* this too guarantees the existence of a utility function consistent with the observed behavior and hence transitivity of revealed preferences. This latter issue is the classic problem of integrability of the demand functions, i.e., the question of whether a given set of demand functions is capable of being generated by some utility function.

Let us formally state the strong axiom of revealed preferences, due to H. S. Houthakker.t

The strong axiom of revealed preference. Let the bundle of goods purchased at price vector p' be denoted x'. For any finite set of bundles (x^1, ..., x*), if x^1 is revealed preferred to x^2, x^2 revealed preferred to x^3,..., x*"1 revealed preferred to x*, or algebraically, if p'x^1 > p'x^2, $p^2x^2 > p^2x^3$,..., p^'x*"1 > p*~'x*, then *p*k\k < p^x^0; that is, x* is *not* revealed preferred to x^1.

Theorem. Individual demand functions x, = *x^(p\ ,...,/?„, M), i* = 1,..., *n,* that are consistent with the strong axiom of revealed preference are derivable from utility analysis. That is, there exists a class of utility functions *F(U(x*\,..., *„)), where *F* is

. S. Houthakker, "Revealed Preference and the Utility Function," *Economica,* 17:159-174, May 1950.

> any monotonic transformation, which, when maximized subject to the budget constraint Y^ Pi*i = M, results in those particular demand functions.

This "theorem" is subject to certain technical mathematical conditions concerning differentiability and other details (hence the quotation marks). In essence, however, the strong axiom is equivalent to the utility maximization hypothesis; either one implies the other. The proof of this theorem is unfortunately beyond the scope of this book. The interested reader should consult Houthakker's original paper and the later literature.

The strong axiom is a straightforward generalization of the weak axiom. It merely extends the notion of the weak axiom to a chain of more than two consumption bundles. In general, pairwise comparison of consumption points is too weak a basis for making statements about multidimensional curvature properties of functions. Suppose, for example, a consumer possesses well-defined indifference curves, all nonconcave, etc., for two commodities JCI

and x_2. Likewise, assume a similarly well-behaved indifference map between x_2 and some other good x_3 and another well-behaved set for x_3 and *x*. Are these separate indifference maps consistent with an overall utility function *U(x\,* x_2, x_3*)7* Not necessarily. No such integral function need exist. The indifference maps may all be well behaved taken alone, but they may be inconsistent with each other algebraically or they may allow the intransitivity demonstrated in the previous example. Suppose, for example, at a given point, this consumer's MRS of apples for oranges is three apples for one orange. And suppose the consumer will trade one orange for two pears and two pears for four apples. These marginal rates of substitution could not be generated by a three-dimensional utility function, for the consumer would spiral around the original point and wind up being indifferent between the original bundle of goods and one that had more of one good and the same amount of the others. Yet it is perfectly easy to draw these indifference curves in two-dimensional space.

In addition, the usual convexity of the separate two-dimensional curves is insufficient to guarantee usual convexity of the three-dimensional indifference surfaces, assuming it exists. If each two-dimensional curve is convex to the origin, the bordered Hessians of the form

$$\begin{vmatrix} U_{ii} & U_{ij} & -P_i \\ U_{ji} & U_{jj} & -P_j \\ -P_i & -P_j & 0 \end{vmatrix}$$

are all nonnegative. However, even if they are all positive, the full bordered Hessian

$$H = \begin{vmatrix} U_{11} & U_{12} & U_{13} & -P_1 \\ U_{21} & U_{22} & U_{23} & -P_2 \\ U_{31} & U_{32} & U_{33} & -P_3 \\ -P_1 & -P_2 & -P_3 & 0 \end{vmatrix}$$

need not have the appropriate sign (nonpositive). The resulting indifference surface can, at least locally, be concave to the origin at some point, even though all two-dimensional projections of that surface exhibit strict quasi-concavity. These are all subtle geometric issues. It is remarkable that as simple a statement as the strong axiom of revealed preference contains the same behavioral implications as the quasi-concavity of a multidimensional utility function.

Integrability

Suppose an econometrician estimates a set of demand relations x, = x^(pi, ..., p_n, M),i = 1, ..., n, and asks you to check whether these estimated functions are capable of being derived by utility analysis. That is, is it possible (and if so, how) to determine that a given set of demand functions is consistent with standard utility analysis? Consider the two demand functions

$$\frac{M}{2pi} \qquad \frac{M}{2p_2} \qquad (11\text{-}15)$$

Let us make such a determination here (even though we know the answer, these demand functions having been derived in Chap. 10). Both functions are clearly homogeneous of degree 0 in all prices and money income. Also,

$$P\backslash X\backslash + p_2x_2 = \frac{M}{} + \frac{M}{} = M$$

Thus, the budget constraint is satisfied identically. Let us calculate the matrix of Slutsky terms s_{ij} (remember, we cannot yet call these pure substitution effects, since that notion has not yet been established). We have

dxi dx$_r$ M M 1 M

9^ 3M ~ 2p\ + 2/^2^ ~ Jp{

and hence s\\ < 0 as needed. Similarly, s_{22} = —M/Ap\ < 0. For the cross-effects,

dx$_x$ dx$_x$ M l M

dp$_2$ dM 2p$_2$2p\ ^P\l

and

8x$_2$ dx$_2$ M l M

S2\ = ~--1- x\ —— = 0

dpi dM 2pi 2p$_2$ \p\p$_2$

Thus, ^12 = 52i, also as needed. The last requirement on these s_{ij} 's is that their matrix be negative semidefinite; that is, s_n, s_{22} < 0 (already shown) and $s_n s_{22}$ — $s\backslash_2$ = 0. For the latter,

M \(M \ (M V

= 0

Thus these demand functions exhibit all the usual properties. But is that enough? How can we be sure that there are not other conditions that must be satisfied in order for a utility function to exist? Let us try to find a utility function (if it exists) that would generate these demand functions. Since the demand functions are solutions to first-order conditions (partial derivatives of a Lagrangian function), this problem is known as *integrating* back to the utility function. We proceed as follows. If a utility function $U(x\backslash, x_2)$ exists for these demand functions, then along any indifference curve

$$dU = U_x dx_x + U_2 dx_2 = 0$$

or

$$dx\backslash \text{-}\backslash\text{---} \frac{U_2}{} dx_2 = 0 \qquad (11\text{-}16)$$

This is equivalent to

$$\frac{}{dx_x} = -\frac{}{U_2} \qquad (11\text{-}17)$$

This is the familiar statement that at any point, the MRS between two goods equals the ratio of the marginal utilities of the two goods. However, at the chosen point,
$U\backslash/U_2 = p\backslash/p_2$- For these demand functions, $p\backslash = M/2x\backslash$, $p_2 =$ $M/2x_2$- Hence,

$$\frac{U\backslash}{l\sim>2} \quad \frac{p\backslash}{P2} \quad \frac{x_2}{X[}$$

The differential Eq. (11-17) then becomes

$$\frac{dx_2}{dx\backslash} \qquad \frac{x_2}{x\backslash} \qquad (11\text{-}18)$$

This can be integrated by separating variables:

$$dx2 \qquad dx$$

Integrating gives

$$\log x_2 = \text{—} \log xi + \log$$

$F(U)$ or

$$F(U) = x_{IX2} \qquad (11\text{-}19)$$

where the constant of integration $F(U)$ is the arbitrary indifference level chosen for the slope element $dx_2/dx\backslash$. This situation is depicted in Fig. 11-3.

Several things happened to go right in this puzzle. There was no problem expressing the slope element $dx_2/dx\backslash$ in terms of consumption variables $x\backslash$ and x_2 only, and the differential equation itself was easy to integrate. The resulting utility function is that which was used in earlier chapters to derive the preceding demand functions. Let us investigate these matters more closely.

The first step was to express the slope element in terms of JCI and x_2. This involved inverting the demand functions, which were originally functions of the

FIGURE 11-3
Integrability: The Two- Variable Case. The differential equation (11 -16) or (11 -17) defines a direction at each point *(x\X2)* of the *x\X2* plane. These directional elements are depicted in the graph as short line segments through each point. The problem of integrability is to find a function that links up these directions, for constant (the arbitrary constant of integration) levels of the functions. In utility analysis, these level-curve linkups of the slope elements given by the differential equation (11-16) are the indifference curves. For two commodities, this integration is always possible.

prices and income, for functions of the quantities. In most cases, this can be done, with one qualification. The demand functions x_i = *x^(p\ ,...,/?„, M)* are homogeneous of degree 0 in prices and money income. Therefore, it is clearly *not* possible to write p_i = *p*(x\, ...,* x_n*),* since any given consumption bundle is associated with an infinity of price vectors, all multiples of each other. However, we can expect to solve for the x_i 's in terms of *relative* prices, or prices relative to income. Using the homogeneity property, we have

x?itp$_u$ ..., tp$_n$, tM) = xf{p$_u$ ..., p$_n$, M)

If we let *t =\/M,* the demand function can be written

*u ..., p$_n$, M) = x**(?±, ..., El, | | = g$_i$(r$_u$ ..., r$_n$* (11-20)

where r, = *pi/M, i* = 1,..., *n.* The r,'s represent that fraction of a consumer's income necessary for the purchase of one unit of x,. In general, we can expect the Jacobian matrix of the g_i relative price demand functions to have a nonzero determinant and to be able to solve for these relative prices in terms of the *x_i's,* or

n = hi(x\, ...,„)* (11-21)

Then, since *Pi/Pj = n/fj,* the slope elements *dxt/dxj = —pj/pi* are expressible in terms of the quantity variables, using the inverted demand functions (11-21). This was accomplished in the above example, in the differential Eq. (11-18).

for the two-variable case, $r_1 = h_1(x_1, x_2)$, $r_2 = h_2(x_1, x_2)$, it remains to solve the differential Eq. (11-16) or (11-17),

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2} = -\frac{h_1(x_1,x_2)}{h_2(x_1,x_2)}$$

or

$$h_1(x_1, x_2)\,dx_1 + h_2(x_1, x_2)\,d$$

x (11-22)

These differential equations are not, in general, easy to solve. However, for the *two-variable* case only, a solution is assured by a well-known mathematical theorem.

How is Eq. (11-22) to be integrated, i.e., solved? Remember, this differential equation is supposed to represent the total differential of a utility function along an indifference curve; that is, $dU = 0$. Then for $U = U(x_1, x_2)$, from Eq. (11-16) we must have $U_1 = h_1, U_2 = h_2$. Moreover, since cross-partials are invariant to the order of differentiation, we must have $U_{12} = U_{21}$, or

$$\frac{\partial h_1}{\partial x_2} = \frac{\partial h_2}{\partial x_1} \qquad (11\text{-}23)$$

This condition happened to be satisfied for the demand functions in the preceding example. There, $h_1 = 1/2x_1$, $h_2 = 1/2x_2$, $\partial h_1/\partial x_2 = \partial h_2/\partial x_1 = 0$. Whereas it is clear that $\partial h_1/\partial x_2 = \partial h_2/\partial x_1$ is a *necessary* condition that must exist if an integral function $U(x_1, x_2)$ is to exist, it is also the case that this condition is *sufficient* for the existence of such an integral function, by a well-known theorem of differential equations. Hence, since $\partial h_1/\partial x_2 = \partial h_2/\partial x_1$ in the preceding example, some integral utility function $U =$

$U(x_1, x_2)$ necessarily existed, with $\partial U/\partial x_1 = h_1$, $\partial U/\partial x_2 = h_2$.

The point of the preceding discussion is that if one starts with an arbitrary set of demand functions $x_i = x_i(p_1, \ldots, p_n, M)$, $i = 1, \ldots, n$, satisfying the usual budget and homogeneity conditions, it will be rather fortuitous if the resulting differential generalization of Eq. (11-22) has a solution:

$$h_1(x_1, \ldots, x_n)\,dx_1 + \cdots + h_n(x_1, \ldots, x_n)\,dx_n = 0 \qquad (11\text{-}24)$$

In general, a solution to this differential equation does not exist; i.e., there may be no utility function $U(x_1, \ldots, x_n)$ such that $\partial U/\partial x_i = h_i$, $i = 1, \ldots, n$. Special restrictions on the h_i's must be imposed in order to guarantee a solution.

Curiously enough, however, for the two-variable case, the differential Eq. (11-22) or (11-24) *always* has a solution. That is, starting with two demand functions $x_1 = x_1(p_1, p_2, M)$ and $x_2 = x_2(p_1, p_2, M)$ which satisfy the budget and homogeneity condition, the resulting $h_x(x_1, x_2)$ and $h_2(x_1, x_2)$ are always integrable. The resulting differential expression

$$h_1(x_1, x_2)dx_1 + h_2(x_1, x_2)dx_2 = 0$$

may not in fact exhibit $dh_1/dx_2 = dh_2/dx_1$. However, in this two-variable case, it happens that there will always be an *integrating factor* r G(JCI, x_2) such that

$$G(x_1, x_2)h_1(x_1, x_2)dx_1 + G(x_1, x_2)h_2(x_1, x_2)dx_2 = 0$$

is integrable, i.e., that $\partial(Gh_1)/\partial x_2 = \partial(Gh_2)/\partial x_1$. The proof of this nontrivial theorem is available in most calculus texts and will not be reproduced here. Notice, though, that $dx_2/dx_1 = -Gh_1/Gh_2 = -h_1/h_2$, and hence the G function corresponds to $F'(U)$, where $F(U)$ is any monotonic transformation of the utility function.

The reason the two-variable system is always integrable can be seen with the help of Fig. 11-3, and at the same time the relationship of integrability and revealed preference will be more clearly exhibited. The differential Eqs. (11-16) or (11-17) serve to define at each point (x_1, x_2) of the x_1x_2 plane a slope element, or direction, $-h_1/h_2$. Every point has such a direction defined. Some of these are exhibited by the short line segments drawn through each point. The integral function $U(x_1,x_2)$ connects up these directional elements for constant functional values. Under some technical mathematical conditions, this can always be done, guaranteeing the existence of a solution to the differential Eq. (11-16) for the two-dimensional case.

In the three-good case, however, the story is different. In each of the x_1x_2, x_2x_3, and x_3x_1 planes, a slope element $dx_j/dx_i = -h_i/h_j$ will be defined. These will be the indifference elements described in the revealed preference section. However, there is no guarantee that these directional elements will all link up with each other; the possibility remains that a consumer can spiral around some point and reach a point that is indifferent to the original while having more of one good and not less of the others. The situation leaves open the possibility of the nontransitive behavior exhibited earlier. The force of the integrability condition $\partial h_i/\partial x_j = \partial h_j/\partial x_i$ is to guarantee that such behavior does not exist; like the strong axiom of revealed preference, integrability rules out nontransitive behavior by guaranteeing the existence of a utility function. This is illustrated in Fig. 11-4.

The consequences of this analysis with regard to revealed preference theory are that for the two-commodity case, the *weak* axiom is in fact sufficient to guarantee the existence of a utility function. The type of intransitive behavior exhibited earlier cannot occur with only two commodities if the weak axiom is satisfied. The fundamental difference between the two- and many-commodity situations is that for two commodities there is only one relative price. With only one price, the consumer cannot circle around the original point to a new position of greater commodity levels while remaining on the same indifference curve, as is possible in three dimensions if only the weak axiom is asserted.

What conditions on the demand functions themselves lead to integrability? Although it cannot be shown here, unfortunately, as advanced techniques are required, the symmetry of the Slutsky terms $s_{ij} = \partial x_i/\partial p_j + x_j \partial x_i/\partial M$ is sufficient to guarantee the existence of a utility function from which the demand curves are derived. Given this symmetry, the terms s_{ij} are interpretable as the slopes of compensated demand curves, i.e., the partial derivatives $\partial x_i^u(p_1, \ldots, p_n, U)/\partial p_j$. Since $\partial x_i^u/\partial p_j = \partial x_j^u/\partial p_i$, the differential expression $x_1^u dp_1 + \cdots + x_n^u dp_n$ is exact; i.e., it is integrable. Since by the envelope theorem we know that the expenditure, or cost, function $M^*(p_1, \ldots, p_n, U)$ has the property $\partial M^*/\partial p_j = x_j^u$, clearly, the above differential expression is simply

$$dM^* = \frac{\partial M^*}{\partial p_1}dp_1 + \cdots + \frac{\partial M^*}{\partial p_n}dp_n = x_1^u dp_1 + \cdots + x_n^u dp_n$$

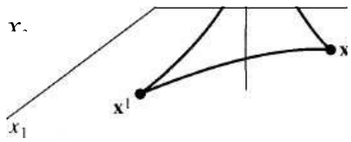

FIGURE 11-4
Integrability: The Three-Commodity Case. Consider some point x° = (xj\ *x®, x®).* A slope element in the X1X3 direction, *—h\/hi,* is defined by the differential equation (11-24). Integrating in that plane, parallel to the JCI , X3 axes, we come to some other point x^1. At x^1, a directional element *—h\/h₂* is defined in an *x\X2* plane parallel to the JCI , *X2* axes. Integrating along some level curve there, we can get to another point x^2 at which the original *x* value is restored. At x^2, a directional element — / 13//12 is defined in the *x₃xj,* plane. Integrating along that level curve, we may get back to a point such as x^3 which has the same level of *x* and *x₂* as x^1 but has more of the third commodity JC3. This spiraling process

is what is ruled out by the integrability conditions. In order for a well-defined indifference *surface* to exist, relating to a utility function *U{x\, x_2,* X3), a point such as x^3 cannot occur but must coincide with x°. Then the path x° —>■ x^1 —>■ x^2 —>■ x^3 represents a movement along one indifference surface. A path like x° —>■ x^1 —>■ x^2 —> x^3 in the diagram is what can occur if only the *weak* axiom of revealed preference is asserted. The numerical example illustrated this. The strong axiom, by asserting that the last point in the chain will not be revealed preferred to the first, effectively eliminates the situation depicted in the diagram.

Hence, *M*(p\,* ..., *p_n, U)* is the integral of this expression and is known to exist, since *d^2M*/dpidpj* = *dx^/dpj* = *dx^/dpi* = *d^2M*/dpjdp$_t$*. Thus, the expenditure function is well defined if the Slutsky terms are symmetrical. From the discussion in the previous chapter, the corresponding utility function must exist also.

Let us check that the two-variable case is always integrable. Consider two demand functions *x\ = xf*(p\, P2, M)* and*2 = *x^(p\, pi, M).* Assume that

P = (

i

*

,

+

and (1

, t That is, the demand functions satisfy the budget and homogeneity conditions. If these demand curves are integrable, i.e., if there exists some utility function that generates these functions, then we should find 512 = ^21 • Let us see if this is the case. Superscripts will be omitted to save notational clutter.

From homogeneity and (11-25), using Euler's theorem, we get

dp\ dp₂ oM

since $M = p\backslash X\backslash + p_2x_2$. Collecting terms gives

P2

dx\ dx\ \ (dx\ dx\ \ + X ₍)
*+) + P 2 \ - ᴶ - + X ₂ ...~ + P 2 *
=0 i

dMj \dp₂

or

+ *P2S12* = 0 (11-27a)

In like fashion, applying Euler's theorem to $x_2(p\backslash, p_2, M)$ gives

Pis₂\ + P2S22 = 0 (11-27b)

Equations (11-27) generalize to n commodities in a straightforward manner. In general,

PjSij=0 $i = l, \ldots, n$ (11-28)

7=1

Now consider the budget relation (11-25). Differentiating with respect to $p\backslash$ yields

Pi^dx\ + /?2 dx₂ =
-xi
dpi dpi

Differentiating (11-25) with respect to M yields

P\ dxi -----h Pi dx₂ --- =
1

Multiplying this expression by $-x\backslash$ and substituting into the preceding equation leads to

dxi dx₂ dxi dx₂
+
dpi dpi

Combining terms, we have

Pi /3JCI — + *i 3XI \ ) = 0 {dx₂ + Pi dx₂\ — + xi*
\dpi dMJ \dpi dMJ

or

Pisₙ + $p_2s_2\backslash$ =0 (11-29a)

In like fashion, by differentiating the budget relation with respect to p_2, one can derive

$$+ P2S22 = 0 \qquad (11\text{-}2\%)$$

In general, for *n* commodities,

$$\sum_{i=1}^{n} p_i S_{ij} = 0 \qquad j = 1, \ldots, n \qquad (11\text{-}30)$$

Note that in Eq. (11-28) the sum runs over *j,* whereas in (11-30) it runs over *i*. These relations say that the weighted sum of the Slutsky terms for any good or for any price equals zero, where the weights are the prices. These relations were derived without any reference to utility theory; the only assumption was that the consumer had well-defined choice functions satisfying the budget and homogeneity conditions.

It is apparent from Eqs. (11-27a) and *(11-29a)* that for the two-commodity case, $S_{12} = S_{21}$. Hence, it is indeed always possible to find a utility function $U(x_1, x_2)$ which generates the money income demand curves $x_i = x_i^M(p_1, p_2, M)$, $i = 1,2$. If the additional property of negative semidefiniteness is imposed on the Slutsky terms, that is, $s_{11} < 0$, $s_{22} < 0$, $s_{11}s_{22} - s_{12}^2 = 0$ following from either Eqs. (11-27) or (11-29), then the utility function will have the usual convex indifference curves of consumer theory.

The importance of these results is that they demonstrate that negative semidefiniteness and symmetry of the Slutsky terms constitute all the implications of utility theory. Since it is possible to work backward from these assumptions and demonstrate the existence of quasi-concave utility functions, there can be no other independent results of utility theory. Any other results, e.g., Le Chatelier effects, can be derived from these assumptions as well as from utility theory. In addition, since the strong axiom of revealed preference also guarantees the existence of utility functions, these approaches are all equivalent aspects of consumer theory.

11.3 THE COMPOSITE COMMODITY THEOREM

One feature of economic systems is the interplay of a large number of variables. The number of commodities produced in a modern economy runs into the millions or billions; the variety of tasks, skills, and capital is enormous. Indeed, analysis of such systems would be impossible for most minds without some simplification or abstraction from reality. In the first chapter we discussed the role of assumptions in science, in particular the necessary simplification in order to make a theory tractable. In this section we shall investigate one aspect of this procedure, the lumping together of many commodities into one composite commodity.

Most textbooks and articles in economics generally reduce the world to two commodities or two factors of production, etc. One is usually the good under analysis, and the other is usually labeled "all other goods," or all other "closely related" goods. To what extent is this procedure justifiable? Under what circumstances can all other goods be treated as one good?

It might be appropriate at the start to recognize that what is in fact called a *commodity* is not a technological datum. Most commodities have several characteristics, each of which presumably generates utility to consumers. Yet usually, only one of these characteristics is used to label the commodity. Consider the example of eggs. Eggs come in various volumes, weights, colors, and degrees of firmness of yolk and white. The fact that egg sizes are by weight rather than volume is due to the relative ease, i.e., lower cost, of measuring that dimension than, say, volume. (The last characteristic, firmness, is the one used by the U.S. Department of Agriculture in grading. It is difficult to measure in a nondestructive manner.) Diamonds, on the other hand, are extensively measured. They are classified by color (white, blue-white, yellow, etc.), various degrees of departure from flawless crystal structure, shape of cut (round, marquise, emerald, etc.). Each of these characteristics is carefully measured, and prices vary accordingly. Diamonds are so extensively measured and categorized because, given the "high" price of the basic material of diamonds, measurement is relatively cheap. Hence, a great deal of measuring is done on diamonds, and relatively less measuring is done on lower-valued commodities. As a last example, much produce is sold by the piece in season and by weight out of season. When the produce is in season, i.e., in relatively greater supply, its price is lower. The cost of measuring, e.g., weighing at the checkout counter, or bunching together uniform packages, is relatively high. Hence, less measuring is done, and consumers are left to do whatever measuring they please on their own. Thus, the units of the commodity are apt to be different at different times or even at different retail establishments, depending upon the level of retail services offered. The notion of a commodity is thus not a technological datum but dependent in large part on the economic costs of characterization of the good.

We shall ignore these matters, however, in the forthcoming discussion. Assume that there are n well-defined commodities, $x\backslash, \ldots, x_n$, which the consumer purchases in positive amounts at prices $p\backslash, \ldots, p_n$, respectively. Suppose now that the prices of some subset of these commodities all change simultaneously in the same proportion. Mathematically, let p° = $(p®, \ldots, p®)$ be the initial price vector. Suppose, by suitable relabeling of the commodities that $pk+\backslash, \bullet \blacksquare \bullet, p_n$ all vary in the same proportion; that is, $p_{k+i} = tp^\circ_{+l}, \ldots, p_n = tp^\circ_n$, where initially $t=\backslash$. (If, for example, t became 2, then all prices would have doubled.) How would the consumer react to changes in t, that is, proportionate changes in $n - k$ of the prices?

As we have indicated in the previous sections, the behavioral implications of utility theory are summarized and exhausted in the statement that the compensated, or Slutsky, changes s_ij form a symmetric negative semidefinite matrix. These pure substitution effects are derivable from the cost minimization problem:

minimize

$$\sum^{n} p_i x_i = M$$

subject to

$$U(x_u, \ldots, x_n) = U^\circ$$

The expenditure function $M^*(pi, \ldots, p_n, U^\circ)$ has the property that $dM^*/dpi - M^* = x^\wedge ipi, \ldots, p_n, U^\circ), M^* = dx^\wedge/dpj$. In the present problem, a new parameter t is introduced. To derive the compensated changes in the x_i's when t changes,

one must consider the
model: minimize

(11-31)

$i=k+l$

subject to

(11-32)

The objective function (11-31) can be
written minimize

(11-33)

where $y = YTi=k+\backslash\ Pi^xt>$ ^e total expenditure on commodities $k + 1$ through n, the ones whose prices are all changing proportionately. When $t = 1$,(11-33) is identical to (11-31); hence the same demand point results. We can thus analyze a change in t from 1 by use of the duality results of Chap. 7. Since $dM^*/dt = M^* = y^u =$

$n°x^u$

Pi^xi '

$j = '!, \ldots, k$

The $(k+ 1) \times (k+ 1)$ matrix of terms,

dpi d dp d i ni "x d k y^u dp d $x d \ dt dt dt I

(11-34)

must be negative semidefinite. The factor of proportion t enters this matrix exactly as do the prices $p_1, \ldots, p_k$, and total expenditure y^u on $x_{k+1}, \ldots, x_n$ enters just like any other quantity variable. Thus this system of compensated changes is no different from any other well-behaved set of $k + 1$ compensated demand functions. Therefore, when prices of several commodities vary simultaneously, in the same proportion, total expenditures on those commodities behave exactly like any other commodity. This new variable y is called a *composite commodity,* and was introduced first by John R. Hicks in *Value and Capital.* The composite commodity is just like any other decision variable; e.g., the response to a change in its own "price" t is negative:

$$\frac{\partial y^u}{\partial t} < 0 \tag{11-35}$$

Also, from the symmetry of the cross-partials in (11-34)

$$\frac{\partial y^u}{\partial p_i} = \frac{\partial x^u}{\partial t} \qquad i = 1, \ldots, k \tag{11-36}$$

in addition to the usual reciprocity conditions ^

This important theorem justifies the use of two-dimensional graphical analysis in much of economic theory. It is easy to imagine that in many empirically important cases where a single, outstanding price change takes place in the economy, variations in prices of a group of commodities will not vary significantly within the group. The group can thus be regarded as a single commodity. Since one price will have changed, say, this is equivalent, for relative price changes, to a proportionate change in all prices of goods within the group. Thus, the highly convenient simplification of economic analysis by considering the good in question vs. all other goods is at least consistent with utility theory and perhaps empirically sound in many instances.

Shipping the Good Apples Out

Consider now another type of simultaneous price change, that of adding a fixed amount to several prices. That is, consider the effects of changing a parameter t added to $p_1, \ldots, p_k$, yielding prices $p_1 + t, \ldots, p_k + t, p_{k+1}, \ldots, p_n$. Such a situation might occur if the first k goods are subject to the same tax or transportation charge. In this case, the composite good $z = \sum_{i=1}^{k} x_i$ acts as a single good. Again, consider the relevant cost minimization problem:

minimize

$$M = \sum^{k} (p_i + t) x_i + \sum^{n} p_i x_i \tag{11-37}$$

subject to

$$U(x_1, \ldots, x_n) = U^\circ$$

from which the compensated demands

$$x_i = x_i^u(p_1, \ldots, p_n, t, U^\circ) \tag{11-38}$$

are derived. However, M is linear in the parameter t, and letting M^* denote the expenditure function $M^*(p_1, \ldots, p_n, t, U^\circ)$, we have

$$x_y = z^u(p_1, \ldots, p_n, t, U^\circ) \tag{11-}$$

39)

7 = 1

Hence, z^u exhibits the properties of any other good. The matrix of second partials of M^* with respect to just $p_1, \ldots, p_n$ and t is symmetric and negative semidefinite:

/dxl dpi

dx^u_x $dx^u_x\backslash$

dt

(11-40)

dp i

dp_n dt

dz^u dz^u

\ dpi)

dp_n dt

From the symmetry of this matrix,

dz^_dxl

dpi dt (11-41)

From the definition of z^u, and the fact that $s_{ij} = s_{ji}$,

o f / k o „ {/ k k

dt (11-42)

h

The own effect of the composite commodity z^u is negative; i.e.,

dz^u

<0

dt (11-43)

This result is also derivable from the original substitution matrix. Letting $s_{ij} = dx^\wedge/dp_j$, as before, with $s_{it} = dx_i^u/dt$, $s_{ti} = dz^u/dp_i$, by negative semidefiniteness we have

(11-44)

***i=l* 7 = 1**

where the equality holds when the /z, 's are proportional to the prices. Let h_i, $h_j = 1$, $i = 1, \ldots, k$, and h_i, $h_j = 0$, $i, j = k + 1, \ldots, n$. Then (11-44) becomes

k k

which is Eq. (11-43), assuming all prices are not the same number (all proportional to unity).

Let us now investigate the empirical effects of this type of price change more closely. Using the composite commodity theorem, we can consider a three-good world, *x\,X2,x?,,* where JC_3 is the composite commodity of the previous section. Suppose goods 1 and 2 are transported from another location, with *Xj,* produced locally, producing a set of prices *p\ + t,* p_2 *+ t,* p_3*,* where *pi* and p_2 are the point-of-origin prices of *x*\ and *xi,* respectively.t The transportation charge is apt to produce a predictable change in the mix of goods consumed in the origin versus the destination of the goods. Consider the following complaint mailed into the "Troubleshooter" column of the *Seattle Times* by an irate consumer:*

> Why are Washington apples in local markets so small and old-looking? The dried-up stems might seem they were taken out of cold storage from some gathered last year.
>
> Recently, some apple-picking friends brought some apples they had just picked, and they were at least four times the size of those available for sale here.
>
> Where do these big Delicious apples go? Are they shipped to Europe, to the East or can they be bought here in Seattle?
>
> M.W.P.

An answer from a trade representative allowed that "itinerant truckers" (price cutters) were at fault:

> The apples [she] is seeing in her local markets may have been some left from the 1974 crop, or could be lower-grade fruit sold store-to-store by itinerant truckers.

The textbook answer was supplied by this economist several days later:§

Comparing apples to apples

> Regarding M.W.P.'s complaint (Sunday, October 19) that all the good apples were being shipped to the East, you might be interested to know that "shipping the good apples out" has been a favorite classroom and exam question in the economics department at U.W. for many years.
>
> It is a real phenomenon, easily explained:
>
> Suppose, for example, a "good" apple costs 10 cents and a "poor" apple 5 cents locally. Then, since the decision to eat one good apple costs the same as eating two poor apples, we can say that a good apple in essence "costs" two poor apples. Two good apples cost four poor apples.
>
> Suppose now that it costs 5 cents per apple (any apple) to ship apples East. Then, in the East, good apples will cost 15 cents each and poor ones 10 cents each. But now eating two good apples will cost three—not four poor apples.
>
> Though both prices are higher, good apples have become relatively cheaper, and a higher percentage of good apples will be consumed in the East than here.
>
> It is no conspiracy—just the laws of supply and demand.

Other examples of this reasoning are as follows. Cheap French wines (often sold in cans in France) are never exported to the United States. There probably are French

^This situation was first analyzed by A. Alchian and W. Allen in their principles texts, *University Economics,* and its condensation, *Exchange and Production,* Wadsworth Publishing Company, Belmont, CA, 1969.

* *Seattle Times,* Sunday, Oct. 19, 1975.

§*Seattle Times,* Tuesday, Oct. 28, 1975.

consumers irate about the best French wines being shipped out. In the 1920s, there were two grades of tobacco: A pack of premium-grade cigarettes sold for 10 cents, while a pack of standard-grade cigarettes sold for 5 cents. When the government levied a 10-cent per pack tax on cigarettes, the relative price of the premium grade became 4:3 rather than 2:1 and the standard-grade tobacco disappeared from the market. Today, some states tax wine *by the bottle,* i.e., by volume only. This has the effect of lowering the relative price of better wines and encouraging their sale. Restaurants buy most of the beef rated as prime by the U. S. Department of Agriculture. This could be a conspiracy of restaurateurs, but it is more likely a result of substitution toward the lower-priced good. At home, a prime steak might cost $4 to produce vs. $2 for a USDA choice steak. With $20 of added restaurant amenities, the relative price becomes 24:22 instead of 2:1. It is for this reason that we rarely see hamburgers on the menus of expensive restaurants. Likewise, one rarely sees expensive homes built on cheap lots; the more expensive lot makes the higher-priced house relatively cheaper.

There are a few caveats to all this. The apple example is pretty clean because consumers get no utility from the transportation cost. Restaurant amenities, on the other hand, or a nice lot with a view are goods from which a consumer derives utility and which, in principle could be purchased separately. In this case, the theorem does not directly apply. We would have to ask, for example, whether the cook's services are complementary to the beef or perhaps a substitute for poor beef.

John Gould and Joel Segall further challenged the Alchian and Allen substitution theorem on the grounds that with three or more goods, interactions with the third good could destroy the effect.t Gould and Segall used the example of having to go to Maine to get a truly good lobster. This raises another interesting question: Does it matter whether the goods are shipped to the people or the people are shipped to the goods? (If good lobsters were only available in Maine due to spoilage en route, this would not be relevant to the current discussion.) Some departments give faculty members a $10 stipend to go out to dinner with faculty recruits. This subsidy should make faculty substitute for *less* expensive restaurants, but is the subsidy really tied to this choice? What if we were inclined to eat out anyway? In that case this would be a simple cash transfer, with no implications other than a weak income effect. The theorem should imply that tourists in Maine on average consume higher-grade lobsters than do the natives. Recently Eric Bertonazzi, Michael Maloney, and Robert McCormick confirmed this idea. They found that ticket holders who came from far distances to Clemson University's home football games bought better, more expensive seats than did the locals.* One could argue that the transportation cost was sunk once the fans arrived in Clemson, but it seems these consumers effectively bundled the game and the trip together. Let us analyze the situation mathematically.

^John Gould and Joel Segall, "The Substitution Effects of Transportation Costs," *Journal of Political Economy,* 130-137, 1968.
*Eric Bertonazzi, Michael Maloney, and Robert McCormick, "Some Evidence on the Alchian and Allen Theorem: The Third Law of

Demand?" *Economic Inquiry,* 31:383-393, July 1993.

Alchian and Allen's thesis is that if JCI is the premium quality good (higher-priced) and x_2 the inferior quality, then

(Superscripts will now be dropped. The student must remember that these are all compensated changes. The introduction of income effects complicates the analysis in predictable ways; viz., income effects are always indeterminate.) Expanding the quotient in (11-45)* gives

d(x\/x₂) / *dxi* *dx₂*

dt V"" *dt* "' 3? *) x*

FromEq. (11-42), *dx\/dt = sn + s_{i2}, dx_2/dt = s_{2i} + s_{22}*. Thus,

**2/V /-V \ v / p c r o **

O^Xi/X₂y)Aj *I S* *S\₂* *S₂* *S₂₂ *

O? X₂ \ Xj Xj *X₂* *X₂ J*

Let us convert these terms to elasticities. Letting e,; = *(Pj/Xi)*

(dXi/dpj), we have *d{xₓ/x₂)*

_ ^ i _ / n _ £^2 _

£2i_ _ ^22

dt *x₂* V *p* *p₂* *P*

p₂

However, from the homogeneity of the compensated demand, X^=i *Pj^s u* — 0 and hence by dividing by x_t yields

C1l +^12+^13 =0

and

621 + €22 + e_{23} = 0

for this three-good model. Using these expressions to substitute for e_{12} and 6₂2 in the preceding equation, we have

[V *Pi* *Pi)* *P* \ *Pi* *Pi)*

or

(11-46)

Pi P2/ \Pl ' '

Let us examine Eq. (11-46). Since JCI is the higher-quality good, *p\ > p₂,* and thus *l/pi — l/p₂* < 0. Also, *€ᵤ* < 0 and e₂i > 0 since two qualities of the same good are presumably substitutes. Alchian and Allen's thesis is that *d{x\/x₂)/dt* > 0; the

^Thomas Borcherding demonstrated this algebra. See Thomas Borcherding and Eugene Silberberg, "Shipping the Good Apples Out: The Alchian and Allen Substitution Theorem Reconsidered," *Journal of Political Economy,* 86:131-138, February 1978.

first compound term in (11-46) confirms this. In a two-good world, this would be the entire expression, and then Alchian and Allen would be entirely correct. The last term, e_23 — f 13, however, is indeterminate. If, however, we assume that the lower- and higher-quality good interact in the same manner with the composite good x_3, that is, that 613 = 6_23, then the hypothesis will be valid. The hypothesis becomes invalid only in the asymmetrical case, where, say, the premium good is a much closer substitute for the third good than the inferior good (€13 > 623). Then when $p\backslash$ and p_2 are both raised, say, to $p\backslash + t$ and $p_2 + t$, respectively, the consumer substitutes x_3 for $x\backslash$ in greater proportion than x_3 for x_2, confounding the hypothesis. This asymmetry seems to be empirically insignificant to these casual observers.

A similar result can be derived for the *difference,* as opposed to the ratio of consumption of Xi to x_2, when t changes. Letting $p\backslash = p_2 + k$, $k > 0$, from homogeneity we get

$$(p_2 + k)s_n + p_2s_{l2} + p_3s_{x3} = 0$$
$$(p_2 + k)s_2i + P2S_{22} + P3S_{23} =$$
$$0$$

Since $dx\backslash/dt = s_u = s_n +$ ^12 and $dx_2/dt — s_{2t} = s_2\backslash + s_{22}$,

$p_2s_{Xt} + ks_n + p_3s_{13} = 0$ $p_2s_{2t} + ks_2\backslash + P3S23 = 0$

Subtracting gives

$$- s_{2t}) = -k(s_n - s_2\backslash) + p_3(s_{23} - 5{,}3) \qquad (11\text{-}47)$$

Assuming that the lower- and higher-quality goods are substitutes for each other (otherwise the whole exercise is meaningless), $s_2\backslash > 0$. Thus the first term on the right side of (11-47), $—k(s_n — s_2\backslash)$, is positive. This tends to confirm the idea that an increase in transport cost will raise the absolute level of consumption of the premium good relative to the lower-quality good. The validity of the inference in a three-good model boils down to the term (^23 — £13), a term similar to that appearing in Eq. (11-46), dealing with the ratio $X\backslash/x_2$. If these interactions with the third good are similar, then the higher-quality good will be shipped to distant places in greater amounts than the lower-quality good.

It should be noted that a higher-quality good and lower quality of the *same good* should be fairly close substitutes. Therefore, as an empirical matter one should expect relatively high absolute values of *Sn*, s_{i2}, and s_22, or the corresponding elasticities. This will make the first term in Eqs. (11-46) and (11-47) relatively large. And if these goods are not closely related to the composite commodity, S13 and 523 should be fairly small, even if not approximately equal. Hence, as an empirical matter, the Alchian and Allen hypothesis that the higher-quality good will tend to increase relative to the lower-quality good when like transport (or other) costs are added to each item might be expected to be true for most commodities.

In general, simultaneous price changes of the form /?< = /?? + $pi(t)$, $i = 1,\ldots, k$, $k < n$, with $pi(0) = 0$ can be defined. These changes will in general not produce interesting comparative statics theorems. The resulting composite commodities will be complicated expressions involving the derivatives of $pt\{t)$. The empirical usefulness of such constructions is likely to be small.

11.4 HOUSEHOLD PRODUCTION FUNCTIONS

In 1965 and 1966, in two related articles, Gary Becker and Kelvin Lancaster introduced the concept of household production functions.t In these models, instead of receiving utility directly from goods purchased in the market, consumers derive utility from the attributes possessed by these goods, and then only after some transformation is performed on those market goods. For example, although consumers purchase raw foods in the market, utility is derived from consumption of the completed meal, which has been produced by combining the raw food with labor, time, and, perhaps, other inputs.

Many goods produced in a modern economy appear to serve similar purposes. For example, there are wide varieties and qualities of the same foods, and likewise for clothing, housing, etc. Consumers appear to select only one or a few of these different qualities and forgo completely the consumption of the others. In the previous section, we analyzed the effects of adding a lump-sum tax or other cost to two different "qualities" of the same good. In fact, standard utility theory provides no mechanism for identifying two goods as different qualities of the same good vs. two separate goods altogether. The algebra of the previous section applies to any two goods, labeled "JCJ" and "X2" The analysis applies equally to apples and oranges, or for that matter, apples and typewriters, as to red and golden Delicious apples. We seem to feel comfortable speaking of beef and pork as substitutes, and pencils and paper as complements; yet such pronouncements are based on the technology of using these particular goods, i.e., the way we combine these goods with other goods and inputs in order to produce utility. Standard utility theory provides no clues as to why food is different from clothing, shelter, etc.

In order to remedy this, Lancaster postulated that the vector of goods, x, purchased in the market at price vector p, are transformed by $z = g(x)$ into attributes z which produce utility. In a very general sense, therefore, the model is

maximize

$$U = U(z)$$

subject to

$$z = g(x) \tag{11-48}$$

*Gary S. Becker, "A Theory of Allocation of Time," *Economic Journal,* 75:493-517, September 1965; and Kelvin J. Lancaster, "A New Approach to Consumer Theory," *Journal of Political Economy,* 74:132-57, April 1966.

and

$$px = M$$

where M is the consumer's income. Combining the transformation function and the utility function,

maximize

$$U = U(g(x)) = V(x)$$

subject to

$$px = M \tag{11-49}$$

It is apparent that at this level of generality, the Lancaster model is equivalent to the standard utility model, assuming the V(JC) function exhibits the same curvature properties as utility functions. Assuming interior solutions to (11-49), the refutable implications will consist of the usual properties of the "compensated" demands $x = x^v(p, V°)$, defined as the solutions to, minimize $M = px$ subject to $V(x) — V°$, a constant. The partial derivatives of these demand functions are not really "pure substitution effects" in the traditional sense, since production changes [i.e., changes in the z's through $g(x)$] may take place as prices change. However, the statement that the matrix (dx^v/dp) is negative semidefinite still comprises the complete set of refutable implications; thus at this level of generality, the model is indistinguishable from the standard theory.

In order to be useful, that is, to provide insights or propositions beyond that of ordinary utility analysis, some sort of *observable* structure must be imposed on the transformation function $g(x)$. Lancaster assumed that $g(x)$ is linear, i.e., $z = Bx$, where B is some matrix of (constant) technological coefficients. Lancaster further postulated that B was constant across consumers; i.e., the technology for converting market JC'S into attributes z is the same for all consumers. If the matrix B differs for each consumer, there is little likelihood that the model will be operational. To attain the utility-maximizing z, say z^*, the consumer would necessarily have to purchase the market JC'S that produced z^* at least cost; i.e., the consumer would have to solve the "linear programming problem," minimize px subject to $Bx > z^*$.

Linear models of this type will be analyzed in more detail in the chapter on linear programming. Suffice it to say here that the feasible region, i.e., the set of attainable z's, will now consist of a (^-dimensional) convex polyhedron, with many corners and faces, rather than the "flat" budget hyperplane. If changes in technology lowered the cost of producing some attribute z,, a change to some new market good or goods would likely be the least-cost means of producing the utility-maximizing attributes. This seems in conformance with observation. Consider that as the prices of electronic calculators and computers have decreased, consumers have gradually shifted from hand calculations on simple calculators to extensive calculations often made on sophisticated machines. The utility-producing attribute would be "calculations"; changes in the technology for producing calculations induce more calculations, on successively more powerful calculating machines. The idea of a "new commodity," always troublesome in traditional utility analysis, is also more easily accomplished with Lancaster's framework. In the traditional framework, a new utility function must be asserted. With Lancaster's model, the invention of new computers, for example, does not cause a rearrangement of preferences but merely a new solution to a cost-minimizing problem involving the attribute "calculations."

All this being said and done, it still remains that empirical implementation of the Lancaster model in a truly observable manner is not straightforward. Identification and measurement of "attributes" may be more difficult than measurement of market goods. Even with relatively few variables, measurements and predictions of qualitative changes in the purchases of market goods, as the technological coefficients change, are apt to be quite difficult, as familiarity with the complex nature of solutions to just three linear equations in three unknowns would indicate. It remains the case that for "compensated" changes, *dxjdpi* < 0; however, this is no improvement over traditional utility theory. The model has been most successful when applied to goods whose attributes are additive and nonconflicting, e.g., the nutrient values for foods.t

In his related article, Gary Becker sought to incorporate decisions concerning the use of time into the standard utility framework. By considering the cost of time in terms of its forgone use in producing income, Becker provided a basis for explaining some changes in consumption as wage income changes, in terms of substitution effects, which have known sign, rather than through ad hoc income effects. If the increase in income is produced by an increase in wages, this represents an increase in the marginal value of leisure. We should therefore expect to see the consumer substituting away from time-intensive goods (goods whose consumption involves relatively heavy use of time) and toward those goods for which the time cost is relatively less. In this way, changes in consumption that were once considered on an ad hoc basis, by asserting a change in tastes or a sign for an income effect, could be interpreted as consequences of the law of demand.

Like Lancaster, Becker assumes that utility is a function of a vector of attributes z, i.e., $U = U(z)$. However, Becker adopts a very simple structure for production of attributes. For each *zt,*

$$Xi=biZi \qquad \textbf{(11-506)}$$

where t_t is a parameter indicating the per-unit consumption of time for each z,-consumed, so the total time spent consuming some amount $z_{(}$- is $T_{(}$, and b_t is a parameter indicating the amount of market good $x_{(}$ required per unit z,. Those attributes with relatively high values of t_t are called *time-intensive.*

tSilberberg, showed that as incomes increase, the fraction of the food budget allocated to pure nutrition (as opposed to tastiness) falls, as diminishing marginal productivity of nutrition would suggest. See Eugene Silberberg, "Nutrition and the Demand for Tastes," *Journal of Political Economy,* 93(5):881-900, October 1985.

Consumers are postulated to maximize utility of attributes consumed, subject to a market budget constraint and a time constraint. Let T represent the total time available for all activities (i.e., 24 hours per day), and let T_w = amount of time spent working at some constant wage rate w. Assume also that the individual has available nonwage income in the amount Y. Then we can write

maximize

$$U = U(z u \ldots, z_n)$$

subject to

$$Y\hat{}ptXi = wT_w + Y$$

and

$$Y = T - T_w$$

However, since time and market goods are inextricably linked by the production Eqs. (11-50), the two constraints can be combined. Replacing T_w in the income constraint with 7[1] — ^ 7} from the time constraint yields the single constraint

£>*,.= $_w$ (r-£ 7})+ r

or

Substituting 7} = *Un* and $x_t = b_t Zi$ yields Becker's

basic model maximize

subject to

J2 *Y* (11-51)

We can interpret the value 7r(= *pibi* +wt_t as the "full price" of consuming z,.t When one unit of some attribute *zt* is consumed, it entails the cash expenditure of *Pibi* (dollars) *plus* the time expenditure of t_t (hours). This time could have been

^The implicit price of any *n* is independent of the final choice of *n* 's only because of special assumptions regarding the technology of household production. Specifically, one must assume that $z = g(x)$ exhibits constant returns to scale and no joint production. This is satisfied in Becker's simple linear technology. See Robert A. Pollak and Michael L. Wachter, "The Relevance of the Household Production Function and Its Implications for the Allocation of Time," *Journal of Political Economy,* 88(2):255-277, April 1975, for a more complete discussion of the theoretical limitations of these models.

used to produce income in the amount *wti,* and so represents an opportunity cost of consuming z,. The sum of these full prices times quantities of attributes consumed equals an individual's full income, consisting of nonwage income plus the amount of income that would be earned if the entire day were spent at work.

In this model, idle time (and sleeping) are attributes, i.e., part of the set of Zi's. They perhaps involve no cash expenditure, in which case Z?, would be zero. All of this time is valued at some constant wage rate w; thus it is assumed that the individual has available as much work as he or she desires at that wage. The total time spent consuming all attributes is T_c — T — T_w = *J2 T,-*

Assuming the sufficient second-order conditions hold, the solutions to the first-order equations yield the Marshallian demand functions

$$H = zj(ni, \ldots, n_n, w, Y) = {}_zJ(p, b, t, w, Y) \qquad (11\text{-}52)$$

where *p, b,* and *t* are the vectors of prices, technological coefficients, and time intensities, respectively. Using Eqs. (11-50), the demands for the market goods x_t and time spent 7} on each attribute are immediately derivable.

Comparative Statics

The purpose of this model is to shed light on the use of time. In particular, we are interested in characterizing consumers' responses to changing wage levels and changing technological coefficients. As in the standard utility maximization model, the parameters in the Becker model all enter the constraint, and thus, as usual, no refutable implications can be derived on the basis of the maximization hypothesis alone. We thus consider the pure substitution effects. The Hicksian demands are derived from the expenditure minimization model,

minimize

subject to

Assuming the first- and second-order conditions hold, the Hicksian demands are

$$Zi = Z\hat{}7T_U \ldots, 7T_n, W, U^\circ) = Z?(_P, b, t, W, U^\circ) \qquad (11\text{-}53)$$

The structure of this model in 7T, and *Zi* is formally identical to the standard expenditure minimization model; thus 3z"/37T, < 0 is implied. Also, since parametric changes in either /?,-, b_i, or r, increase 7r, by a proportional amount, it follows that *dz^/dpi* < 0, *dzf/dbt* < 0 and *dzf/dti* < 0 also. From the technological relations (11-50), and defining the Hicksian demands for the market goods and time spent on each good as *xj*7 and *Tj*U, respectively, it follows that

dpi *dpi*

dpi

"

£ = „*£ <o

d

The effects of changes in b_t on JC,, and t_t on 7}, are, however, ambiguous: *j = d(b*$_l$*z")/db*$_l$ *= bidzY/dbi+z"* = zf(l+^), where e_h is the elasticity of consumption of z, with respect to changes in the coefficient $b_{\{}$ linking x_t with z,-. This elasticity is necessarily negative; however, the sign of the entire expression depends on its magnitude relative to unity. A similar expression holds for 7): *dT^/dtj* = 7^(1 +6,), where e_t is the elasticity of consumption of z,- with respect to changes in the time coefficient t_t. We can understand these results as follows. Suppose some *tj* increases. An increase in t_t means that consuming a given amount of z, now requires greater time. This raises the full price of that z,-, which is therefore reduced in consumption. Only if the reduction in z(is in greater proportion than the increase in *t(* will the total amount of *time* spent on that attribute be reduced. However, since *Xj = bjZi,* a decrease in z; must lead to less consumption of the market good $x_{\{}$ from which it derives.

The analysis of changes in wage rates is somewhat more problematic. The parameter *w* enters the full price of each and every z, for which time is consumed. A change in *w* therefore necessarily changes many prices simultaneously, precluding application of the law of demand. That is, since *w* appears in many, if not all, first-order equations, a refutable hypothesis for the compensated demand functions concerning this important parameter is not possible even in this simple model. Becker argues that as the wage increases, consumption will in general switch to goods that are relatively less time-intensive. This seems plausible enough, but additional assumptions regarding the values of the various parameters in the model are required in order to rigorously derive such a result.

The pure substitution effect regarding the total number of hours worked, however, does have a determinate sign. Using the relation ^7} = T — T_w, we can express the expenditure minimization model in terms of the n + 1 variables z i, ..., z_n and T_w, and two constraints:

minimize

$$Y = \hat{}PibiZt - wT_w$$

subject to

and

$$\hat{}tiZi + T_w = T$$

Recall from the theorems on general methodology, as long as the first- and second-order conditions are assumed satisfied, the comparative statics theorems for parameters appearing in only the objective function are the same as for unconstrained models. Here, the parameter w does not appear in the constraints; it enters the objective function in the particularly simple form $—wT_w$, i.e., as a price of T_w. Since this is a minimization problem, the expenditure function is concave in $—w$, that is, $3(—T\hat{})/dw < 0$, or $d(T\hat{})/dw > 0$, where, of course, $T\hat{}$ denotes the compensated demand for hours worked. "Leisure" in this model really means the total time spent consuming the z,'s; thus $dT\hat{}/dw = d(T — T\hat{})/dw < 0$. Thus, as in the simpler model of labor-leisure choice, a compensated increase in wages is an increase in the opportunity cost of leisure and leads to a decrease in leisure consumed and a corresponding increase in the number of hours worked.

The theory of household production, as outlined here, concerns an important aspect of human behavior. The economic theories of family structure, birthrates, participation in the labor market, etc., proceed from this model. Higher market wages for women, for example, raise the opportunity cost of children and other homemak- ing tasks. Thus, even though "children" are most likely a noninferior good, higher incomes are associated with smaller families, if that income is derived from wages as opposed to inheritance. The increased consumption of "convenience foods" by families with two wage earners can be attributed to higher market wages of the homemaker in those families. Higher-wage families are predicted to purchase "higher-quality" items, when the quality attribute reduces the amount of time required for repair, etc. The theory enables us to think more rigorously about some important choices and provides a framework for replacing explanations based on tastes with explanations based on changing opportunities.

11.5 CONSUMER'S SURPLUS

One of the most vexing problems in the theory of exchange has been the measurement, in units of money income, of the gains from trade. Consider Fig. 11-5, in which the consumer is initially at point x° = *(x^x®)* on indifference curve $U°$, having faced prices of $p°_x$, $p\backslash$ and money income $M°$. Suppose that $p\backslash$ is now lowered to $p\backslash$, the consumer moving to point $x^1 = (x\{, x\backslash)$ on indifference curve U^l. How much better off is the consumer at x^1 compared with being back at x°? One answer might be to ask how much income can be taken away from the consumer and still leave him or her no worse off than before, at point x°. This represents a parallel shift of the budget line from x^1 to a point $\backslash^a$ on the original indifference curve $U°$. This amount of income is the maximum amount the consumer would be willing to pay for the right to face the lower price of $X\backslash$; it is called a *compensating variation.* Call this amount M^{la}. Now consider another answer: How much income must this consumer be given at the original prices to be as well off as with the lowered price of $x\backslash$? This amount, call it M^{Ob}, is the

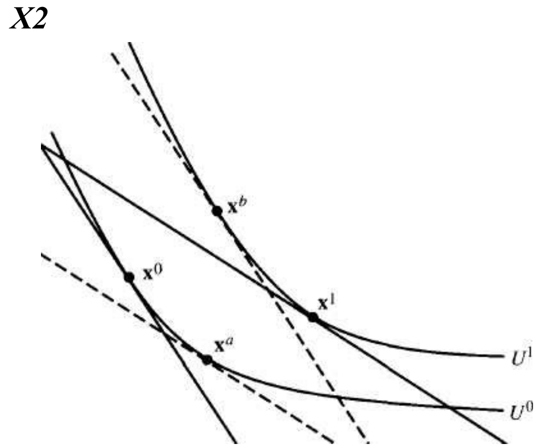

FIGURE 11-5
Two Possible Measures of the Gains from Exchange. Suppose the consumer is initially at point x°, at prices p° = *(p®, p®)*- If *p* is lowered to *p*, the consumer moves to point x^1. The maximum amount this consumer would pay for the *right io* face this lower price is the amount of income M^{1a} that would shift the budget plane from x^1 back to x^a which is on the original indifference surface *U*°. This amount is known as the *compensating variation for a fall in price.* In that case, the consumer would be indifferent between consuming the original bundle x° and facing the lower price but consuming bundle $\backslash^a$. Similarly, if the consumer already has the right, i.e., sufficient income, to consume x^1, raising *p* from *p* to *p* would move the consumer back to x°. The consumer will have to be paid at least the amount of income M^{0b} needed to shift the budget plane from x° to x^b on U^l in order to face the higher price of *x* voluntarily. For then, the consumer will be no worse off than at x^1. This amount of income is known as the *equivalent variation;* it is a *compensating variation for a rise in price.* These two measures of the gain in going from x° to x^1 will not in general be equal. If *x* is a normal good, then $M^{1a} < M^{0b}$ (why?).

amount of income needed to shift the budget line parallel to itself from point x° to a point x^ on *U* since the consumer is indifferent between x^h and x^1. This amount, M^{0b}, is the amount the consumer would have to be bribed to accept the higher price $p°_x$ of JCI voluntarily instead of the lower price. It is usually called the *equivalent variation,* but it is just a compensating variation for a rise in price.

These are two plausible measures of the gains from going to x^1 from x°. The problem arises because these two measures, M^{1a} and M^{0b} (and others that could be considered), are not in general equal. The consumer might be willing to pay $10 to face a lower price of some good; having achieved that point, however, the consumer might be unwilling to relinquish it for the original situation for any payment less than $15. Having achieved a higher indifference level (an increase in real income), if the good is not inferior, the consumer will value it more; hence, more will have to be paid to make the consumer give up the good than to get more units starting at the lower real income. (The reverse is true for inferior goods.) What to do?

The gains received by consumers, which are derived from the opportunity to purchase a good at its *marginal* rather than its average value (in which case no gain

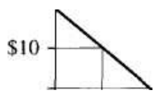

FIGURE 11-6
Marshallian Consumer's Surplus. If a consumer would pay $10 for 1 pound of tea, $9 for the next pound, and so on, he or she would pay 10 + 9 + 8 + 7 + 6 + 5 = $45 for 6 pounds, rather than go without any tea. Marshall concluded that the consumer's surplus was $15, since at price $5, 6 pounds would be purchased for only $30. This argument, however, ignores the income effects resulting from charging the consumer the intramarginal values for each successive unit.

could occur, since if we paid an average value per unit, our total payment for all units would, by definition, be the total value, leaving no gains), is termed *consumer's surplus.* If consumer's surplus is to be a useful construct, however, it must be capable, at least in principle, of being identified with some observable real-world problem or experiment. That is, knowledge of the value of consumer's surplus must imply something operational about the consumer's responses to price or quantity changes (or anything else affecting consumer welfare). Measures that correspond to no such operational experiment are useless.

The first systematic analysis of this problem, utilizing the modern concept of demand, was undertaken by Alfred Marshall, in his *Principles.* Marshall reasoned as follows. At any quantity consumed of some good, the height of the demand curve represents the consumer's marginal valuation of that good in terms of other goods forgone. Consider Fig. 11-6, showing the demand for tea, to use Marshall's example. At a price above, say, $ 10, the consumer purchases no tea at all, but when p = $ 10, he or she (she, for convenience) purchases 1 pound. When the price is $9, she purchases 2 pounds, at $8,3 pounds, etc. According to Marshall, this consumer is willing to pay $10 to obtain 1 pound, $9 to obtain a second pound, $8 to obtain the third pound, and so on down the demand curve (assumed linear in this discussion for computational ease.)^

If the market price of tea is $5, this consumer will purchase 6 pounds, at a total expenditure of $30. However, the total value of 6 pounds of tea to this consumer

ignore the errors associated with treating discrete purchases as a continuum.

is evidently $10 + $9 + $8 + $7 + $6 + $5 = $45, the sum of the marginal evaluations of each succeeding pound. This total value is the area *under* the demand curve. Subtracting the rectangle representing the consumer's actual expenditure on the good leaves the triangular area *to the left* of the demand curve, above the market price, as the gain to the consumer from purchasing 6 pounds of tea at $5 each, rather than at her successive marginal evaluations. Marshall called this area *consumer's surplus,* but added a caveat. In the text, he qualified his analysis as requiring one "to neglect for the moment the fact that the same sum of money represents a different amount of pleasure to different people." In the mathematical appendix (Note VI), Marshall identified the "total utility of the commodity" with the area under the demand curve, defined by an integral, and restated the above qualification by saying "... we assume that the marginal utility of money to the individual purchaser is the same throughout."^

The meaning of these phrases is anything but clear, and they have led to considerable confusion since publication. The text phrase seems to indicate that interpersonal comparisons of utility are a necessary prerequisite for the use of consumer's surplus; in the appendix, Marshall's concern is that as more of a commodity is purchased, money will yield less satisfaction to the consumer, destroying any linear relationship between money and utility. In 1942, Paul Samuelson further pointed out and analyzed the ambiguity surrounding the phrase, "constant marginal utility of money."* To Marshall, money provided no direct utility to the consumer; it was a device solely for lowering the transaction's cost of exchange. The concurrently developed general equilibrium theory of Walras, however, treated money as that one good which happened to have the additional property of serving as the medium of exchange, a *numeraire* commodity whose price was unity. Let us analyze these puzzles.

Returning to Fig. 11-6, we did not specify exactly what kind of demand curve this is, i.e., what is being held constant as the price of tea changes. If $10 represents the maximum a consumer would pay for 1 pound of tea, *and she is in fact charged that entire amount for that unit,* then she must be no better or worse off having made the purchase. The *maximum* a consumer is willing to pay in order to acquire some good is, by definition, the amount that leaves the consumer indifferent to the new versus the old situation, i.e., on the same indifference level. If a consumer is actually charged the maximum amounts she is willing to pay for succeeding units of a good, then these marginal values must represent points along a Hicksian, or compensated, utility-held-constant demand curve. These are the demand curves derived from minimize $M = Yl\ Pi^{x}i$ subject to $U\{x\backslash, \ldots, x_n) = U°$, where the associated expenditure function $M^*(p\backslash, \ldots, p_n, U°)$ — $YlPi^{x}?$ indicates the minimum cost of maintaining utility level $U°$. Only for these demands is Marshall's reasoning appropriate.

^Alfred Marshall, *Principles of Economics,* 8th ed., Macmillan, London, 1920, mathematical Note VI. ^Paul A. Samuelson, "Constancy of the Marginal Utility of Money," *Studies in Mathematical Economics and Econometrics, in Honor of Henry Schultz,* O. Lange et al. (eds.), University of Chicago Press, Chicago, 1942.

Algebraically, the area to the left of a consumer's demand curve, for a reduction in price, is *—fxjdp;.* The units of this integral are that of money income, being price times quantity. By the envelope theorem, the Hicksian demand functions are the first partials of the expenditure function. Therefore, the area to the left of these demand curves is simply a change in the value of the expenditure function:

$$_{P_i} = M^*(p^0, U^0) - M^*(p\backslash U^0) \qquad (11\text{-}54)$$

where p° and p^l are the initial and final price vectors over which the integral is taken. The areas to the left of Hicksian demand functions therefore represent changes in expenditure holding utility constant. A moment's reflection reveals that these areas therefore indicate the amount a consumer would be willing to pay (or have to be paid) to willingly accept some change in property rights, e.g., a change in the purchase price of some good. If we interpret our numerical example as a Hicksian demand curve, the consumer would be willing to pay up to $15 to be able to purchase tea at $5 per pound, rather than be faced with a price in excess of $10. Similarly, she would be willing to pay $9 to face the price $5 rather than $7.

If more than one price were to change, the demand curve for any one good will start to shift, as the price of some other good changes. How could one calculate the amount a consumer would pay to have the prices of two interrelated goods, say, $x\backslash$, and x_2, each decrease by some amount? One could start by calculating the area to the left of the demand curve for x_{15} holding p_2 constant. The demand for x_2 would then shift to some new position. Then, the area to the left of x_2 could be calculated and added to the previous area. Will this give us the desired answer? What if we had started by changing p_2, calculating the area to the left of the demand for x_2, allowing the demand for $x\backslash$ to shift, and then adding to that the area to the left of the resulting demand for $x\backslash$? Would we get the same answer?

For the case of the Hicksian, or compensated demands, we will get the same answer no matter what order or "path" of price changes we choose. For multiple price changes, letting p° represent the initial price vector and p^l the final price vector, the sum of the areas to the left of the Hicksian demand functions is still simply the change in the value of the expenditure function between the initial and final prices:

$$= M^*(p^\circ, U^\circ) - M^*(p\backslash U^\circ) \qquad (11\text{-}55)$$

Equation (11-55) shows that the sum of all these areas is simply the difference in the minimum expenditures necessary to reach the indifference level U° at the alternative price levels. The difference in the value of the expenditure function at the final versus the initial prices indicates how much a consumer would be willing to pay (or have to be paid) to face the final, rather than the initial, prices.

To sum up, the areas to the left of the Hicksian demand functions are *always* interpretable as the amounts consumers would be willing to pay to face a lower

price, or, if the price is to be raised, by how much they must be compensated in order to voluntarily accept the higher price. These amounts, often called *compensating variations,*^ are always well defined, without the need for further assumptions about the shape or functional form of the utility function. These areas are geometric representations of changes in the value of the expenditure function, which is well defined for all utility functions satisfying the standard curvature properties, i.e., strictly increasing and quasi-concave.

The area to the left of a Marshallian demand function, however, has no such easy interpretation. Unlike the Hicksian demands, the Marshallian demand functions, derived from utility maximization subject to a budget constraint, are *not* in general the partial derivatives of some integral function, e.g., total expenditure or utility. Therefore, the integrals of the Marshallian demands are not expressible in terms of changes in some well-defined function of the initial and final prices and income levels.

From Roy's equality, the Marshallian demands are the first partials of the indirect utility function *divided by the marginal utility of income.* Thus,

Equation (11-56) says that the area to the left of a Marshallian demand curve is a sum (integral) of changes in utility *(dU*/dpi),* as some price /?, changes, multiplied by a factor l/X^M that converts the change in utility into units of money. The conversion factor itself varies as p_t changes; that is, as price changes, a dollar, at the margin, is worth differing amounts of utiles. Although the integral in (11-56) takes on some value, it is not identifiable with any operational experiment concerning consumer behavior.

In the case of multiple price changes, the value of the integral depends on the order in which prices are changed. That is, even for specified initial and final price and income vectors, the value of the integral is not unique but dependent on the path of prices between the initial and final values. Therefore, without further assumptions on the shape of the indifference curves, there is no obvious way to evaluate, in some useful sense, the gains or losses derived from one or more price changes using the Marshallian demand functions alone.

If, however, the marginal utility of money term is "constant," it can be moved in front of the integral sign. This expression can then be integrated to yield a function of the endpoint prices (and money income):

$$J \qquad d \quad = \hat{}[U(p^l, M) - U(p^\circ, M)] \qquad (11\text{-}57)$$

t Following Hicks, for price *increases,* these areas are often referred to as *equivalent variations;* however, they are conceptually identical to the compensating variations.

In this case, the area to the left of the Marshallian demand function equals a change in utility divided by the marginal utility of money. Equation (11-57) in some sense rescues Marshall's claim that the area to the left of a demand curve is interpretable as a change in utility under the assumption of constant marginal utility of money, though how much of the preceding discussion he had in mind can easily be debated.

We have, however, glossed over the meaning of "constancy" of k^M, the marginal utility of money. In fact, k^M cannot literally be a "constant," i.e., some numerical value, say, 3 utiles per dollar, for all prices and income. Recall that k^M is homogeneous of degree -1 in prices and money income. From Euler's theorem,

$$\sum \frac{\partial k^M}{\partial p_i} p_i + \frac{\partial k^M}{\partial M} M = -k^M$$

Therefore, k^M cannot be independent of all its arguments; this would make the left-hand side of this expression vanish, while leaving the right-hand side at some nonzero, negative value. The marginal utility of income, k^M, can, for example, be independent of all prices, but not income also, or it can be independent of money income and up to $n - 1$ prices.

What meaning, therefore, can be given to the concept "constant marginal utility of money," and what implications does it have for the analysis of consumer's surplus? Since $\partial U^*/\partial p_i = -k^M x_i^M$ and $\partial U^*/\partial M = k^M$, applying Young's theorem on invariance of partial derivatives to the order of differentiation yields (omitting superscripts)

$$-\left[k\frac{\partial x_i}{\partial M} + x_i \frac{\partial k}{\partial M}\right] = \frac{\partial k}{\partial p_i} \qquad (11\text{-}58)$$

Suppose $\partial k^M/\partial p_i = 0$, $i = 1, \ldots, n$, in which case k^M can be moved outside the integral, as in Eq. (11-57). Then $(M/x_i)(\partial x_i/\partial M) = -(M/k)(\partial k/\partial M)$ for $i = 1, \ldots, n$, i.e., the income elasticities are all equal (necessarily to unity, from the budget constraint); thus the utility function must be homothetic. Denoting the Marshallian area C, we have $C = (1/k^M)[U^*(p^1, M) - U^*(p^0, M)]$. Thus for homothetic utility functions, where the indifference curves are all radial blowups of each other, the Marshallian area represents the unique monetary equivalent of a change in utility; the coefficient that converts utiles to money income is invariant over the price change.

Suppose now that k^M is a function only of one price, say p_n. Then from Eq. (11-58), $\partial x_i/\partial M = 0$, $i = 1, \ldots, n - 1$. Since there is no income effect for goods 1 through $n - 1$, the Marshallian demands for goods 1 through $n - 1$ coincide with the Hicksian demands. This is the situation produced by "vertically parallel" indifference curves. (See Prob. 20, Chap. 10.) Therefore, the interpretation of the area to the left of any of these Marshallian demand curves is identical to the case of the Hicksian demands, i.e., the willingness to pay to face the lower price. The areas to the left of these Marshallian demand curves have meaning only because they are also the areas to the left of the Hicksian demands.

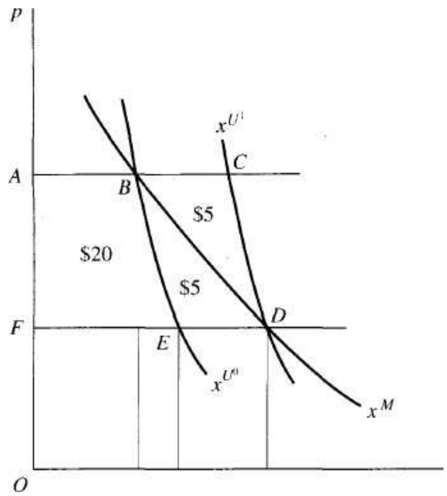

FIGURE 11-7
The Various Consumer's Surpluses. Initially, at price *OA,* the consumer purchases *AB*. When the price is lowered to *OF,* she moves down the Mar-shallian demand curve *BD* and purchases *FD*. *BE* is a section of the Hicksian (utility-held-constant) demand curve at the initial utility level; *CD* is a Hicksian demand curve at the level achieved when *FD* is purchased at price *OF.* Then *ABEF* is the amount the consumer would be willing to pay to face price *OF* instead of *OA; ACDF* is the amount the consumer would have to be paid to voluntarily accept the higher price *OA,* given the preexisting right to face *OF.* The area to the left of the Mar-shallian demand curve, *ABDF,* has no operational meaning.

Example

Consider Fig. 11 -7 in which various demand functions for some good x are displayed. At price *OA,* a consumer purchases *AB(= OB');* the Marshallian demand curve for x passes through points *B* and *D*. The consumer attains some utility level U° at point *B*. The curve passing through *B* and *E* is the Hicksian demand for JC, i.e., the demand for x derived from cost minimization, holding utility constant at $U = U^\circ$. When the price is lowered to *OF,* the consumer increases consumption of x to *FD*. The pure substitution effect of this price change is *B'E';* the income effect (assumed positive) is *ED (= E'D')*. At point *D,* the consumer achieves utility $U^1 > U^\circ$. The curve passing through *CD* is the Hicksian demand for x holding utility constant at U^1.

Suppose now area *ABEF* = \$20, *BDE* = \$5, and *BCD* = \$5. Thus, between prices *OA* and *OF,* the area to the left of the Hicksian demand curve at the initial utility level U° is \$20, the area to the left of the Marshallian demand curve is \$25, and the area to the left of the Hicksian demand at the final utility level U^1 is \$30. Here's the question: These values, \$20, \$25, and \$30 are all well defined mathematically, but what are the questions they answer? That is, what operational, i.e., observable (even if hypothetical), experiments involving consumer behavior are answered by these values?

The area to the left of x^u , \$20, is the amount the consumer would be willing to pay in order to face price *OF* instead of *OA*. At *OF,* the consumer can attain utility U° with a total expenditure \$20 less than at price *OA*. This would be indicated by the change in the value of the expenditure function between these two prices. Likewise, the area to the left of the Hicksian demand curve x^u , \$30, is the amount the consumer would have to be paid, or compensated, in order to voluntarily accept the higher price *OA*. This higher value assumes that the consumer already has the right to face the lower price *OF.* As the diagram makes clear, with normal (noninferior) goods, the compensating variations are necessarily larger for price increases between two price levels than for price decreases between those same two price levels. If the initial price is *OA,* the consumer will pay at most $20 to get the price lowered to *OF.* If the price is already *OF,* the consumer is purchasing more of the good and is on a higher indifference curve; now the consumer requires a $30 bribe to face the original higher price *OA*!

The amount $25, on the other hand, answers no operational question at all. That is, there is no finite experiment involving this consumer for which $25 represents some revealed value of an outcome. This area is best viewed as an approximation to the areas to the left of the compensated, or Hicksian demands; it has no other meaning.

Empirical Approximations

Although the areas to the left of the Marshallian and Hicksian demand curves are conceptually different, it is obvious that the extent to which such areas would differ from each other in practice depends mainly on the income elasticity of the good in question, and the size of the price change. If the Marshallian demand functions are more easily estimated in practice (since the Hicksian demands require that utility be held constant), it would be handy to be able to approximate the Hicksian areas from knowledge of the Marshallian demands. In 1976 Robert Willig presented some formulas in this regard.J Referring to Fig. 11-7, and using Willig's notation, let C = area *ABEF* (the compensating variation at the initial utility level), *A* = *ABDF* (the Marshallian "surplus"), and *E* = *ACDF,* the compensating (or equivalent) variation at the final price level. Willig derived, using a Taylor series approach, the following approximation formulas:

ri°\A\ C-A rJ'\A

2M ~ \A\ ~ 2M

and

rj°\A\ A - E r]'\A

2M ~ \A\ ~ 2M

where *rf* and *r\x* are, respectively, the smallest and largest values of the income elasticity of the good in question within the region of the price change. If, for example, the income elasticities are near unity and the area to the left of the Marshallian demand function is approximately 5 percent of money income *M,* then these areas are within a few percent of each other. If, however, one is estimating a "welfare loss" triangle rather than the entire trapezoidal area to the left of a demand curve, the percentage

^Robert D. Willig, "Consumer's Surplus Without Apology," *American Economic Review,* 66:589-597, September 1976.

impact of the income effect might be much more significant, since the comparisons will be made among areas of similar magnitude.

It should also be noted that for price changes which are not small, the differences between the areas to the left of the Hicksian and Marshallian demands can get quite large. Consider, for example, the demand curves associated with the simple Cobb-Douglas utility function *U — Xix*$_2$. The Marshallian demand for jq is jCj^{M} = *M/2p* the Hicksian demand is *x* = *(Up*$_2$*/p\)*$^{il/2)}$. Both of these demand curves are asymptotic to the horizontal axis *(x\)*. However, below any arbitrary price *p°,* the area to the left of the Marshallian demand curve tends to infinity, whereas the area to the left of the Hicksian curve is finite. That is, as *p* —> 0, *J*^{Pf} *xfdpi* —► oo, whereas *J*$^{\wedge I}$ *x\dp* = *M*(p®, p*$_2$*,U)*. Thus, if one were interested in using these areas to measure how much a consumer would be willing to pay to face a price of zero rather than any finite price *p°*$_x$*,* no matter how small, the difference between the Marshallian and Hicksian areas would become unboundedly large. The area to the left of this particular Marshallian demand curve loses all empirical meaning, as one of the endpoint prices tends to zero.

Finally, we note that since the Marshallian and the Hicksian demand functions are related by the Slutsky equation [or, more precisely, by the fundamental identity (10-35)], it is always possible, in principle at least, to calculate either demand function from the other. If, for example, a system of Marshallian demand functions has been empirically estimated, one could in principle integrate back to the utility function and then derive the Hicksian demands, using the expenditure minimization hypothesis. However, this procedure is apt to be intractable for even the simplest demand systems (though it was done earlier for the Cobb-Douglas case). A single linear demand equation in one price has been analyzed by Jerry Hausman, using the Roy identity. * However, it is clear that the task will in general be complex, though given the advances in computer technology, suitable approximation procedures may someday become available.

The phrase "consumer's surplus" is used in two contexts. As a tool of positive economic analysis, consumer's surplus is simply another term for the gains from trade. The law of demand implies that individuals participating in voluntary trade will always pay less for a total quantity of goods than they would if that quantity were offered on an all-or-none basis. It follows that individuals can be expected to devote some resources to enlarging this gain for themselves. The concept of consumer's surplus should therefore be the behaviorial basis for theories of the formation of monopolies and cartels and the political economy of legislation aimed at altering the terms of trade, i.e., the property rights, of the participants in exchange.

tjerry Hausman, "Exact Consumer's Surplus and Deadweight Loss," *American Economic Review,* 71(4):662-676, September 1981.

The most widespread use of an explicit concept of consumer's surplus, however, has been in the area of welfare economics and social policy. In this context, a function measuring the "welfare loss" due to, for example, a set of excise taxes is formulated to measure the costs, in terms of forgone opportunities to trade (sometimes called *deadweight loss)* of a given tax policy. This loss function is construed as a function of the deviations of prices *pi* from marginal costs or, symbolically,

$$5E = f(p_1-MC_1,\ldots,p_n-MC_n)$$

Similarly, the areas to the left of demand curves are used to measure the potential gains from erecting various public works, e.g., dams, to lower the marginal cost of some good.

It is now well established that the only meaningful measures of consumers' benefits are changes in the value of the expenditure function. Such calculations require no exotic assumptions about the utility function. Their shortcoming is that they depend on a single indifference curve, and thus do not measure "benefits" per se, but rather amounts consumers would be willing to pay (or be paid) to face different constraints. Welfare loss functions such as the above represent attempts to generalize this concept to the case where marginal costs are not constant. They can be used to calculate (in principle) the amounts consumers would be willing to pay to avoid monopolies, distortionary taxes, or other policies that cause deviations from marginal cost pricing. We shall return to these matters in the chapter on welfare economics.

11.6 EMPIRICAL ESTIMATION AND FUNCTIONAL FORMS

In previous chapters we investigated the properties of the Cobb-Douglas and CES production functions and their associated cost functions. We also briefly investigated the generalized Leontief cost function. These specifications have been useful in cost and production theory and are also used in the empirical estimation of consumer demands. We shall now briefly analyze the CES and other functional forms that have been found to be useful in estimation of empirical demand relations.

Linear Expenditure System

The Linear Expenditure System (LES) is a generalization of the Cobb-Douglas utility function. It was developed by Klein and Rubin (1947-1948) and Samuelson (1947-1948) and investigated empirically by Stone (1954) and Geary (1950), and it is sometimes referred to as the Stone-Geary function. The function is basically the Cobb-Douglas function with the origin translated to a point *(fti*, *fc)* in the positive quadrant:

$$U(x_u x_2) = a, \log (*, - ft) + a_2 \log (x_2 - fa) \qquad \text{(H-59)}$$

where x_t — /?, and a_t (i — 1, 2) are positive and $OL\backslash + a_2 = 1$. Maximizing (11-59) subject to the budget constraint yields the first-order conditions

$$\cancel{x^{a}*} \quad - *pi = 0$$

$$—\hat{}— - A./72 = 0$$

$$M - p_x xi - p_2 x_2 = 0$$

From the first two equations we get $p_t Xi = Pifii + a, \blacksquare /X,$ and substituting them into the third equation yields $X = 1/(M — p\backslash fi\backslash —$ $Pifii)$. Therefore the demand functions are

$$x? = A + -(M - {}_{Pl}fa - P2P2) \qquad i = 1, 2 \qquad (11\text{-}60)$$

$$Pi$$

If we write the demand functions in expenditure form:

$$Pix? = Pi\ Pi + caiM - p_x p_x - p_2/3_2) \qquad i = 1, 2 \qquad (11\text{-}61)$$

we see that the expenditure on each good is linear in all prices and in income—hence the name Linear Expenditure System.

The Cobb-Douglas utility function may be regarded as a special case of the LES, with all the /Ts equal to zero. In fact economists working with the LES often describe consumers as first buying subsistence quantities of each good $(fi\backslash, f3_2)$ and then dividing the remaining expenditure among the goods in fixed proportions («i, a_2)- Since the *marginal* budget shares are constant, the LES has linear Engel curves, although preferences are not homothetic. These income-expenditure lines all pass through the point $\{fi_t, fi_2)$.

The indirect utility function corresponding to the LES can be derived by substituting the demand functions (11-60) into the direct utility function (11-59):

$$U* = ari \log \frac{}{P\backslash}(M - pi\hat{}i - p_2 fi_2) + a_2 \log \frac{}{Pi}(M - piPi - p_2 p_2)$$

$$= \log\text{--------------;—--------------} \qquad (11\text{-}62)$$

$$Pi\ Pi$$

Since $a.\backslash + a_2 = 1$ and the indirect utility function is invariant to monotonic transformations, we can exponentiate (11-62) and delete the constant a"¹ a"² from it. Such an operation yields

$$= \frac{\underline{M - p_x p_x - p_2 p_2}\text{-------}\hat{}rz\hat{}2\text{-----}}{P\backslash\ Pi} \qquad (\text{u-63})$$

Equation (11-63) can be inverted to get the expenditure function

$$M* = Up\hat{}pl^{l} + p!/5i + {}_{P2}/3_2 \qquad (11\text{-}64)$$

The Marshallian demand functions can be obtained by applying

Roy's identity to (11-63); applying Shephard's lemma to (11-64) yields the Hicksian demands.

CES Utility Function

In direct analogy with production theory, the CES utility function (Arrow et al., 1961) has the form

$U(xi,x_2)$ = (a,Jtf+a₂x₂ᴾ)¹ᐟᴾ $P < 1$ (11-65)

The first-order conditions for utility maximization are

a 1*1 {oc\X\ +0(2X2) — Xp = 0

p—\ I p 1 P\({—P)/P
i r

0,2X2 (a 1*1 + «*2) — Xp = 0

M — /?i*i — /?2-^2 = 0

The first two equations can be combined to get

^ f ^ i) ' / " " ' > (H-66)

x₂ \0np2

The elasticity of substitution, *a*, is

ff = _aiog(*r/*2)=_i_ (11

d\og(pi/p₂) 1 - / 0

The greater the value of the parameter p, the greater the degree of substitutability between the commodities.^

The Marshallian demand functions corresponding to the CES utility function can be obtained by substituting (11-66) into the budget constraint, which gives

M - pA ---- x₂ - p₂x₂ = 0
\u\P2J

or

M------ !----^—!----x₂ = 0 (11-68)

Thus,

M __ ______________
(a° p\~ᵃ + a? £>9~ᶠᶠ)

(11-69)

J

Since the preferences represented by the CES utility function are homothetic, the Marshallian demand functions (11-69) are linear in income.

$p = 0$, the CES function becomes the Cobb-Douglas function.

To derive the indirect utility function, substitute (11-69) into (11-66) and note that pa = p/(1 — p) — a — 1:

{a₂/p₂)°M

$$\left[\left(\alpha_1^{1+\rho\sigma} p_1^{-\rho\sigma} + \alpha_2^{1+\rho\sigma} p_2^{-\rho\sigma}\right)\left(\frac{M}{(\alpha_1^{\sigma} p_1^{1-\alpha} + \alpha_2^{\sigma} p_2^{1-\sigma})}\right)\right.$$

$$(\alpha_1^{\sigma} p_1^{1-\sigma} + \alpha_2^{\sigma} p_2^{1-\sigma})^{1/\rho} \frac{M}{(\alpha_1^{\sigma} p_1^{1-\sigma} + \alpha_2^{\sigma} p_2^{1-\sigma})}$$

M (11-70)

The expenditure function is obtained by inverting the indirect utility function:

M* = $U(a?p\backslash\text{-}^{a} + a°p\backslash\text{-}^{a})^{1/(\{-a)}$ (11-71)

Indirect Addilog Utility Function

When a utility function is specified, it is in principle possible to derive the commodity demand functions by maximizing the utility function subject to the budget constraint; however, a closed-form solution is not always available. Duality theory suggests that an alternative is to specify an indirect utility function. Any function that is (1) nondecreasing in income, (2) nonincreasing and quasi-convex in prices, and (3) continuous and homogeneous of degree 0 in prices and income is a legitimate, indirect utility function that corresponds to some consumer preferences; and the commodity demand functions can be readily derived by using Roy's identity.

A useful functional form is the "addilog" indirect utility function introduced byHouthakker(1965):[t]

$=_{ai}(\text{—})$ $P\backslash J$ $+a_2$ M (11-72)

The demand functions obtained from the addilog are

$$M \_ \frac{-du^*/d_{Pi}}{8U^*/dM}$$

(11-73)

direct utility function and the cost function corresponding to the addilog have no closed-form solutions.

If we divide *xf*[1] by *x*™ the result will be log-linear in income and in the relative price of xi and x_2:

g -fe- R—-
QI2P2P
2

= log -^ - (ft + 1) log p_1 + (ft + 1) log p_2 + (ft - ft) log M
<*ft

(11-74)

The parameters of the system of demand equations can therefore be estimated by the least-squares method.

Translog Specifications

The translog indirect utility function (Christensen, Jorgenson, and Lau, 1971, 1975) has been one of the most widely used functional forms in empirical demand analysis. One advantage of the translog is that it is a flexible functional form: It can be a second-order local approximation to an arbitrary indirect utility function. The basic translog specification is given by

log *U*(pi, ..., p_n, M)* = — *2_,^{a}i* log ~ *z_^£_lfi^{k}J* ^°g — ^°g ~
(11-75)

where *J2j ^{a}j* — 1 ^{an}d *fikj* = *Pjk* for all *k* and *j.*

It is often more convenient to work with the expenditure share equations rather than the demand equations when using translog specifications. Note that

$$\frac{-d \log U79 \log pi}{a \log *y*/a \log M} \sim \frac{-dU*/dpi\ p_i/U*}{du*/dM\ M/U*} \_ \frac{ptxf^1}{M}$$

Also, the translog specification can be alternatively written as

1

log *U** = log *M* — V^ *aj* log *pj* Y^ V^ ft^ (log p_k — log *M)* (log *pj* — log *M)*

(11-77) Therefore the shares &>, can be obtained upon logarithmic differentiation of (11-77):

, . ^- **J J J 2 *—Jit**
1 + \ *Zk Zj Pkj \og(pj/M)* + *\Zk Zj Pkj \og(p_k/M)*

/ = ! , . . . , « (11-78)

A special case of the basic translog is the homothetic translog, which is obtained by imposing the restrictions

Given these n restrictions the indirect utility function and the share equations are

$$\log U^* = \log M - \sum_i \alpha_i \log p_i - \frac{1}{2}\sum_i \sum_k \beta_{ik} \log p_k \log p_i \qquad (11\text{-}79)$$

$$\theta_i = \alpha_i + \sum_j \beta_{ij} \log p_j \qquad i = 1, \ldots, n \qquad (11\text{-}80)$$

Equations (11-80) show that the expenditure shares are independent of income, which confirms that preferences are homothetic. Also note that the indirect utility function (11-79) can be inverted to obtain the homothetic translog expenditure function:

$$\log M^*(p_1, \ldots, p_n, u) = \log u + \sum_j \alpha_j \log p_j + \frac{1}{2}\sum_k \sum_j \beta_{kj} \log p_k \log p_j \qquad (11\text{-}81)$$

The expenditure function (11-81) is frequently used in empirical studies of production. Interpreting M^* as total cost C^*, the factor share equations are readily obtained using Shephard's lemma:

Almost Ideal Demand System

A theoretically plausible system of demand equations may be derived from an expenditure function as long as the expenditure function is (1) continuous and non-decreasing in prices and utility and (2) concave and homogeneous of degree 1 in prices. An example is the almost ideal demand system (AIDS). It is obtained from the (logarithmic) cost function:

$$\log M^*(p_1, \ldots, p_n, U) = a(p_1, \ldots, p_n) + U b(p_1, \ldots, p_n) \qquad (11\text{-}83)$$

where

$$a(\cdot) = a_0 + \sum_j \alpha_j \log p_j + \frac{1}{2}\sum_k \sum_j \gamma_{kj} \log p_k \log p_j \qquad (11\text{-}84)$$

ht is also possible to specify a translog profit function; but the translog profit function and the translog cost function will in general correspond to different technologies.

$$Ykj = Yjk$$

Using Shephard's lemma the share

equations are 9 log M*

CO; =

3 log *pi*

ii + > *Yi j* log p / + *piU* —

— fAlog(M/P) i = 1, . . . , n

where log *P* = *a(-)*. Deaton and Muelbauer argue that *P* can be considered as a price index and it may be approximated by *J2j* ^ 1°§ P« • Given this approximation the system of demand equations are linear in the logarithm of prices and real income and can be estimated easily.

PROBLEMS

1. Consider the class of utility functions that are *additively separable,* i.e.,

For this class of utility functions, show:

1.328 At most one good can exhibit increasing marginal utility and in that case *(i)* that good is normal whereas all remaining goods are inferior and *(ii)* that good is a *net* substitute *(dxf/dpj* > 0) for the remaining goods whereas the remaining goods are all complementary to each other.

1.329 If all goods exhibit diminishing marginal utility, *(i)* all goods are normal and *(ii)* all goods are net substitutes.

2. For demand functions derived from additively separable utility functions, show that

(a)

$$\frac{\partial x_i/\partial p_k}{\partial x_j/\partial p_k} = \frac{\partial x_i/\partial M}{\partial x_j/\partial M}$$

$$(b)\quad s_{ij} = -k\frac{\partial x_i^M}{\partial M}\frac{\partial x_j^M}{\partial M}$$

3. Consider the indirect utility function $U^*(p_x, \ldots, p_n, M) = U^*(p_x/M, \ldots, p_n/M, 1) = V(r_u \ldots, r_n)$, where r, = *pi/M.* Show that if *V* is additively separable in r, ..., /-,,, then

dxf/dpk _

$_{x_i}\ dx f/dp_k$
$\sim Xj$

1.330 Using the results of the previous two problems, show that if $U(x)$ and $V(r)$ are both additively separable, then $U(x)$ is homothetic.

1.331 Consider a set of demand functions $x_i = x^*(p_1, \ldots, p_n, M)$ whose only known properties are homogeneity and satisfying a budget constraint; i.e.,

$$\sum_{j=1}^{n} \frac{\partial x_i^*}{\partial p_j} p_j + \frac{\partial x_i^*}{\partial M} M = 0 \qquad i = 1, \ldots, n \tag{1}$$

and

$$\sum_{i=1}^{n} p_i x_i^* = M \tag{2}$$

Show that under (1) and (2) alone, Hicks' third law holds, i.e.,

$$\sum_{j=1}^{n} p_j S_{ij} = 0 \qquad \text{where} \qquad S_{ij} = \frac{\partial x_i^*}{\partial p_j} + x_j^* \frac{\partial x_i^*}{\partial M}$$

1.332 Suppose a consumer's utility function is additively separable and, in addition, the marginal utility of money income is independent of prices. Show that the elasticity of each money-income demand curve is everywhere unity.

1.333 The development of utility theory can be regarded as the attempt to provide a theory that explains the phenomenon of downward-sloping demand curves. It was soon discovered that the Hicksian pure substitution terms were symmetric, a result, said Samuelson, "which would not have been discovered without the use of mathematics."

1.334 Explain why it is something of a non sequitur to assert the symmetry of the substitution terms.

1.335 What behavioral differences are there, if any, in terms of observable price-quantity combinations, between a theory of the consumer that includes such symmetry and one that does not require such symmetry?

1.336 (A Yiddish parable, with love to Milt Gross.) Morty, de one in de schmatah beezness by Coney Highland (sotch a mensch), is exessparaded from de rise in de price from gebardins (is something tarrible), from \$200 to \$300 itch suit. Silskin cuts, denks Gut, is de same at \$300. Voise, some doidy gonif didn't pay all de bills he chodged opp. Is diss a system? Meelton, dot spuxman from de right, lest year bought

three of itch, for his trip
witt spitches from China. Diss year, de books show one gebardin and four silskins. Leo,
dot odder beeg shot, lest year bought four gebardins and three silskins, and diss year de
books show five gebardins and two silskins. So, Nize Baby, I hesk you, who is de gonif?

1.337 Which of the following sets of observations of price-quantity data are consistent with
utility maximization?

(a)

(			x^1 =	2
	=		x^2 =	2
	=		x^3 =	2
b			x^1 =	1
	=		x^2 =	3
p^3	=	(5,3,	x^3 =	2
p^1	=	(4,	x^1 =	2
p^2	=	(5,	x^2 =	3
p^3	=	(5,	x^3 =	3

10. A certain consumer is observed to purchase bundles x' at prices p':

$p^1 = (4,2,3)$ $x^1 = (1,3,3)$ $p^2 = (3, 2, 3)$ $x^2 = (2, 3, 2)$ $p^3 = (2, 3,3)$ $x^3 = (.5, 1,5)$

What is the sex of the consumer?

11. Consider the two demand functions

$$X\backslash = \frac{P_2}{Pi} \qquad X_2 \frac{M}{Pi} \text{---} 1 \qquad T \quad A4 \quad Pi < M$$

Integrate these demand functions to find the class of utility functions from which they are derived.

12. Answer the previous question for the demand functions

$$*i = \frac{p_2 M}{;\text{—}2 + pi}$$

$$X_2 = \frac{P\backslash M}{P\backslash Pl + P_2}$$

13. A consumer faced with prices p_x — 9, p_2 = 12 consumes at some point x°, where JCI =4, x_2 = 7, $U(x°)$ — 10. When p_x is lowered to $p\backslash$ — 8, the consumer would move to point x^1, where %\ =6, x_2 = 6, $U(x^l)$ = 15. From these data, estimate the following values:

1.338 How much would the consumer be willing to pay to face the lower price of x_x ?

1.339 How much would a consumer initially at x^1 have to be paid to accept the higher price of X! voluntarily?

1.340 Are your answers to *(a)* and *(b)* exact calculations of these values, or are they approximations? If the latter, is the direction of bias predictable?

1.341 How much better off is the consumer at x^1 than at x°?

14. In his *Principles,* Marshall gave the following definition of consumer's surplus:

1. The amount a consumer would pay over which he does pay for a given amount of a good rather than none at all.

Several other consumer's surplus measures have been proposed, e.g.:

1.342 The amount the consumer would pay for the *right* to purchase the good at its market price rather than have no good at all.

1.343 The amount the consumer would have to be paid to voluntarily forgo entirely consumption of the good at its present level.

1.344 The monetary equivalent of the gain in utility that the consumer receives by being able to purchase the good at the market price rather than purchase none at some higher price.

1.345 The monetary equivalent of the fall in utility a consumer would experience if the right to purchase the good at the market price were taken away.

1.346 Discuss the relationship, if any, between these measures.

1.347 Show that measure 2 is greater than measure 1.

1.348 Show that if the good is normal over the whole range of consumption, then measure 3 is greater than measure 2.

(d) Would knowledge of measures 1 to 3 enable one to determine which of two mutually exclusive projects would result in maximizing the consumer's utility? 15. Show that if a consumer's income consists of a numeraire commodity that enters the utility function, then the line integral generating consumer's surplus measures will be path-independent only if all nonnumeraire commodities have zero income elasticities.

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CHAPTER

12

INTERTEMPORAL CHOICE

12.1 w-PERIOD UTILITY MAXIMIZATION

The analyses of previous chapters have all concerned choices among contemporaneous commodities. An important class of choices made by consumers, however, relates to consumption over time, that is, how one allocates income earned in different time periods to consumption. We notice, for example, that college students are in general poor, that earnings are highest during a person's middle age, and earnings fall after retirement; the typical response to this pattern of income is to borrow when one is young and lend (e.g., in the form of investing in a retirement fund) during middle age. It seems that when income is earned in an uneven pattern, individuals attempt to "smooth out" their consumption through borrowing and lending. In this way, people's consumption varies less than their income. Is there some systematic basis for this behavior?

We begin this discussion by considering consumption in just two time periods. Denote the present as period 1 and the future (next year) as period 2, and consumption in periods 1 and 2 as *x*\ and x_2. Suppose a person earns *x*® in the present (this year) and *x*® m me future (next year). Suppose also that this individual can borrow and lend in the "capital market" at interest rate *r*. What this means is that any income *y* not spent this year can be loaned to others, in return for which the consumer receives some greater amount *y* + *ry* = *y(l* + *r)* next year. Alternatively, the consumer can increase present consumption by some amount *y* and repay *y(* + *r)* next year. The opportunity cost of consuming income *y* this year is thus forgoing consumption of *y{* + *r)* next year. The *price* of present consumption is thus 1 + *r* units of future consumption; alternatively, the price of future consumption is 1/(1 + r) units of present consumption. We commonly say that the *present value* of *%y* 1 year from

= w

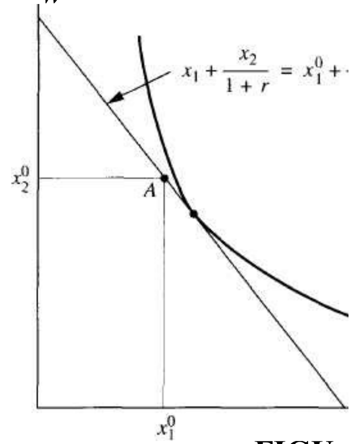


FIGURE 12-1 *Maximization of Utility Subject to a Wealth Constraint.* On a general level, utility maximization subject to a wealth constraint is structurally the same as any such problem in which endowments are brought to the market. In this diagram, the

tan-gency occurs at a point of net borrowing, since jtf* > x®, x^ < x\. Roy's identity states that 9t/*/3(l/(l + r)) = ~k^M(x\ - x?). In this case, since an increase in the interest rate will rotate the budget line clockwise through A, the consumer will be worse off.

he r w ea lth , i.e ., 1 =

now is $y/ (l + r); this is merely the quantity y times its price in terms of present consumption. The interest rate, to quote Irving Fisher, is the *premium for earlier availability of goods)*

Wealth W in the present is defined as the present value of current and future income. The consumer's budget constraint is that he or she cannot spend more than his or

W (1

The consumer maximizes *U{x\, xj)* subject to (12-1). Though we are using "income" and "consumption" interchangeably as arguments in the utility function, it is well to remember, as pointed out by Fisher, that "income" really consists of *consuming* something. "Saving" (or dissaving) is just a way of rearranging consumption over time. Income is realized when it is consumed.

The model is depicted in Fig. 12-1. The budget line has slope $dx_2/dx\backslash$ = — (1 + r), the price of *X[* in terms of x_2, and passes through the endowment point A, (xj\ x°). An increase in the interest rate represents an increase in the price of present cons umption, and has the effect of rotating the wealth constraint clockwise through A.

The Lagrangian for this problem is

$$= U(x_u x_2$$

t Irving Fisher, *The Theory of Interest,* New York, August M. Kelley, 1970. (First edition, The Macmillan Co., New York, 1930.)

producing the first-order conditions

(\2-3a)

____ r 0
1 +

x - x_2
°
____

and the constraint

x° — x 1 +r Combining (12-3a) and *(\2-3b)*
yields

Ul -

Equation (12-4) says that the consumer's marginal value of present consumption, $U\backslash/U_2$, equals the opportunity cost of present consumption in terms of future consumption forgone. It will simplify the algebra if we let $p = 1/(1 + r)$, the price of future consumption. Assuming the sufficient second-order conditions hold, the first-order conditions can be solved for the Marshallian demand functions

Xi = $_x$?(p, x°$_x$, JC°) / = 1, 2 (12-5)

It is apparent from the previous analyses of the demand for leisure and the "general equilibrium" demands that refutable implications cannot be derived from this model; like those other models, the parameters all enter the constraint. However, from the envelope theorem, *dU*/dp* = A *(x^* — xf). If the individual i s a net borrower in the present so that *x®* — *x^* < 0 and thus *x®* — *x^* > 0, then an increase in the interest rate (which decreases *p)* makes the individual worse off, since now a greater amount of future goods must be forgone in order to finance current consumption. Likewise, increases in the interest rate increase the achievable utility level for net lenders.

We can gain greater insight into the model by deriving the Slutsky equation, separating out the substitution effect and the wealth (income) effect.

The Hicksian demands can be derived minimizing the endowment in either period so as to achieve some arbitrary indifference level *U°*. We can therefore state the model as

minimize

subject to

The Lagrangian for this problem is then

£' = $_{Xl}$ + *p{x_2* - JC$_2$^{0}) + *k(U° - U(x_1,x_2))*

Assuming the first and sufficient second-order conditions hold, the implied first-order equations can be solved for the Hicksian demands

$$Xi =$$

Substituting these demands into the objective function produces a minimum "expenditure" type of function

$$x^*(p, U°) = < + p(x\% - x°_2) \qquad (12\text{-}6)$$

The fundamental identity linking the Marshallian and Hicksian demands is therefore

$$xV(p, U°) = xF(p, x\backslash(p, U°), x°_2) \qquad (12\text{-}7)$$

producing a Slutsky equation t

$$Bx^u \quad {}^{-2\,f}\left(\partial x_1^0 \right)/dr^M \backslash$$

If the interest rate increases, the price of future consumption, *p,* decreases. This produces a pure substitution effect toward less present and greater future consumption: $dx_2/dp < 0$. However, a change in the interest rate produces an attendant wealth effect. An increase in the endowment of present income is the same as an increase in wealth from any source, since income can be traded back and forth across time periods. Assume that consumption in both time periods enters the utility function as normal goods so that *dx^/dx®* > 0. The income, or, more properly, the wealth term on the right-hand side of the Slutsky equation, indicates that if, for example, the consumer is a net borrower in period 2 so that $x_2 — x_2 < 0$, the substitution effect will be reinforced by the wealth effect. In this case, an increase in the interest rate, in addition to making present consumption relatively more expensive, also lowers the consumer's wealth, producing an additional reduction in present consumption. If the individual is a net lender in period 1, the wealth and substitution effects oppose one another: An increase in the interest rate raises present wealth and leads to greater present consumption.*

Time Preference

The preceding discussion is formally identical to any utility maximization problem in which the consumer brings endowments to the market. What additional assumptions are appropriate if this is to be interpreted specifically as modeling consumption over time? We wrote utility as any well-behaved (strictly increasing and quasi-concave)

tSee the derivation of Eq. (10-75).

^However, if the interest rate has risen due to an increase in future prospects (see the next section),
present consumption may rise due to the implied wealth effect.

function $U(x\backslash, x_2)$. However, suppose we wish to specify that the individual's tastes do not change over time. In that case, the trade-offs a consumer would be willing to make, with regard to present versus future consumption, should not depend on the *date,* i.e., the time identifier. That is, an individual's marginal willingness to sacrifice a unit of present consumption in return for some amount of future consumption should depend only on the levels of consumption in each time period, and not whether this evaluation is taking place in 2000, 2005, or 2010. We can incorporate this assumption by specifying the utility function as $V(xi, x_2) = U(x\backslash) + U(x_2)$, with the same function U in each time period.^ This utility function is *additively (or strongly) separable* in $X\backslash$ and x_2; moreover, the separate parts are functionally identical. This utility specification would rule out "becoming accustomed" to some level of, say, luxury. The utility received in any one time period is independent of either past history or future prospects.

Irving Fisher wrote that people were "impatient" (he in fact included it in the subtitle of his book), meaning they preferred present consumption to the same amount of future consumption. If wealth can be costlessly stored, it is of course always preferable to have wealth now, say, in the form of money, rather than in the future, simply as a consequence of more being preferred to less. If one has money now, one can always choose not to consume it for a while; the reverse is not true. The set of opportunities for consumption is necessarily larger if the money is in hand, as opposed to becoming available in the future, assuming there is no cost of insuring against theft, etc. *Impatience* means something else: It refers to *preferences,* not opportunities. Impatience means that a given level of income v will generate less utility if it is consumed in the future rather than in the present.

We can express impatience by writing the utility function as

$$p > 0 \qquad (12\text{-}9)$$

$$i + p$$

For n time periods, this utility function is

$$U(x_t)$$

Thus, consumption in the future is given less weight than consumption now, with proportionate decreases in weight, the further into the future the consumption takes place.

Though we tentatively allow for it, the existence of time preference is in fact controversial, and empirically unconfirmed. It implies a "myopia" concerning the future. If we know the future will arrive (and uncertainty about the future is assumed

tOf course, any monotonic transformation of this function would work as well. Note also that it would be incorrect to use the same symbol, U, to mean both a function of two variables and a function of one variable.

not to be the source of time preference), why should the future count for less than the present in our utility? Having shifted consumption earlier, will we not regret having done so when the future arrives, and can we not anticipate this regret?

The general properties that are important in utility analysis, i.e., that *V(x)* be strictly increasing and quasi-concave, allow an infinite variety of "discounting" schemes by which the "goods" x_t are given successively less weight as * increases. However, we mean to interpret this function as the utility derived from consuming the *same* good, "consumption," in succeeding time periods. Robert Strotz argued compellingly that if, in some succeeding year, an individual could be predicted to change the weighting scheme for future years, then the original n-period utility function would essentially be inconsistent with itself and irrelevant.^ Suppose, for example, an individual were to decide right now, in the present, that he or she would consume wealth evenly for 2 years, and then in year three, consume one-half the remaining wealth, with constant consumption thereafter. Suppose 2 years pass, and year three is now "the present." Will the individual go forward with the original plan? Quasi-concavity of the utility function, by itself, does not rule out this behavior. However, such a consumption plan implies an inexplicable change in tastes. Suddenly, in a given year, the consumer is willing to sacrifice a much greater amount of future consumption than previously (or henceforth) in order to obtain a given amount of "present" consumption. It would be inconsistent with other applications of utility theory and the general paradigm of economics to allow such arbitrary taste changes over time. Therefore we would in general wish to impose this important property, commonly referred to as *dynamic consistency,* on intertemporal utility functions: specifically, that *the marginal value of consumption in period i in terms of forgone consumption in period j be independent of the date,* i.e., dependent only on the consumption levels in the two time periods.*

The utility function (12-10) has this important property. The marginal rate of substitution (marginal value of x, in terms of *Xj)* is

$$\frac{dxj}{dxi} \quad \frac{-V_t}{Vj} \quad \frac{-V+P)^{i}\text{-}^{i}U'_{i}(Xi)}{Ufa)} \quad Ufa)$$

Equation (12-11) says that the marginal value of consumption in period /, in terms of forgone consumption in period *j,* depends only on the *levels* of consumption in those two periods, and not *which* two time periods are involved, since the function $U(x_t)$ is the same for all / = 1, ...,«, and, moreover, only on the *number* of time periods separating the two periods, not *when* the time periods occur. Dynamic consistency

^Robert Strotz, "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies, 23(3):* 165-180, 1956.

* Strotz went on to say that if such changes in marginal rates of substitution between two time periods were anticipated, a consumer might rationally plan ahead to prevent these changes in plans, by, for example, tying up his or her wealth in trusts containing penalties for changing the original consumption plan. We shall not explore this

aspect of the problem here.

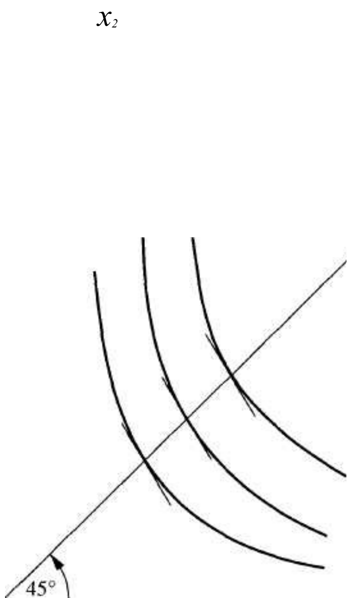


FIGURE 12-2
Indifference Curves for Additively Separable Utility Functions with Impatience. Indifference curves for $V(x_1, x_2) = U(x_1) + U(x_2)/(1+\rho)$ are displayed, with $\rho > 0$. Along the 45° ray, where $x_1 = x_2$, the slopes are $-(1 + \rho) < -1$. Since the slope of the budget (wealth) line is $-(1 + r)$, $x_1^* > x_2^*$ if $\rho > r$.

is thus assured. This utility function is depicted for two time periods in Fig. 12-2. Along the 45° ray from the origin, $x_1 = x_2$, and thus all indifference curves cut through this line with slope $-(1 + \rho) < -1$. That is, the absolute slope is the rate of time preference 1 + *p* and is greater than or equal to unity. If no "impatience" is assumed, the indifference curves have slope — 1 along the 45° ray.

Maximizing the utility function (12-10) subject to the wealth constraint

E ((1 +*r)'~'* ^ (1 +T)'-1

produces the tangency condition, for consecutive time periods *i, j,*

1

or

1

+

r

U

From this condition we can see how consumption of income is affected by the relation between the consumer's *preference* for earlier availability, as measured by 1 + *p,* and the market price of earlier availability, measured by 1 + r. Suppose, initially, that the consumer is not impatient so that *p* = 0. Then since we know the indifference curves have slope — 1 along the 45° ray and since the wealth constraint has the steeper slope — (1 + r), it must be the case that the tangency lies above the 45° ray so that *x*™ > *xf.* Given no impatience and a positive premium for earlier availability of goods, the consumer shifts consumption to the future. If the rate of time preference *p* is positive, but less than the market premium for earlier availability r, then obviously the same result will occur: The consumer will consume more income in the future than in the present. If, however, the rate of time preference exceeds the interest rate, then consumption will be shifted forward to the present, and we will find $x,^{M} > x^*f$.

The sufficient second-order conditions for utility maximization include, for all consecutive time periods/ and *j, j = l+i, — [(l+p)/(\ +r)*^{2}*]U''*(x_i) *— U''(XJ)* < 0. If *Xj ^=Xj* and *r ^ p,* these conditions do *not* imply diminishing marginal utility in each time period. Using the results of Chap. 11, Prob. 1, there can be (locally) increasing marginal utility in at most one time period, say period /. That is, there may be a convex portion of *U(Xj)* occurring in a neighborhood of some particular consumption level *x**. In that case, an increase in wealth could, locally at least, produce an increase in consumption in period *i* and a decrease in consumption in all other time periods. Thus, using only the assumptions of quasi-concavity and strong separability, one could not rule out an individual spending an unexpected windfall entirely in the year it was received. Typically, *U''* < 0 is asserted for all consumption levels, eliminating this possibility.

Since *V(x\, ...,* x_n*)* represents intertemporal utility, consumption takes place in the order *x\,* x_2*,* ... etc., unlike the model of contemporaneous consumption, where all goods are consumed together. With intertemporal utility, consumption levels in the past are fixed at whatever values were chosen. As time passes, additional x_t 's become fixed. The Le Chatelier results for consumer models say that the Hicksian demand functions become more inelastic as additional "goods" are held fixed. The model predicts, therefore, that individuals become less responsive to changes in relative prices as they age. This perhaps confirms the casual empiricism that young people often regard their elders as rigid and conservative. (Of course, as we age, the payoff from experimenting with new procedures is less, due to the smaller number of years left to enjoy the possible benefits.)

Let us now explore the regularity stated at the beginning of this chapter, the tendency of consumers to even out the flow of consumption. Assume for the moment that the consumer's rate of impatience equals the market interest rate, i.e., *p = r.* In this case, from Eq. (12-13), *xf = x^;* i.e., consumption must be the same in any two adjacent time periods. Thus income will be consumed at a constant rate. There is no analogy to this result in the utility theory of consumption of contemporaneous goods; we never purport to demonstrate that *x** = *x**. The result appears here because of the additional structure imposed on the utility function, in particular, the assumptions of dynamic consistency.

The tendency to even out the flow of consumption is illustrated further in Fig. 12-3. Suppose, for convenience, that *p — r* = 0. The curve labeled *x^* is the Hicksian demand curve for present consumption; on the vertical axis is the "price" of that good. The height of the demand curve, as always, is the marginal value, in this case, of present consumption in terms of future consumption forgone; thus the subjective price of present consumption along the demand curve represents the amount of future consumption the individual is willing to trade in order to acquire an additional increment of present consumption.

Suppose the individual has the option of consuming *x®* in each of two time periods vs. consuming *x®* + *Ax* in the first time period and then *x®* — *Ax* in the second period; i.e., let us compare the relative merits of steady consumption vs. "feast and famine." During the time of feast, the marginal value of present consumption is some relatively low value *c;* during famine, the marginal value of present consumption

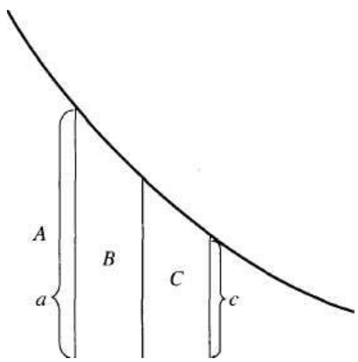


FIGURE 12-3
The Gain from Even Flows of Consumption. The total value of consumption, measured by the area under the compensated demand curve, is greatest when consumption is even. Neglecting interest and time preference, transferring a dollar of consumption from x° + *Ax* to *x®* — *Ax* increases total value by *a* — *c.* Alternatively, consuming *x®* twice yields total value *2(A + B)* > 2A + *B* + C, the amount the consumer would pay for the combination (x° + Ax, *x®* — *Ax).*

is relatively high, *a.* If the consumer can trade a unit of income from the time of feast to the time of famine, he or she will experience a net gain of *a* — *c* by converting relatively low-valued consumption into higher-valued consumption. As such transfers of consumption take place, the respective marginal values converge on *b,* the marginal value of present consumption when consumption is steady. Recall that at maximum utility, the marginal values of goods are in proportion to their price. In this scenario, where (1 + r)/(1 + *p)* = 1 and the individual can rearrange consumption over time by either borrowing or lending, the gains will be a maximum when jcf[7] = *x%* = *x®.*

Another way to view the gains from even consumption is to consider the "total" benefits of consuming various levels of present consumption. These total benefits are measured by the area under the compensated (Hicksian) demand curve. These areas represent the amounts of future income the consumer would be willing to pay to consume the specified level of present consumption. Denote the areas under the demand curve up to *x°* — Ax, between JC° — *Ax* and *x®,* and between *x®* and *x®* + *Ax* as *A, B,* and *C,* respectively. Then the total benefit from consuming *x®* for 2 years is *2A + 2B.* On the other hand, the total benefits from the feast-famine pattern are *(A + B + C)* + *A* = *2A* + *B* + *C* <2A + *2B,* since *C* < *B* due to the negative slope of the Hicksian demand curve. Thus income is valued more highly if it is consumed at an even rather than an uneven rate.

The analysis must be modified slightly if positive time preference or market interest rates are present and not equal to each other. Instead of constant consumption, consumption will either rise steadily or fall steadily at some rate given by the solution to Eq. (12-13). It is still the case that consumption will not be erratic, in the sense of varying up and down over time, but it will not be literally constant. If income is expected to be uneven, e.g., highest during a person's middle years, individuals will, even under these more general assumptions, endeavor to even out consumption by, in this case, borrowing when they are young and lending during middle age, in anticipation of retirement.

The increase in the value of goods resulting from even vs. uneven consumption explains why resources are spent to store seasonably produced goods for future use. Although apples, for example, are all harvested in the fall, it is possible, through controlled climate storage, to spread their consumption throughout the year. Other procedures, such as canning and freezing, accomplish the same end. It is worthwhile for producers to engage in these costly activities only because consumers place a sufficiently higher value on these goods when they can be consumed over the entire year rather than all at once. (Of course, storage encourages greater production for the same reasons.)

"Speculators" include people who purchase goods now for later resale. If the supply of oil, say, is interrupted, these individuals will purchase oil now for storage, further reducing the present supply and thus increasing the price above what would currently exist without this activity. The motives of the speculators are, of course, simply to buy low and sell high. However, an individual can only earn (produce) income in this manner if something of value is being produced for consumers. What is in fact being produced is the smoothing out of consumption. Although speculators always get a bad press regarding their withholding of goods from the market in the present, what is usually not noted is that when they inevitably resell those goods, the supply will be *greater,* and therefore the price lower, than if those goods had not been originally withdrawn. The price, and the flow of consumption, will be more even; it is this increase in the value of goods that speculators produce. It also follows that if an anticipated supply interruption does not materialize, then the preceding activity will not produce a valuable service for consumers, and at least some of the individuals engaging in this unproductive activity will suffer a loss in wealth.

The desire to even out consumption is the basis of the *permanent income hypothesis,* due to Milton Friedman^ and the *life cycle hypothesis,* developed by Franco Modigliani and others.* If income can be costlessly traded across time at the prevailing interest rate, changes in income are identical to changes in wealth. Therefore, a temporary (one-period) increase in income is apt to have a relatively small effect on current consumption, since that wealth increase will be spread over all time periods. It would be odd if the utility function were such that the income (wealth) elasticity of current consumption were close to unity, and close to zero for future consumption. Of course, as one got older, leaving fewer years to consume income, changes in income could be expected to have relatively larger effects on current consumption. This result, however, depends upon an assumption that the utility of one's heirs is weighted less than one's own utility. If one derives utility

^Milton Friedman, *A Theory of the Consumption Function,* National Bureau of Economic Research, Princeton University Press, Princeton, NJ, 1957.

*See, in particular, Franco Modigliani and R. Blumberg, "Utility Analysis and the Consumption Function: An Interpretation of Cross Section Data," in *Post Keynesian Economics,* K. Kurihara (ed.), Rutgers University Press, New Brunswick, NJ, 1954; and Menahem

Yaari, "On the Consumer's Lifetime Allocation Process," *International Economic Review,* 5:304-317, 1964.

from the anticipated future consumption of one's heirs equally to one's own utility, then there would be no effect of age on the propensity to consume increases in wealth. Suppose, for example, in a given year, two individuals each have an income of $30,000, but individual *A* has this income every year, whereas individual *B* usually earns $20,000 but had unusually good fortune this year. Which individual is likely to save more (or dissave less)? Individual *B* has had a temporary increase in income. Assuming capital markets are available so that he or she is able to transfer this income to the future, this increase in wealth will be spread out over many time periods. The individual will do this by saving. Saving can take many forms, e.g., purchasing bonds, or perhaps purchasing consumer durables, such as a house or car, or investing in one's education. Thus we would expect persons with temporary increases in income to have greater savings rates than those consuming near their "permanent" or normal income.t

Fisherian Investment

We now consider a problem analyzed by Irving Fisher, among others. Suppose you own a tract of trees that you are raising for production of pulp for paper mills. Each year, the trees grow at some rate, perhaps initially very quickly, but then at a slower rate as the trees mature. When do you cut down the trees? When they have stopped growing altogether? Or, perhaps you bought some fine wine a few years ago and are aging it in your wine cellar. Each year, the wine gets a little better. When do you decide to drink the wine?

Let us represent the value of the above mentioned trees at time t as $g\{t)$. We assume $g'(t) > 0$ so that the trees are indeed growing, and while $g''(t)$ may be initially positive so that the trees grow at an increasing rate, eventually $g''(t) < 0$. A person who grows trees (or anything else) has to consider the alternative use of the funds tied up in the trees. We always have the alternative of investing our money on other projects of like risk; let the interest yield on such investments be r. Recall from Chap. 2 that with continuous compounding of interest, the present value P of any future amount F V paid or received at time t is $P = FVe{\sim}^{rt}$. The present value of the trees when harvested at time t is

$$P = g(t)e{\sim}^{rt}$$

To maximize wealth, we set $dP/dt = 0$:

$$dt$$

^Friedman never provided a precise conceptual definition of "permanent" income. M. J. Farrell defined "normal" income as that constant flow which had a present value equal to the individual's wealth. However, Friedman meant to allow the possibility that permanent income might rise (or, in general, change) over time. See M. J. Farrell, "The New Theories of the Consumption Function," *Economic Journal,* 69(276):678-696, 1959.

Canceling the $e^{\sim rt}$ terms and dividing by *g{t)* yields

$$\wedge \qquad r \qquad (12\text{-}14)$$

The term *g'(t)/g(t)* is the percentage rate of growth of the trees at time *t.* Wealth maximization therefore says that the trees should be harvested when the percentage rate of growth of the trees (or any other asset whose value increases over time) equals the alternative cost of the invested funds measured by the interest rate. Thus if alternative investments of this nature could yield 10 percent interest per year, then if the trees are growing at 15 percent per year, it pays to leave them as trees. When the rate of growth slows to 10 percent, it becomes time to harvest them. We can solve (12-14) for $t = t^*(r)$; it is an easy exercise to show that $dt^*/dr < 0$ so that an increase in the alternative cost of funds causes the tree owner to cut the trees earlier, when the percentage rate of growth of trees is higher. If the premium for earlier availability decreases, we will let the bottle of wine age a bit longer.

The above analysis assumes the trees can be planted only once; it is as if the world ends when the trees are harvested. In fact, the land under the trees has an alternative use in that trees, or some other crop, can be replanted after the previous harvest. If we imagine we can immediately replant the same amount of trees, we get an infinite series of harvests. However, we can simplify the analysis by noting that after the first harvest, the value of the land will be its current value but discounted to the present for *t* years. Thus with crop rotation, the objective function becomes

$$P = g(t)e^{\sim rt} + Pe^{\sim rt}$$

Solving for P,

$$p = \frac{\quad}{1 - e^{\sim rt}}$$

Using the quotient rule,

$$\frac{dP}{dt} \quad \frac{(1 - e^{-rt})[-rg(t)e^{-rt} + e^{-rt}g'(t)] - g(r)g(+rg)}{(l-e^{-rt})^2} =$$

Dividing by $g(t)e^{\sim rt}$,

$$\frac{g'(0}{8(0} = r + r\frac{e^{\sim rt}}{1-e-''} \qquad (12\text{-}15)$$

This equation says that the trees should be harvested when the percentage rate of growth of the trees equals the alternative interest yield *r* plus the interest income on the land, i.e., the opportunity cost of not replanting. This is the Faustmann solution to the crop rotation problem, first published in 1849; it is the Fisherian solution with the addition of the opportunity cost of the land. Although the algebra is complicated, we can again show that an increase in the interest rate will shorten the time to harvest. It is also apparent that with crop rotation, harvest will occur earlier than with the single-period model, assuming, as needed for the second-order conditions, that $g''(t) < 0$. For crops such as trees that take perhaps decades to mature, the difference between the harvest times implied by (12-14) and (12-15) is not great (try substituting in some plausible values for r and *t),* but for shorter-lived crops it can be significant. In areas of the country where two seasons for crops are possible, the effect is probably important.

Last, policy makers often talk about maximizing the "average sustained yield" of some renewable resource, especially ocean fish, which are subject to overfishing due to the common property problem (lack of private ownership of these resources). One could imagine staggering the plantings of trees (or production in private fish hatcheries) so that a constant amount of the resource is harvested each year. Maximizing the average yield is maximizing the quantity *g(t)/t.* The solution to this, which is to harvest when the average rate of growth equals the marginal rate, will not in general be wealth-maximizing, since it pays no attention whatsoever to the alternative cost of the funds used in producing this resource.

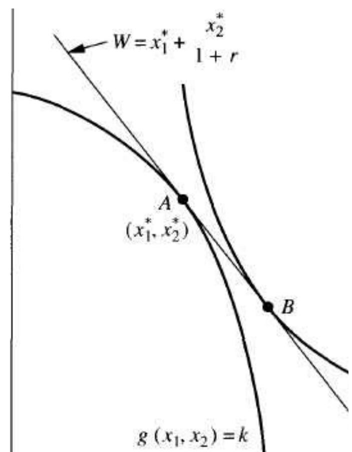
The Fisher Separation Theorem

In the earlier analysis we have treated the income received by an individual as exogenously fixed in each time period. Suppose now consumers can choose among alternative income plans so that the income earned in a given year is part of the utility maximization decision. For example, individuals make career choices in which patterns of income often differ substantially. A person could enter the labor force right after high school and immediately start earning income in some trade. Alternatively, the individual can attend college and perhaps a graduate or professional school, e.g., law or medicine. In that case, income will be very low in the present but eventually higher than that produced with no post-high school training. Or, several business investments might be possible, with varying "cash flows." What strategy is consistent with utility maximization?

Consider Fig. 12-4, in which a production possibilities frontier $g(xi, x_2) = k$ is indicated. This function represents the locus of alternative income streams available to an individual. Whatever point the person chooses along this frontier, say point A = (x*, JC|), income can then be transferred across time by either borrowing or lending at the market interest rate r. Thus a wealth line *W* = *x** + *xy{*\ + r) is implied, with slope —(1 + r). The consumer chooses a point along this line that maximizes utility of consumption.

It is geometrically obvious that under these conditions (most crucially, the assumption of borrowing and lending at the same interest rate), *the consumer will achieve the highest indifference level by first choosing that income stream which maximizes wealth and then rearranging consumption so as to maximize utility.* This famous result is known as the *Fisher separation theorem,* after Irving Fisher, who first propounded it.[T] The algebraic analysis of this proposition is formally identical to that

^See Fisher, *The Theory of Interest,* op. cit.



$V(x_1, x_2)$

FIGURE 12-4 *The Fisher Separation Theorem.* Assume an individual can produce a combination of consumption points *g(xi, X2)*

= k. With efficient capital markets, the bundle produced need not be the bundle consumed. The consumer maximizes utility by first maximizing wealth, and then by borrowing or lending (at some unique interest rate) moves along the budget (wealth) line to some point of tangency B.

presented earlier, in Chap. 10, on Adam Smith's famous dictum, "the division of labor is limited by the extent of the market." This algebra therefore will not be repeated. With extensive markets, increases in income or wealth are always desirable. We must, however, assume that an individual does not have specific preferences with regard to the various investment or income streams available. In the case of career choice particularly, th e theorem must be applied with caution, since individuals commonly are not indifferent to the nonpecuniary aspects of the various alternatives. However, the theorem still applies in a marginal sense: As the pecuniary wealth of some career increases relative to others, more individuals will be attracted to it.t

Last, the preceding argument assumes that the individual can borrow and lend at the same rate. Typically, however, individuals borrow at a higher rate of interest than that at which they lend. If the individual wishes to

transfer income from the future to the present, he or she can do that at price 1 + r_h ("ft" for borrowing); transfers of income fr om the present to the future are transacted at price 1 + r_l ("l" for lending). The effects of this are shown on Fig. 12-5. Point *L* on the production frontier is where the wealth line is tangent at the lending price $1+r_l$. The individual can move "northwest" along the line *LL'*, i.e., toward more future consumption and less present consumption. The extension of the line beyond the tangency point *L* is not available. Likewise, the consumer can transfer income to the present along the steeper line *BB'* but could not lend along the extension of this line beyond *B*. The set of feasible consumption points is that contained by the boundary *L'LBB'*.

t Adam Smith first spoke of

"compensating differentials" in wages to offset undesirable nonpecuniary aspects of jobs, and predicted that garbage collectors would likely be paid as well as teachers for that

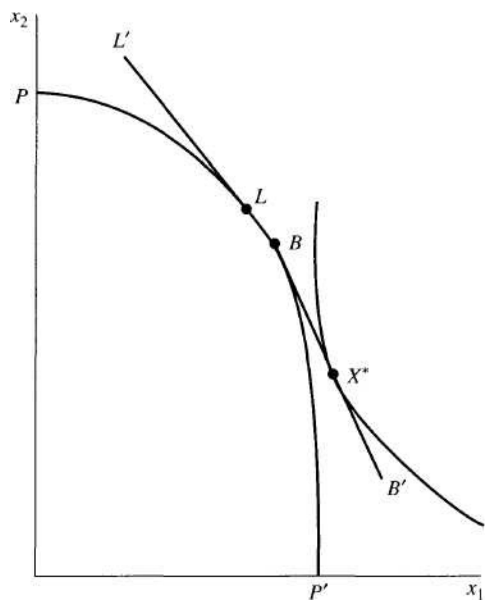


FIGURE 12-5
Utility Maximization When the Borrowing Rate Exceeds the Lending Rate. When the borrowing rate exceeds the lending rate, the Fisher separation theorem may not hold. Maximization of wealth may not lead to utility maximization.

It is obvious that without more detailed knowledge of the individual's utility function, in particular, knowledge of the marginal value of present consumption in the neighborhood of section *LB* of the production frontier, it is not possible to determine which segment of *L'LBB'* the consumer will choose. As drawn, the utility maximum occurs at X^* along *BB'*\ however, it could have as easily been drawn to occur along *L'L.* The individual's present wealth at X^*, evaluated using the higher interest rate r_b, is $W^* = x^* + x\%/(l + r_b)$. This might in fact be a lower number than some point along *L'L,* where a higher price of future consumption, $1 + r_h$ prevails. Thus the exact correspondence of wealth and utility maximization is not present when borrowing and lending rates diverge.

Real Versus Nominal Interest Rates

The previous discussion was entirely in terms of trading off one good for another, i.e., some real amount of consumption in one time period for some real amount of consumption in another time period. Most commonly, however, borrowing and lending contracts are stated in *nominal* terms, i.e., the monetary unit of account. Such contracts are of course always *forward-looking,* meaning repayment of a loan will take place in the future, not in the past. If loan contracts are stated in dollars, say, the borrowers and lenders will attempt to incorporate into the contract any anticipated change in the value of dollars relative to goods. (If the contract were specified in any unit at all, say gold, the parties would attempt to build in anticipated changes in the price of that unit, i.e., gold.) We say *anticipated* changes, since when the contract is formed, the actual change in the value of the monetary unit is not known. The rate of interest fully adjusted for any changes in the unit of account of a loan contract is called the *real* rate of interest; it is what we have been dealing with thus far.

With continuous compounding of interest (see Chap. 2, Sec. 2.3), a loan of initial principal P, earning r percent per year, will have a future value after t years of Pe^{rt}. However, suppose inflation is anticipated to occur at g percent per year. Any nominal amount P today would depreciate at that rate; in t years, its value would be Pe^{-gt}. The combined effect of real interest and inflation would produce a future value of $Pe^{rt}e^{-gt} = Pe^{(r-g)t}$. To offset the effect of anticipated inflation, interest would be set at $r + g$; in that case, the future value would be restored to Pe^{rt}, if the rate of inflation actually were g. The nominal rate of interest, i, is thus

$$i = r + \text{anticipated rate of inflation} \qquad (12\text{-}16)$$

Equation (12-16) is generally known as the *Fisher equation.* Letting p be the general price level, the anticipated rate of inflation is generally written $E[(1/p)(dp/dt)]$, where E is the mathematical expectation operator. ^ In terms of discrete time, Eq. (12-16) is only an approximation. The combined effect of the real interest rate r and anticipated inflation rate g would yield a nominal interest rate of $i = (1+r)(1+g) = 1 + r + g + rg$. However, for small values of r and g, the term rg is negligible, and Eq. (12-16) can be safely used. Estimation of the real rate of interest has been the subject of substantial research; it has been variously estimated in the 1 to 2 percent range, though the extent of its variation over time is the subject of debate.*

Market rates of interest also incorporate a premium for the riskiness of the loan. Risk increases the variability of income; the previous analysis suggests that there should be such a risk premium in the market to compensate for this lowering of the total value of consumption. The premium for risk is evident in the market for capital. For example, bonds issued by corporations promise higher interest than bonds issued by the U.S. government for the same time period. Corporate bonds are also rated by various rating services such as Moody's and Standard & Poor's; the interest that corporations must promise on their bonds generally increases as their ratings worsen, and the realized yields (lower than the promised yields due to occasional defaults) are also higher to compensate for the greater variability in outcomes as risk increases. Last, the average long-term yield on equity capital, i.e., stocks, where dividends are contingent on the existence of corporate profits, is larger than the average long-term yields on bonds, reflecting the greater risk of stocks versus bonds. Thus we could write the Fisher equation as

$$i = r + E\left[\left(\frac{1}{p}\right)\left(\frac{dp}{dt}\right)\right] + \text{risk premium} \qquad (12\text{-}17)$$

The measurement of these quantities is the subject of much current research.

t\Ve will deal with uncertainty and mathematical expectation in the next chapter.

*See, for example, Ibbotson Associates, *Stocks, Bonds, Bills and Inflation: 1997 [Annual] Yearbook,*
Ibbotson Associates, Chicago, 1997.

12.2 THE DETERMINATION OF THE INTEREST RATE

The rate of interest is a price that appears in the market. Probably no other price has engendered as much public and private hostility as this particular price, at which consumption is traded across time. Its persistence through history, in spite of religious and secular laws forbidding or restricting loans at interest, is testimony to the importance of its function.

It is perhaps best to first inquire as to why interest rates are positive and whether they could reasonably be negative. Of course, the prospect of inflation and the presence of risk both increase the observed rates. We mean to inquire as to the existence of a positive *real* rate of interest. Following Fisher and Bohm-Bawerk, we shall consider, in turn, the effects of time preference, economic growth, and the conditions of production, i.e., the ability of society to increase future consumption by producing goods in the present that enhance future productivity, t

Consider first whether a negative real interest rate can prevail. Consider a pure trading economy, such as a World War II prisoner of war camp, in which no production takes place and "income" arrives periodically in the form of Red Cross packages. Suppose, in addition, that it is impossible to store wealth for anything but a brief time period. The individuals in the camp will all attempt to even out their flow of consumption. Suppose now it becomes known that incomes will be declining in the future. In that case, each person would try to trade some present consumption for future consumption. Such trades could be made by sacrificing presently available goods in return for sharing the other person's Red Cross parcels in the future. In the absence of preference for earlier consumption, the simultaneous efforts of all individuals to transfer consumption to the future would lead to a negative interest rate. Crucial to this outcome is the assumption that wealth cannot be costlessly stored. If wealth can be costlessly stored, the real interest rate could never be negative. One would never in that case loan, say, $100 today in return for $95 next year; it would suffice simply to store the $100 for a year. Thus a negative real interest rate could only exist if sufficient individuals wished to shift income to the future, for example in anticipation of falling incomes, and if it is costly to store wealth over that time period.

Suppose, instead, that economic growth is taking place so that individuals anticipate that future incomes will be higher than present income. In this case, the desire to even out consumption would lead individuals to contract with each other to shift future consumption to the present. Any one person can do this by promising to trade some amount of future income to another person in return for receiving income in the present from that person. However, whereas it is possible for some of the individuals in the economy to accomplish this, it is impossible for all to do so.

^These three reasons for the existence of a positive interest rate can be found in Bohm-Bawerk's *Capital and Interest,* translated by William Smart as *The Positive Theory of Capital,* Books for Libraries Press, Freeport, NY, 1971.

Individuals are not really consuming next year's income; they are simply trading with other persons in the same economy. Next year's income cannot be consumed until it is produced. The simultaneous effort to transfer future income to the present will create a premium for sacrificing present consumption. As this premium increases, more individuals will be willing to make the sacrifice. Exhaustion of the gains from trade will lead to that premium, i.e., price of present consumption in terms of future consumption forgone, for which all individuals' marginal values of present consumption are the same.

Similar reasoning applies in the case where individuals have positive rates of time preference so that the indifference curves cut through the 45° ray in Fig. 12-2 at a slope of $-(1 + p)$. This assumption is simply another reason why consumers would wish to transfer future income to the present. The same analysis as in the preceding paragraph applies; the simultaneous efforts to accomplish this will create a positive price for earlier availability of goods.

The foregoing analysis takes future income as exogenous; the individuals could do nothing to affect the levels of future income that would become available. Let us now incorporate this important aspect into the analysis. It is possible to increase future income by diverting present income to the production of "capital goods," which yield no consumption in and of themselves but which increase the marginal product of other inputs in production so that larger incomes can be produced in the future. Diverting resources into the production of tractors, computers, education, and the like costs society present consumption but leads to higher future incomes. The ability to accomplish this "roundabout" production affects the rate of interest.

The simplest theoretical device along these lines is perhaps that used by the distinguished theorist Frank Knight. Knight contemplated Robinson Crusoe, stranded on an island, with a food supply consisting of an edible *Crusonia* bush, which grows exogenously at some constant rate r, say 10 percent per year. It in fact grows at this constant rate no matter how small or large it is, that is, no matter how much of the bush Crusoe should partake of at any given time. The situation is depicted in Fig. 12-6. Current consumption is plotted along the horizontal axis; future consumption is plotted vertically. The entire bush consists of a level of consumption C_1, which if consumed would lead to starvation, and if none is eaten, $C_2 = (1 + r)C_1$ in the next time period. In this case, Crusoe's budget constraint is the straight-line production frontier defined by the edible bush, with slope $-(1 + r)$. No matter what rate of time preference Crusoe might have, as long as his utility function is strictly increasing (Crusoe is not sated) and quasi-concave and ruling out boundary solutions, at the utility maximum tangency point his marginal value of present consumption must be $-(1 + r)$.† In this case, production conditions completely determine the real rate of interest. With pervasive technology of this sort, so that any number of individuals could trade present for future consumption along this

† If the bush in fact *shrank* at some rate *s* and any part spoiled once removed from the bush so that storage was impossible, Crusoe would face the negative real rate of interest *s*.

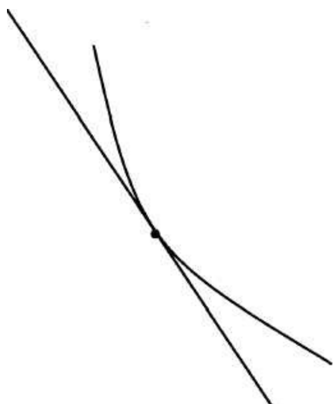
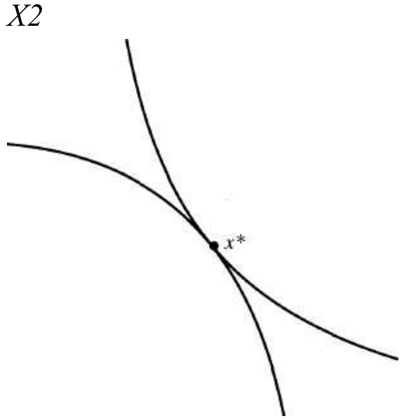


FIGURE 12-6 *Intertemporal Consumption in a Crusoe Economy with Constant Marginal Cost of Present Consumption*. If technology is such that one unit of present consumption can always be transfor

frontier, the rate of interest in a many-person economy would be the constant growth rate, r.

Assume now that production is such that current and future consumption can be produced along some concave frontier, as shown in Fig. 12-7. In a single-person economy, the rate of interest will be determined by the tangency representing the utility maximum subject to the production constraint. Both the rate of time preference, if present, and the (changing) marginal cost of transforming present consumption into future consumption determine the interest rate. In a many-person economy, the production frontier defines the marginal cost (supply) curve of present consumption in terms of future consumption forgone; the preferences of individuals would



V $g(x_1,x_2) = k$

$(x_1,x_2) = U$

FIGURE 12-7 *Market Determination of the Interest Rate.* In a society where present consumption can be transformed into future consumption according to some concave production frontier, the market rate of interest will be determined where consumers' marginal value of present consumption equals the marginal opportunity cost of present consumption.

determine the demand for present consumption. Exhaustion of gains from exchange would lead to the establishment of a single market price such that for all participants in the market, the marginal subjective value of present consumption in terms of forgone future consumption would equal the marginal opportunity cost, in terms of future consumption forgone, of providing that level of present consumption.^

12.3 STOCKS AND FLOWS

For the most part, goods that provide income in the future are *durable.* Capital can consist of intermediate goods, which are used only once in some productive process, but interest rates and time preference are not directly relevant for such goods. Most interest (no pun intended) in capital goods focuses on those goods that last over several or many time periods. In that case we have to distinguish the physical item, called the *stock,* from *the flow* of service (per unit time) derived from the stock.

A convenient illustration of these concepts is the distinction between a house and the flow of housing services we derive from owning or renting a house. When we rent housing, we purchase the service flow for periods of time. If we purchase the house itself, we are purchasing the stock. Owning the stock (the house) entitles you to consume the entire future service flow, for the duration of the existence of the stock, and also obligates you to pay any costs associated with ownership, e.g., property taxes, maintenance, and the like. The price of the stock is therefore the present value of the anticipated net rents (value of service flow, per unit time, net of costs) for the indefinite future. Let t = time, and suppose the stock lasts from $t = 0$ to $t = T$. If the net rental value is some constant R, the price of the stock is

$$P = \quad Re^{-rt}dt = \mathrm{I} - \backslash R(\backslash - e^{-rt}) \qquad (12\text{-}18)$$

In general, R varies over time: $R = R(t)$. In that case, all anticipated net rents are incorporated into the price of the stock, P. With efficient markets, if some news occurs which changes the value of R at some future time, that news will quickly be "capitalized" into the price P. In the case where $T \longrightarrow \infty$,

$$P = - \qquad (12\text{-}19)$$

$$r$$

With, say, \$1000 in hand and a permanent interest rate of 0.10, interest of \$100 can be withdrawn every year. The opportunity cost of consuming the principal (the stock)

^In the second chapter on general equilibrium, we shall explore a model in which two factors, labor and capital, are used to produce two "goods," capital and a consumption good. In that case, the implied rental rate of capital, R, equals the marginal product of capital, MP/c times the price P of capital; by division, the interest rate $r = R/P = MPK$. In that model, the interest rate is determined by the relative price of the capital and consumption goods and the relative intensities of use of labor and capital in the production of those goods. See Chapter 18.

is the forgone perpetual flow of $100 per year. Alternatively, the ("permanent") rental rate on some asset divided by the price of the asset is the implied interest rate: $r = R/P$.

In fact, the present value of rents beyond a generation or two is very small for common levels of interest rates. The value of the stock, taken from $t = T$ to $t = \infty$, is

$$P = \int_T^\infty Re^{-rt}\,dt = \left(\frac{1}{r}\right)Re^{-rT} \qquad (12\text{-}20)$$

If, for example, $T = 50$ and $r = 0.10$, less than 1 percent of the value of the asset is accounted for by the indefinite future past 50 years.[†] Thus it is often possible to approximate closely the present value of any long-lived asset with the simple formula (12-19), $P = R/r$. In present value calculations, the interest rate is often referred to as the *discount* rate, since using it has the effect of arithmetically lowering the nominal value of future income.

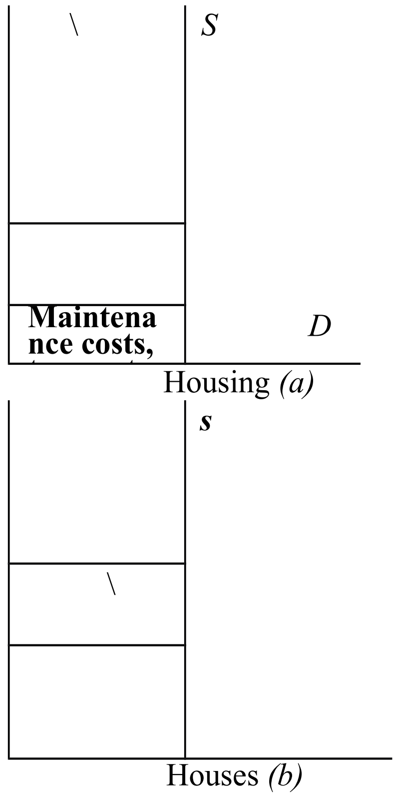
The proper interest rate to use in these formulas is the rate that reflects the opportunity cost of the funds in terms of their use either in alternative investment projects or in the production of present consumption. The risk associated with the level of future rents, based on actual uncertainty about the future, is part of the opportunity cost of using present funds to build capital which produces income in the future. Risk means that there is some chance that part or all of the current consumption sacrificed to build an asset may be for naught; the anticipated future income may never fully materialize. Thus the appropriate interest rate in these calculations would include the real rate plus the risk premium. The ratio of the price of publicly traded corporate stocks to current dividends is reported on the stock market tables as the "price-earnings" ratio. Typical numbers are 8 to 15; 5 is considered low, and 20 is high. If the current earnings are the expected average future earnings, the reciprocals of these numbers may approximate "rates of return on equity capital," i.e., the interest rate reflecting the opportunity cost of investing in those corporations, taking into account their riskiness.

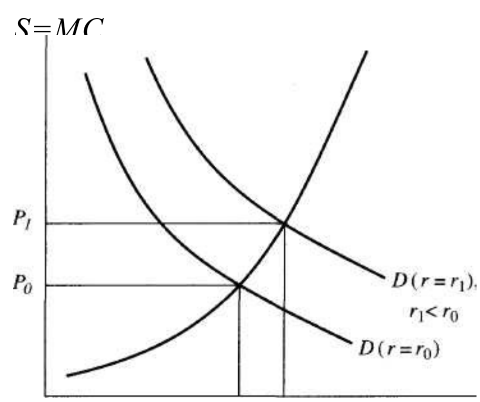
Let us now investigate how the prices and quantities of stocks and flows, and the rate of investment, are affected by changes in interest rates.* In Fig. 12-8, panel *(a),* the supply and demand for *housing,* the service flow of shelter, is indicated. The supply curve is drawn vertically, to represent that the supply of housing units is very large and cannot change a great deal in the short run. Of course, at any moment of time, some houses are being demolished and others are being built. For simplicity,

[†]The famous French mathematician Blaise Pascal once argued that, considering the infinity of afterlife, prudence dictated participation in religion. He neglected to take discounting to present value into account, which is why the young frequently ignore his advice and the old take it. See Corry Azzi and Ron Ehrenberg, "Household Allocation of Time and Church Attendance," *Journal of Political Economy,* 83:27-56, February 1975.

*This discussion is adapted from James Witte, Jr., "The Microfoundations of the Social Investment *Function? Journal of*

Political Economy, 71:441-456, October 1963.


S
Maintena
nce costs,
D
Housing (a)
s
Houses (b)


S=MC
P_1
P_0
$D(r=r_1)$,
$r_1 < r_0$
$D(r=r_0)$

predictable effect on the demand for housing relative to other contemporaneous goods. A fall in the interest rate is a decrease in the price of present relative to future consumption; the *mix* of present goods, i.e., the amount of shelter versus food and clothing demanded, should not be affected in any particular way. Thus we would expect no change in the rental price *R.* In panel *(b),* however, the demand and supply of *houses* is shown. The market price *P* is the present value of net rents:

$$P = \frac{R - C}{r} \qquad (12\text{-}21)$$

where *C* = annual obligations incurred by the owners of housing, such as maintenance costs and taxes. Costs *C* are represented by the shaded part of total annual rents in panel *(a).* A fall in *r* means the premium for present consumption has fallen; future consumption is now relatively more highly valued. Therefore, any asset that generates future income is now more highly valued. The demand for owning such an asset therefore shifts out, as shown in panel *(b).* The price *P* of the house (the stock), representing the present value of all anticipated future housing services, therefore increases.

An increase in the current price of houses has an obvious effect on the market for new houses. In panel *(c),* the demand and supply *of new* houses is shown. When the interest rate falls, the demand for new houses shifts out, since owning an asset that provides income in the future is now relatively more valuable. Therefore, the prices of new (and old) houses increase; production will increase until the marginal cost of producing new houses equals consumers' marginal values of new houses. The housing stock will start to rise above its previous level. In the short run, there will be a negligible effect on rentals, as the housing stock is already very large. However, over time, the housing stock will increase above its former level, increasing the supply of housing services and thereby lowering housing rental prices. If the new rate of interest is permanently lower, some new, larger, steady state stock of houses and housing supply will exist.

This analysis shows the futility of policies to "make housing more affordable" by attempting to lower the real interest rate. Even if the monetary authorities could actually do that (problematical, at best), the short-term effect is simply to raise the price of houses. Rental values are unaffected by such a policy; the only change will be in the mix of interest and principal in the mortgage payment. In the long run, when a larger housing stock prevails, some benefits might accrue to future home purchasers. However, if this result were achieved through the distortion of capital prices, i.e., if the price of present consumption were subsidized from other types of income, the economy as a whole would suffer some loss of efficiency, i.e., lost gains from trade. It is always possible to make certain individuals better off by subsidizing the interest *they* would have to pay on some loan contract, e.g., a home mortgage; enhancing everyone's wealth by such a procedure is another matter entirely.

Other questions can be addressed using the same framework. In 1978, voters in California passed the famous Proposition 13, which drastically lowered property taxes. It was sold to the public partly as a means to reduce rental prices. Of course, considering panel *(a)* above, it could do no such thing, at least in the short run, since neither the supply nor the demand for housing services was affected by the law. However, by lowering the costs *C* of housing ownership, net rents to property owners were increased. The price of houses would therefore rise, as shown by Eq. (12-21), producing increased new construction of houses. In the long run, rental prices would decrease, but only due to the larger supply of housing services resulting from the larger housing stock, t

PROBLEMS

1. Consider the utility functions

$c?$ $0 < a < 1$

and assume an individual with these preferences and endowments $x®, \ldots, x®$ maximizes utility subject to the wealth constraint, with interest rate *r*. Prove that consumption in any time period (beyond the first) is a constant times consumption in the previous time period. Show that this constant is greater (less) than unity if $r > p$ ($r < p$).

2. Suppose you own an apartment complex that generates *R* per year in rentals. It can be
sold outright for some price *P;* alternatively, shares of ownership can be sold.
1.349 Plotting "present consumption" on the horizontal axis and "future consumption"
(assumed infinite) on the vertical axis, indicate the feasible consumption points
generated by selling ownership shares.
1.350 Suppose you hear that the value of the complex has fallen in half. This could be due
to either a doubling of the interest rate or that half the complex has burned down.
Under what conditions, if any, would you be indifferent to the cause of this wealth
loss?[1]

3. Currently, in the United States, interest on a home mortgage is deducted from income
when calculating federal income taxes. How would the removal of this provision affect
the wealth of
1.351 Current home owners?
1.352 Prospective home owners?
1.353 People in the construction business?

4. Suppose sheep could reproduce so as to increase the stock of sheep 10 percent per year,
forever. Would the real interest rate become 10 percent?

tOn the other hand, many local property taxes are directly tied to local community services, such as
schools. People move to certain cities and neighborhoods specifically to pay those taxes, i.e., to be able

to consume the local services they provide. In that case, defeat, say, of a school levy might actually depress house prices, if the main purpose of moving to that locality was to consume the local school district.

* Adapted from A. Alchian and W. Allen, *Exchange and Production,* 3d ed., Wadsworth, Belmont, CA, 1983.

1.354 "Invest in land. Population grows steadily, and land is relatively fixed, so rentals will
increase over time. Therefore, land will yield a profitable return." Evaluate. How does
your answer depend, if at all, on the validity of the premises of population growth and
fixed land?

1.355 Consider that current U.S. tax law requires corporations to value assets based on *historical*
costs, not replacement value.

1.356 How would inflation (on this account) affect the present value of future profits and
the price of the stock?

1.357 Consider that gold and owner-occupied houses are not depreciable for tax purposes
under U.S. law. How would an unanticipated increase in inflation affect the price of
these assets relative to others?

1.358 "Capital gains" (increases in the value of some asset) are taxed on a nominal ba
sis; inflation thus creates spurious taxable capital gains. Owner-occupied houses,
however, are exempt if a new and more expensive house is purchased within a year.
On this basis, how does (unanticipated?) inflation affect the relative price of these
assets?

1.359 If home mortgages are based on fixed monthly payments, how does an increase in inflation
affect the real burden of repayment over time? How might this affect a person's ability
to secure such loans?

1.360 The voters in a certain urban area enact "greenbelt" legislation, which restricts develop
ment of surrounding rural land for housing development. What is the effect on

1.361 Current and future rental rates?

1.362 Current and future prices of houses?

9. Suppose cars are on average driven 10,000 miles per year, and gasoline retails at $1.00
per gallon. How much is it worth to consumers and to car manufacturers to increase the
mileage of cars by 1 mile per gallon? Assume various initial mileages, e.g., 15 mpg, 20
mpg, 25 mpg.

10. Prove the following famous "Rule of 72": If an amount grows at g percent per year, the original principal will double in approximately $12/g$ years.

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CHAPTER

13

BEHAVIOR UNDER UNCERTAINTY

13.1 UNCERTAINTY AND PROBABILITY

Uncertainty is a pervasive fact of life. A mathematical analysis of behavior under uncertainty requires use of the concept of probability. We first give three examples of how the probability of an event may be assigned.

1.363 There are two possible outcomes in a coin toss: a head or a tail. If it may be assumed that the two events are *equally likely* to occur, they must have the same probability. The probability of a head and the probability of a tail will each be 0.5.

1.364 In the United States roughly 8 out of every 1000 people die each year. We can say the probability that a person in the United States will die within 1 year is 0.008. Here the probability of an event is equal to the *relative frequency* with which the event occurs when the experiment is repeated a large number of times under similar conditions. Of course, people of different age, sex, or state of health represent dissimilar conditions. If we know that the person is, say, over 85 years old, the probability of death will become 0.14 because people in that age group die with that relative frequency.

1.365 When an entrepreneur introduces a new product into the market, the concepts of equally likely events and relative frequency are not very helpful. However, as long as a person's preferences for actions with uncertain outcomes satisfy some consistency conditions, his or her *subjective probabilities* of different possible events can be determined.T For example, the entrepreneur may assign a probability of, say, 0.71, that the new product is a flop and a probability of 0.29 that it is a success. If we are interested in predicting behavior, it is these subjective probabilities that matter.

The preceding examples correspond to three different interpretations of probability. Regardless of which interpretation one adopts, the mathematical theory of probability is the same. Suppose *S* is the set of all possible outcomes and £ is a subset of *S.* Denote the probability of event *E* by Pr(£). A probability function is a function that assigns real numbers to subsets of *S* and that satisfies the following conditions:

1.366 For any event *E,* Pr(£) > 0.
1.367 Pr(5) = 1.
1.368 For any finite or infinite sequence of mutually exclusive events
E\, E_2, ...,
Pr(£, U E_2 U • • •) = Pr(£i) + Pr(£₂) • • •.

In this book space limitations dictate that we provide only the most cursory introduction to the concepts of probability, random variable, mean, and variance. The reader should consult any of the various textbooks on probability or mathematical statistics for a more detailed treatment of this important theory.

Random Variables and Probability Distributions

A random variable is a function that maps an outcome to a real variable. In a coin toss, for example, we can define a random variable *X* such that *X* = 47 if the coin lands on a head and *X* = 35 if it lands on a tail. Associated with each random variable *X* is a (cumulative) *distribution function F* such that

F(x) = Pr[X < *x]* Continuing our example,

if the coin is a fair coin, the distribution of *X* is given by

(I for*x* > 47
F(x)=\0.5 for47>x>35 (13-1)
[0 for* < 35

Note that a distribution function must have the following properties:

1.369 F(oo) = 1.
1.370 F(-oo) = 0.
1.371 *F(x)* is monotonically nondecreasing in *x.*

The distribution given in Eq. (13-1) obviously satisfies these conditions.

L. J. Savage, *The Foundations of Statistics,* John Wiley & Sons, Inc., New York, 1954.

Random variables can be discrete or continuous. If a random variable *X* is discrete, it can only take on a finite or countably infinite number of values, say, *X\,X2,x3,*___The *probability function f*Associated with *X* is

fix) = Pr[X = *x]* Clearly, the

probabilities must be nonnegative and sum to 1. Therefore, we have:

1. 1 > *fix)* >
0. **2- £*/(*) -**
I-

When a random variable is continuous, the probability that it is (exactly) equal to a prespecified number is zero. We can nevertheless find the probability that the random variable lies in a small interval, Pr[x < *X* < *x* + *h]*. Dividing this probability by the length of the interval and taking the limit as the length goes to zero, we obtain the *probability density function:*

?r[x <X <x+h]
/(x) = lTo-------------1---------
—'hm- *F(x + h) - F{x)*
h^o *h*

The probability density function must also satisfy two

conditions: 1. *f(x)* > 0.

Note that the value of *f(x)* can be greater than 1 since it is not a probability.

Mean and Variance

A random variable can be completely characterized by its distribution function. However, it is often useful to summarize the central tendency or the average behavior of a random variable by a real number. The most important measure of central tendency is the *mean.* The mean or the expected value of a random variable *x,* denoted *E[x],* is defined by

xf (x) if JC is discrete
E[x] = *I* *x*
/ *xf(x)dx* if *x* is continuous
. J —oo

Example 1. Consider the following gamble. A fair coin is flipped until a tail appears. You win $1 if it appears on the first toss, $2 if it appears on the second, $4 if it appears on the third toss, and, in general, *$2"~x* if it appears on the *nth* toss. Letting the random variable *x* denote your winnings, the probability of winning $2"" is *{\)n*. Thus, the expected value of x is

1' " 2

_ 1 1 1

= oo

This gamble is known as the St. Petersburg paradox. It is a paradox because people do not seem willing to pay a large sum of money for the right to play this gamble, even though the expected value of the winnings is infinite. One way to reconcile this paradox is to propose that individuals are risk-averse. This is the approach taken by the eighteenth century mathematician Daniel Bernoulli^ and we will discuss it in detail later. Another way is to suggest that your opponent does not have infinite wealth. Suppose your opponent possesses only the modest amount of $1 billion ($10^9$). If a tail appears at or before the 30th toss, your opponent will still be able to pay you the promised amount. After the 30th toss, he will only be able to pay you $1 billion. The probability that you will get $1 billion is / ($10^9$) = *(\f¹* + *(\f²* + • • • • = (|)[30]. The expected value of a St. Petersburg gamble given this wealth constraint is less than $16:

30 /1\« /1 x3°

=1+10"'1

= 30(0.5) + 10^9(9.3 x

10~[10]) = 15.93

Example 2. Suppose some event has a probability of p of success (and thus a probability 1 — p of failure). For example, suppose a firm has a constant probability p of going bankrupt each year. That is, if the firm is in business at the beginning of any year, the probability that it will be in business a year later is 1 — p. How many years do we expect the firm to survive? The expected number of years is

E(n) = l - p + 2p(1 *-p)* + 3/?(1 -

p)[2] + ■ ■ ■ To evaluate this infinite series, note the

identity

1 - *Z*

from the formula for the sum of an infinite geometric series, with \z\ < 1. Since this is an identity, we can differentiate both sides, yielding

1 + *2z* + *3z*[2] + ■ ■ ■ =-

tSee Daniel Bernoulli, "Exposition of a New Theory on the Measurement of Risk" (1738), trans, by L. Sommer, *Econometrica,* 22:23-36, 1954.

Applying this to the expression for $E(n)$, with $z = (1 — p)$, yields

p^2 J p

Thus if there is a constant probability of 0.1 that a firm will go out of business from one year to the next, we would expect the average lives of such firms to be 10 years.

The formula for expected value extends naturally for functions of random variables. If $u(x)$ is a function of random variable x, then $u(x)$ is itself a random variable and its expected value is given by

$$E[u(x)]=l \quad \begin{cases} \hat{} \; u(x)f(x) & \text{if } x \text{ is discrete} \\ {}_x & \\ / \quad u\,(x)\,/\,(\mathrm{JC})\,dx & \text{if JC is continuous} \\ , J —oo & \end{cases}$$

In general, $E[u(x)] =/= u(E[x])$ unless u is linear in x. If $u = a + bx$, where a and b are constants,

$$E[a + bx] = a + bE[x]$$

The linearity of expected value also applies to two or more random variables. Thus, for any two random variables x and y,

$$E[x + \mathrm{y}] = E[x] + E[y]$$

regardless of whether JC and y are independent. Two random variables JC and y are independent if and only if Pr[jc < JC°, y < y°] = Pr[jc < x°] Pr[y < y°] for all JC° and y°. If JC and y are independent, we also have

$$E[xy] = E[x]E[y]$$

Whereas the expected value of a random variable is a measure of its central tendency, the variance indicates the degree of its variability. The variance of a random variable JC, denoted var[jc], is defined as

$$\mathrm{var}[\mathrm{x}] = E[(x - /\mathrm{x})^2] \qquad (13\text{-}2)$$

where $\backslash i = E[x]$. Equation (13-2) can also be

expressed as var[x] = £[JC^2 — $2X/JL$

+ /x^2]

= £[JC^2] - $2fiE[x]$ + ii^2 =■ $E\backslash x\,]$ — 2/x

+ $\backslash JL = E[x^2]$ - /Lt^2 Two facts about the

variance are worth mentioning.

Fact 1. For any constants a and b,

$$\mathrm{var}[\mathrm{a} + bx] = b^2\,\mathrm{var}[\mathrm{x}]$$

Proof. Let *fi* = *E[x].* Then

var[a + *bx] = E[((a + bx) - (a + bfx))*^{2}]

= b^2*E[(x-*$_f$*i)*^{2}]

= b^2 var[x]

Fact 2. If *X]*, ..., x_n are independent random variables,

var[x, H- -\-x_n] = var[x,] -\- h var[xj (13-3)

Proof. Let ^, = *E[Xj]* for *i* = 1, ... , *n*. Then

x_n] = £[(*, + • • • + * „) - (MI + • • • + /-0J^2

^ - M,)^2]) + 2]>](£[*,•*,•] - *frfjLj)*

(13-4)

If x, and *Xj* are independent, the second sum is equal to zero and Eq. (13-3) follows.

The term *E[xtXj]* — /x,/x$_7$ in Eq. (13-4) is equal to the *covariance* between x_t and *Xj.* Thus, in general,

var[xi + x_2] = var[jti] + var[x_2] + 2 cov[xi, x_2]

Example 3. Let *x* be the value of one share of the stock in a firm. If a^2 is the variance of x, an investment portfolio consisting of *n* shares in the firm will have a variance of var*[nx]* = n^2a^2. On the other hand, suppose a person invests one share each in *n* different stocks, *X*\,..., x_n, whose returns are independent. If all stocks have a common variance of a^2, the variance of the diversified portfolio is only var[xi + • • • +x_n] = no^2.

13.2 SPECIFICATION OF PREFERENCES State Preference Approach

Consumer theory as developed in earlier chapters can be readily generalized to cover behavior under uncertainty. Just as an apple consumed today is different from an apple consumed tomorrow, ice cream on a hot day is a different commodity from ice cream on a cold day. In intertemporal problems, the same physical good consumed at different dates is treated as different commodities. In problems related to uncertainty, we can treat the same physical good available at different states of the world as distinct commodities. Using such an approach, utility is defined as a function of *state-contingent commodities.* A state-contingent commodity is a good that can be consumed only if a specified state of the world obtains. An example is a contract that offers to deliver ice cream if the temperature is above 80°F (and nothing otherwise). Suppose there are only two possible states, and let *W*\ and W_2 denote the amounts of commodities contingent upon state 1 and state 2, respectively. *W*\ and W_2 can be vectors representing bundles of commodities, but very often they are simply scalars representing wealth or composite consumption. If the probabilities of state

1 and state 2 are *n* and n_2, respectively *{it* + n_2 = 1), a consumer's preferences can be represented by a utility function,

$$U(Wi, W_2; 7ri, 7t_2) \qquad (13\text{-}5)$$

Here utility is defined over the contingent consumption plan *(*W_u W_2*)*. The probabilities *ii* and n_2 are included as parameters of the utility function because the value of a state-contingent commodity depends on how likely the state is to occur. If there are complete markets where one can buy state-contingent commodities at exogenous prices, the analysis of consumer's choice under uncertainty is formally equivalent to the certainty case. The consumer will choose *W* and W_2 so as to maximize utility subject to a budget constraint. The resulting demand functions for state-contingent commodities will satisfy all the theorems about demand functions derived in earlier chapters. Whereas the state preference approach is very general, the requirement of complete markets in state-contingent commodities is hard to satisfy. If there are *n* different goods and *s* possible states of the world, there have to be *ns* separate markets. Arrow^ has shown that trade in state-contingent claims (i.e., financial contracts that yield different amounts of money under different states of the world) can be substituted for trade in state-contingent commodities. Then a complete set of markets requires only *n* goods markets plus *s* securities markets. It may well be the case that contingent markets are particularly costly to organize because specification and measurement of states are difficult. When state-contingent commodities are not traded in the market, the state preference approach is of limited empirical applicability.

The Expected Utility Hypothesis

The utility function shown in expression (13-5) is very general. It is possible to impose more structure on the utility function if preferences satisfy some additional axioms. Among the more important axioms are the following:

1.372 *State independence: An uncertain prospect consisting of x in state 1 and y in state 2 is equally preferred to a prospect ofy in state 1 and x in state 2 if the probability of receiving x in both prospects is the same.* State independence means that preferences depend on the probabilities of the states of the world but not on the states themselves. Whether this is reasonable is a matter of the context of the problem. In medical insurance problems, for example, preferences may depend on one's state of health even though all medical expenses are fully covered.

1.373 *Reduction of compound lotteries: If x is an uncertain prospect consisting of y and z with probabilities iz and* 1 — *n, then a prospect consisting* *ofx and z with*

^See Kenneth J. Arrow, "The Role of Securities in the Optimal Allocation of Risk Bearing," *Review of Economic Studies,* 31:91-96, 1964.

probabilities ft and 1 — *ft is equally preferred to a prospect of y and z with probabilities TTTT and* 1 — *nft.* This axiom asserts that a consumer's preferences for uncertain prospects depend only on the probabilities of receiving the various prizes, not on how these probabilities are formed. The axiom will be violated if, say, the consumer has a love for suspense.

1.374 *Continuity: Ifx is preferred to y andy is preferred to z, there exists some probability value n such that y is equally preferred to an uncertain prospect consisting of x and z, where x is realizable with probability TT and z with probability* 1 — *TT* . It can be argued that the continuity axiom will not hold when *x* is \$2, *y* is \$ 1, and *z* is death. On the other hand, people do often take the risk of jaywalking to gain a few seconds.

1.375 *Independence of irrelevant alternativies: If x is preferred to y, then for any z an uncertain prospect consisting of x and z with probabilities TT and* 1 — *TT will be preferred to an uncertain prospect consisting of y and z with the same probabilities.* In the certainty case the independence axiom is a strong assertion, because there can be all sorts of complementarity and substitutability relationships between two goods when they are consumed simultaneously or in a temporal order. In the case of preferences for uncertain prospects, however, the individual will never get *x* and *z* together or *y* and *z* together. Thus, it is unlikely that the presence of *z* will affect the preferences for *x* and *y.*

Given these postulates it can be proved that preferences for uncertain prospects can be expressed in terms of expected utility. If a prospect consists of prizes *W* and *Wi* with probabilities *TT* and $7r_2$, respectively, we can find a utility function *u(-)* such that

$$U(W_U\ W_2;\ TT_U TT_2) = nMWi) + TT_2 U(W_2)$$

The function «(•) is often called a *Von Neumann-Morgenstern utility function,* after their pioneering work on decision theory.* Note that preferences are now expressed as the expected value of a utility function. This representation of preferences is simple because utility is additively separable in *W* and W_2 and is linear in *TT* and TT_2. Separability is a result of axiom 4, and linearity is a result of axiom 2. If preferences are not state-independent but the other axioms still hold, utility can be expressed as $U = TT\backslash U^x(W\backslash) + TT_2U^2(W_2)$, where w^1 and u^2 are different functions.

Cardinal and Ordinal Utility

As in the certainty case, the utility function for uncertain prospects is just a convenient way of representing preferences. If prospect JC is preferred to or indifferent to

D. Luce and H. Raiffa, *Games and Decisions,* John Wiley & Sons, Inc., New York, 1957. *See J. Von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior,* Princeton University Press, Princeton, NJ, 1944.

prospect *y* whenever *U(x) > U(y),* then *U(-)* is a valid utility function. Since *U(x) > U(y)* implies *F(U(x)) > F(U(y))* for any monotonically increasing transformation *F, F(U(-))* is also a valid utility function. In other words, utility is still an *ordinal* concept in the analysis of behavior under uncertainty.

Example 1. Let *x* be an uncertain prospect consisting of prizes $W_{\{}$ and W_2 with respective probabilities *it* and n_2. If preferences can be represented by the utility function

$$U(x) = 7\text{r}, \log W_x + Jt_2 \log W_2 \qquad (13\text{-}6)$$

then

$$V(x) = e^{U(x)} = W''^{j}\, W_2^{2} \qquad (13\text{-}7)$$

is also a valid utility function.

In this example, however, there is an important difference between Eqs. (13-6) and (13-7). If we let *u(W)* = log *W,* Eq. (13-6) satisfies the expected utility property, whereas it is impossible to express (13-7) as the expected value of a utility function. In general, the expected utility property will not hold under an arbitrary monotonic transformation of the utility function. To preserve the expected utility property, the transformation has to be linear. This claim is easily verified. Suppose $U — n_x u(W\backslash) + TT_2U(W_2)$, and we subject it to a linear transformation $V = a + bil$ (with $b > 0$). Since $Ti\backslash + \mathrm{TT}_2 = 1$, we get

$$V = a + b(7tMWi) + 7T_2u(W_2))$$
$$= n_l(a + bu(W_{\{})) + n_2(a + bu(W_2)) \qquad (13\text{-}8)$$

Equation (13-8) satisfies the expected utility property with the Von Neumann-Morgenstern utility function equal to *a + bu(-).* It is important to distinguish clearly between the utility function for an uncertain prospect, *U(x),* and the Von Neumann-Morgenstern utility function, *u(W).* Whereas any monotonic transformation of *U* is a valid utility function representing the same preferences for uncertain prospects, an arbitrary monotonic transformation of *u* will not necessarily produce a valid Von Neumann-Morgenstern utility function that represents the same preferences. Von Neumann-Morgenstern utility functions are unique only up to linear transformations.

Example 2. Suppose preferences are represented by Eq. (13-6) in Example 1. The Von Neumann-Morgenstern utility function is *u(W)* — log *W.* If we subject *u* to the monotonic transformation $v = e^u$ and treat v as a Von Neumann-Morgenstern utility function, then the preferences for uncertain prospects are given by

$$V = JZ_xW_x + TT_2W_2$$

which is clearly different from the original preference structure shown in (13-6) or (13-7).

An index that is unique up to positive linear transformations is sometimes called a *cardinal* index. Once the origin and the interval of increments are determined, the cardinal index is uniquely determined. Temperature is an example of a cardinal scale; so is Von Neumann-Morgenstern utility. A fu nction subject to a linear transformation has the property that the sign of its second derivative is unchanged. Suppose W stands for wealth and $u''(W)$ is negative so that the marginal utility of wealth is decreasing. Since

$$\frac{d^2(a+b\,u(W))}{dW^2} = b\,u''(W)$$

any increasing linear transformation of u will preserve the property of diminishing marginal utility of wealth. As we will see in the next section, whether the Von Neumann-Morgenstern utility function exhibits increasing or decreasing marginal utility has important implications for behavior toward risk. However, it cannot justify the claim that changes in the level of subjective satisfaction can be compared, because the Von Neumann-Morgenstern utility function is only one (convenient) way to represent consumer preferences.

13.3 RISK AVERSION

In the certainty case, convexity of preferences implies a preference for variety. Figure 13-1 shows the indifference curve for a consumer who is indifferent between *(a)* two apples and no orange and *(b)* no apple and two oranges. Since the indifference curve is convex to the origin, the combination of one apple and one orange is strictly preferred to options *(a)* or *(b)*. Similarly, in the theory of intertem-poral consumption, convexity of indifference curves implies that a smooth path of consumption over time is preferred to an erratic path. When we analyze consumer behavior under uncertainty using the state preference approach, indifference curves can be drawn for state-contingent consumption. If we relabel the axes in Fig. 13-1 as "income in state 1" and "income in state 2," the diagram indicates that a sure income of $1 in either state is preferred to an uncertain income prospect of $2 in one

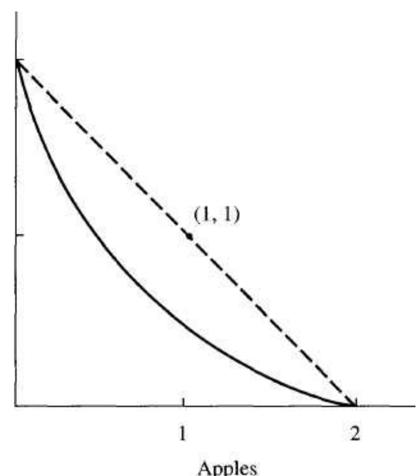


FIGURE 13-1
Risk Aversion. When the Von Neumann-Morgenstern utility function is concave, the marginal utility of income is decreasing. Individuals with such utility functions will be *risk-averse,* in the sense that they will refuse fair gambles. Such behavior is equivalent to convex indifference curves between state-contingent commodities.

state and nothing otherwise. In other words, the assumption of convex indifference curves implies that consumers are risk-averse.

Let us now consider the relationship between the convexity of indifference curves and the shape of the Von Neumann-Morgenstern utility function. Along an indifference curve, expected utility is constant. Thus, the indifference curve is defined by

$$TZ_xU\{W_x) + 7t_2u(W_2(Wi)) = U^\circ \qquad (13\text{-}9)$$

Differentiating (13-9) with respect to W_l, the slope of the indifference curve is

$$\underline{dW_2}$$

$K_2U'(W_2)$ If indifference curves are convex everywhere, then the second derivative,

$$\frac{d^2W_2 \qquad 7T_lu''(W_l)(n_2u'(W_2))^2 \qquad\qquad j^l}{dW_x^2 \sim \qquad\qquad \{n_2u'\{W_2)f}$$

is positive for all Wi and W_2. In particular, for $W\backslash = W_2 = W$, the second derivative is

$$\frac{d^2W_2}{dW^2} \qquad \frac{K_l7t_2(7T_l + 7T_2)U''(W)U'(W)^2}{(n_2u'(W)y} \qquad (13\text{-}10)$$

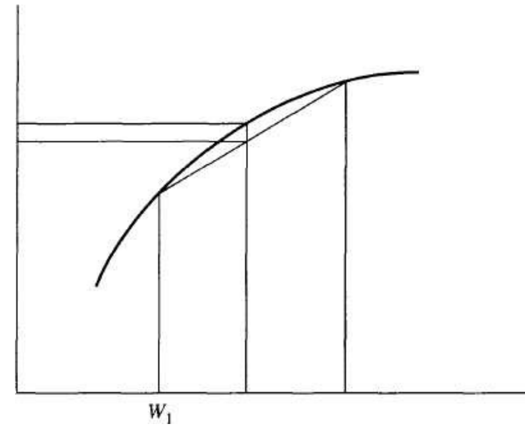
Expression (13-10) is positive if and only if $u''(W)$ is negative. The assumption that indifference curves are everywhere convex to the origin is equivalent to the assumption that the Von Neumann-Morgenstern utility function is concave.

When the Von Neumann-Morgenstern utility function is concave, marginal utility of income is decreasing. If an individual with a concave utility function is given a 50-50 chance of losing or winning \$1, we can predict that the individual will not take the gamble. Loosely speaking, this is because the gain in utility as a result of winning \$1 is less than the utility loss from losing the gamble, although we cannot attribute any psychological significance to comparing changes in utility levels. In general, for any individual with a concave utility function, a sure income prospect is preferred to an uncertain income prospect with equal expected value. This is a consequence of *Jensen's inequality,* which states that for any random variable Wand any strictly concave function $u(W)$,

$$E[u(W)] < u(E[W])$$

Jensen's inequality is illustrated in Fig. 13-2. Expected utility is given by the height of the chord at £[W], whereas the utility of expected wealth is given by the height of the arc at £"[W]. On the other hand, if the utility function is convex, the chord will lie above the arc, and the individual will be risk-loving. An individual will be risk-neutral if and only if the utility function is linear in income.

Example 1. Suppose a person's utility function is $u(W) = \log W$. Since $u''(W) = — l/W^2 < 0$, the person is risk-averse. We have already seen in Sec. 13.1 that the expected value of a St. Petersburg gamble is infinite. However, the expected utility of



such a gamble is finite

when the person is risk-averse.

00,jxn

Xr = log2 IV ^/i

112 + 8 = log2

In other words, if the person starts with no initial wealth, the certainty equivalent of a St. Petersburg gamble is worth $2.

Measures of Risk Aversion

We have seen that convex indifference curves imply risk aversion. A natural measure of the degree of risk aversion is therefore the degree of convexity of the indifference curves. In Eq. (13-10), the magnitude of the second derivative of the indifference curves along the 45° certainty line is proportional to — $u''(W)/u'(W)$. We call this **quantity the** *coefficient of absolute risk aversion* or **the** *Arrow-Pratt measure of absolute risk aversion,* after Kenneth Arrow and John Pratt.t The coefficient of

*See Kenneth J. Arrow, *Aspects of the Theory of Risk Bearing,* Yrjo Jahnssonin Saatio, Helsinki, 1965; and John W. Pratt, "Risk Aversion in the Small and in the Large," *Econometrica,* 32:122-136, 1964.

absolute risk aversion has implications for the willingness of individuals to accept risk. Suppose an individual has initial wealth *W. A* risk-averse individual will not be willing to take a fair gamble. The risk premium $P_X(W)$ is defined as the amount a person is willing to pay to avoid a fair gamble x (with mean 0 and variance *a^).* Mathematically, we can write

$$u(W - P_X(W)) = E[u(W + \text{JC})]$$

Taking a first-order Taylor series approximation on the left and a second-order approximation on the right, we obtain

$$u(W) - P_x(W)u'(W) \text{ « } E\backslash U(W) + XU'(W) + -x^2u''(W)$$

$$u(W) + -o^2_x u''(W)$$

and therefore

$$2 \;^x \quad u'(W)$$

Thus, the higher the coefficient of absolute risk aversion, the higher the risk premium the individual is willing to pay.

A related measure of risk aversion is the coefficient of *relative* risk aversion, — $Wu''(W)/u'(W)$. Let $P_X(W)$ be the proportional risk premium corresponding to a proportional risk$x$ (with mean 0 and variance a_x). Then $P_X(W)$ is defined by the relation

$$u(W - WP_X(W)) = E[u(W +$$

Wx)] Taking Taylor approximations on both sides,

$$u(W) - WP_x(W)u'(W) \text{ « } E\backslash U(W) + Wxu'(W) +$$

$\sim W^2x^2u''(W)$ and therefore

$$P_X\{W) \text{ « -« } \qquad \frac{2\text{-}Wu''(W)}{2\;^x \quad u'\{W)}$$

Again, the relative risk premium is higher as the coefficient of relative risk aversion is higher.

Mean-Variance Utility Function

The expected utility hypothesis suggests that preferences toward uncertain prospects can be represented by the expected value of a Von Neumann-Morgenstern utility function *E[u(W)],* where *W* is a random variable that represents the income from an uncertain prospect. Expected utility in general depends on the form of the function «(•) and on the distribution of *W.* Suppose the distribution of *W* can be completely characterized by a vector of parameters *a*. In particular, let *W* be distributed on the real line with a probability density function *f(W; a)*. Then

$$E[u(W)] = \int_{-\infty} u(W)f(W;a)dW$$

The integral on the right-hand side of this equation is a function of *a)* If we let this integral be represented by *U(a),* then *U(a) = E[u(W)]* is a valid representation of preferences.

Many problems in the economics of uncertainty are related to the trade-off between the average level of income and its degree of riskiness. Since the mean is a summary measure of average and the variance is a summary measure of risk, it will be particularly convenient to represent preferences by a function of the mean and variance of the income distribution. Unfortunately, this is not always possible, because in general the mean and variance do not completely determine the distribution of a random variable. There are many income streams that have the same mean and variance but different probability distributions. The expected utility associated with these income streams are different. Although *U(a)* is a valid representation of preferences, the vector *a* generally contains more than two parameters. Thus a utility function that depends only on mean and variance can at best be viewed as an approximation to expected utility.

There are some special cases, however, when a function involving only the mean and variance of the income distribution can be used to represent preferences. For example, suppose an uncertain income prospect *W* is normally distributed with mean *m* and variance *v.* Its probability density function is

$$\underline{*} \quad e^{-(1/2v)(W-m)^2}$$

This function is completely determined by the values of *m* and *v,* so expected utility may be written as *U(m, v)*.

A further simplification is possible if the Von Neumann-Morgenstern utility function takes the form

$$u(W) = -e^{-rW}$$

This function exhibits constant absolute risk aversion, because the degree of absolute risk aversion is a constant equal to r:

$$\frac{r^2e^{-rW}}{u'(W)} \sim re^{-rW} \sim$$

Given a utility function with constant absolute risk aversion and an income prospect

†It is not a function of *W* because *W* is just the variable of integration.

that is normally distributed, expected utility is

poo i

U(m, v) = / -e-rW—L=e-^-m)2/2v dW

$$\frac{1}{\text{jjy}} \quad {}_p\text{-}(\backslash/2v)[\{W\text{-}m)^2+2rvW]$$

/

Completing the square, the term in brackets is equal to

$$(W - m)^2 + 2rvW = [(W - mf + (rv)^2 + 2rv(W - m)]$$
$$-(rv)^2 - 2rv(W - m) +$$
$$2rvW = [(W - m) + rv]^2 +$$
$$2rvm - \mathrm{r\ V} = [W - im - rv)]^2 +$$
$$2rvm - r^2v^2$$

Therefore,

$$U(m,v)= \quad / \quad {}_p\text{-}\{\backslash/2v)[W\text{-}(m\text{-}rv)]^2{}_p\text{-}\{\backslash/2v)$$
$$\{2rvm\text{-}r^2v^2)$$
$$J\text{-}oo \quad \backslash j2llV$$

$$— \_{}_e\text{-}(rm\text{-}r^2v/2) \quad / \quad {}_L \quad {}_c\text{-}(l/2v)\backslash(W\text{-}(m\text{-}rv)$$
$$J \quad /$$

The integral in the last expression above is the integral of a normal probability density function with mean *m — rv* and variance *v.* Since probability density functions integrate to 1, we have

$$U(m, v) = \_{}_e\text{-}(\text{™-'-V2})$$

Let

$$Vim, v) = m\text{---}\frac{r}{2}v$$

Then *U* = —*e~rV*. Notice that *dU/dV* = *re~rV* > 0. Therefore *V(m, v)* is a mono-tonic transformation of *U(m, v).* Maximizing expected utility *U(m, v)* is equivalent to maximizing the function *V(m, v).* Thus the function *V(m, v)* is a valid representation of preferences. This mean-variance utility function is often used in applied work because of its simplicity: It is a linear function of the mean and variance. Furthermore, the marginal rate of substitution between expected income and risk is a constant:

— *V* *r*

***y* ~ 9**

The higher the degree of absolute risk aversion, the more expected income one is willing to give up in order to reduce the exposure to risk.

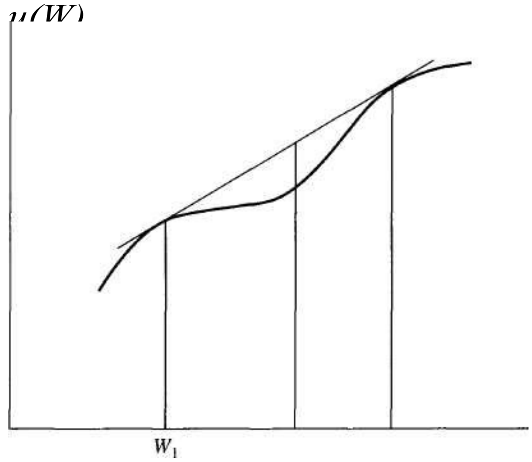


FIGURE 13-3 *The Friedman-Savage Proposition.* In 1948, Milton Friedman and L. J. Savage proposed a utility function with a convex section to explain why an individual might buy insurance and lotteries at the same time. However, such an individual would take large gambles to leave the convex section and then behave as a risk averter. Gambling can be explained by its entertainment value, consistent with the observation that people divide their stakes into small bets.

Gambling, Insurance, and Diversification

In the absence of restrictions on the shape of the utility function, the expected utility hypothesis is consistent with both risk-taking and risk-avoiding behavior. Friedman and Savage[t] argue that if the utility function is shaped like the one shown in Fig. 13-3, an individual may buy insurance and lotteries at the same time. However, there are two problems with the theory that gambling is a result of nonconcavity of the utility function:

1.376 Since it is relatively inexpensive to effect a gamble, any person with initial wealth falling into the nonconcave range of the utility function will take gambles to leave that range. In Fig. 13-3, an individual with initial wealth *E[W]* will take even enormous gambles and end up at either W_1 or W_2. Enormous gambles are not common, and once people have taken such gambles they will behave as risk averters.

1.377 Most gambles have odds that are worse than fair. If gambling is for maximizing expected utility of wealth, the optimal strategy is to place the entire stake in one gamble. The observation that most people divide their stakes into small bets is consistent with the theory that people gamble because of its entertainment value.

When individuals have concave utility functions, they will take steps to reduce their exposure to risk. One approach is to buy market insurance. Suppose an individual has initial wealth *W*. There is a chance of losing JC with probability *n* due to,

Milton Friedman and L. J. Savage, "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy,* 56:279-304, 1948.

say, theft. Assume the person can buy actuarily fair insurance at a premium of *n Q* dollars for *Q* dollars of coverage. He or she can choose the amount of coverage *Q* to maximize expected utility:

$$\max\ nu(W -x -nQ + 2) + (1 - n)u(W - nQ)$$

The first-order condition is

$$nu'(W - x - JTQ^* + Q^*)(1 - n) + (1 - n)u'\{W -nQ^*)(-n)$$

$=$ 0 that is,

$$u'(W\text{-}x -nQ^* + Q^*) = U'(W\text{-}TTQ^*) \quad (13\text{-}12)$$

For w(-) strictly concave, (13-12) implies $W — x — TTQ^* + Q^* = W — nQ^*$, or

$$Q^*=x$$

Thus, a risk-averse individual will buy full insurance if it is available at an actuarily fair premium. Very often, however, the probability and the amount of damage are not fixed. If efforts to reduce the chance and the extent of damage are costly to observe, buying insurance will reduce the individual's incentive to supply such efforts. This is known as *moral hazard.* Methods to mitigate moral hazard include coinsurance and deductibles, but these are beyond the scope of this chapter.^

Another way to reduce exposure to risk is diversification. If an individual invests in one risky project X, Eq. (13-11) shows that the risk premium is approximately *\o^a,* where *a* is the coefficient of absolute risk aversion. On the other hand, if the individual invests in *n* different projects, with a1/« share in each, the risk premium *P* for each project is given by

$$u(W - P) = \underset{l}{E} \setminus u (W + -x$$

Taking Taylor approximations on both sides and rearranging, we get

If the returns to the *n* projects are independent, the total risk premium is

$$nP \sim \frac{\quad}{2\ n}\text{-}a$$

which is only *\ln* of the risk premium for the undiversified investment.

^Moral Hazard in a principal-agent model is discussed in Chap. 15.

13.4 COMPARATIVE STATICS

Allocation of Wealth to Risky

Assets

Most decisions are made under conditions of uncertainty. Economists postulate that individuals make choices so as to maximize expected utility. Let us begin with a problem in the allocation of wealth between risky and safe assets. Suppose an individual has initial wealth *W,* which is to be divided between a safe asset (say, money) whose rate of return is zero and a risky asset whose rate of return is a random variable *R*. If he or she invests *x* dollars in the risky asset, final wealth will *be, (W — x)* + *x(l + R)* = *W* + *xR*. The individual chooses *x* so as to maximize expected utility of wealth:

$$\max_{X} E[u(W + xR)]$$

When the utility function is well behaved, we can differentiate inside the expectation operator^ to get the first- and second-order conditions:

$$E[u'(W + xR)R] =$$

$$0 \; E[u''(W + xR)R^2]$$

$$< 0$$

The assumption that the individual is risk-averse (i.e., $u'' < 0$) ensures that the second-order condition is satisfied. The first-order condition defines the amount of investment in the risky asset as a function of initial wealth, $x = x^*(W)$. Substituting $x^*(W)$ for x in the first-order condition and differentiating with respect to *W,* we obtain

$$E[u''(W + xR)\{\backslash + Rx^{*\prime}(W))R] = 0$$

Using the additive property of the expectation

operator,

$$E[u''(W + xR)R] + E[u''(W + xR)R^2 x^{*\prime}(W)]$$

= 0 Therefore,

$$=$$

$$E[u''(W + xR)R^2]$$

Since the denominator is negative, the sign of $x^{*\prime}(W)$ is the same as the sign of the numerator. It turns out that the numerator is positive if the coefficient of absolute risk aversion is decreasing in wealth. When absolute risk aversion is decreasing, we have

$$u''\{W+xR) \qquad \sim u''(W)$$

$$u'(W + xR) \quad \sim \quad u'(W) \qquad < \qquad \text{for } R > 0$$

$$u\{W)$$

$$> \quad (W) \; \text{for } R < 0$$

$$u'(W+xR) \quad \sim \quad u'(W)$$

^Think of this as differentiating inside an integral sign.

Multiplying both sides by $-u'(W + xR)R$ (which is a negative number for the first inequality and a positive number for the second inequality), we get

$$u''(W + xR)R > \frac{u''(W)}{u'(W)} u'(W + xR)R \qquad \text{for all } R$$

Taking expectations on both sides,

$$E[u''(W + xR)R] > \frac{u''(W)}{u'(W)} E[u'(W + xR)R]$$

The right-hand side of this inequality is equal to zero by the first-order condition. Hence, $x^{*\prime}(W) > 0$. If absolute risk aversion is decreasing in wealth, a rise in wealth will raise the amount of investment in risky assets.

Output Decisions Under Price Uncertainty

In the previous example we derived a typical comparative statics result concerning the effect of a change in a nonrandom parameter. Under uncertainty, however, the exogenous factors affecting choice are often random. Instead of asking how changes in the *value* of a random variable will affect choice, we have to ask how changes in the *distribution* of the random variable affect behavior. We illustrate this with a model of the competitive firm under price uncertainty. Suppose a risk-averse, price-taking firm has to make output decisions before the price of the product is known. The objective of the firm is to maximize expected utility of profits:

$$\max_y E[u(py - c(y))]$$

where p is a random variable denoting the price of the product, y is the output of the firm, and $c(y)$ is the cost function. Differentiating with respect to y, we obtain the conditions for a maximum:

$$E[u'(py - c(y))(p - c'(y))] = 0$$
$$D = E[u''(py - c(y))(p - c'(y))^2 - u'(py - c(y))c''(y)] < 0$$

As in the previous analyses, we assume the strict inequality for the second-order conditions.

It is instructive to compare the level of output under price uncertainty to the certainty case. Let $\bar{p}$ be the mean of the random variable p, and write the first-order condition as $E[u'(py - c(y))p] = E[u'(py - c(y))c'(y)]$. Then, subtracting $E[u'(py - c(y))\bar{p}]$ on both sides, we get

$$E[u'(py - c(y))(p - \bar{p})] = E[u'(py - c(y))(c'(y) - \bar{p})] \tag{13-13}$$

The left-hand side of Eq. (13-13) is the covariance between price and marginal utility. When price is high, profits are high and (because of diminishing marginal utility) marginal utility is low. Similarly, marginal utility is high when price is low. The covariance term is thus negative. Consequently, the right-hand side of (13-13) is also negative, which implies

$$c'iy) < P$$

In other words, output under price uncertainty is characterized by marginal cost being less than the expected price. If marginal cost is increasing in output, then for the same expected price, output under price uncertainty is lower than for the certainty case.

To derive comparative statics results, first note that output y^* is a function of the *distribution of p.* We cannot ask how y^* changes as p varies because p is itself a random variable. To do comparative statics we have to change the parameters of the distribution *of p.* For example, since the mean of p is p, we can write $p = p + e$, where e is a random variable with mean zero. Then the first-order condition can be written as

$$E[u'((p + e)y^*(p) - c(y^*(p)))«P + {}^{e}) \sim c'(/(p)))]$$

$= 0$ Differentiating with respect to p, we get

$$\frac{dy^*}{dp} \quad \frac{yE[u''(py - c\{y))\{p - c'(y))]}{-D} \quad \frac{E[u'(py -}{-D}$$

The second term is clearly positive; it is the substitution effect. The sign of the first term depends on the degree of absolute risk aversion. Let x be the level of profits when $p = c^f(y)$ (x is nonrandom). If absolute risk aversion is decreasing, then

$$\frac{-u'Xpy - c(y))}{u'\{py - c(y))} < \frac{-u\backslash x)}{u'\{x)} \quad \text{for } p > c(y)$$

$$\frac{-u'Xpy - c(y))}{uXpy - c(y))} > \frac{-u''\{x)}{u'(x)} \quad \text{for } p < c(y)$$

Multiplying both sides by $—u'(py — c(y))(p — c'(y))$, we have

$$u'Xpy - c(y))(p - c'(y)) > -—\hat{}uXpy - c(y))(p - c'(y)) \quad \text{for all } p$$
(13-15)

Taking expectations on Eq. (13-15), it can be seen from the first-order condition that the right-hand side has expected value zero. Thus, the first term of Eq. (13-14) is positive. That term represents the wealth effect. As expected price increases, wealth rises and (assuming decreasing risk aversion) the firm is willing to take greater risk by increasing production. The wealth effect reinforces the substitution effect to give a positive response of output to expected price.

Increases in Riskiness

In models of decision making under uncertainty, the choice variables are functions of the distribution of random variables. We have already seen how one can derive comparative statics results for changes in the mean of the distribution. Very often it is also interesting to analyze the change in behavior as the distribution becomes

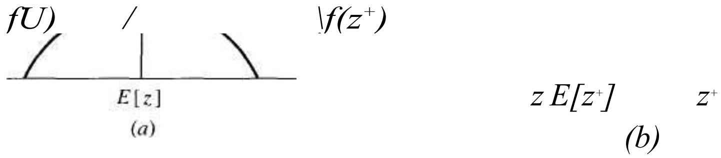


FIGURE 13-4
Mean-Preserving Spread. The density function shown in panel *(a)* is subjected to a mean-preserving spread, shown in panel *(b)*. The distributions have the same mean, but in panel *(b)* added weight is given to outcomes further from the mean.

more "risky," with the mean remaining unchanged. One way to do this is to perform comparative statics for the scale parameter of the distribution. For example, if z is a random variable with mean z and standard deviation o_z, we can let z = z + $<J_z e$, with e being a random variable with zero mean and unit variance. Substituting z + $o_z e$ for z in the first-order condition for maximization and differentiating with respect to a_z, we obtain the optimal response to an increase in riskiness. An increase in the scale parameter is one way to increase the riskiness of the distribution of a random variable; it makes the probability distribution more "stretched" around a constant mean. A more general and more useful representation of increases in riskiness is "mean-preserving spreads." If a random variable z is replaced by z^+ = z + e, where e is a random variable with conditional mean equal to zero, then z and z^+ have the same mean, and it is natural to say that z^+ is more risky than z. It turns out that adding noise to a random variable (i.e., replacing z with z^+) is equivalent to moving some of the probability mass from the center part of the density out to the tails. Figure 13-4 shows the probability density functions of z and z^+■ The random variable z^+ is called *a mean-preserving spread* of z.

The notion of mean-preserving spreads is useful because if z^+ is a mean-preserving spread of z, then for any concave function «(•),

$$E[u(z^+)] < E[u(z)] \qquad (13\text{-}16)$$

Equation (13-16) follows from Jensen's inequality. Using the method of iterated expectation,

$$\begin{aligned} E[u(z + e)] &= E[E[u(z + e) \mid z]] \\ &< E[u(z + E[e \setminus z])] = E[u(z)] \end{aligned}$$

Thus, if an income prospect becomes more risky in the sense that its probability distribution undergoes a mean-preserving spread, the expected utility to a risk-averse individual will fall. Similarly, since $E[u(z^+)] > E[u(z)]$ for any convex function $u(-)$, a risk-loving individual will prefer income prospects that are more risky.

It is also convenient to do comparative statics using the concept of mean-preserving spreads. Suppose an individual chooses x to maximize the objective function $E[f(x, z)]$, where z is a random variable. The sufficient conditions for maximization are

$$E[fAx,z)] =$$
$$0$$
$$E[f_{xx}(x,z)]<0$$

Now let a be a parameter that represents a mean-preserving spread to the distribution of z. The first-order condition defines a choice function $x = x^* (a)$. A change in a will affect the value of x^* and the value of $E[f_x(x, z)]$ directly as well. Differentiating the first-order condition with respect to a, we get

$$\text{^}f\text{-}E[f_{xx}(x, z)] + \text{^-}E[f_x(x, z)\backslash = 0 \qquad (13\text{-}17)$$
$$da \qquad\qquad da$$

If $f_x(x,z)$ is a concave function in z, then $E[f_x(x, z)]$ will decrease as z undergoes a mean-preserving spread. Thus, the second term of (13-17) is negative. Since $E[f_{xx}(x, z)]$ is negative by the second-order condition, $dx^*/da < 0$. Similarly, $dx^*/da > 0$ if $f_x(x, z)$ is a convex function in z.

Let us illustrate the method with a simple model of investment under uncertainty. Suppose the production function is $f(K, L)$, where K is capital, L is labor, and / is homogeneous of degree 1. For simplicity, assume that capital lasts only one period so that we do not have to consider the dynamic aspect of the problem. The producer has to choose K before output price is known. After K is chosen, output price is revealed, and the producer determines the number of workers to work with his capital. The wage rate for labor is w and the cost of capital is given by a convex function $c(K)$. We assume the producer is risk-neutral. For any given K, the producer will choose L so as to maximize profits. We can define the indirect profit function by

$$v(p, w, K) = maxpf(K, L) - wL$$

In earlier chapters we have shown that the indirect profit function is convex in p and w. It is linear in K if / is homogeneous of degree 1. Thus, we can write $v(p,w, K) = Ki)(p,w)$. Since the producer is risk-neutral, he or she will choose K to maximize expected profits:

$$msLxE[Ki>(p,w)-c(K)]$$

The sufficient conditions are

$$E[v(p,w)]-c'(K) =0$$
$$0$$

The assumption that marginal cost of capital is increasing ensures that the second-order condition is satisfied.

To see how the amount of investment will change as the distribution of output price becomes more variable, let a be a parameter that represents a mean-preserving spread to the distribution of p. Differentiating the first-order condition with respect to a, we get

$$\frac{4E[\Pi p, w)]}{da} \quad c(K)^{\wedge} \frac{}{da} = 0 \qquad (13\text{-}18)$$

Since $v(p, w)$ is convex, a mean-preserving spread will increase the value of $E[v(p, w)]$. The first term of (13-18) is positive, and therefore $dK^*/da > 0$. If the amount of labor cannot be adjusted after output price is revealed, expected profits will be unaffected by changes in the price distribution as long as the mean price remains unchanged. In this model, however, the producer can hire more workers when output price is high. Consequently, the increase in profits will be more than proportional to the increase in price. On the other hand, when output price is low, the producer can reduce the number of workers so that the fall in profits will be less than proportional to the fall in price. As a result, the expected return to investment will be higher as output price becomes more variable, and the amount of investment will increase.

PROBLEMS

1.378 Show that the coefficient of absolute risk aversion is invariant to linear transformations of the utility function.

1.379 Let u and v be two utility functions, with $v(W) — f(u(W))$, where/is concave. Prove that the coefficient of absolute risk aversion for v is greater than that for u.

1.380 *(a)* Verify that the function $u(W) = W^{1}\sim^{a}/(l —a)$ has a constant coefficient of relative risk aversion equal to a.

(b) Verify that the function $u(W)$ = log W has a constant coefficient of relative risk aversion of 1.

4. *(a)* Suppose the utility function is given by $u(W) — aW — bW^2$ (with a and b both positive). Does the function exhibit increasing or decreasing risk aversion?

1.381 If the rate of return on risky assets is a random variable R with mean $R > 0$ and variance $o\backslash$, and if the individual's initial wealth is W, what is the optimal amount of investment in risky assets?

1.382 Show that the optimal amount of risky investment is a decreasing function of wealth.

5. If the utility function is $u(W) = — e\sim^{aW}$ so that the absolute risk aversion is constant, show that the amount of investment in risky assets is independent of initial wealth.

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CHAPTER

14

MAXIMIZATION WITH INEQUALITY AND NONNEGATIVITY CONSTRAINTS

14.1 NONNEGATIVITY

In the previous pages we have largely ignored the issues raised by constraining the variables in a maximization model to be nonnegative. In the model of the firm, for example, we did not consider the possibility that simultaneous solution of the first-order equations might lead to negative values of one or more inputs. Such an occurrence would nullify the condition for profit maximization that wages be equal to marginal revenue product. In a more general sense, there are many factors of production that a firm chooses not to use at all. Similarly, consumers choose to consume only a small fraction of the myriad of consumer goods available. It is possible to characterize mathematically the conditions under which nonnegativity becomes a binding constraint. It might be remarked first, however, that since the refutable comparative statics theorems are concerned with how choice variables change when parameters change, the comparative statics of variables *not* chosen is fairly trivial. In a local sense (the evaluation of the partial derivatives of the choice functions at a given point) these variables continue not to be chosen; that is, $dx^*/daj = 0$ for these variables. In a global sense, e.g., price changes of finite magnitude, factors or goods previously not chosen may enter the relevant choice set. For these situations, more powerful assumptions must be made to yield refutable theorems than in our previous discussions, where strictly *local* phenomena were analyzed.

Consider the monopolist of the first chapter. A profit function of the type

TT(JC) = *R(x)* - *C(x)* (14-1)

is asserted to be maximized, where *R(x)* and *C(x)* denote, respectively, the revenue and cost associated with a given level of output *x*. (We are ignoring the tax aspect of the model, as it is not germane to this discussion.) The first-order conditions for a maximum of *n(x)* are

TT'(X) = *R'(x)* - *C\x)* = 0 (14-2)

However, this condition is meant to apply only to those situations where the solution to (14-2) is nonnegative. The firm might choose to produce zero output, however, if, for example, *R'(x)* < *C'(x)* for all *x* > 0. In that case, where the marginal revenue is less than marginal cost, increasing output *reduces* profits 7r (JC). The existence of a *maximum* of profits (not necessarily *positive* profits, another issue entirely) at some *positive* level of output JC* presupposes that for some 0 < *x* < JC*, MR > MC; that is, *R'(x)* > *C'(x)* so that it "paid" for the firm to start operations in the first place. The only reason the profit *maximum* would occur at JC = 0 is that MR(0) < MC(0). That is, *if* maximum *TT* occurs at *x* = 0, then *IT'* = *R'(x)* — C'(JC) < 0 at JC = 0. The converse is not being asserted; it is in fact false. If *R'(x)* — *C'(x)* < 0 at JC = 0, this does not imply that an interior maximum cannot occur at some *x* distant from the origin. Again, the only aspect of the firm's behavior under consideration here is the attainment of *maximum* profits, not whether the firm shall exist or not [presumably dependent upon 7T(JC) > 0].

Let us summarize this condition for maximization of functions of one variable. Consider some function *y* = /(JC). Then the first-order condition for *f(x)* to achieve a maximum *subject to the nonnegativity constraint* JC > 0 is

/'(*) < 0 **if/'(*)<O** then = 0 (14-3)

Alternatively, one can express the same idea as

fix) < 0 (14-4a)

*/'(*) = 0 (14-4&)

Geometrically, the situation is as depicted in Fig. 14-1. In Fig. 14-la the usual, *interior* maximum is illustrated. This solution is called an interior maximum because the value of JC that maximizes /(JC) does not lie on the boundary of the set over which *x* is defined (here, the nonnegative real axis; its only boundary is the point *x* = 0). The set of positive real numbers is the *interior* of this domain of definition of JC: hence the terminology. In Fig. 14-1/? and c, corner solutions are depicted. That is, the maximum value of /(JC), for JC > 0, occurs when JC = 0. (The fact that the function in Fig. *14-lb* achieves a regular maximum at a negative value of JC is irrelevant.) When the maximum occurs at *x* = 0, it is impossible to have /'(JC) > 0 there. If /'(O) > 0, increasing JC would increase / (JC) and /(0) could not be a maximum. However, it is possible that /'(()) = 0, as in Fig. 14-lc. There, the *fix)*

O *(a)* *x*

fix) has an interior maximum; that is, *x* > *0,f'(x)* = 0

fix)

O *(b)*

fix) has a corner solution, with/'(x) < 0, *x* = 0

fix)

f'iO) = 0

O *(c)*

fix) has a corner solution, with/'(;c) = 0, *x* = 0

FIGURE 14-1
(a) f (x) has an interior maximum; that is, *x* > 0, *fix)* = 0. *(b)f(x)* has a corner solution with *fix)* < 0, x = 0. fc) /(x) has a corner solution, with *f{x)=0, x = 0.*

nonnegativity constraint is nonbinding. That is, the maximum *f(x)* would occur at *x* = 0 anyway, even without the restriction *x* > 0. Thus, if a maximum occurs when *x* > 0, /'(JC) = 0. If the maximum occurs when JC = 0, then necessarily *f'(x)* < 0. This condition is expressed in relation (14-3) or, equivalently, (14-4).

These more general first-order conditions can be derived algebraically by the device known as adding a *slack variable.* The constraint JC > 0 is an elementary form of the more general inequality constraint *g(x)* > 0. By converting this inequality to an *equality constraint,* ordinary Lagrangian methods can be used to derive the first-order conditions.

The constraint $x > 0$ is equivalent to

$$x-s^2 = 0 \qquad \text{(14-5)}$$

where s takes on any real value. When s =£ 0, an interior solution is implied, since x = $s^2 > 0$. When $s = 0$, a corner solution is present.

We can now state this as the constrained maximum problem:

maximize

$$y = fix)$$

subject to

$$x - s^2 = 0$$

The Lagrangian for this problem is

$$X = f(_x) + X(x - s^2) \qquad \text{(14-6)}$$

Taking the first partials of i£ with respect to JC, *s,* and A. gives

$$2_X = /'(*) +A. = 0 \qquad \text{(14-7fl)}$$

$$\% = -2ks = 0 \qquad \text{(14-1 b)}$$

$$£g_A = JC - s^2 = 0 \qquad \text{(H-lc)}$$

From Eq. *(14-1 b)* we see that if s =*fc* 0, that is, an *interior* solution is obtained, then $X = 0$ and hence from *(14-la),* /'(JC) = 0. Thus, as expected, the usual condition $f'(x)$ — 0 is obtained for noncorner solutions. Using the second-order conditions for constrained maximization, we can show that A. > 0. The second-order condition is that

$$£_{xx}h^2_x + 2£_{xs}h_xh_s + <£_{ss}h] < 0 \qquad \text{(14-8)}$$

for all h_x, h_s satisfying

$$g_xh_x+g_sh_s=0 \qquad \text{(14-9)}$$

where $g(x, s) = x — s^2$, the constraint. From the Lagrangian (14-6), $! £_{xx} — f''(x)$, $!£_{xs} = 0$, $!£_{ss}$ — —$2k$. From the constraint $g(x, s) = x — s^2$, $g_x = 1$, $g_s = —2s$. Hence, (14-8) and (14-9) become

$$f''(x)h^2_x-2Xh] <0 \qquad \text{(14-10)}$$

for all h_x, h_s satisfying

$$h_x - 2sh_s = 0 \qquad (14\text{-}11)$$

We already know from Eq. (14-7&) that if s ^0, then A. = 0. Suppose now that $s = 0$. Then from Eq. (14-11), $h_x = 0$, but no restriction is placed on h_s. When we use $h_x = 0$, h_s = anything, Eq. (14-10) becomes

$$-lkh^{l}_{s} < 0$$

implying, since $h^2 > 0$,

$$X > 0 \qquad (14\text{-}12)$$

We now have a complete statement of the first-order conditions for maximizing *f(x)* subject to $x > 0$. From *(\4-7a),* since A > 0,

$$fix) < 0 \qquad (14\text{-}13)$$

If $f'(x) < 0$, then A > 0. From (14-76) $s = 0$ and thus $x = 0$ from (14-7c). Therefore,

if

$$f'(x) < 0 \qquad x = 0 \qquad (14\text{-}14)$$

Equations (14-13) and (14-14) are equivalent to

$$f(x) < 0 \qquad (14\text{-}15)$$

$$xf'(x) = 0 \qquad (14\text{-}16)$$

commonly written

$$f'(x) < 0 \qquad \text{i f} <, x = 0$$

Notice that if the maximum occurs at $x = 0$, no restrictions on *f"(0)* are implied. In Fig. *14-lb, f(x)* could be either convex (as drawn) or concave, and the maximum would still occur at *x* — 0.

These conditions can also be derived using the determinantal conditions on the bordered Hessian of second partials of if:

cp eg
<=^xx °*^x **> 0**
<g <g

8s 0

Using the values previously calculated for these partials, we have

0 /"(*) 0 1 \SE\ = (14-17)
0 -2A. -25 1
-25 0

From (14-17), if *s* — 0 (corner solution), *2X* > 0; hence A. > 0. Thus, from (14-7a), *fix)* < 0.

The first-order conditions for obtaining a *minimum* value of *f(x)* subject to $x > 0$ are obtained in a similar manner. One quickly shows that these conditions are

$$f'(x) > 0 \qquad \text{i f} >, x = 0 \qquad (14\text{-}18)$$

That is, if a minimum occurs at $x = 0$, it must be the case that /(JC) is rising (or horizontal) at JC = 0. Otherwise, i.e., if the function were falling at $x = 0$, making x positive would lower the value of /(JC) and /(JC) could not have a minimum at **jt = 0.**

Functions of Two or More Variables

The principles just delineated for maximization of functions of one variable generalize in an obvious manner to functions of two or more variables. Consider the problem

maximize

$$z = f(x_x, x_2)$$

subject to

$$JCI > 0 \qquad JC_2 > 0$$

Let us now add slack variables s^2, $s\backslash$ in the manner of the first example. The problem then becomes one of maximization subject to two *equality* constraints:

maximize

$$y = f(x_u x_2)$$

subject to

$$g^1(x_U Si) = *! \ \text{-jf} = 0$$

$$g^2(x_2, s_2) = x_2 -$$

$s^2 = 0$ The Lagrangian for this problem is

$$\# = f(xi, x_2) + k_x (JCI - s^2) + A_2(x_2 -$$

$s^2{}_2)$ The first-order conditions for maximization are

$$2,, = /i + A.! = 0 \qquad (14\text{-}19«)$$

$$\wedge = /_2 + A._2 = 0 \qquad (14\text{-}1\%)$$

$$= 0 \qquad (14\text{-}19c)$$

$$= 0 \qquad (14\text{-}19J)$$

$$se_{XI} = *, - s? = 0 \qquad (i'$$

$$£,_2 = x_2 - s^2{}_2 = 0 \qquad (14\text{-}19/)$$

From Eqs. (14-19c) and (14-19J), if either constraint is nonbinding, i.e., if $si =£ 0$ or s_2^0, then, respectively, $k\backslash = 0$, $X_2 = 0$. In that case

(JCI > 0, $JC_2 > 0$), the ordinary first-order relations $f\backslash = 0$, $f_2 = 0$ obtain.

We can show that $\lambda_1 > 0$, $\lambda_2 > 0$ by using the second-order conditions. For a constrained maximum,

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \mathcal{L}_{x_i x_j} h_i h_j + 2\sum_{i=1}^{2}\sum_{j=1}^{2} \mathcal{L}_{x_i s_j} h_i k_j + \sum_{i=1}^{2}\sum_{j=1}^{2} \mathcal{L}_{s_i s_j} k_i k_j < 0 \qquad (14\text{-}20)$$

for all values h_1, h_2, k_1, k_2 such that

$$g^1_{x_1} h_1 + g^1_{s_1} k_1 = 0 \qquad (14\text{-}21a)$$

$$g^2_{x_2} h_2 + g^2_{s_2} k_2 = 0 \qquad (14\text{-}21b)$$

By inspection of the Lagrangian [or Eqs. (14-19)] we have

$$\mathcal{L}_{x_i x_j} = f_{ij} \qquad i, j = 1, 2$$

$$\mathcal{L}_{x_i s_j} = 0 \qquad i, j = 1, 2$$

$$\mathcal{L}_{s_i s_j} = \begin{cases} -2\lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$g^i_{x_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$g^i_{s_j} = \begin{cases} -2s_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Relations (14-20) and (14-21) therefore become

$$\sum_{i=1}^{2}\sum_{j=1}^{2} f_{ij} h_i h_j - 2\lambda_1 k_1^2 - 2\lambda_2 k_2^2 < 0 \qquad (14\text{-}22)$$

for all h_1, h_2, k_1, k_2 such that

$$h_1 - 2s_1 k_1 = 0 \qquad (14\text{-}23a)$$

$$h_2 - 2s_2 k_2 = 0 \qquad (14\text{-}23b)$$

We already know that if $s_i \neq 0$, then $\lambda_i = 0$. Suppose therefore that $s_i = 0$. Then from (14-23), $h_i = 0$. Then Eq. (14-22) becomes

$$-2\lambda_1 k_1^2 - 2\lambda_2 k_2^2 < 0$$

This must hold for all k_1, k_2. Setting $k_1 = 0$, $k_2 = 0$ in turn therefore yields

$$\lambda_1 > 0 \qquad (14\text{-}24a)$$

$$\lambda_2 > 0 \qquad (14\text{-}24b)$$

From the nonnegativity of the Lagrange multipliers, Eqs. (14-19a) and (14-19Z?) become

$$/i < 0 \qquad f_2 < 0$$

And if $f_i < 0$ (meaning $\lambda_i > 0$), then from (14-19c) and (14-19d), $s_i = 0$, and hence $x_i = 0$. Thus the first-order conditions for a maximum subject to nonnegativity constraints are

$$f_i < 0 \qquad \text{if} <,\ x_i = 0 \qquad i = 1, 2 \qquad (14\text{-}25)$$

This reasoning generalizes to functions of n variables in a straightforward manner, yielding analogous results. The first-order conditions for

maximize

$$Z = f(x_1, \ldots, x_n)$$

subject to

$$x_i > 0 \qquad \text{some or all } i = 1, \ldots, n$$

are

$$f_i < 0 \qquad \text{if} <,\ x_i = 0 \qquad (14\text{-}26)$$

for variables constrained to be nonnegative, and simply

$$f_i = 0$$

for variables not constrained to be nonnegative.

Let us see what these conditions imply for the profit-maximizing firm. We previously considered the model

maximize

$$\pi = pf(x_1, x_2) - w_1x_1 - w_2x_2$$

Let us now specify explicitly that the factors x_1 and x_2 can only be employed in positive amounts, as physical reality would dictated With $x_1, x_2 > 0$, the first-order conditions for profit maximization become

$$\begin{aligned} \pi_1 &= pf_1 - w_1 < 0 \qquad \text{if} <,\ x_1 = 0 \\ \pi_2 &= pf_2 - w_2 < 0 \qquad \text{if} <,\ x_2 = 0 \end{aligned} \qquad (14\text{-}27)$$

Equations (14-27) say that if the profit maximum occurs at zero input of some factor, then the value of the marginal product of that factor is less than its wage. This is in accord with intuition. If the marginal value product were initially greater than the

*Some general mathematical treatments of the firm treat inputs as negative outputs. This type of black box approach to the theory of the firm generates a mathematical symmetry that is convenient in some analyses. Also, in more sophisticated models of the firm involving physical stocks of certain inputs, drawing down of some such stock (disinvestment) can be regarded as negative accumulation but probably still positive service flow from that stock.

wage of some factor, the firm could increase its profits by employing that factor in positive amounts.

Notice carefully the direction of implication intended by Eqs. (14-26) and, for the firm, (14-27). These relations do *not* say that if the marginal value product is initially, i.e., at $x_t = 0$, less than the wage of some factor, that factor will not be used. We might *initially* find $pf_i <$ w_h but, as $x_($ increased, f_t might increase and then decrease, yielding pf_t = w, at some finite, positive value of *,■. The "law" of diminishing returns is in fact usually stated to allow this possibility; the usual assertion is that f_t declines after some level of use of x_t (holding the other factors constant). The preceding first-order equations say only that *if* the maximum of profits (or anything else) is observed to occur when $pf_t <$ w,, *then* it must be the case that that input is not used; that is, $x_t = 0$. The converse of this statement is *not* implied by this analysis and will in general be false. These are strictly local conditions around the maximum position.

To illustrate this important point, consider a farmer who has to decide which of two tractors, a large model x_L or a small one x_s, to purchase. Either one alone may yield positive profits, with a marginal value product initially greater than the rental wage. This particular farmer would never find it profitable to use two tractors. It turns out, say, that using only the smaller tractor yields the highest profits. At zero (or small) input levels of the other tractor, the marginal value product of either tractor is greater than the rental wage. But at *maximum* profits, $x_s > 0$, $x_L = 0$; at that point, $pf_{x_l} < w_L$. But the nonuse of some factor does *not* imply that the value of the marginal product of that factor is always less than its wage.

The generalized first-order conditions, while providing a conceptual generalization of the conditions for a maximum, are not useful for actually finding that maximum. As the previous paragraph indicates, these conditions describe the maximum position after the fact. They don't tell us in advance which variables will equal zero at the maximum position. Consider, for example, that firms usually employ only a few of the hundreds or thousands of potential factors of production available to them. Firms typically reject one type of machinery in favor of another, they set skill levels for employees, etc., rejecting certain factors outright. The preceding first-order conditions merely indicate that for the rejected factors, the marginal value product must have been less than the wage, even at zero input levels. But that is precious little to go on in predicting in advance exactly which factors will be employed and which factors will not.

More importantly, as indicated earlier, the only interesting refutable comparative-statics relations are those which predict a direction (or magnitude, if possible) of change in a choice variable as parameters change. The comparative statics of variables not chosen is rather elementary: *dx*/doij* = 0 for all x, not chosen, by definition. Hence the meaningful results that are forthcoming with mathematical model building will de facto be derived from the classical maximum conditions of first-order *equalities.* Models involving nonnegativity (or other inequality constraints) will in general require an *algorithm* for solution. That is, some iterative trial-and-error process will be required to see which, if any, constraints are in fact binding. In Chap. 17 on linear programming, an example of such an algorithm will be presented.

14.2 INEQUALITY CONSTRAINTS

Let us now consider the imposition of an inequality constraint *g(x\, x₂)* > 0 in addition to the nonnegativity constraints in a two-variable problem. That is, consider

maximize

$$Z = f(Xi,X_2)$$

subject to

$$g(xi, x_2) > 0 \qquad \text{and} \qquad JCI > 0, x_2 > 0$$

(No loss of generality is involved by writing the constraint as > 0; multiplying the constraint by —1 reverses the sign.) Again, we first convert these inequalities to equalities, yielding the constrained maximum problem

maximize

$$z = f(x_u x_2)$$

subject to

$$g(x_u x_2) \ -x\backslash = 0$$

$$g^l(x_u si) = x_i - s^2{}_l = 0 \qquad g^2(x_2,\ s_2) = x_2 - s\backslash$$

$= 0$ Here the slack variables are x_3, $s\backslash$, and s_2. The

Lagrangian is

$$2 = fix_u x_2) + X(g(x_u x_2) - x]) + X_x\ (x_{\{} - s^2) + X_2(x_2 - sj) \qquad (14\text{-}28)$$

The first-order conditions for a maximum are thus

$$2*. = /!+*\$!+A., = 0 \qquad (14\text{-}29a)$$

$$\hat{}_2 = /_2 + A.g2 + A.2 = 0 \qquad (14\text{-}29Z>)$$

$$££_{X3} = -2Xx_3 = 0 \qquad (14\text{-}30a)$$

$$£_{sx} = -2X_{lSl} = 0 \qquad (14\text{-}30Z?)$$

$$5E_{S2} = -2k_2 s_2 = 0 \qquad (14\text{-}30c)$$

and the constraints

$$X_k = g(x_u x_2) - xl = 0 \qquad (14\text{-}31a)$$

$$X_M = x_l - s^2{}_l = 0 \qquad (14\text{-}316)$$

$$£_{k2} = x_2 - s\backslash = 0 \qquad (14\text{-}$$

3lc)

Using exactly the same reasoning as before, we note from Eqs. (14-30) that if any constraint is nonbinding (holds as a strict inequality), then the associated Lagrange multiplier is 0. Suppose, at the maximum point, X\,x2 > 0, and g{x\,x2) > 0; then all these constraints turn out to be completely irrelevant. From Eqs. (14-30), A. = A.i = A.2 = 0, and Eqs. (14-29) become the ordinary equations for unconstrained maximum, fi=f2 = 0. If in fact g(x\, x2) = 0, that is, the constraint is binding, and x\,x2 > 0, then Eqs. (14-29) give the ordinary first-order conditions for a constrained maximum, ^Ef — f\ + kgi = 0, ££2 — fi + kg2 = 0.

It must also be the case that k, ki, k2 > 0- The second-order conditions for constrained maximum are

7 = 1 i = 1 7 = 1 (= 1 7 = 1 i = \

for dX\.h\,h2,h^,k\, k2 satisfying

+ ^3*3=0 (14-33fl)

g2xh2 + g2sk2 = Q (14-33c)

Now

a? °^xiXj — Jij i A-Sij ° *^ij °^XiSj — . i = 1 2 3 r I i , (f cp n J A, Z

2X3X3 = -2k %XiX3=0 if / / 3

0 if

i Then the relations (14-32) and (14-33) become

y - 2kh\ - 2kxk] - 2k2k\ < 0 (14-34)

7 = 1

for al\hi, h2, h^, ki, k2 such that

gihi + £2^2 — 2x3/i3 =0 (1

hi -2siki =0 (14-35b)

h2 — 2s2k2 = 0 (14-35c)

Again, we already know that if $s\backslash$, s_2^=0, then $k\backslash, k_2$ = 0, respectively. Also, if JC_3 / 0, then k — 0, from (14-30a). Therefore, suppose $s\backslash$ = s_2 = 0. Then, as before, from (14-356) and (14-35c), $h_1 = h_2 = 0$. Then (14-34) becomes

$$-2kh\backslash\ -2k_xk\backslash-\ 2k_2k\backslash < 0$$

Letting any *two* of h_3, $k\backslash$, and k_2 = 0 [this is valid since (14-34) must hold for *all* /j,'s and &,'s] yields

A > 0 A i > 0 $A_2 > 0$

The first-order Eqs. (14-29) to (14-31) therefore can

be stated as

$£_{x}$. = *fr*+ *Xgi* < 0 if <, *Xi*• = 0 (14-36)

$£_{k}$ = *g(xi,*x_2*)>0* if >, A. = 0 (14-37)

and we note that A. > 0.

These conditions generalize in a straightforward fashion to the case of *n* variables and *m* inequality constraints. In general, consider

maximize

Z = *f(*x_U . . . , * „)

subject to

g \ x_u . . . , x_n*)* >0

g^m (x_1, . . . , x_n*)>0*
X\, . . . , x_n > 0

There is no a priori need to restrict *m* to be less than *n* (as might be the case with *equality* constraints), since some (or all) of these constraints may turn out to be nonbinding.

Define the Lagrangian

X = *f(*x_u . . . , x_n*)* +

7 = 1

Then the first-order conditions for a maximum are

- ° if <•*«■= 0

2xj=g^j*>0* *i f* > , *X j* = *O* (14-39)

These relations are known as the *Kuhn-Tucker* conditions for a maximum subject to inequality constraints.* Again, these conditions are not very useful for determining

tThe original paper is H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," in J. Neyman (ed.), *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability,* University of California Press, Berkeley, 1951.

the actual solution of such a problem. They are descriptions of the maximum position, after the fact, so to speak. *If* it turns out that at the maximum position $f_(+ kg_t < 0$, *then* $Xi = 0$. Nothing more is implied.

The conditions for a constrained *minimum* are similarly derived. Consider the problem

minimize

$$z = f(x_u \ldots, x_n)$$

subject to

$$g^J(x_u \ldots, x_n) < 0 \qquad j =$$
$$1, \ldots, \text{ra } X, > 0$$
$$i — I, \ldots, n$$

The constraints are written as < 0 to preserve symmetry. No loss of generality is involved; merely multiplying any constraint by — 1 reverses the sign of any constraint. Again, the Lagrangian, as before, is

$$\hat{\_} \qquad X\backslash, \blacksquare \blacksquare \blacksquare, X_n)$$

The first-order conditions are then

$$\overset{m}{\sim} \qquad \text{i f>, j t, = O,} \qquad * = 1, \ldots, n$$

$$\text{i f>, A}_; = 0, \qquad j = l, \ldots, m$$

Writing the constraints as $g^j < 0$ ensures that $kj > 0$.

Let us illustrate these Kuhn-Tucker conditions using the model of a consumer who maximizes his or her utility $U(x\backslash, x_2)$ subject to a budget constraint. Let us now assume that the consumer need not spend all of his or her money income. The model then becomes

maximize

$$U(x_u x_2$$

) subject to

$$PiXi \text{ 4- } p_2x_2 < M \qquad x\backslash, x_2 > 0$$

The Lagrangian for this problem is

$$SB = U(x\backslash, x_2) + k(M — p]X\backslash — p_2x_2)$$

The constraint has been incorporated in the Lagrangian in the form $M — P2X2 > 0$, to conform with the previous analysis.

The first-order conditions are thus

$£_x = U_x$ - $kp_x < 0$ if <, JCI = 0 (14-40a)
$£_2 = U_2$- $Xp_2 < 0$ if <, $x_2 = 0$ *(\4-40b)*
$£_x = M$ - $p_x x_x$ - $p_2 x_2 > 0$ *if> , k = O* (14-40c)

The Lagrange multiplier *X* represents the consumer's marginal utility of money income. Briefly, suppose *x\,* $x_2 > 0$. Then *U* = *Xp\,*U_2 = Xp_2, and

k=£l = ^2
P\ Pi

The term *U\lp* represents the marginal utility, per dollar, of income spent on JCI. Likewise, U_2/p_2 represents the marginal utility of income spent on x_2. At a constrained maximum, these two ratios are equal, their common value being simply the marginal utility of money income.

Consider the last condition (14-40c). This can now be interpreted as saying that if the budget constraint is not binding, that is, *piX* + p_2x_2 < *M* (the consumer doesn't exhaust his or her income), then *X,* the marginal utility of income, must be 0. The consumer is *satiated* in all commodities. This is confirmed by (14-40a) and (14-40&). If *X* = 0, then *U* = U_2 = 0; that is, the marginal utilities of both goods are 0. Hence, the consumer would not consume more of these goods even if they were given outright, i.e., free. This consumer is at a *bliss point.*

Now consider the situation where *X* > 0 (the consumer would prefer to have more income) and x_2 = *x^* > 0, but at the maximum point, *U* — Xp_x < 0 so that *X* = *x** = 0. Assuming positive prices, we have at JC* = 0, *x^* > 0,

A = ^> *El*
Pi
P

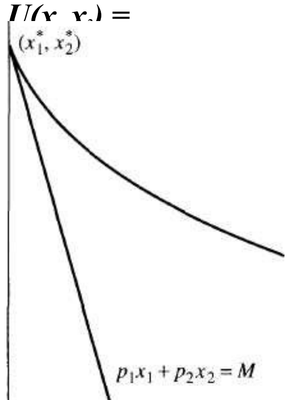
Rearranging terms gives

U_2 Pi

This situation is depicted in Fig. 14-2. At any point, the consumer's subjective marginal evaluation of Xi, in terms of the x_2 the consumer would willingly forgo to consume an extra unit of *x*, is given by *U* / U_2, the ratio of marginal utilities. This is the (negative) slope of the indifference curve at any point. If the consumer chooses to consume no *x* at all at the utility maximum, then the consumer's subjective marginal evaluation must be less than the value the market places on x_x. The market will exchange x_2 for *x* at the ratio *p* /p_2. If, for example, *p* = $6 and p_2 = $2, the market will exchange three units of

x_2 for one unit of *x*\. At zero *x*\ consumption, a consumer valuing *x*\ at only two units of x_2 would not be purchasing any JCI at all at the utility maximum. In Fig. 14-2, this situation is represented by having the budget line cut the vertical x_2 axis at a steeper slope than the indifference curve

U(x x) =



xl

O

*i

FIGURE 14-2
Maximization of Utility at a Corner. A consumer achieves maximum utility when $x^* = 0$, $x_2 > 0$. The consumption of *x* is 0 because *U* — *kp* < 0. Assuming positive prices, this inequality is equivalent to A. > *U\ Ip*, since x_2 is consumed in positive amounts. That is, for x_2, the marginal utility of income is the marginal utility per dollar spent on x_2. However, the marginal utility per dollar spent on xi is less than that spent on x_2 at the utility maximum; hence xi = 0. Combining these two relations gives *U2/pi* > *U\/p* or *U1/U2* < *P\l'P2,* as exhibited in this diagram, where *U1/U2* represents the slope of an indifference curve (the consumer's marginal evaluation of x_1) and *p\/p2* represents the market's evaluation of *x*. As depicted, with convexity, $U\backslash/U_2$ < *P\lPi* all along the indifference surface. This consumer, no matter how little *x* is consumed, always values *x* less than the market does. Hence, no *X* is consumed.

U{x.\, x_2) — *U°,* where *U°* is the maximum achievable utility. That is, $U[/U_2$ < ***P\lPi*** at ***x*** — **0, *x* > 0.**

14.3 THE SADDLE POINT THEOREM

Let us now return to the first-order conditions for the problem

maximize $z =$

subject to

$g^1(x_1, \ldots, x_n) \geq 0$

$g^m(x_1, \ldots, x_n$

)
>
0
X
i
,
.
.
.
,
x
n
>
0

For the Lagrangian

$$kjg^J(Xi, \ldots, *_{„}$$

the Kuhn-Tucker conditions are, again,

i f<, x , = 0 (14-38)

and

i f> , k j = O (14-39)

Noting the direction of the inequalities, we see that these conditions are suggestive of the *Lagrangian* function *SE(x\, ..., x_n, X*, ..., *X_m),* achieving a *maximum* in the *x* directions and a *minimum* in the *X* directions. That is, consider the Lagrangian above as just some function of x_t 's and *Xj* 's. If ££ achieved a *maximum* with regard to the x,'s, the first-order necessary conditions would be Eqs. (14-38). Likewise, if ££ achieved a *minimum* with respect to the *Xj*'s, the first-order necessary conditions would be precisely Eqs. (14-39).

A point on a function which is a maximum in some directions and a minimum in the others is called a *saddle point* of the function. The terminology is suggested by the shape of saddles: in the direction along the horse's backbone, the center of the saddle represents a minimum point, but going from one side of the horse to the other, the center of the saddle represents a maximum.

Consider a function /(x_1, ..., x_n, *y*, ..., y_m), or, more briefly,/(x, y), where x = (JCI , ..., x_n), y = (*yi*, ..., y_m). The point (x°, y°) is said to be a saddle point of

/(x,y)if

$$/(x, y^o) < /(x^o, y^o) < /(x^o, y)$$

Let us now apply this concept to the Lagrangian above. *If* the Lagrangian *X = f(x_u … ,x_n) + Yl"j=\ ^jg^j(^xi, •-•■>Xn)* nas a saddle point at some values $x_(= x*, i = 1, \ldots, n;$ *Xj = X*, j* = 1, ..., *m* (briefly, at x = *x*, X = X*),* then, as a necessary consequence, the relations (14-38) and (14-39) are implied. That is, *if*

(x, A*) < £6(x*, A*) < , A) (14-41)

then it is being asserted that ££(x, A) has a maximum in the x directions and a minimum in the A directions. The first-order necessary conditions for such an extremum ofi£(x, A) are

0 *if<,x_t =*0

and

if >, A.,- = 0

However, the mere fact that two assertions [constrained maximum of/(xi, ..., x_n) and saddle point of 5£(x, A)] imply the same conditions [Eq. (14-38) and (14-39)] does *not* imply that those two assertions are equivalent or that a particular one implies the other. It *is* the case, however, under fairly general mathematical conditions, that the saddle point criterion implies that / (x) has a constrained maximum. The converse is *nottmo.,* however, unless stronger conditions are attached. If it assumed, in addition, that (1) /(x) and the g^7 (x)'s are all concave functions and (2) there exists an x° > 0 such that g^j (x°) > 0, *j* — 1, ..., *m* (this condition is known as Slater's *constraint qualification),* then if (x*, A*) is a solution of the constrained maximum problem, (x*, A*) is also a saddle point of the Lagrangian function.

This theorem is known as the *Kuhn-Tucker saddle point theorem* (there are actually many variants of it). Part of the proof appears in the Appendix to this chapter. Vector notation will be used throughout.

Suppose (x*, A*) is in fact a saddle point of i£(x, A). Then, by definition, for x > 0, A > 0,

$$fix) + A^*g(x) < f(x^*) + A^*g(x^*) \qquad (14\text{-}42)$$

and

$$f(x^*) + A^*g(x^*) < f(x^*) + Ag(x^*) \qquad (14\text{-}43)$$

where Ag(x) means ^7=i ^yg^7 (x), the inner product ofthe vectors A = (A_{l5} ..., k_m) and g(x) = (g^1 (x), ..., g^m (x)). From (14-43), after canceling *fix*)* from both sides and rearranging, we have

$$(A - A^*)g(x^*) > 0 \qquad (14\text{-}44)$$

Since (14-44) must hold for an_y A, by hypothesis, for sufficiently large A, A — A* > 0 and hence

$$g(x^*) > 0 \qquad (14\text{-}45)$$

Thus we have shown that x* is *feasible;* i.e., it satisfies the constraints of the maximum problem. Moreover, we can set A = 0 in (14-44) (again, since this must hold for *all* A), obtaining, after multiplying by —1,

$$A^*g(x^*) < 0 \qquad (14\text{-}46)$$

However, A* > 0, g(x*) > 0. Therefore, in order to satisfy (14-46), it must be that

$$A^*g(x^*) = 0 \qquad (14\text{-}$$

47)

Now consider the first inequality, (14-42), which refers to the maximum in the x directions. When we use Eq. (14-47), (14-42) becomes

$$fix^*) > fix) + A^*g(x) \qquad (14\text{-}48)$$

However, A* > 0, and for any *feasible* x, that is, an x which satisfies the constraints, g(x) > 0. Therefore, **A*g(x)** > 0, and thus

$$/(x^*) > f(x) \qquad (14\text{-}49)$$

for any feasible x. Therefore, x* maximizes/(x) subject to the constraints g(x) > 0. We have therefore shown that the saddle point condition implies that a constrained maximum exists.

To repeat, the converse of the preceding is in general false. If conditions 1 and 2 above are added, viz., that/(x) and $g^j(x)$, $j = 1, \ldots, m$, are all concave and that there exists an x° such that $g^j(x°) > 0$, $j = 1, \ldots, m$, then the "converse" follows. The proof of this proposition unfortunately requires more advanced methods of linear algebra dealing with convex sets. It is presented in the Appendix to the chapter. Note, however, that the right-hand part of the saddle point inequality follows readily from the assumption of a constrained maximum. If x*, A* are the values that maximize /(x) subject to g(x) > 0, then

However, from the first-order conditions, A*g(x*) = 0. Hence,

$$S6(x^*, A^*) = f(x^*)$$

By definition

$$56(x^*, A) = fix^*) +$$

Ag(x*) But g(x*) > 0, and A > 0 by assumption;

thus

$$£g(x^*, A^*) = f(x^*) < fix^*) + \mathbf{Ag(x^*)} = £$$

(x*, A) i.e., the right-hand part of the relation (14-41).

Example. We shall show by example that achieving a constrained maximum does not imply that the Lagrangian has a saddle point there. Consider a consumer who maximizes the utility function $U = X\backslash X_2$ subject to the constraint $p\backslash X\backslash + p_2x_2 < M$. Since the level (indifference) curves of the utility function, $x_xx_2 = U^o$, never cross the axes and $U_u\ U_2 > 0$ for all positive x, the consumer will in fact spend his or her entire income; that is $M — p\backslash X_x — p_2x_2 = 0$. Thus, the problem is solved by formulating

$$ig = {}_{Xl}x_2 + XiM - p_xx_x - p_2x_2) \qquad (14\text{-}50)$$

with first-order equations

$$\begin{aligned} &\hat{} = JC_2 - Xp_x = 0 \\ &\$_{2=XJ}-Xp_2 = 0 \qquad (14\text{-}51) \\ &\&x = M — p\backslash X\backslash — p_2x_2 = 0 \end{aligned}$$

The consumer's demand functions are found by first eliminating A.:

$$x_2 = Xp_1 \qquad x_1 = Xp_2$$

and thus

fi - *El*

X *Pi*

or

$p_x x_x = p_2 x_2$ Substituting

this relation into the budget constraint *(S£$_x$* = 0)

gives

$p_x x_x + p_x x_x$ —

M and thus

x = — (14-

52a)

Similarly,

*x** = ^~ (14-

52Z?)

2p$_2$

Also,

-v- * v * ***H/f***

(14-53)

Pi *Pi*

2p$_x$p$_2$ We therefore find

££(x*, *X*) = x\x*$_2$ + X*(M - p$_x$x\ - p$_2$x\)* (14-
54)

However, the budget constraint is satisfied by *x\, x\,* and thus

,*•) = —— =---- f/ MM M^2

=X^A ,A) —

By definition

££(x*, *X) — U(x*, x$_2$) + X{M —*

p$_x$x — p$_2$x$_2$)* Since the budget constraint is binding

at *x*, x$_2$,*

iC(x*, *X) = U(x* x*)* $\frac{M^2}{4p_xp_2}$ = = £E(x*, A*)

Hence, the right-hand side of the saddle point is satisfied as an equality,

The left-hand side of the saddle point condition is *not* satisfied, however:

££(x, A.*) = *U(x$_x$, x$_2$) + X*(M — p$_x$x$_x$ — p$_2$x$_2$)*

= $x_x x_2 + \frac{M}{2p_xp_2}$(M - $p_x x_x$ - $p_2 x_2$)

If we let $x_x — x_2 = 0$,

$$\text{ig}(x, A^*) = \frac{M^2}{2p_x p_2} > \frac{M^2}{\hat{} P \backslash Pi} = ££(x^*, A^*)$$

The saddle point condition is violated because although *U* — $x \backslash x_2$ is quasi-concave in $x \backslash$ and x_2, it is not *concave.* Thus the mere attainment of a constrained maximum is *not* sufficient for the Lagrangian to possess a saddle point at the maximum position.

14.4 NONLINEAR PROGRAMMING

The general class of problems involving maximization of a function subject to inequality and nonnegativity constraints is called *nonlinear programming problems.* These problems, of the form

maximize

$$y =$$

subject to

$$g \backslash x i, \ldots, x_n) > 0$$

$$g^m (x_u \ldots, x_n) > 0$$
$$X \backslash, \ldots, x_n > 0$$

do not contain specific enough structure to permit description of the solution. The determination of exactly which constraints will be binding and which will not makes this class of problems significantly more complex than the classical problem of maximizing a function subject to equality constraints with nonnegativity not imposed. Once it is shown which constraints are binding, the preceding problem reduces to a classical maximization problem, solvable *{in principle*—the equations may admit of no easily expressible solution) by standard Lagrangian techniques.

Solutions to nonlinear programming problems will be found only by some *iterative* procedure, i.e., an *algorithm* which leads one toward the maximum in a stepwise fashion. In general, such algorithms begin with an arbitrary *feasible* point, i.e., an x = *{x*\, ..., x_n) which satisfies all the constraints including nonnegativity. Then in the neighborhood of that point some evaluation is made of how /(x) could be increased, e.g., by decreasing some JC,-'S and increasing others. When a new point is reached, the evaluation is repeated. A successful algorithm is one which leads to the maximum position in a finite (but not astronomically large) number of steps.

A number of algorithms have been developed, assuming various specific structures on the/and g^j functions. The most famous is the *simplex algorithm,* developed by George Dantzig in 1947 for solving the class of *linear* programming problems.^

^G. Dantzig, "Maximization of a Linear Function of Variables Subject to Linear Inequalities," in T. C. Koopmans (ed.), *Activity Analysis and Allocation, Cowles Commission Monograph* 13, John Wiley & Sons, Inc., New York, 1951.

This type of problem results when/and the g^{j}'s are all linear functions, or maximize

$$y = Yl^{PiXi} \; 1=1$$

subject to

$$£ \qquad i = 1, \dots, m$$

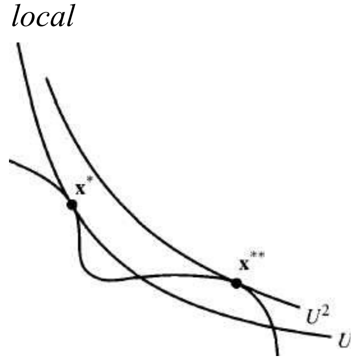
$$Xj > 0 \qquad / = 1, \dots, n$$

This class of problems will be investigated in Chap. 17 on linear general equilibrium models.

No general algorithm for all nonlinear programming problems exists. The specific algorithms that exist for some nonlinear problems are not of central interest to most economists and are outside the scope of this book. We shall only briefly indicate some structures for which algorithms have been more successful.

One of the

central problems encountered in nonlinear programming problems is the determination of whether a *local* solution is in fact the *global* solution of the problem. That is, suppose /(x*) > /(x) for all x in some neighborhood of x*. Then x* is a *local* maximum. How can we be sure that x* is the *global* solution, that is, /(x*) > /(x), for all feasible x? In general, of course, one can't be sure, but under certain structures local solutions are in fact global solutions. Let us explore these circumstances. Consider Fig. 14-3, in which a consumer attempts to maximize some utility function *U(x\, x₂)* whose indifference curves U^1 and U^2 are shown. Suppose, contrary to the usual assumptions, that the budget constraint is *not* the usual linear form, $P\backslash X\backslash + p_2x_2 < M$, but the area bounded by the curved line *MM'*. Given this situation, *two* local constrained maxima exist: x* and x**. At x*, *U(x*) > U(x)* for all x in *some* neighborhoods of x*. An iterative procedure which led to x* as the solution



F *Ut*

ximization Subject to Nonlinear Budget Constraint, x* and x** are both *local* maximao to this problem might be insufficiently powerful to indicate that if the neighborhood is made large enough, some x's will be found for which $U(x) > U(x^*)$. In the given example, x** is the *global* maximum, since clearly $U(x^{**}) > U(x)$ for all other x in the budget set.

The problem of nonglobal maxima occurs here because points connecting x* and x** lie outside the *feasible region,* i.e., the set of all feasible x's. That is, a straight line joining x* and x** contains points not admissible under the conditions of the model. A very important construct in analyses of nonlinear programming problems is therefore that of a *convex set.*

Definition. A set S is said to be convex if, for all x^1 *e S,* x^2 e 5 (the symbol "e" means "belonging to" or "is an element of"), the points $x = kx^l + (1 - k)x^2$ belong to *S,* for all $0 < k < 1$.

Geometrically, a convex set is one such that all points along a straight line joining any two points in the set also belong to the set. The straight line joining any two points in the set never leaves the set. All squares, triangles, circles, spheres, and parallelograms are convex sets; sets like that depicted in Fig. 14-3, the points bounded by the axes and the curve *MM',* are nonconvex. The principal result on local versus global maxima is, as indicated in the above discussion, the following theorem.

Theorem. Let /(x), x = (JCI ,..., x_n) be a quasi-concave function defined over some convex set S. Then if $f(x^*)$ is a unique local maximum in *S,* it is in fact the *global* maximum.

Proof. Suppose there exists an x** such that /(x**) > /(x*).Then,

by quasi-concavity, $f(kx^* + (1 - k)x^{**}) >$ /(x*)

$0 < k < 1$

By choosing k arbitrarily close to 1, the point $(kx^* + (1 - k)x^{**})$ becomes arbitrarily close to x*; yet, the function has a value there greater than or equal to its value at x*, a unique local maximum. This contradiction demonstrates the result.

Heuristically, if x* and x** are any two finitely separated points of local maxima, the chord joining them must lie in the convex set *S.* The function evaluated along that chord must be at least as large as the smaller of $f(x^*)$ and $f(x^{**})$. If, say, /(x*) > /(x**), points x arbitrarily near x** must also yield /(x*) > /(x**), contradicting the assumption that x** is a local maximum. If /(x*) = /(x**), the function must be constant along the chord joining x* and x**. It follows, therefore, that if a local maximum is unique, it is the global maximum over the convex set *S.*

Under what conditions will the set of variables over which a maximum problem is posed be convex? That is, under what conditions is the feasible region of a nonlinear programming problem a convex set? It is easy to show that if the constraints are all concave functions, the feasible region is in fact convex.

Consider the set S defined as the x = (jti, ..., x_n) such that $g(x) > a$, where a is any real number. Then S is convex; for consider any x^1, x^2 for which $g(x^l) > a$, $g(x^2) > a$. From concavity

$$g(kx^x + (1 - k)x^2) > kg(x^l) + (1 - k)g(x^2) > ka + (\backslash\text{-}k)a = a \qquad 0 < A:$$

< 1

Therefore the point $kx^1 + (1 - k)x^2$ lies in the set, and S is convex. If some functions $g^1(x), \dots, g^m(x)$ are all concave, then the set of x's that satisfy

simultaneously clearly also constitutes a convex set, as would be the case if nonnega-tivity constraints are added. (The intersection of convex sets is a convex set.) Hence, if the constraints of a programming problem are all concave, the feasible region will be a convex set. If the objective function is also concave, we can be assured that any local maximum is the global maximum of the model.

The principal application of the above theorem, to be discussed in the next chapter, is in the theory of linear programming in which $f(x), g^1(x), \dots, g^m(x)$ are all linear functions. In that case, the feasible region is convex, and an efficient algorithm for finding the solution to the problem has been developed.

14.5 AN "ADDING-UP" THEOREM

Many economic models have the general structure maximize

$$y = f(x_1, \dots, x_n)$$

subject to

$$g^m(x_1, \dots, x_n) \le b_m \qquad x_1, \dots, x_n \ge 0$$

Let us now assume that $f, g^1, \dots, g^m$ are all homogeneous of the same degree r. Assume that the problem admits of a solution found by standard Lagrange-Kuhn-Tucker techniques. The Lagrangian is

$$\mathcal{L} = f(x_1, \dots, x_n) + \sum_{j=1}^{m} \lambda_j (b_j - g^j(x_1, \dots, x_n)) \tag{14-56}$$

The first-order conditions are therefore

$$f_i - \sum_{j=1}^{m} \lambda_j g_i^j \le 0 \qquad \text{if} <, \; x_i = 0 \tag{14-57}$$

and

$$b_j - g^j \ge 0 \qquad \text{if} >, \; \lambda_j = 0 \tag{14-58}$$

Alternatively,

m
f.x* — S^ X*eJx* (14-59)

and

bjk) = gjX* (14-60)

Let us now sum (14-59) over *i* and (14-60) over/ This yields

n m n

E l l^i / *** i / /S/' -^»-------\ 1 '—Ol / u I / J J / J '-' t I
(= 1 7 = 1 (= 1

and

m m
^ ^ (14-62)

Now let us use Euler's theorem. Since/and g^l, ..., g^m are all homogeneous of degree r, *J2* /JC, = r/, ^ g/jt, = rg7, and hence from (14-61), letting _y* = /(x*), we have

*ry** = r/(x*) = £A.Jr^(x*) = *r*
7=1 7=

or

Now from general envelope considerations,

***k*j* = —**

If the constraint g^j (x) < *bj* is thought of as a *resource constraint,* where *bj* represents the amount of some resource used by the economy, *X* = dy*/dbj* represents the *imputed rent,* or shadow price, of that resource, measured in terms of *y.* In other words, *k*bj* can be thought of as the total factor cost of some factor associated with some resource allocation. Equation (14-63) then says that under these assumptions, the output being maximized can be allocated to each resource, with nothing left over on either side. This type of adding-up, or exhaustion-of-the-product, theorem appeared in the chapters on production and cost, when linear homogeneous production functions were involved. The preceding is a generalization of those results.

Moreover, consider the indirect objective function

Since $y^* =]T™_{=1} k)bj$ and $k^* = dy^*/dbj = d<f>/dbj$,

$$£\text{-}bj \qquad (14\text{-}64)$$

Therefore, under these conditions, the indirect objective function is homogeneous of degree 1 in the parameters $b\backslash, \ldots, b_m$, from the converse of Euler's theorem.

PROBLEMS

1.383 Explain the error in the following statement: For a profit-maximizing firm, if the value of the marginal product of some factor is initially less than its wage, the factor will not be used. State the condition correctly.

1.384 Consider the constrained *minimum* problem

minimize

$$Z = f(Xi, X_2)$$

subject to

$$g(x\backslash, x_2) < 0 \qquad x i, x_2 > 0$$

Derive the Kuhn-Tucker first-order conditions for a minimum.

3. Consider the cost minimization problem

minimize

$$C = W\backslash X\backslash +$$

w_2x_2 subject to

$$f(X], x_2) > y \qquad X\backslash, x_2 > 0$$

Derive and interpret the first-order conditions for a minimum. Under what conditions on the production function will the Lagrangian have a saddle point at the cost-minimizing solution?

1.385 Consider a consumer who maximizes the utility function $U = x_2e^{XI}$ subject to a budget constraint. Characterize the implied demand levels via the Kuhn-Tucker conditions; i.e., indicate when positive demand levels are present for both commodities, etc.

1.386 Consider the quadratic utility function $U — ax\backslash + 2bx\backslash x_2 + cx\backslash$. Discuss the nature of the implied consumer choices for this utility function in terms of the values *a, b,* and *c.*

1.387 Find the solution to the following nonlinear programming problem:

maximize

subject to

$$JCI + X_2 < 10 \qquad JC, +2*2 < 18 \qquad Xi, x_2 > 0$$

7. Consider the nonlinear programming problem maximize

$$y =$$

**2 subject to

$$10 \qquad x_2 < k \qquad X], x_2 > 0$$

What is the maximum value of k for which that constraint is binding? 8. Solve

minimize

$$y = Xi +$$

$2x_2$ subject to

$$\& \quad X\backslash > 5 \quad \hat{}1, X_2 > 0$$

9. Solve Prob. 8 with $x\backslash < 5$ replacing Xi > 5.

10. An individual has the utility function $U = x_{(} x_2^{/}\sim$ for consumption in two time periods, with $x\backslash$ = present consumption, x_2 = next year's consumption. This person has an initial stock of capital of $10, which can yield consumption along an "investment possibilities frontier," given by $2x] + x\backslash — 200$. The person can, however, borrow and lend at some market rate of interest r to rearrange consumption.

1.388 Explain why maximization of utility requires a prior maximization of wealth W, where $W = X\backslash + x_2/(l + r)$. That is, explain why if Wis not maximized, $U\{x_x, x_2)$ cannot be maximized.

1.389 Suppose the consumer can borrow or lend at r = 30 percent. Find the utility-maximizing consumption choices. Is the consumer a borrower or a lender?

1.390 Suppose the consumer can lend money at only 20 percent interest and can borrow at no less than 40 percent interest. What consumption plan maximizes utility, and what is the present value of that consumption?

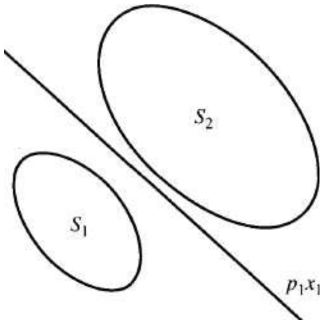
APPENDIX

The proof that if $/(x), g^{l}(x), ..., g^{m}(x)$ are all concave functions, then

$$£(x, A^*) < £\{x\backslash X) = f\{x^*)$$

where x* solves the maximum problem, is based on a famous theorem of convex set analysis. Consider two nonintersecting (disjoint) convex sets $S\backslash$ and 52. It is geometrically obvious, though messy to prove, that a *hyperplane* (the generalization of a *line* in two dimensions, *plane* in three dimensions, etc.) can be passed between Si and 5_2. This proposition is known as the *separating hyperplane* *theorem.* The theorem also holds if the sets are tangent at one point.

Consider Fig. 14-4; 5i and 5_2 are two convex sets that do not intersect. It is therefore possible to pass between them a line *p,x* + *P2X2* = *k* or, in vector notation, px = *k.* Figure 14-5 shows why such may not be the case if the sets are nonconvex.



$+ p_2x_2 =$

o

x,

FIGURE 14-4
A *Separating Hyperplane.* Sets *S* and 52 are both convex, and they are *disjoint;* i.e., they have no point in common. Under these circumstances it is always possible to pass a hyperplane (in two dimensions, a straight line) between the two sets. The equation of this hyperplane in two dimensions is *p \ x \ +* $p_2x_2 = k$. Since S2 lies above this plane, for all x^2,x^2 in S_2, *p\x\ +* p_2x^ — *k.* Similarly, for all *x\,x* in Si, *p\x\ +* p_2x\ < *k.* Hence, the separating hyperplane theorem says that if Si and S_2 are disjoint convex sets in *n* space, there exist scalars *p\, ...,* p_n not all 0, such that ^" p,x/ < J^"=1 *Pixf,* for all x^1 in Si and x^2 in S2. The theorem also holds if the sets intersect at only one point; that is, Si and S2 are tangent to each other.

Since all the points in 52 lie "above" the hyperplane, for all x^2 e ^2, $px^2 > k$. (The weak inequality is used since the hyperplane might be tangent to 52.) Similarly, since *S* lies "below" the hyperplane, for all x^1 G 5i, $px^1 < k$. Therefore, for any two disjoint convex sets *S* and 52, there exist scalars *p*, p_2 not both 0 such that

$$px^1 < px^2$$

The direction of the inequality is actually arbitrary. Reversing the signs of *p\,* p_2 changes the direction of the inequality. The theorem generalizes to *n* dimensions. If *S* and 52 are any two disjoint convex sets in Euclidean *n* space, for any x^1 *e Si*, x^2 e 5_2,

x_2

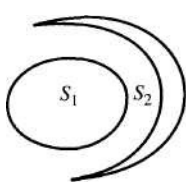


FIGURE 14-5
Nonconvex Sets. It is not always possible to sepa-rate nonconvex sets with a hyperplane.

there exist scalars *p*\, ..., p_n, not all zero, such that

(=1 *i=l*

Let us return now to the saddle point problem. We are assuming that x* maximizes /(x) subject to $g^j(x)$ > 0, *j* — 1,..., *m,* x > 0. We shall also assume Slater's constraint qualification that there exists an x° > 0 such that $g^j(x°)$ > 0, *j* = 1, ..., *m.* For any given x, there exist the *m* + 1 values/(x), $g^l(x)$, ..., $g^m(x)$, an (m + 1)-dimensional vector.

1.391 Define the set 5i as the vectors U = $(U_o, U_u ..., U_m)$ such that U_o < /(x), *Uj* <
$g^j(x)$, *j* = 1, ..., *m,* for all feasible x.
1.392 Define S_2 as the vectors V = (V_o, V*i, ..., V_m) such that V_o > /(x*), *Vj* > 0,
***j* = \, . . . , *m*.**

The sets 5i and S_2 are convex, disjoint sets. *Si* is convex because *f*, g^x, . . . , g^m are all concave functions. The results at the end of Sec. 14.4 imply convexity for Si; S2 is convex because S_2 is essentially the positive quadrant in *m* + 1 space, except that the first coordinate, Vo, starts at /(x*). Finally, since /(x*) > /(x) and since Vo > /(x*), there can be no V vector that lies in Si. The first coordinate, V_o, violates the definition of Si.

Since Si and S_2 are disjoint convex sets, by the separating hyperplane theorem there exist scalars $A._o$, A.i,..., X_m such that

7=0 7=0

for all U e Si, V G S_2. Moreover, although the point (/(x*), 0, ..., 0) is not in S_2, it is on the boundary of S_2, and hence the theorem applies to that point as well. The point (/(x), $g^l(x)$, ..., $g^m(x)$) is in Si. Hence, applying Eq. (14A-2) gives

J V > - lo∧(x*} (14A"3)
7 = 1

It can be seen from Eq. (14A-2) that A_o, *k*\, ..., X_m are all nonnegative. The vectors U include the entire negative "quadrant," or orthant, of this *m* + 1 space. Any of the *Uj*, *s* can be made arbitrarily large, negatively. Note that V_{ℓ}, ..., V_m are all greater than 0. If any k_{jt} *j* = 1, ..., *m,* were negative, making that *Uj* sufficiently negative would violate the inequality (14A-2). Last, since /(x*) > /(x) and since x* maximizes/(x), $A._o$ > 0 for essentially the same reasons.

Therefore, all the A.'s in (14A-3) are nonnegative. Moreover, given the constraint qualification, $A._o$ > 0; for suppose $A._o$ = 0; then (14A-3) says that

However, since the separating hyperplane theorem says that not all the *kj* 's are 0 and the constraint qualification says that *g^j(x°)* > 0, *j* — 1, ..., *m,* it must be the case that at x°

contradicting the preceding. Hence, $A_o > 0$. We can therefore divide (14A-3) by k_0, and if we define

* —

ʲ Eq. (14A-3) becomes

m

fix) + ^ A;V(x) < *fix*)* (14A-4)

When x = x*, Eq. (14A-4) yields

but since *k** > 0, *g^j\x*)* > 0, *j* =

1, ..., *m,* Defining the Lagrangian,

7 = 1

we find, with x > 0, A > 0,

££(x*, A*) =

f(x)* and therefore

m

2(x, A*) = *fix)* + *J2* *-**jg^j(x)* < /(x*) = ^(x*, A*) (14A-5)

satisfying the saddle point criterion. We showed in the chapter proper that

i£(x*,A*) <££(x*,A)

Hence, if *fix), g^l* (x),..., *g^mix)* are all concave and if there exists an x° such that *g^j\x°)* > 0, *j* — 1, ..., *m,* solving the constrained maximum problem implies that the saddle point condition will be satisfied.

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CHAPTER

15

CONTRACTS AND INCENTIVES

15.1 THE ORGANIZATION OF PRODUCTION

Standard producer theory is concerned with how prices determine the optimal choice of inputs and outputs. Input and output choices, however, are merely one aspect of production decisions. The owners of various factors of production have to be motivated to contribute in various ways to the production process. In a world with no information cost, each and every dimension of these input contributions could be correctly measured, and the efficient organization of production could be achieved through a system of prices. When information is costly, alternative methods of organization may be more economical than using prices. For example, a manager may just tell the secretary what to do, instead of paying a price for every phone call she receives and a price for every page she types.* Because individuals care about their self-interest, any method of organizing production must ultimately rely on incentives. The secretary who is told what to do does not follow orders blindly; she may be motivated by promotion prospects or by the threat of dismissal. The incentives that are used in organizing production are sometimes spelled out in an explicit contract and are sometimes left implicit. This chapter examines how these implicit or explicit contracts affect behavior and how people choose the form of contracts they use.

Before we discuss the specific models in detail, it is useful to consider some of the potential problems that may arise in a typical contracting situation. Suppose

tWe generally use gender-neutral terminology, but to avoid excessive linguistic clutter in this chapter, we arbitrarily made all the principals men and their agents women.

x units of inputs cost $C\{x)$ dollars and yield a total benefit of $B\{x)$ dollars. The objective is to maximize the value of net benefits

$B(x)$ - $C(x)$ The optimal amount of

input, denoted by $x°$, satisfies the first-order condition

$$B'\{x°) - C'(x°) = 0$$

Suppose the input costs are incurred by one person, while the benefits accrue to another, and let the price of the input be $p°$, where $p° = B'(x°) = C'(x°)$. At this price, the supplier of input will choose x to maximize $p°x — C(x)$. The solution to the first-order condition $p° — C'(x) = 0$ is $x = x°$. At the same price, the buyer of the input will choose x to maximize $B(x) — p°x$. The solution to the first-order condition $B'(x) — p° = 0$ is again $x = x°$. Thus the optimal level of input can be implemented by a decentralized price system. The use of the price system, however, is not without problems. Although we assume the input amount x is a sealer, a typical productive input has many attributes that contribute to output. Accurately measuring each of these attributes can be costly. Furthermore, setting the correct price $p°$ requires knowledge about the benefit and cost functions, but the transacting parties may possess private information that they have no incentive to reveal.

Because of the costs of using prices, alternative forms of contracts are sometimes used. For example, instead of paying the input supplier on the basis of the amount of input x, reward can be given on the basis of the total benefits $B(x)$ or on the basis of the input costs $C(x)$. Frequently, even the benefits and costs are hard to measure, and pay has to be made on the basis of some proxies for performance. These alternative incentive systems are often associated with direct monitoring that rewards performers by promotion and punishes nonperformers by dismissal. There is indeed a huge variety of contractual forms used in the organization of economic activities. Only a few will be discussed in this chapter.

15.2 PRINCIPAL-AGENT MODELS

Agency relationships arise whenever the person who undertakes an action (the agent) is not the same as the person who bears the consequences of that action (the principal). Principal-agent models are also called *hidden-action* models, because the action taken is assumed to be unobservable by the principal. When the agent's action cannot be observed and directly specified in a contract, she may not have the incentive to undertake the appropriate actions for the principal. Problems of this kind are known as *moral hazard.* This term originates in the insurance industry. Moral hazard is said to occur when a person fails to exert effort to reduce the probability of an insured loss. In this usage, the insured person is the agent and the insurance company is the principal. Today the term moral hazard is used generally in economics to refer to incentive problems that arise when productive actions taken by one person cannot be observed by another person or be verified by some third party.

The basic insights of principal-agent models can be captured in a simple setting in which the agent has only two actions to choose from: a high-cost (for the agent) action $x = x_H$ and a low-cost action $x = x_L$. We designate these costs *C(x_H)* and *C(x_L),* respectively, with *C(x_H)* > *C(x_L)*. The question is how to motivate the agent to choose the more costly action when her action is not directly observable.

Let *B{x)* represent the value of output when the action taken is *x.* If output is a one-to-one function of the action taken by the agent, then observing *B(x)* is the same as observing *x,* and there will be no information problem to overcome. We assume instead that output depends on random factors as well as on *x.* Specifically, suppose output can take *n* different values, *b\, ...,* b_n. Let the probability that output equals *bj* be given by *n_H{bi)* when $x = x_H$, and let this probability be given by *n_L(bi)* when $x = x_L$.

Although payment to the agent cannot be a function of the unobservable action *x,* it can be made contingent on the observed output. Let w, be the transfer payment to the agent when b_t is observed. If the principal wants to implement the action x_L, he can simply pay the agent a fixed wage because the agent has no reason to choose anything other than the low-cost action. The problem becomes interesting only when the principal wants to induce the more costly action x_H. When the action taken is x_H, the relevant probability function for the various outcomes is *n_H(-).* The principal chooses the wage payments *w\,* ..., w_n corresponding to the different possible observed values of output to maximize his expected net gain. This problem can be stated as

maximize

subject to

^ i) - C(x_H) > UQ

J2^H(bi)u(Wi) - C(x_H) > ^TtdbiMwi) - C(x_L)
i i

where *u(-)* is the Von Neumann-Morgenstern utility function of the agent and *UQ* is her reservation utility level.

We assume that the agent is strictly risk-averse and, for simplicity, that the principal is risk-neutral. The first inequality above is a *participation constraint.* It states that the agent's expected utility from working for the principal must exceed her reservation utility. The second inequality is an *incentive compatibility constraint.* Since the principal cannot observe the agent's action, he must design a contract such that it is in the agent's self-interest to carry out the action which is to be implemented. Therefore, if the principal wants to implement action x_H, choosing x_H must give the agent a higher expected utility than choosing the other feasible action x_L; the agent must prefer working to shirking. More generally, the incentive compatibility constraint requires that the action which the principal wants to induce must be the solution to the utility maximization problem for the agent given the terms of the contract.

The Lagrangian for this maximization problem is

$$- C(x_H) -$$

$$w_i) + \lambda_1 \left(\sum_i \pi_H(b_i) u(w_i) \right.$$

$$+ \lambda_2 \left(\sum_i [\pi_H(b_i) - \pi_L(b_i)] u(w_i) \right.$$

$$-[C(x_H) - C(x_L)]$$

where $X\backslash$ and X_2 represent the Lagrange multipliers associated with the participation constraint and the incentive compatibility constraint, respectively. The first-

$$\cdot 4'(Wi)) = 0$$

for i — 1,..., n. The above expression can be rearranged to get

(15-1)

To interpret this first-order condition, first suppose that X_2 = 0. Equation (15-1) then implies that $\backslash/U'(WJ) = X\backslash$ for all i. For any two realized output levels bj and bj, the corresponding payments to the agent are w, and Wj. Since the utility function is strictly concave, $\backslash/U'\{WJ)$ and $\backslash/U'(WJ)$ are equal to the same X^ if and only if Wj = Wj. In other words, the wage payment does not vary with output if X_2 = 0. Having X_2 = 0 means that the incentive compatibility constraint is not binding. When there is no need to provide incentives for the agent to choose the more costly action, the only consideration in the choice of the payment scheme is risk sharing. Since the principal is risk-neutral, the optimal arrangement is for him to offer full insurance to the agent through a fixed wage. However, if the agent receives a constant wage, she will always choose the less costly action x_L. In other words, the second inequality constraint will be violated. We therefore conclude that X_2 must be strictly positive. With X_2 > 0, the agent's payment w, will vary with the output bj, trading off some risk-sharing benefits for incentive provision.

The optimal payment to the agent increases with the value of the likelihood ratio n_H (bj) $/n_L$ (bj). If this ratio is large, the first-order condition (15-1) requires that $\backslash/u'(Wj)$ be large. Since $u'' < 0$, this implies that w, is large. This payment structure reflects the logic of statistical inference (although strictly speaking the principal already knows that his payment scheme will induce the agent to choose x_H). The observed output b_t contains information about the action taken by the agent. A high value of $n_H(bj)/7T_L(bj)$ is evidence in favor of the hypothesis that the action taken is x_H rather than x_L. Thus the agent is rewarded by being paid a high w_t whenever the value of this likelihood ratio is large.

Although the above model assumes that w, is contingent only on output, a more elaborate model can allow w, to be made contingent on other signals as well.

I

Given the assumed linear payment schedule, the agent's net income from choosing input level x is

$$y = w- C(x) = \text{or} + PB(x) - C(x) + fie$$

Expected income is therefore $E[y] = a + /3B(x) — C(x)$, and the variance is var[v] = p^2a^2. With a mean-variance utility function, the agent chooses x to maximize

$$a + pB(x) - C(x) - rp^2a^2$$

The first-order condition for utility maximization

is

$$PB\backslash x) - C'(x) = 0 \qquad (15\text{-}$$

2)

Equation (15-2) implicitly defines the agent's input as a function of the strength of the incentives, i.e., $x = x*(P)$. Note that unless $ft = 1$, the amount of input supplied by the agent will not be optimal. Standard comparative statics analysis yields

$$\begin{matrix} dx* & \_ & -B' \\ \sim df5 & \sim & fiB'' - C'' \end{matrix} >$$

Since the input x is unobservable to the principal, it cannot be directly specified in the contract. However, the principal can indirectly influence input supply by manipulating the strength of incentives. The principal is assumed to be risk-neutral. The optimal contract specifies a fi that will maximize his share of the expected output,

$$-a + (1 - 0)B(x)$$

subject to the participation constraint and the incentive compatibility

constraint $a + 0B(x) - C(x) - rp^2o^2 = u_o x =$

$x*(ft)$

After substituting out these two constraints, the principal's problem can be written as

maximize

$$B(x*(P)) - C(x*(P)) - rp^2a^2 -$$

u_0 The first-order condition for this problem

is

$$(*'(**) - C'(x*))\%- - 2rPa^2 = 0 \qquad (15\text{-}$$

4)

$$dp$$

where $dx*/dfi$ is given by Eq. (15-3). Once the optimal incentive parameter $p*$ is determined from Eq. (15-4), the fixed wage $a*$ can be determined from the participation constraint.

The first term of Eq. (15-4) is the marginal gain from raising *P*, and the second term is the marginal cost. For $p < \backslash$, Eq. (15-2) implies that the input supplied is below the fully efficient level. Therefore, raising the input level x by raising yS will contribute to greater efficiency. On the other hand, a contract with a greater incentive pay component is also more risky, and will tend to lower the expected utility of the agent. If the agent is risk-neutral (r = 0), the marginal cost of raising /? is zero, and therefore the marginal benefit must also be zero. From Eq. (15-2), *B'(x*) — C'(x*)* = 0 implies that *ft* = 1. In other words, when there is no need for risk sharing, the optimal contract will make the agent the full *residual claimant* to output (the agent's marginal share of output is 100 percent). When r > *0,B'(x*) — C'(x*)* > 0, and */3* will be less than 1. Incentives are diluted to reduce the risk exposure for the agent, and the amount of input supplied by the agent will be less than the fully efficient level. If we differentiate Eq. (15-4) with respect to r and use the second-order sufficient condition for maximization, it is can be shown that *dfi*/dr* < 0 and $d/3^*/da^2 < 0$. These comparative statics results establish that the strength of incentives in the optimal contract is decreasing in the agent's degree of risk aversion and in the degree of output variability involved. We leave the derivations as an exercise for the student.

Example. Let *B(x) = px* and $C(x) = ex^2$. Then, the agent maximizes $a + /3px — ex^2 — r(3^2a^2,$ and the solution is *x*(/3) = ftp/2c.* Substituting this value of x into the objective function for the principal, we have

maximize

The solution value for *{}** is

$$\frac{P^2}{p^2 + Aero^2}$$

In addition to the usual comparative statics results for r and a^2, this example also allows us to derive comparative statics for p and c. Direct differentiation shows that *d^*/dp* > 0. A large value of p indicates that the marginal product of the input is high. In this case, underprovision of input (shirking) would be relatively costly. Thus, providing incentives is more important than providing insurance, and the principal chooses a large /J*. It is also straightforward to show that *d/3*/dc* < 0. A high value of c indicates a steep marginal cost curve. When the marginal cost curve is steep, large increases in ^ would result in relatively minor increases in x. Therefore, the marginal benefit from raising the strength of incentives is low, and the contract would specify a low *ft** for greater insurance.

Multitask Agency

Consider an extension of the principal-agent model in which the agent performs multiple tasks instead of a single task. Let output be B $(x$ i, $x_2)$ = *PiX\ + pix^+e*, and let cost be given by a convex function *C{x\, X2)*. Assume that output is not directly observable (the variance of e is infinitely large). Instead, the principal observes imperfect signals of the effort devoted to the two tasks. In particular, these two signals are $t\backslash = x\backslash + \epsilon\backslash$ and $t_2 = x_2 + \epsilon_2$, where 6, and ϵ_2 are independent random variables with mean equal to zero. Let the variances of $\epsilon\backslash$ and e_2 be *erf* and *o\,* respectively. The payment to the agent is assumed to be a linear function of the signals: $w = a + /3\backslash t\backslash + fi_2 t_2$.

The agent's expected utility is assumed to take the mean-variance form. Therefore, she chooses $x\backslash$ and x_2 to maximize

$$a + \mathrm{ft}^*, + P_2x_2 - C(x_u x_2)$$

-The first-order conditions are

Equations (15-5) implicitly define the agent's optimal effort levels in the two tasks as functions of the contract parameters. In particular, assuming the sufficient second-order condition holds,

— Oil —O 19

"' = _c -C =* °

v_^ | £*^ 22

Standard comparative statics analysis yields $dx^*/dfii = C_{22}/H\backslash > 0$, $dx\text{^}/dfc = Cu/Hi > 0$, and $dx^*/dfi_2 = dxf/dfii = -C_{i2}/H_x$. Notice that the sign of the last comparative statics result depends on whether the two tasks are complements or substitutes. If $C\backslash_2 > 0$ so that the tasks are substitutes (increasing the effort level in one task makes performing the other task more costly), then increasing the reward for one task will reduce the incentive for the other task.

In the second step of the analysis, we assume that the principal is risk-neutral. He chooses the contract parameters so as to maximize the expected value of output less wage payment, subject to the participation constraint and the incentive compatibility constraint. This problem is equivalent to

maximize

$$P\backslash X\text{^} -\backslash- p_2X\text{^} — (-^* (Xj , \text{^}2) — f \backslash P\backslash \&\backslash \sim\backslash\sim P_2®_2)$$
$$— \text{^}0$$

where $x\backslash$ and $x\%$ satisfy the first-order condition for the maximization of the agent's expected utility.

The choice variables of this maximization problem are *($\backslash* and / 3_2. The first-order conditions are

0*Ji* 1 *QJC'-t*

dp$_2$ *dp*$_2$

Using Eqs. (15-5) and the comparative statics results derived earlier, these conditions can be rewritten as

^ r, x —C\2

(pi - Pi)^r

(15-6)

(Pi ~ (Pi ~ = 0

Equations (15-6) form the basis for deriving comparative statics results for the optimal contract parameters. If $C(x\, x_2)$ is a quadratic function in Xi and x_2 or if it can be closely approximated by a quadratic function, then C_u, Cn, and C_22 do not depend on $X\$ and x_2. Under this simplication, the determinant of the Hessian matrix of second-order derivatives is

Hj = $—C_{22}$ - 2ra$_2^2$

- 2rcr^2

-C

1 C

Cn Hi

$-C_{22}$ Cn

C n $-C_u$

CnC_{22} —

H,

Since $H_2 > 0$ and since the diagonal elements of the Hessian matrix are also negative, the second-order sufficient conditions for maximization are satisfied.

Consider the comparative statics for cr^2. Differentiating the system of first-order conditions (15-6) with respect to this parameter and using Cramer's rule, we have

dcrf -Ci <0

This conclusion is hardly surprising. As in the single-task agency model, optimal incentives for effort in the first task is reduced for greater insurance when the signal for that task becomes more noisy. We can also derive the effect of an increase in a^2 on by

H_2 H_x

This derivative is negative if $C_{i2} > 0$. The reason is that an increase in *af* reduces /}*. As *fi* falls, *x* will rise because task 1 and task 2 are substitutes. With a higher level of effort in task 2, additional incentive provision becomes less important than additional insurance, so the principal responds by lowering the strength of incentive for effort in task 2. Another way of interpreting this result is that there are two ways to induce more effort in task 1 when the two tasks are substitutes: raising /3i or lowering *j3₂*. When of rises, it becomes more costly to induce effort in task 1 by raising *(5* because the risks associated with the signal noise become large. Hence, the principal provides incentive for task 1 by lowering the incentive for the competing task instead.

Consider next the effect on the optimal contract when one of the tasks becomes relatively more important than the other. This can be represented by an increase in marginal product of, say, task 1. Comparative statics analysis yields

apr **1**

C11C22 — C_n *(C22*

+ 2ro-|$C_2$2 *h*

\ *Hi* >0

dpi *H_2 [\HiJ*

Hi 1

i>-

An increase in pi will raise *fi**. Its effect on /?| depends of the sign of C_{i2}. If C_{J2} > 0 so that the two tasks are substitutes, *dfi^/dpx* < 0. By lowering the incentives for task 2, the principal can induce the agent to spend more effort on the competing task 1, because *dx*/d^2* < 0. Thus, lowering incentives for task 2 becomes more attractive as the competing task becomes more productive.

15.3 PERFORMANCE MEASUREMENT

The models described above assume that output is measurable and can be used as a basis to reward input supply. However, the output of a production process, just like the input, is often multidimensional and hard to measure. Farming yields a crop, but it also affects soil quality and equipment depreciation. Sales agents generate revenue for the firm, but they also have an effect on the firm's reputation. When the principal's objective cannot be directly specified in the incentive contract, imperfect performance measures must be used. The choice of alternative performance measures, as well as the design of an optimal contract given such measures, then becomes a central problem in agency theory.

Let the principal's objective function be *B(x, e),* where *x* denotes the agent's action and *e* is a set of random factors that characterizes the state of the world. In contrast to the principal-agent model discussed above, we do *not* make the simplifying assumption that output is additively separable in *x* and *e.* Writing the objective function in the form *B(x, e)* allows the marginal product of *x* to depend on *e,* which in turn implies that the optimal action will in general depend on the realization of the state of the world.

In the performance measurement model, the principal's objective is not con-tractible. A performance indicator *M(x, e)* is used in place of the objective in the incentive contract. Again, the function *M(x, e)* is not necessarily additively separable. This specification allows the marginal effect of *x* on the performance indicator to depend on *e,* so the agent's incentive to take costly action also varies with realization of the state of the world.

An important set of assumptions of this model is related to the informational structure. Unlike the principal-agent models described in the earlier section, we assume that the agent is asymmetrically well informed about the state of the world. Neither the principal nor the agent knows *e* before signing the contract, but the realization of *e* is known to the agent before she chooses her action. Since the marginal product of *x* may depend on *e,* the principal would not know whether the agent's action is optimal even if the agent's action can be observed. Indeed, even if the action *x* is costless, incentives must be provided to induce the appropriate actions to be taken at the appropriate circumstances.

Given a linear incentive payoff structure, the agent's payoff is

$$a + fiM(x,e) - \mathrm{C(JC)}$$

She chooses *x* to maximize her payoff after observing the realization of *e.* The first-order condition for maximization is

$$PM_x(x,e) - C'(x) = 0 \qquad (15\text{-}7)$$

This equation implicitly defines the input choice function $x = x^*(fi, e)$. Differentiating (15-7) with respect to *P* gives

$$\wedge \qquad \sim^{M} \qquad 0 \qquad (15\text{-}8)$$
$$dp \qquad PM_{XX} - C''$$

For simplicity, assume that both the principal and the agent are risk-neutral. In designing the contract, the principal chooses *ft* to maximize the expected value of output minus payment to the agent. This problem is stated as

maximize

$$E[B(x,e) - a - fiM(x,e)]$$ subject to

$$E[a + pM\{x, e) - C(x)] = u_0 \quad x = x^*(P,e)$$

After substituting the two constraints into the objective function, this problem amounts to

maximize

$$E[B(x^*(P,e),e) - C(x^*(P,e)) - u_0]$$

The first-order condition for this problem is

$$\frown \backslash O \qquad (15\text{-}9)$$

Using Eqs. (15-7) and (15-8), Eq. (15-9) can be rewritten as

$$E\backslash(B_X - fiM_x)---\hat{}-\hat{}--- = 0$$

If we take a second-order Taylor approximation of $M(x, e)$ and $C(x)$, then the term 1 / (/J M_{xx} — $C'')$ can be taken out of the expectation operator because it is independent of e. The solution for /? is

$$= \frac{E[B_X M_X]}{E[M^2_X]} = \frac{E[B_x]E[M_x] + cow[B_x, M_x]}{E[M]^2 + \quad [M]}$$

Example 1. Suppose true output is nonstochastic but is measured with noise. In particular, let $M(x, e) = B\{x) + ex$, where e has mean zero and variance a^2. Then the formula for fi^* reduces to

Since the principal's objective is nonstochastic, the optimal input level should not vary with e. Given the imperfect performance measure, however, the agent would increase her input when e is high and reduce it when e is low. Such behavior is wasteful, and the optimal contract constrains it by reducing the strength of incentives.

Example 2. Suppose the marginal product of the input varies with the state of the world, but such dependence is not reflected by the performance measure. In particular, let $B(x, e) = M(x) + ex$, where the expected value of e is zero. Then Eq. (15-10) implies /3* = 1. At ft^* = 1, the agent would choose her input such that, on average, the marginal product equals the marginal cost. Choosing the right input level on average, however, means that x is too high when e is low and x is too low when e is high. The contract does not achieve full efficiency even though there is no systematic underprovision of effort.

In the performance measurement model, a fully efficient choice of input x must satisfy

$$B_x(x,e) - C(x) = 0 \qquad (15\text{-}11)$$

Comparing Eq. (15-11) to Eq. (15-7), it is clear that implementing the fully efficient outcome requires $fi^*M_x = B_x$ for all realizations of e. Indeed, even if input is observable, the fully efficient outcome cannot be achieved unless M_x is perfectly correlated with B_x. Note further that the agent's input level under the optimal contract is not always below the fully efficient input level. Inefficiency in this model arises not because the agent has insufficient incentives to provide effort, but because the performance indicator is not perfectly aligned with the principal's true objective.

Choosing the Performance Measure

When there are several possible performance indicators available, and when it is too costly to use all of them in the contract, optimal contract design involves not just the choice of the parameter *B,* but also the choice of which performance measure to use. To analyze this problem, one approach is to use the fully efficient outcome as the benchmark. Let *x°{e)* be the optimal action in the absence of informational problems. Then the efficiency loss resulting from using an imperfect performance measure is

$$A(e) = [B(x°(e), e) - C(x°(e))] - [B(x*(B*, e), e) - C(x*(6*, e))]$$

$$« (x° - x*)(B_x - C) + \{x° - x*)^2(B_{xx} - C") \qquad (15-12)$$

Since B_x — C = 0 at $x = x°$, the first term in Eq. (15-12) can be eliminated. To further simplify the expression, we use the approximation

$$[B_x(x°, e) - C'(x^0)] - [B_x(x\ e) - C'(x*)] « (x° - x*)(B_{xx} - C")$$

The first term in brackets is 0 by Eq. (15-11), and the second term in brackets is equal to B_x - $p*M_x$ by (15-7). Thus

$$B_{xx} - C"$$

Substituting (15-13) into (15-12) and taking expectation, the expected efficiency loss is

$$E[A(e)] = \frac{E[(B_X - P*M_X)^2]}{2(B_{XX} - C")}$$

$$\frac{E[Bl] -2/ \qquad ^2 [^2]}{2(B_{XX} - C")}$$

Since $B* = E[B_XM_X]/E[M^2]$ from Eq. (15-10), we can eliminate $E[B_XM_X]$ from the numerator in the expression for $E[A(e)]$ above to get

$$= \frac{E[B_X]^2 + var[B_x] - 6*^2(E[M_x]^2 + var[M_x])}{2(B_{XX} - C")} \qquad (15-14)$$

Alternatively, we can eliminate $E[M^2]$ from the numerator to get

$$= \frac{E[B_X]^2 + var[^] - B*(E[B_X]E[M_X] + cov^, M_x])}{2\{B_{XX}-C")}$$

Holding $vax[M_x]$ constant, a higher value of $cov[B_x, M_x]$ increases $8*$ and therefore reduces the expected loss using Eq. (15-14). Holding $cov[B_x, M_x]$ constant, a higher value of var[M_x] reduces $8*$ and therefore increases the expected loss using Eq. (15-15). Thus, a principal tends to choose performance measures which are highly correlated with his objective function and which have a low idiosyncratic noise.

15.4 COSTLY MONITORING AND EFFICIENCY WAGES

Agency problems may arise because either inputs or outputs are unobservable. However, observability is seldom an all-or-none matter. Observability can typically be improved by spending resources on measurement or monitoring. Instead of inducing an agent to behave properly by just offering her financial rewards, an alternative is to directly monitor her behavior. Both methods are costly, and the principal's problem is to find the cost-minimizing combination of these two approaches.

Consider an employer who wants to induce a worker to supply JC units of effort. A workers who supplies less than the agreed-upon effort level will be detected with probability *n*. The contract is characterized by a standard wage *w* and a penalty wage wo- If the worker does not shirk, or if she shirks but is not detected, her compensation is w. If she shirks and this is detected by the employer, she is paid vv$_0$ instead (w$_0$ may be negative). For a sufficiently low w$_0$, the expected cost of shirking can be very large. The employer would then be able to induce the worker to supply the desired level of effort with a probability of detection that is arbitrarily close to zero. This is known as *a forcing contract.* However, forcing contracts are not always feasible, if only because there are limits on how low wo can be. For example, the maximum penalty for shirking may be dismissal, which corresponds to $w_0 = 0$. Even when the worker is required to compensate the employer when she is found shirking, the compensation cannot exceed the worker's wealth.

Let *K(TT)* be the expected costs of monitoring. We assume that these costs are increasing and convex in *TT.* Suppose both the employer and the worker are risk-neutral. To induce effort level JC, the employer chooses *TT,W,* and w$_0$ to minimize total (wage and monitoring) costs. This problem is stated as

minimize

$$w +$$

K(jt) subject to

$$w - C(x) > u_0 \tag{15-16}$$

$$w - C(x) > 7\mathrm{TWo} + (1 - n)w \tag{15-17}$$

$$\mathrm{w_o} > \mathrm{O} \tag{15-18}$$

The first inequality is a participation constraint. It says that wage minus the cost of effort must be at least as great as the worker's reservation utility u_0. The second inequality is an incentive compatibility constraint. If the worker shirks and supplies zero units of effort, expected payment is TTW_0 + (1 — *n)w,* while the cost is C(0) = 0. This constraint says that the worker must prefer supplying JC units of effort to supplying no effort. The third inequality imposes a lower bound on the penalty wage, which we conveniently set at zero.

Note that inequality (15-17) can be written as *TT(W — w_0)* > *C(x)*. From this, we can conclude that the probability of monitoring *TT* must be strictly positive. Furthermore, the standard wage *w* must be strictly greater than the penalty wage w_0.

The Lagrangian for this minimization problem is

i£ = *w* + *K(TT)* — *X\(w* — *C(x)* — «o) — k_2*(w* —

C(x) — The first-order conditions for *w,* w_0, and *n*

are

X_w = 1 - A.i - $k_2 n$ = 0 (15-19)

$5e_{wo}$ = $X_2$7T - A.3 = 0 (15-20)

<£$_{Jl}$ = $K^f(n)$-$k_2(w$-$w_0)$=0 (15-21)

and the inequality constraints hold with complementary slackness.

From Eq. (15-21), k_2 = *K'/(w* — wo) > 0. Therefore the incentive compatibility constraint (15-17) binds. Furthermore, since k_2 > 0 and *TT* > 0, Eq. (15-20) implies that k_3 > 0. Therefore, the constraint on the magnitude of the penalty wage (15-18) also binds. We conclude that

wo = 0

and

$$71 — \frac{C(x)}{W}$$

Notice that the optimal vv_0 involves the maximum possible penalty. Also note that *TT* and *w* are substitutes: If a higher wage is paid, a less intense monitoring is required to induce the worker to supply effort.

Substituting *n* = *C(x)/w* into the objective function, the employer's problem is to choose w to minimize

K(W,X) = *w* + *K[* ——) (15-22)

\ w J

subject to (15-16). It can be shown that *K(W, X)* is convex in *w.* Thus, if *dic/dw* > 0 at the boundary of constraint (15-16), then the optimal wage is at the corner solution, that is,

w = *uo* + *C(x)* Otherwise, the optimal

wage is given by the solution to the first-order condition:

1 - ^ = 0 (15-23)

In this latter case, raising the wage above the reservation level «o + *C{x)* is desirable because an increase in *w* will reduce *TT* according to the incentive compatibility constraint. As long as monitoring costs are sufficiently high, the increase in direct wage cost is offset by the reduction in monitoring cost, and the worker's participation constraint (15-16) will not bind. Such a wage policy, where the employer pays the worker more than her reservation wage, is known as an *efficiency wage* policy. When the participation constraint does not bind at the cost-minimizing solution, workers strictly prefer working to their next best alternatives, but wages do not fall because lower wages would necessitate much higher costs of monitoring. Efficiency wages therefore bring about a whole set of issues related to the nonclearing of the labor market, and this is the subject of active research in labor economics and in macroeconomics.

The presence of monitoring costs also has implications for the choice of input supply. Let *K*(X)* be the solution to the cost-minimizing problem (15-22). To the employer, the cost of input is given by *K*(X)* and the benefit is *B(x)*. Suppose first that the participation constraint does not bind. From (15-22), we can use the envelope theorem to get

d

x w

Notice that the fir

st-order condition (15-23) implies that $K'/w = w/C$. Furthermore, since n — C/w, this implies $K'/w = 1/TT$. Thus $dK^*(x)/dx = C'(x)/n > C'(x)$. Consider next the case where the participation constraint binds. Then, we substitute $w = UQ + C(x)$ into the objective function in (15-22) to get

$C(x)$

Taking the derivative with respect

to x, $dK^*(x)$

$dx = C'(x)$

$C(x)J$

$K'C'UQu$

$+$

c

Obviously, the second term in the above is positive. Therefore, regardless of whether the participation constraint binds, we have $dK^*(x)/dx > C'(x)$. An employer who maximizes $B(x) - K^*(X)$ will choose a lowers* than if he were to maximize $B(x) - C(x)$. Monitoring is costly not only because it directly consumes resources but also because it leads to an input choice that is below the fully efficient level.

15.5 TEAM PRODUCTION

More often than not, production involves the cooperation of several input owners: Clinics are run by doctors and nurses, and law firms consist of attorneys and assistants. In neoclassical economics, it does not matter whether doctors hire nurses or nurses hire doctors. Yet we usually observe that it is the more productive workers (doctors, attorneys) who are employing the less productive ones (nurses, secretaries). What determines the kind of contracts governing the relationship between cooperating input owners?

We use the term *team production* to refer to productive activities in which inputs are provided by several persons. The gains from team production may stem from specialization, and we assume that any contracting problem is not severe enough to induce the individuals to revert to autarky. The value of team output is a function of the level of inputs provided by each team member. If B is output and x_1 and x_2 are the input levels in a two-person team, then $B = B(x_1, x_2)$. The costs of inputs are borne privately and are given by $C_1(x_1)$ and $C_2(x_2)$ for person 1 and person 2, respectively. We assume that inputs are not observable and cannot be specified in a contract.

In the single-agent case, if the agent is risk-neutral or if there is no uncertainty in output, full efficiency can be achieved by making the agent the full residual claimant to output. When production involves a team, it is impossible to make every contributing agent a full residual claimant since output has to be shared among the different members. Since agents receive only a fraction of their contribution to output, their incentives to provide inputs are diminished. Indeed, there is no way of fully allocating the joint output so that the resulting equilibrium is fully efficient. To see this, let $s_1(b)$ and $s_2(b)$ be the output shares of person 1 and person 2 such that, for all levels of output b, there is budget balance

$$s_1(b) + s_2(b) = b \tag{15-24}$$

The payoff to person i ($i = 1, 2$) is $s_i[B(x_1, x_2)] - C_i(x_i)$. The first-order condition is

$$s_i' B_i - C_i' = 0 \tag{15-25}$$

On the other hand, full efficiency implies that

$$B_i - C_i' = 0 \tag{15-26}$$

Consistency of (15-25) and (15-26) requires that $s_1' = s_2' = 1$. However, this contradicts budget balance, since differentiating (15-24) implies $s_1' + s_2' = 1$.

Although full efficiency is not attainable when inputs are not contractible, the loss from shirking can be minimized by an appropriate choice of contract. Consider a linear sharing rule in which person 1 receives $\alpha + \beta B(x_1, x_2)$ and person 2 receives $-\alpha + (1 - \beta)B(x_1, x_2)$. Each person maximizes his share of the output less the input cost. The first-order conditions for x_1 and for x_2 are

"i = 0 (15-27)

These two equations show that there is a *double* moral hazard problem. For any $0 < \beta < 1$, *both* persons will supply fewer inputs than the level that would equate marginal benefits to marginal costs. Equations (15-27) define the equilibrium input supplies x_1^* and x_2^* as functions of the sharing parameter β. Differentiating this system with respect to β, the following is obtained:

$$dx^*_{\,(}\backslash$$

$$\begin{pmatrix} \beta B_{11} - C_1'' \\ (1-\beta) B_{12} \end{pmatrix}$$

B_2 dx^*

Let H be the determinant of the square matrix above. Then, since $B(xi,x_2)$ is assumed to be concave,

$$H = [fiB_u - C'\{W- \sim P)B_{22} - C'H - /\mathrm{HI} - P)B^2{}_n$$

> 0 Solving

by Cramer's rule, we have

$$\frac{dx^*}{dp} = \frac{-\mathrm{fr}((1 - P)B_{22} - C'j) - PB_2B_{12}}{H}$$

(15-28)

$$\frac{dx^*}{dp} = \frac{B_2(PB_n - C'Q + (1 - P)B_lB_{12}}{H}$$

When $B_n < 0$, these comparative statics results are unambiguous. In this case, Eqs. (15-28) imply that $dx^*/dp > 0$ and $dx^\wedge/dp < 0$. That is, an increase in the share of output given to person 1 will increase the input supply from person 1 but will reduce the input supply from person 2. There is a trade-off between shirking by one team member and shirking by another member.

The optimal sharing rule maximizes the net value of production by balancing the cost of shirking by one team member against the cost of shirking by another member. Let

VG8) - $B(xi(P),$ Then the

condition for the optimal share satisfies

$$(*! - \mathrm{CJ})\frac{dx^*}{op}\mathrm{j} + (B_2 - C'_2)\frac{dx^*}{op}\pm = 0 \qquad (15\text{-}29)$$

The first term in (15-29) can be interpreted as the marginal gain from increasing p. A higher p tends to raise x^*. Since x^* is below the fully efficient level (that is, 5i — Cj > 0), a higher $X\backslash$ will improve efficiency. The second term in (15-29) is the marginal cost of increasing p, as a higher p tends to reduce JC| and lower efficiency. Using Eqs. (15-27) and (15-28), Eq. (15-29) becomes

$$[(1 \quad P)B\backslash\{\{\backslash - P)B_{22} - C_2') + PB^2{}_2(PB_n - C'\{)] = 0 \qquad (15\text{-}30)$$

ti

To derive comparative statics results, suppose $B(X\backslash,XT) = f(x\backslash,x_2) + p\backslash X\backslash$ and consider the effect of a rise in $p\backslash$. Equation (15-30) defines $P = P^*(p\backslash)$. Differentiating this with respect to $p\backslash$, we get

$$y\backslash P)^d\text{-}i\sim + \wedge[-2Bdl - P)(\{\backslash - P)B_{22} - C_2')] = 0$$

Since *V"(fi)* < Oby the second-order condition for *fi* and since (1 — *fi) Bn — C'^* < 0 by the second-order condition for x_2, we have *dfi*/dpi* > 0. The interpretation of this result is straightforward. When the marginal product of *x* is increased, shirking by person 1 becomes more costly relative to shirking by person 2. Therefore the optimal contract will provide person 1 with greater incentives to supply inputs by allocating person 1 a larger marginal share of the output, while the other member will receive a smaller marginal share. This model predicts that, within a team, the less productive (in the sense of low *marginal* productivity) member will face a relatively fixed pay. This agent's pay will be rather unresponsive to output; changes in the value of team output are largely borne by the more productive member of the team. Thus the pay structure for the less productive member resembles that of an employee, and the pay structure for the more productive member resembles that of a residual claimant.

15.6 INCOMPLETE CONTRACTS

Production relationships are typically very complex. Cooperating inputs involve a large number of attributes that are difficult to measure. The range of possible actions taken by the input owners are hard to conceive. Furthermore, different states of the world often require different actions, and it will be prohibitively costly to write a contract that prescribes how individuals will behave under every possible contingency. For these reasons, contracts have gaps and ambiguities. When contracts are incomplete, ownership matters. The owner of an asset can decide what to do with the asset as long as it is not inconsistent with customs or the law. Part of these control rights can be transferred to another party by contract, but when the contract is silent, the owner retains the residual right of control. Ownership is therefore a source of power. It tends to enhance bargaining strength and hence increases the incentives to invest in specific assets. Sanford Grossman, Oliver Hart, and John Moore^ developed a theory of property rights based on these ideas.

Consider a model where two persons, 1 and 2, cooperate to produce output in combination with an asset *A. Ex ante,* each person invests in relationship-specific human capital. *Ex post,* each decides whether or not to cooperate. Because of uncertainty and contract incompleteness, however, the terms of cooperation (e.g., how they use the asset, the amount of transfer payment) cannot be specified in advance when the investment decisions are made. Thus, the parties have to renegotiate after uncertainty is resolved. Let *x* and *y* be the levels of human capital investment for person 1 and person 2, and denote their personal benefits (before any transfer payments)

Sanford Grossman and Oliver Hart, "The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration," *Journal ofPolitical Economy,* 94:691-719, 1986; Oliver Hart and John Moore, "Property Rights and the Nature of the Firm," *Journal of Political Economy,* 98:1119-1158, 1990; and Oliver Hart, *Firms, Contracts, and Financial Structure,* Oxford University Press, New York, 1995.

by *B* (JC) and *$B_2(y)$* if they cooperate. If cooperation breaks down, the owner of the physical asset alone will decide how to use the asset. Therefore, the net payoffs to each person will depend on who owns asset *A*. When there is no cooperation, let *b* (JC; *O)* be the net payoffs to person 1 if he owns asset *A,* and let *b\ (x; N)* be his net payoffs if he is not the owner. Define *$b_2(y; O)$* and *$b_2(y; N)$* similarly. We assume that asset *A* and human capital are complementary to each other so that the marginal return to human capital investment is greater the more assets (human or otherwise) there are in the production relationship. For example, the marginal product of *x* is higher when *x* is used alongside with *A* than when JC is used alone. Furthermore, the marginal product of *x* is still higher when it is used together with both *A* and *y.* In other words, for *i* = 1,2,

dBiix) dbiix-O)

dbi(x;N)

dx dx dx

Furthermore, all the benefit functions are assumed to be concave.

Because the human capital investments are relationship-specific, they are more valuable when there is cooperation than when there is not. We assume that *ex post* negotiation is efficient, so cooperation always ensues. Nevertheless, ownership is important because it affects the division of surplus from cooperation. Suppose person 1 owns the asset. Then the default payoffs for person 1 and person 2 are *b\ (x; O)* and *bi(y\ N).* The surplus from cooperation is given by

$$S = Bdx) + B_2(y) - b_{\{}(x; O) - b_2(y; N)$$

How this surplus is divided between the two cooperating parties is the subject of research in bargaining theory. In the early 1950s, the mathematician John Nash proposed an ingenious solution to the bargaining problem. Under certain axioms, Nash proved that the surplus from cooperation will be evenly split between the cooperating parties.^ This result is known as the *Nash bargaining solution.* The Nash bargaining solution has been subsequently refined in different directions, but his fundamental result is still widely used today. *

If we adopt the Nash bargaining solution as the outcome of the bargaining process, the surplus from cooperation *S* will be evenly split between person 1 and person 2. Each person's final payoff is equal to his default payoff (i.e., the payoff that would ensue if cooperation breaks down) plus half the gains from cooperation, 0.55. Let *U\{0)* be the final payoff to person 1 and *$U_2(N)$* be the final payoff to

^The formal statement of these axioms and the proof are beyond the scope of this book. See John Nash, "Two-Person Cooperative Games," *Econometrica,* 21:286-295, 1953; or chap. 2 of Martin Osborne and Ariel Rubinstein, *Bargaining and Markets,* Academic Press, San Diego, 1990.

*John Nash's fundamental result may be generalized to give an *asymmetric* Nash bargaining solution. In this asymmetric solution, one party gets a fraction *a* and the other party gets a fraction 1 — *a* of the surplus, where a is a fixed parameter of the bargaining model. The qualitative results of our model are not affected if we adopt the asymmetric Nash bargaining solution to model the division of surplus.

person 2 when the physical asset is owned by person 1. Then,

$$UdO) = bdx;\ O) + 0.5[B_x\{x) + B_2\{y) - b,(x;\ O) - b_2(y;\ N)]$$
$$= 0.5[Bi(x) + bdx;\ O) + B_2(y) - b_2(y;\ N)] \qquad (15\text{-}32)$$
$$U_2(N) = b_2(y;\ N) + 0.5[\text{fli}(*) + B_2(y) - b_x\{x\backslash\ O) - b_2(y;\ N)]$$
$$= 0.5[B_2(y) + b_2(y;\ N) + B\hat{}x) - b\hat{}x;\ O)] \qquad (15\text{-}33)$$

Similarly, when person 2 owns the asset, the payoffs are

$$U_X(N) = 0.5[Bi(x) + b_l(x;N) + B_2(y) - b_2(y;\ O)] \quad U_2(O) = 0.5[B_2(y) + b_2(y;\ O) + B_x(x) - *,($$

to persons 1 and 2, respectively.

The investment levels are chosen to maximize each person's respective payoffs less his investment costs. Let $x\backslash$ and $y\backslash$ be the investment levels that maximize each person's net payoffs when person 1 owns the asset. That is, $x\backslash$ maximizes $U\backslash(O)—x$, and $y\backslash$ maximizes $U_2(N) — y$, where $U\backslash(O)$ and $U_2(N)$ are given by Eqs. (15-32) and (15-33). These investment levels satisfy the first-order conditions

$$dx \qquad dx$$
$$\underline{db_2(_{yi};N)}l \qquad dy \qquad dy$$
$$– \ d$$

Similarly, if x_2 and y_2 are the investment levels that maximize net payoffs when person 2 is the owner of the asset, they satisfy

$$dx \qquad dx$$
$$J\underline{dB_2(y_2) \qquad db_2(y_2;O)}l$$
$$\bar{\text{L}} \quad dy \qquad dy \quad \text{J}$$

In contrast, the fully efficient investment levels will satisfy $dBi(x°)/dx — 1 = 0$ and $dB_2(y°)/dy — 1 = 0$. Using the assumption made in (15-31), these conditions imply

$$x° > xi > x_2$$

$$y° > yi > y\backslash$$

It can be seen that there is an underinvestment problem under either ownership structure. The cost of human capital investment is sunk. Once the relationship-specific investments are made, they are worth less outside the cooperating relationship than they are worth inside. Part of this surplus will be appropriated by the other party in the bargaining solution. As a result, the incentives to invest in specific human capital are diminished.

It is also observed that underinvestment in human capital is more severe for the person who is not the owner of the complementary physical asset. Human capital investment is more valuable with the physical asset than it is without the asset. Thus the person who does not own the physical asset is in a weak bargaining position and is particularly vulnerable to the appropriation of surplus. If person 1 owns the asset, the problem of underinvestment is more important for person 2 than for person 1. If person 2 owns the asset, the reverse is true.

Factors Affecting Ownership Structure

The optimal ownership structure will minimize the total loss from suboptimal investments. We consider a few parameters that affect the relative size of the loss under alternative ownership structures.

Suppose $B_2(y) = Of(y) + (1 - 6)y$, $b_2(y; O) = Og(y; O) + (1 - 6)y$, and $i>i(y, N) = 6g(y; N) + (1 — 6)y$. If the parameter $6 > 0$ is small, we say that investment by person 2 is *unproductive)* In this example, the optimal investment level $y°$ will maximize $B_2(y) — y = 0(f(y) — y)$. Thus $y°$ is independent of *G*. If person 1 owns the asset, person 2 chooses an investment level *yi* that maximizes her net payoff, $U_2(N) — y$. Substituting the relevant functions in Eq. (15-33), we have

$$U_2(N) -y = 0.5[6f(y) + (l-0)y + 9g(y; N)$$
$$+ (l-0)y + B_1(x)-b_J(x;O)]-$$
$$y = 0.5[0(f(y) + g(y; N) - 2y) + Bdx) - b_{\{}(x; O)]$$

The first-order condition for the optimal choice of *y* does not involve *0*. Thus *y* is independent of *0*. Similarly, if person 2 owns the asset, person 2 chooses an investment level y_2 that maximizes

$$U_2(O) -y = 0.5[0(f(y) + g(y; O) - 2y) + B_{\{}(x) - b_x(x; N)]$$

This investment level y_2 is also independent of *0*. Although the investment *levels* are independent of *0,* the *costs* resulting from underinvestment are not. The loss from underinvestment in 3; when person *i* (*i* = 1, 2) owns the asset is

$$[Of(y°) + (1 - 0)y° - y°] - [6f(_{yi}) + (1 - 0)y_t - y_t]$$

This loss decreases as *0* decreases. As investment by person 2 becomes more and more unproductive *{0* approaches zero), the loss from underinvestment in *y* becomes negligible. On the other hand, the loss from underinvestment in x remains the same.

return from investment, *Biiy) — y = @(f(y) —* j), is increasing in *6*. Therefore a small value of *9* is taken to indicate unproductive investment.

It is then optimal for person 1 to own the asset. Allocating ownership of the asset to the more productive person (person 1) entails a small cost of underinvestment from the unproductive human capital *y,* but it minimizes the cost of the underinvestment in the more productive human capital *x*.

Another factor that affects the ownership structure is the magnitude of the appropriable surplus from the relationship-specific investments. To model this factor, let $b\backslash\{x; N) = b\backslash(x; O) — ax$. The parameter a can be interpreted as the degree of human capital specificity (with respect to the physical asset). A large value of a indicates that human capital is rather unproductive if it is not used jointly with the asset A. To see how this parameter affects the optimal ownership pattern, let $W_2 — B\backslash (x_2) + B_2(y_2) — x_2 — y_2$ be the net value of the cooperative venture if person 2 is the owner of asset A. Then

[B ; fe) 1] < 0

da da

since $B[(x_2) — 1 > 0$ and $dx_2/da < 0$. On the other hand, if person 1 owns the asset, then $dWi/da = 0$, since neither $x\backslash$ nor $y\backslash$ is affected by a. It follows that $d(Wi — W_2)/da > 0$. If a is large and person 1 does not own the physical asset, he becomes vulnerable to surplus appropriation and his incentive to invest in human capital falls. A higher value of a therefore tends to favor ownership of the physical asset by person 1 so as to mitigate his underinvestment problem.

Finally, let $b_x(x; N) = f(x)$, $b_x(x; O) = f(x) + ax$, and $B_x(x) = f(x) + ax + px$. Holding $f(x)$ fixed, an increase in p corresponds to a higher marginal productivity of investment for person 1 within the cooperative relationship. Substituting $db_x(x;N)/dx = /'$, 3&I(JC; $O)/dx = /' + a$, and $dB_x\{x)/dx = f' + a + p$ into first-order conditions for JCI and x_2 and differentiating with respect to p, we have

$$\frac{dxi}{dp} \quad \frac{0.5}{-f''(Xi)}$$

for / = 1, 2. Take a second-order Taylor approximation to $f(x)$: The second derivative /" will be independent of x, and hence $dx\backslash/dp \sim dx_2/dp$. Let W,- represent the net value of the cooperative venture when person / (/ = 1, 2) owns the physical asset. Then

Thus,

$$-W_2)$$

$$- (*i - x_2) - f''(xi - x_2) —$$

$$= 0.5(x, -x_2)$$

> 0

An increase in the productivity of human capital is more valuable the higher the level of investment. Since conferring ownership of the physical asset to person 1 encourages him to invest more in human capital, it helps maximize the gains from the rise in productivity. Thus, if the productivity of human capital investment rises for person 1 relative to that for person 2, ownership of the physical asset by person 1 becomes more favorable relative to the alternative ownership pattern. This complements our earlier result that ownership by person 2 is unattractive if investment by person 2 is unproductive.

PROBLEMS

1. In the principal-agent model discussed in this chapter, use Eq. (15-4) to show that *dfi*/dr* < 0 and *dp*/da*^{2} < 0.

1.393 In the performance measurement model, suppose the agent's action *x* as well as the performance indicator *M(x,e)* can be observed directly. Let payoff to the agent be *a* + /0M(JC, *e)* + *yx.* Derive the optimal values of /3 and *y.* Will the principal be able to always induce the fully efficient outcome?

1.394 Let the monitoring cost function be represented by *K(jt) = f(jr)* + *k7T* and suppose the participation constraint is not binding. Show that the optimal wage paid by the employer increases with monitoring costs. That is, show that *dw*/dk* > 0.

1.395 In the model of the optimal sharing rule with team production, let the cost function for person 1 be represented by C](xi) = *ex*^{2}. Show that the share of output allocated to person 1 (i.e., *fi*)* is decreasing in *c.* Interpret your result.

5. Consider the following model of an employee whose work is monitored by a supervisor. Work effort is measured by an index *e* that ranges between 0 and 1: A value of 0 indicates complete idleness and a value of 1 corresponds to fully effective work. The worker's Von Neumann-Morgenstern utility function takes the additive form: *u(c,e) = f(c)* — *g(e),* where *c* is consumption and *f'(c)* > 0, *f"(c)* < 0, *g'(e)* > 0, and *g"(e)* > 0. Consumption is equal to the individual's income from working. If the employee's work is not checked by a supervisor, the employee is assumed to have worked at maximum intensity *(e* = 1) and is paid w. If the employee's work is checked, then *e* is revealed to the supervisor, and the worker is paid *ew.* The probability that a worker is checked is *n.* This probability is independent of the worker's own behavior.

   1.396 Set up the worker's optimization problem for determining the optimal level of work effort.

   1.397 Will an increase in the probability of being checked increase work effort? Will a higher wage induce more work effort?

1.398 Do the assumptions already made rule out risk-loving behavior? How would risk-loving behavior affect the qualitative answers in part *(b)l* Explain.

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CHAPTER

16

MARKETS WITH IMPERFECT INFORMATION

16.1 THE VALUE OF INFORMATION IN DECISION MAKING

Why do people demand information? For some individuals, knowledge is an end in itself. One can think of knowledge as a good that enters into the utility function. Far more often, however, information is sought for its instrumental value. New information discovered from research may shift the production function, inside information on publicly listed companies may bring about speculative gains, and better information about an uncertain environment generally helps an individual make better decisions. Because information is valuable, people are willing to spend resources acquiring it. Because the acquisition of information is costly, however, information will remain imperfect. The production and use of information, the strategies to cope with imperfect information, and their implications for the operation of the market are the subjects of this chapter.

In the simplest setting, consider an individual whose objective function is $f(x, a)$. Suppose this person is uncertain about the value of the parameter a. Then a risk-neutral person will choose JC to maximize $E[f(x, a)]$. The optimal choice of x, denoted JC, will depend on the probability distribution of a, but not on the unknown value of a itself.

If the individual can buy an information service that accurately reports the true value of a or if the individual can invest in acquiring this information herself, she will choose x after a becomes known. Instead of maximizing the expected value of the objective function, therefore, this person will choose x to maximize $f(x, a)$. The optimal choice function, $x = x^*(a)$, will in general depend on the value of the parameter a. By definition of the optimal choice function, we must have

$f(x^*(a), a) > f(x, a)$. In other words, having information about the true value of the parameter will raise the maximized value of the objective function. Exactly how large this gain is, of course, depends on the actual value of a, which the individual does not know in advance. Nevertheless, since the individual knows the distribution of *a,* the expected value of the gain can be computed. If there is an information service that delivers accurate information about *a,* the maximum amount that the individual is willing to spend to resolve the uncertainty regarding *a* is

$$E[f(x^*(a),a)-f(x,a)]$$

This amount is always nonnegative because $f(x^*(a), a) - f(x, a)$ is nonnegative for all *a.*

In an environment involving more than one decision maker, the analysis of issues related to imperfect information can be quite complex. Not only is information costly to acquire, but situations of asymmetric information may also arise. These are situations in which one individual knows something that other individuals do not know. For example, in a principal-agent model with imperfect performance measurement, the agent knows something about the environment that the principal does not directly observe. The previous chapter discussed how individuals design optimal contracts to minimize the incentive problems arising from imperfect information. In this chapter we discuss other types of behavioral and market responses to situations of imperfect or asymmetric information.

16.2 SEARCH

In a world where information is costless, the law of one price holds: Any firm that charges a price higher than the price charged by another firm will find no customer for its product. Information, however, is not free. Searching for the lowest price requires time and expenses. Because some buyers may want to economize on the cost of acquiring price information, a firm that charges a price above the lowest price in the market will not lose all customers. Indeed, even in markets for homogeneous goods, price dispersion is ubiquitous. In a pair of pioneering articles, George Stigler^ studied the search behavior of buyers when they face a distribution of asking prices and the search behavior of workers when they face a distribution of wage offers. His analysis paved the way for subsequent elaboration and extension into a class of models collectively known as *search theory,* which finds numerous applications in industrial organization, labor economics, and macroeconomics.

Stigler formulated the search problem as the choice of optimal sample size. Increasing the sample size by looking for more price quotations is costly but will increase the likelihood of finding a good deal. Consider a market in which there

^ George Stigler, "The Economics of Information," *Journal of Political Economy,* 69:213-225, June 1961; and "Information in the Labor Market," *Journal of Political Economy,* 70:94-105, October 1962.

is a nondegenerate distribution of prices quoted by sellers. Let $F(p)$ represent the cumulative distribution function of these price quotes, and let $f(p)$ represent the probability density function. A buyer knows the distribution of prices, but she does not know which seller charges which price before making the search. If she has canvassed n sellers, then she only knows the price quotes of these n sellers. Of course, she will make the purchase from the seller who charges the lowest price among these n sellers. It costs c dollars each to canvass a seller. The buyer is risk-neutral, and she plans to buy β units of the good.

The buyer chooses a sample size n to minimize expected total cost:

minimize

$$\beta E[P_{min}(n)] + cn$$

where $P_{min}(n)$ is the lowest price from a sample of n quotations, $P_1, \ldots, P_n$. To analyze this problem, it is necessary to derive the distribution of $P_{min}(n)$. Let the cumulative distribution function and the probability density function for $P_{min}(n)$ be represented by $G(\cdot)$ and $g(\cdot)$, respectively. Then

$$\begin{aligned} G(p) &= 1 - \Pr[P_{min}(n) > p] \\ &= 1 - \Pr[P_1 > p, \ldots, P_n > p] = 1 - [1 - F(p)]^n \end{aligned}$$

Using integration by parts, the expected value of $P_{min}(n)$ is

$$\begin{aligned} E[P_{min}(n)] &= \int_0^\infty p g(p)\,dp \\ &= \int_0^\infty [1-G(p)]\,dp \\ &= \int_0^\infty [1-F(p)]^n\,dp \end{aligned}$$

The marginal benefit of increasing the sample size from $n - 1$ to n is, therefore,

$$\begin{aligned} \beta E[P_{min}(n-1)] - \beta E[P_{min}(n)] &= \beta \int_0^\infty \{[1 - F(p)]^{n-1} - [1 - F(p)]^n\}\,dp \\ &= \beta \int_0^\infty F(p)[1-F(p)]^{n-1}\,dp \end{aligned}$$

Note that the marginal benefit of search is positive and is decreasing in n, while the marginal cost of search is a constant equal to c. Therefore, the buyer will choose an optimal sample size n^* such that

$$\beta \int_0^\infty F(p)[1 - F(p)]^{n^*-1}\,dp \geq c \geq \beta \int_0^\infty F(p)[1 - F(p)]^{n^*}\,dp$$

The first inequality states that the marginal gain from searching the n^*th seller exceeds the marginal cost, so searching n^* sellers is better than searching n^* — 1 sellers. The second inequality states that the marginal gain from searching the $(n^* + 1)$th seller is less than the marginal cost, so the buyer does not expand the sample size from n^* to $n^* + 1$. When both inequalities are satisfied, the buyer cannot gain by deviating from the optimal sample size n^*.

The comparative statics are easy to derive. First, an increase in search cost (c) reduces n^*. Second, an increase in number of units purchased (/3) increases n^*. The latter result reflects one important property of information: The benefit from information increases with the intensity of use, but the cost does not. A frequent buyer has more to gain from price information than does an infrequent buyer. For example, tourists tend to get a bad deal not only because their costs of search are relatively high but also because they have less incentive to search.

Sequential Search

The analysis above assumed that people follow a particular search rule—they determine the number of price quotations to collect before conducting the search, and they always keep on sampling until that number is fulfilled. Although the fixed sample size rule has some intuitive appeal, it does not optimally utilize the information gathered during the search process. For example, if a buyer is lucky enough that she receives the lowest possible quotation from the first seller she visits, obviously there is no point in continuing the search regardless of the initial sample size that she intends to collect. As another example, consider another buyer who has received the highest possible price quotation from all the first n sellers she has canvassed. According to the fixed sample size rule, she should terminate the search if n is the predetermined optimal sample size. However, if she is convinced that her estimate of the distribution of price quotations is the true one, then her incentive to search after receiving the n high price quotes is exactly the same as her incentive to search before receiving those quotes. After all, the search cost already incurred is sunk. This buyer should continue to gather more quotations.

To optimally utilize the information collected during search, the buyer should adopt a sequential strategy: After receiving each price quote, the buyer evaluates whether she should continue to search or stop searching and accept the quoted price. Suppose the price quote she receives is x, and let the expected gain from another search be $H(x)$. If she looks for another quote and the quoted price p is greater than x, she gains nothing. If the quoted price p is less than x, she gains $fi(x — p)$. Therefore the expected gain is

$$H(x) = /3 \int f\backslash x - p)f(p)dp = p \int [^{X}F(p)dp \qquad (16\text{-}1)$$

where the second equality is obtained through integration by parts. Since $H'(x) = P\ F\ (x) > 0$, the expected gain from further search is lower, the lower the current price quotation received. When the current price quote is sufficiently low, the expected gain from search falls below the cost of search. The optimal policy therefore has a *reservation price* property. If the price quote x is above some reservation price p^*, then continue the search; otherwise, stop. This reservation price is defined by the condition

$$\int^{p} \qquad c \qquad (16\text{-}2)$$

Equation (16-2) implicitly defines the reservation price as a function of the parameters c and *(3*. Differentiating this equation with respect to c, we get

$$\qquad >0$$
$$dc \qquad 0F(p^*)$$

A buyer with high search costs sets a high reservation price. She is more likely to accept the prices quoted by sellers than are buyers with lower search costs. Similarly, the comparative statics for the parameter *ft* can be obtained:

$$dp^* = \frac{-\int_o^{p^*} F(p)dp}{} \wedge Q$$

Frequent purchasers or bulk purchasers tend to set more stringent (i.e., lower) reservation prices.

The reservation price also depends on the form of the distribution function F. In Chap. 13 we introduced the notion of increases in riskiness. Suppose the price quotes p are replaced by p^+ such that $p^+ = p + e$, where e is a random noise with conditional mean equal to zero. Clearly the distribution of p^+ is more risky or more dispersed than that of p. The variable p^+ is said to be a *mean-preserving spread* of p. We showed in Chap. 13 that $E[u(p^+)] > E[u(p)]$ for all convex function «(•). Now let a be a parameter that represents a mean-preserving spread to the price distribution. Since the reservation price is characterized by the condition $H(p^*) = c$, differentiating with respect to a gives

$$\mathbf{W} \mid \qquad + \qquad \wedge \mathbf{O} \qquad (16\text{-}3)$$
$$da \qquad da$$

FromEq. (16-1) we see that $H(p^*) = pE[max\{p^* - p, 0\}]$. The function $max\{p^* - p, 0\}$ is convex in p (see Fig. 16-1). Therefore, by the result from Chap. 13, $dH\{p^*)/da > 0$. Since //'(/?*) is also positive, Eq. (16-3) then implies that $dp^*/da < 0$. A buyer always has the option of continuing the search if she obtains a high price quote. Because finding a high price quote is not costly (the buyer does not have to buy at the high price) while finding a low price quote is beneficial, an increase in the dispersion of the price distribution (which increases the probability of finding very high and very low price quotes) raises the expected gains from search. The buyer therefore searches more intensively by setting a low reservation price.

Although the choice variable in the sequential search model is the reservation price and not the sample size, it is straightforward to derive the *expected* sample size in the sequential model. On each search, the probability of successfully finding a satisfactory price is $F(p^*)$. Therefore, the expected number of attempts required to find a price lower than the reservation price is $E[n] = l/F(p^*)$. Since $dE[n]/dp^* < 0$, *max[p* -p,* 0}

FIGURE 16-1
Convexity of the Gains from Search. The gain from getting a price quote of p is $p^* - p$ if p is less than p^*. The gain from getting a price quote higher than p^* is zero, because the buyer can simply ignore this high price and search again. As a result, the gain function is convex in p.

comparative statics for expected sample size simply have signs that are opposite to the comparative statics for reservation price. Notice that if buyers are restricted to conducting at most one search per period, expected sample size can also be interpreted as the expected duration of search. Therefore, an increase in search cost will shorten the search duration, while an increase in the number of units purchased or an increase in dispersion of the price distribution will lengthen it.

Equilibrium Price Dispersion

The search model is a partial equilibrium model. It takes the distribution of prices as given and analyzes buyers' optimal response to the lack of perfect information. The search behavior of buyers, however, will affect equilibrium sales at different prices. For example, we showed that a more dispersed price distribution will lead to a more intensive search. But as search becomes more intensive, sellers who quote very high prices will face a lower demand and may not survive. What determines the equilibrium degree of price dispersion? A general equilibrium analysis requires that buyers' search behavior and sellers' incentives to quote different prices be studied jointly. Many of these models are quite complex. Here we study a relatively simple model adapted from Salop and Stiglitz^ to illustrate how a nondegenerate price dispersion can indeed be supported in equilibrium.

Consider a market in which all consumers are willing to purchase one unit of a good as long as the price does not exceed /?. A consumer knows the distribution of prices in the market. Instead of a fixed sample size rule or a sequential search rule, we assume a very simple search setting. A consumer may incur a cost of c for information that allows her to purchase at the lowest price store. This may be interpreted as the cost of buying and reading a newspaper that carries the price quotations of all sellers. Alternatively, a consumer may just make the purchase at a randomly selected store. Different consumers have different search costs. The cumulative distribution of c is described by the function $G(c)$.

Firms are risk-neutral. There is free entry in this industry, and firms have access to the same technology. This technology is represented by a U-shaped average cost function $A(q)$. Let $q°$ be the output level that minimizes average cost, and let $p = A(q°)$. As long as some consumers choose to acquire price information, the lowest price in this market must be equal to p. Otherwise, firms could enter profitably by offering a lower price to consumers with price information. Firms that charge a price above the competitive price p will cater only to consumers with no price information. Because consumers with no information choose randomly, all firms that charge a price higher than p will serve the same expected number of customers. These high-price firms will maximize profits by charging the maximum price p. In equilibrium, therefore, there are two prices in the market: p and/?. By the zero profit condition, a low-price firm will sell $q°$ units and a high-price firm will sell q' units, where/? = $A(q')$. See Fig. 16-2.

With two possible prices in the market, let x be the fraction of firms charging the low price. Consider the search decision of consumers. A consumer who does not acquire information expects to pay $x\,p + (1 - x)p$ for the good. A consumer who acquires complete information pays the low price /?. Search is worthwhile if $p + c < xp + (1 - x)p$. Let k be the level of search cost that makes a consumer indifferent between acquiring and not acquiring information. That is,

$$k = (\backslash\text{-}x)\{p\text{-}p) \qquad (16\text{-}4)$$

All consumers with search cost $c < k$ will acquire the price information. Consumers with search cost $c > k$ remain uninformed. It is more intuitive to discuss the problem in terms of the fraction of informed consumers, $G(k)$, than in terms of the critical search cost, k. Therefore, we let $y = G(k)$ and transform Eq. (16-4) into

$$y = G((l - x)(p - p)) \qquad (16\text{-}5)$$

t Steven Salop and Joseph Stiglitz, "Bargains and Ripoffs: A Model of Monopolistic Competitive Price Dispersion," *Review of Economic*

Studies, 44:493-510, 1977.

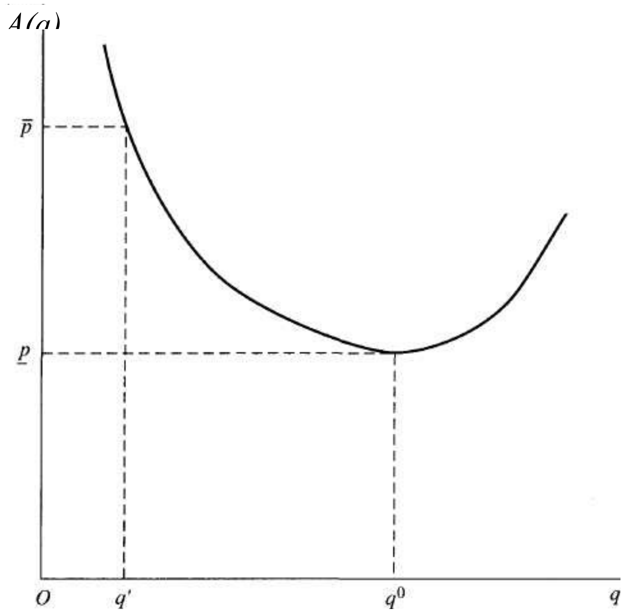

FIGURE 16-2
Two-Price Equilibrium. A low-price firm charges *p* and sells *q°* units. A high-price firm charges *p* and sells *q'* units. Both types of firms make zero profit.

In equilibrium, a fraction 3; of the consumers are perfectly informed. They only make their purchases from the low-price firms. The remaining fraction 1 — _y of the consumers are uninformed. These consumers make their purchases at a randomly selected firm. Because a fraction *x* of the firms are low-price firms, the uninformed consumers will visit the low-price firms with probability *x* and they will visit the high-price firms with probability 1 — *x.* The ratio of purchases made at low-price firms to purchases made at high-price firms is

y+x(l-y)

The ratio of total output produced by low-price firms to total output produced by high-price firms, on the other hand, is

xq°

(l-x)q' In equilibrium, these two

ratios must be equal. After some simplification, we have

y/iX-y)

v – (16-6)

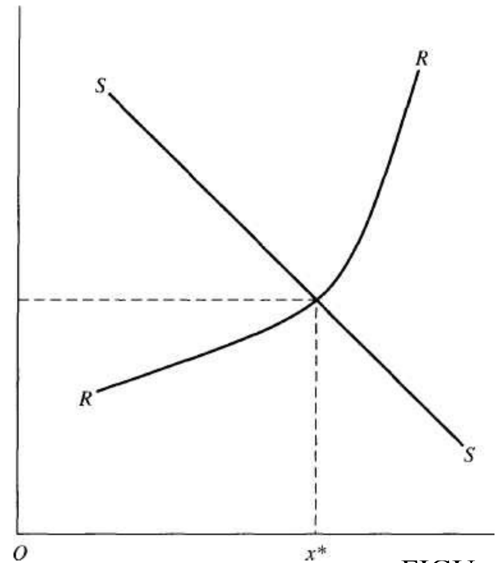

FIGURE 16-3 *Determination of Equilibrium Distribution.* The curve *SS* describes how the fraction of informed consumers *(y)* varies with the fraction of low-price firms in the market (x). The higher is *x,* the lower is the

incentive to acquire information. Hence fewer people become informed. The curve *RR* describes the condition that demand for purchases at low-price firms must be equal to supply. The higher is *y*, the greater is the demand for purchases at low-price stores. Hence there will be more low-price firms. Equilibrium is given by the intersection of these two curves.

The pair of Eqs. (16-5) and (16-6) can be used to solve for equilibrium values of x^* and y^*. This equilibrium is represented in Fig. 16-3. The curve *SS* depicts search decisions specified in Eq. (16-5). This curve is downward-sloping because an increase in the fraction of low-price stores (x) reduces the incentive to search. Hence the fraction of informed customers (v) falls. The curve *RR* represents the equilibrium condition (16-6). This curve is upward-sloping because an increase in y increases the demand at low-price stores. Hence x must increase to accommodate the demand. The intersection of these two curves gives the market equilibrium.

Comparative statics analysis can be conducted by shifting the curves in Fig. 16-3. For example, an increase in p — p shifts the *SS* curve up. Holding other things constant, an increase in the difference between the high and low price in the market will raise the fraction of informed customers because the expe cted gain from search increases. As more customers become informed, this will also raise the fraction of low-price firms in the market. An increase in $q°/q'$ (caused by, say, a change in technology that results in a flatter average cost curve) shifts the *RR* curve to the left. As each low-price firm serves relatively more customers than before, this reduces the fraction of low-price firms. With fewer low-price firms in the market, random shopping becomes less attractive. Thus the fraction of informed customers increases. Finally, consider the effect of a general increase in search costs. Since $G(c)$ is the fraction of consumers with search costs less than c, a general rise in search costs will lower $G(c)$ for any given c. From Eq. (16-5), we can see that the *SS* curve will shift down. Thus, an increase in search costs tends to reduce the fraction of informed consumers as well as the fraction of low-price firms.

16.3 ADVERSE SELECTION

While searching for price information is costly, it is even more costly to ascertain the quality of a commodity. Product quality often involves many dimensions that are difficult to measure. More significantly, the degree of imperfect information may not be symmetric between buyers and sellers. The classic example is the market for used cars, in which existing owners are said to have more accurate assessment of the quality of their cars than prospective buyers have. When buyers cannot assess the quality of individual items of a good, they use market data to form an estimate of average quality. In this case, owners of high-quality items have little incentive to sell their goods because they cannot distinguish their goods from the market average. As high-quality items are withdrawn from the market, average quality further deteriorates. An adverse selection effect obtains: Bad products drive out good products, and the size of the market shrinks. Adverse selection models have been used to study insurance markets, credit markets, and labor markets. This section examines how adverse selection affects equilibrium in a competitive market.

Consider a competitive labor market in an industry with identical firms and het-erogeneous workers. Firms are risk-neutral. They have access to a constant returns-to-scale production function, with labor as the only factor of production. Workers differ in the value of output they can produce if hired by a firm; a worker of type x produces x dollars of output if he works in this industry. Workers also differ in their reservation wages, which we denote by y. A worker's reservation wage in general is related to his productivity. For simplicity, we model this relationship by assuming that y is a function of x, that is, $y = y(x)$. In a more elaborate model, weaker notions of statistical dependence may be used without altering the substance of the argument.

Equilibrium is easy to describe when there is full information about worker productivity. Because worker types are observable, the wage that a worker receives is a function of his type. Under competition, a worker of type x receives a wage equal to his true productivity. The market wage schedule is therefore

$$w(x) = x$$

A worker treats this schedule as given and decides whether to work in this industry. He chooses to work in the industry if and only if the wage offer exceeds his reservation wage; that is, $w(x) > y(x)$. Since $w(x)$ — x, the set of workers employed in the industry in a full information equilibrium is given by

$$S^\circ = \{x : x > y(x)\}$$

Such an allocation of workers is obviously efficient. If a worker in the set S° is removed from the industry, he loses x but gains $y(x)$ from the alternative activity. Since $y(x) < x$ for a worker in 5°, the net gain is nonpositive. If a worker not in the set S° is recruited into the industry, he loses $y(x)$ but gains x. Since $y\{x) > x$ for such a worker, the net gain is negative. No deviation from S° can improve on the allocation of resources.

When there is asymmetric information, equilibrium allocation of resources can be quite different from that described above. Suppose workers know their own types.

Firms are uninformed about the productivity of individual workers; they only know the distribution of types in the market. The proportion of workers with productivity of x or below is given by the distribution function $F(x)$. The corresponding density function is represented by $f(x)$. Since firms cannot differentiate between workers of different types, the market wage cannot be a function of type. Instead all workers are paid the same wage, represented by w. Given this market wage, a worker is willing to accept employment in this industry if and only if the market wage rate is greater than his reservation wage. Denote this set of workers by S. Then

$$S = \{x : y(x) < w\} \tag{16-7}$$

Under competition, expected profits for each firm are driven to zero. Therefore, the market wage rate is equal to the average productivity of workers hired in this industry:

$$w = E[x \mid x \in S] \tag{16-8}$$

Equilibrium is characterized by a wage rate w^* and a set S^* of workers such that Eqs. (16-7) and (16-8) hold simultaneously.

Example 1. Suppose reservation wages are unrelated to productivities so that $y(x)$ is equal to a constant y_0 for workers of all types. Depending on whether y_0 is less than or greater than the market wage rate w, either all workers will choose employment in this industry (if $y_0 < w$) or none will do so (if $y_0 > w$). If all workers are employed (i.e., the equilibrium S^* is the set of all workers), then $w^* = E[x \mid x \in S^*] = E[x]$ is the equilibrium wage rate. This equilibrium obtains whenever the parameter $E[x]$ is such that $E[x] > y_0$. If $E[x] < y_0$, on the other hand, no workers will be employed. The equilibrium S^* is the empty set.

The allocation of resources would be different if productivity were observable by both firms and workers. In that case, the wage offered to a worker would depend on the worker's type, with $w(x) = x$. A worker would choose to work for the industry if and only if $w(x) = x > y_0$. The fraction of workers employed in the industry would be $1 - F(y_0)$, which is different from 0 or 1.

The nature of equilibrium under asymmetric information depends crucially on the properties of the function $y(x)$. A model of adverse selection in the labor market requires $y'(x) > 0$. This assumes that more productive workers have higher reservation wages. For example, reservation wages may reflect forgone earnings in self-employment or in another industry where output is more readily observable. In this case, the assumption that $y' > 0$ holds if labor quality can be ranked on a one-dimensional scale: A worker who is better at one activity is also better at another.

We can define a critical productivity level $c = c(w)$ such that

$$y(c(w)) = w \tag{16-9}$$

That is, the function $c(\cdot)$ is the inverse of $y(\cdot)$. All workers with type x such that $y(x) < w$ are willing to accept employment in the industry. If $y(\cdot)$ is increasing, this condition is equivalent to $x < c(w)$. By the same reasoning, the highly productive workers with type $x > c(w)$ will not participate in the industry. From Eq. (16-9), we have $dc/dw = 1/y' > 0$. Thus, as the market wage falls, the critical type $c(w)$ also falls, and more and more high-type workers will drop out of the market.

Let $A(w)$ be the average productivity level among those willing to work in this industry at wage w. Since the probability density function of x conditional on $x < c(w)$ is $f(x)/F(c(w))$, the conditional mean of JC is

$$A(w) = E[x \mid x < c(w)] = \int_0^{c(w)} x \frac{f(x)}{F(c(w))} dx$$

Clearly $A(w) < c(w)$, because all workers who are willing to accept employment have productivity below $c(w)$. Furthermore, $A(w)$ is an increasing function of w because

$$A'(w) = \left[c(w) \frac{f(c(w))}{F(c(w))} + \int_0^{c(w)} -\frac{x f(x) f(c(w))}{F(c(w))^2} dx \right] \frac{dc}{dw}$$

$$= \frac{f(c(w))}{F(c(w))} \left[c(w) - \int_0^{c(w)} x \frac{f(x)}{F(c(w))} dx \right] \frac{dc}{dw}$$

$$= \frac{f(c(w))}{F(c(w))} \quad \frac{dc}{dw}$$

$$> 0$$

Workers of very high types tend to withdraw from the market because they could only receive a wage equal to average productivity. The withdrawal of these workers reduces average productivity and the market wage. Since $A'(w) > 0$, a fall in market wage leads to a fall in average productivity $A(w)$, which triggers a further fall in wage because $w = A(w)$. The adverse selection effect is therefore cumulative.

Example 2. Consider the example of asymmetric information in used cars due to Akerlof J Suppose potential buyers are willing to pay x dollars for a used car of quality x. Existing owners have a reservation value of $y(x) = 2x/3$ for their cars. Quality is uniformly distributed between 0 and 2. Since $y(x) < x$ for all x, used cars of all quality levels will be traded in a full information equilibrium. When potential buyers cannot observe quality, however, equilibrium would involve a price w that is equal to the expected quality of used cars put up for sale. This expected quality is given by

$$A(w) = E\left[x \,\middle|\, \frac{2x}{3} < w\right] = \frac{3w/2}{2}, \quad \frac{2}{3}$$

George Akerlof, "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics,* 84:488-500, August 1970.

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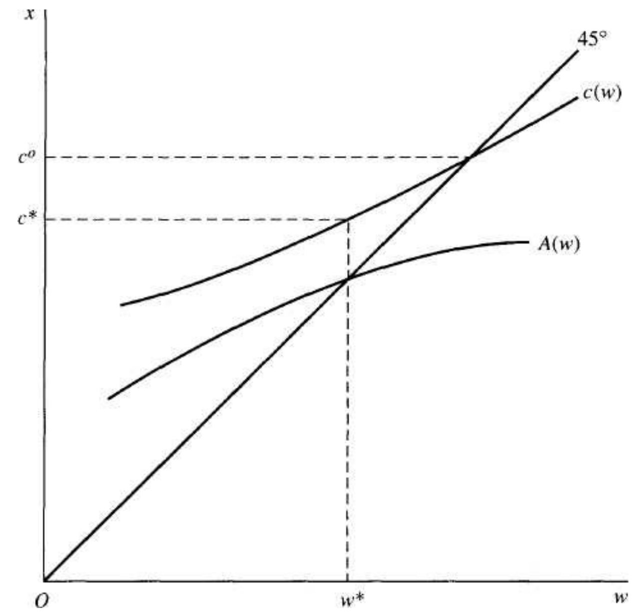

FIGURE 16-4
Equilibrium with Asymmetric Information. The equilibrium wage is given by the intersection of $A(w)$ and the 45° line. Workers of type $x < c^*$ are employed in this industry. In contrast, all workers of type $x < c^\circ$ are employed in the full information solution.

, Since $A(w) = 3w/4 < w$ for any positive price w, no car will be traded in equilibrium. In this example, adverse selection leads to a total collapse of the used cars market.

In an equilibrium with adverse selection, the market wage rate is equal to average productivity. That is, the equilibrium wage satisfies $w = A(w)$. In Fig. 16-4 we plot the graphs of $A(w)$ and $c(w)$. The equilibrium wage w^* is given by the intersection between $A(w)$ and the 45° line. Individuals with type $x < c^*$ are employed in this industry, where $c^* = c(w^*)$. In contrast, if there is no asymmetric information, all workers with type $y(x) < x$ will be employed. Since the function $c(\cdot)$ is the inverse of $y(\cdot)$, the condition $y(x) < x$ is equivalent to $x < c(x)$. In Fig. 16-4, the critical type c° under the full information equilibrium is given by the intersection of $c(w)$ and the 45° line. All workers with $x < c^\circ$ would be employed in the full information equilibrium. As is clear from Fig. 16-4, $c^\circ > c^*$. Adverse selection tends to reduce employment in the labor market where there is asymmetric information.

Favorable Selection

In an adverse selection model, more productive workers drop out of the market because the workers have better outside opportunities than receiving a market wage that reflects average labor productivity. But selection can also work in the opposite direction. Better workers in one activity need not be better workers in another activity.

If the more productive workers tend to have *lower* reservation wages, they are more likely to stay in the industry at any given wage than the less productive workers are. In that case, the selection mechanism will produce an equilibrium quite different from an adverse selection equilibrium.

Let the reservation wage of a worker with productivity x be $y = y(x)$, where $y'(x) < 0$. When productivity is unobservable by the firms, all workers are paid the same wage w. A worker of type x is willing to accept employment if $y(x) < w$. As in Eq. (16-9), define a function c(-) to be the inverse of $y(-)$. When $y(-)$ is a decreasing function, so is its inverse c(-). The condition $y(x) < w$ is therefore equivalent to $x > c(w)$. Unlike the case of adverse selection, it is the *more productive* workers who are more willing to accept employment in this industry. However, because $c'(w) = l/y' < 0$, the selection of more productive workers becomes less pronounced as the market wage rises.

Equilibrium requires that the market wage rate w be equal to the average productivity of those who are willing to work at wage w. Let A(w) represent this average productivity. Then

$$A(w) = E\backslash x \backslash x > c(w)] = \quad \underset{c(w)}{} \quad \frac{fix)}{- F(c(w))} \, dx$$

Clearly $A(w) > c(w)$, because all workers who are willing to accept employment have productivity above $c(w)$. The equilibrium is depicted in Fig. 16.5. The graph of $A(w)$ is downward-sloping because

$$A'(w) = \frac{f(c(w))}{F(c(w))} \quad [A(w) - c(w)] \text{ -/- } < 0 \, dw$$

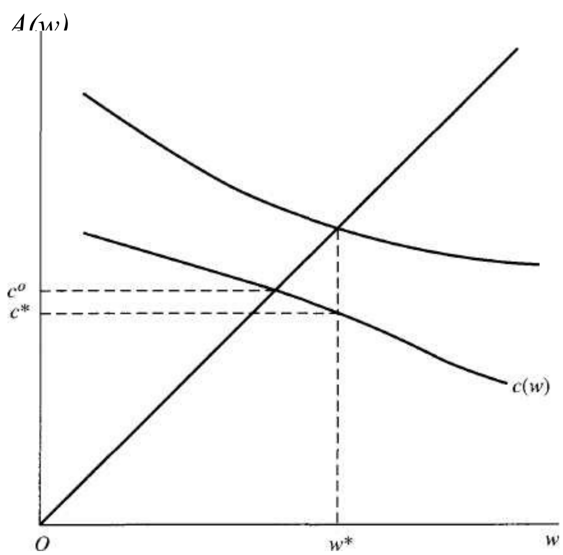

FIGURE 16-5
Equilibrium with Favorable Selection. The equilibrium wage is given by the intersection of $A(w)$ and the 45° line. Workers of type $x > c^*$ are employed in this industry. In a full information equilibrium, in contrast, workers of type $x > c^\circ$ are employed. Whereas adverse selection tends to shrink the size of the market, favorable selection tends to expand it.

Favorable selection tends to select the more productive workers into the industry, thereby raising the industry wage rate above the average productivity level. As the wage rate rises, however, the selection of good workers becomes less pronounced and average labor productivity falls, giving downward pressure on the wage rate. Unlike adverse selection, therefore, favorable selection is self-limiting. The negative slope of the $A(w)$ curve guarantees there is a unique intersection with the 45° line.

If there is no asymmetric information, all workers with $y(x) < x$ are employed. Since $y(x)$ is a decreasing function, this condition is equivalent to $x > c(x)$. In Fig. 16-5, all workers with type $x > c°$ are employed in the full information solution. With asymmetric information, in contrast, workers of type $x > c^*$ are employed. Since $c^* < c°$, as shown in Fig. 16-5, more workers are employed in the asymmetric information equilibrium than in the full information equilibrium. Whereas adverse selection shrinks the market size, favorable selection expands it.

David Hemenway suggested that favorable selection is empirically relevant in insurance markets.^ The conventional view holds that people with bad risks have more incentive to buy insurance. An increase in policy premiums tends to deteriorate the risk pool as individuals with good risks leave the market. The deterioration of the risk pool necessitates a further increase in premiums for the insurance companies to break even. This adverse selection process can lead to very high premiums and the underprovision of insurance. Hemenway argued instead that the insurance market attracts individuals who are relatively risk-averse. Since these individuals also take more measures at self-protection, the average risk among buyers of insurance is lower than the population average. According to this argument, an increase in policy premiums will then improve the risk pool, since only the cautious types remain in the market. When there is favorable selection, the prediction that asymmetric information will lead to underprovision of insurance is no longer valid.

16.4 SIGNALING

In a model of adverse selection, high-quality workers and low-quality workers are paid the same wage if employed because employers are uninformed about worker quality. Instead of receiving a wage that reflects average productivity, a high-quality worker will receive a higher wage under competition if he can reveal his true productivity to potential employers. High-quality workers, however, cannot distinguish themselves from low-quality ones by mere talk, because the latter also have an incentive to (falsely) claim that they are highly productive. By assumption, employers cannot observe worker quality when making the hiring decision. Since all workers have an incentive to claim they are of high quality, such claims are not to be taken

^David Hemenway, "Propitious Selection," *Quarterly Journal of Economics,* 105:1063-1069, November 1990; "Propitious Selection in Insurance," *Journal of Risk and Uncertainty,* 5:247-251, 1992.

seriously. As usual, actions speak louder than words. Signaling models study how individuals undertake costly actions in order to reveal their characteristics to other uninformed individuals.

Consider a simple model of education signaling first proposed by Michael Spence.t He assumed there are two types of workers: Type 1 workers have productivity *v\,* and type 2 workers have productivity v_2, with v_2 > *v*. Workers know their own types, but employers only know that a fraction *n* of the workers are type
1.399 and a fraction n_2 are type 2 (with *ii* + JT_2 = 1). In a competitive labor market
in which employers cannot distinguish between worker types, all workers receive a
wage equal to the average productivity, *7i\V* + JT_2V_2. Since *7T\V* + TT_2V2 < v_2, type
1.400 workers are paid less than their true productivity, and they have an incentive to
signal that they are more productive than the average worker. Spence argued that
education credentials may serve as a signal for worker quality even if education does
not directly raise productivity.

The crucial assumption behind Spence's model is that more productive workers can acquire education at a lower marginal cost than less productive workers can. Suppose it costs type 1 workers *C\e* to attain a level of education indexed by *e,* while it costs type 2 workers c_2e to attain the same education level. Then the crucial assumption is that c_2 < *c*. This specification satisfies the *single-crossing property,* a condition often invoked in the formal analysis of signaling models. The single-crossing property requires that the indifference curves for workers of different types cross at most once. In the present context, type 1 workers' utility function may be written as U^1*(e, w) — w — C\e,* where *w* is the wage received. If we plot *w* on the vertical axis and *e* on the horizontal axis, the slope of the indifference curves is *c*. Similarly, the slope of the indifference curves is c_2 for type 2 workers. Since the indifference curves for type 1 workers are always steeper than those for type 2 workers, the single-crossing property is indeed satisfied.

To see why the assumption of differential costs of education is important, suppose c_2 > *C* instead. Then, whenever high-quality workers have the incentive to invest in education level *e* in order to signal their high productivity, low-quality workers will have the incentive to do the same because their costs of education are lower. Thus there will not be an equilibrium in which employers can distinguish between high-quality and low-quality workers by observing the different education levels they choose to attain.

In an equilibrium in which education is a signal for worker quality, employers expect that workers with education level *e* are type 1, while workers with a different education level e_2 are type 2. Under competition, they pay *V* to the type 1 workers and v_2 to the type 2 workers. Such an equilibrium is called a *separating equilibrium.* A condition for equilibrium is that employers' expectations are confirmed by workers' behavior. This requires that type 1 workers actually choose to obtain education level e_x, while type 2 workers actually choose to obtain education level e_2. The requirement

^Michael Spence, *Market Signaling,* Harvard University Press, Cambridge, MA, 1974.

may be written as

$$Vi - c\backslash e\backslash > v_2 - C\backslash e_2 \qquad (16\text{-}10)$$

$$v_2 - c_2 e_2 > V\backslash - c_2 e\backslash \qquad (16\text{-}11)$$

Inequality (16-10) is a *self-selection* condition for type 1 workers. Any low-quality worker could (falsely) signal a high productivity by choosing a higher education level e_2. Condition (16-10) states that type 1 workers prefer choosing education level $e\backslash$ for wage $v\backslash$ to submitting the false signal for wage v_2. Similarly, a high-quality worker could save some education expenses if he accepts a lower wage $v\backslash$. Condition (16-11) states that type 2 workers prefer the combination (e_2, v_2) to saving the education expenses and receiving the lower wage.

The self-selection conditions (16-10) and (16-11) may be rearranged to yield

$$\frac{\text{---------}}{c_2} > e_2 - e\backslash > \frac{\text{-------}}{c_x} \qquad (16\text{-}12)$$

Note that if $c\backslash < c_2$, then (16-21) cannot be satisfied. Thus a necessary condition for a separating equilibrium is that the cost of education be cheaper for high-ability workers than for low-ability workers. Furthermore, condition (16-12) shows that the difference in equilibrium education levels between workers of different types is bounded above and below. If the difference $e_2 - e_x$ is too great, neither type 1 nor type 2 workers are willing to incur the cost of education signaling. If the difference is too small, on the other hand, both type 1 and type 2 workers would choose e_2, and education would not be a useful signal for differentiating worker quality. Only when $e_2 - e\backslash$ satisfies (16-12) do we have a separating equilibrium whereby workers of different types choose different levels of education and employers correctly infer worker productivity based on observed education levels.

Condition (16-12) does not pin down a unique set of equilibrium values for $e\backslash$ and e_2. However, since education is a costly activity, competition among employers tends to minimize the levels of education needed to achieve a signaling equilibrium. Minimizing $e\backslash$ and e_2 subject to (16-12) implies an equilibrium value of $e\backslash = 0$ and $e^*_2 = (v_2 - V\backslash)/c\backslash$. Any other $(e\backslash, e_2)$ that satisfies (16-12) does not constitute a full equilibrium.^ If any employer offers to pay a wage $V\backslash$ to workers with education level $e\backslash > e\backslash$ and a wage v_2 to workers with education level $e_2 > e\backslash$, another employer can profitably lure all her workers away by paying slightly lower wages but requiring lower education levels (<?*, *e%)* instead.

In a signaling model, high-quality workers invest in costly education to distinguish themselves from less productive ones. Sorting merely results in a transfer

^The formal justification of this conclusion requires a detailed specification of employers' beliefs and equilibrium conditions. We are implicitly adopting the notion of "reactive equilibrium" proposed in John Riley, "Informational Equilibrium," *Econometrica,* 47:331-359,1979. See also In-Koo Cho and David M. Kreps, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics,* 102:179-221, 1987.

of income but does not add to total product. Compared to an economy with full information, there is excessive education in a signaling equilibrium.

A More General Analysis

The analysis of signaling equilibrium does not depend on the assumption that the costly signal (education) is completely unproductive, nor does it depend on a model with only two discrete types of workers. In a more general model, we allow education to have a direct effect on worker productivity. We also assume that worker types are continuously distributed, so the convenient techniques using the calculus can be employed.

Let

$$v = v(a,e)$$

where productivity v is assumed to be an increasing function of innate ability a and education e. Worker types are differentiated by differences in their innate abilities. For each worker, innate ability is fixed but education is an acquired characteristic. Let the cost of education be

$$c = c(a, e)$$

where $c_a < 0$ and $c_e > 0$. That is, more able individuals can acquire education at a cheaper cost, and the cost of education is increasing in the level of education acquired. More importantly, we assume that the cross-derivative c_{ae} is negative; that is, the *marginal* cost of education is lower for people with greater ability. This assumption is related to the single-crossing property discussed in the discrete-type model. Suppose workers' utility function is given by $U — w —c(a, e)$, where w is the wage received. Then the indifference curves in (e, w)-space for a type $a\backslash$ worker has slope $c_e(ai,e)$, and the indifference curves for a type a_2 worker has slope $c(a_2, e)$. If $ai < a_2$, the indifference curves for type a_{ℓ} workers are steeper than those for type a_2 workers at any given $e\backslash$ their indifference curves cross at most once.

Neither productivity nor innate ability is directly observable by employers. Instead, employers use the level of education to infer worker productivity. The wage workers receive depends on their education, according to the market wage schedule $w(e)$. This wage schedule is determined by market competition, as in Eq. (16-14), but each individual treats it as given. Given this wage schedule, each worker chooses an education level e to maximize

$$w(e) — c(a, e)$$

The first-order condition for maximization is

$$w'(e)-c_e(a,e) = 0 \qquad (16\text{-}13)$$

Equation (16-13) defines a choice function $e = e^*(a)$. Comparative statics analysis gives

$$\frac{\mathbf{gf!}}{da} = \frac{c''}{w'' — c_{PP}} > \mathbf{0}$$

since c_{ae} is negative by assumption and the denominator is also negative by the second-order condition for maximization. This establishes that people with greater innate ability invest more in education. Conversely, employers can infer that people who invest more in education are those who have greater innate ability.

The fact that more able people choose to acquire more education allows employers to make valid inferences about innate abilities by observing education credentials. If $a_1 < a_2$, then $e^*(a_1) < e^*(a_2)$. A person with ability a_1 will not find it in his self-interest to misrepresent himself to be of ability a_2 by choosing a higher level of education $e^*(a_2)$, because the marginal cost of education is too high for him. In other words, individuals of different innate abilities self-select themselves through the difference in their choice of educational attainment. Employers can infer that someone with education level $e^*(a_1)$ must be of ability level a_1, and someone with education level $e^*(a_2)$ must be of ability level a_2. In general, such equilibrium inference can be represented by the function $a = a^*(e)$, where $a^*(\cdot)$ is the inverse of the choice function $e^*(\cdot)$. This inverse function exists because $e^*(\cdot)$ is a strictly increasing function. Furthermore, the inference function is increasing because $da^*/de = 1/e^{*\prime} > 0$.

Under competition, the market wage schedule must reflect employers' expectations about worker productivity. For a worker with education level e, employers infer that his innate ability is $a = a^*(e)$. Therefore,

$$w(e) = v(a^*(e), e) \tag{16-14}$$

Equation (16-14) holds for any education level e and is therefore written as an identity. Differentiating (16-14) with respect to e, we get

$$w'(e) = v_a \frac{da^*}{de} + v_e$$

Using Eq. (16-13) to substitute c_e for $w'(e)$, this can be rearranged to

$$c_e - v_e = v_a \frac{da^*}{de} > 0$$

In other words, people invest in education to such an extent that its marginal cost exceeds its marginal benefit. Investment levels are too high compared to a full information equilibrium in which $c_e - v_e = 0$. Such excessive investments in education occur because more education signals greater ability, and greater ability translates into higher earnings.

16.5 MONOPOLISTIC SCREENING

In markets with asymmetric information, the informed individuals have incentives to reveal their true characteristics to individuals on the uninformed side of the market, as in a signaling model. The flip side of this is that the uninformed individuals also have incentives to use various devices to distinguish, or screen, the various types of agents on the other side of the market. The incentives for uninformed individuals to engage in screening is particularly great if they have some degree of market power. Consider, for example, the pricing problem faced by a producer with monopoly power. He has a number of potential customers with different intensities of demand, but he cannot distinguish the high-demand consumers from the low-demand ones. Market research will reduce but not eliminate uncertainty regarding consumer types. This producer faces a dilemma. If he charges too high a price, the low-demand consumers will drop out of the market. If he charges too low a price, he is not maximizing the amount he could extract from the high-demand consumers. Pricing policies that screen consumers into separate market segments may therefore contribute to an increase in profits. For example, the producer may introduce a product line with two varieties of the good. The high-quality (and more expensive) variety caters to high-demand consumers, while the low-quality (and cheaper) variety caters to low-demand consumers. If the price-quality schedule is chosen appropriately, the producer can effectively screen customers into two separate markets without actually knowing who belongs to which type. Screening allows the producer to practise partial price discrimination under conditions of imperfect information.

Suppose a commodity can be produced in a number of varieties. Product varieties are indexed by *q,* with higher values of *q* indicating higher-quality varieties. The unit cost for any variety is $C(q)$, with $C'(q) > 0$ and $C''(q) > 0$. This unit cost is independent of the number of units produced.

There are two types of potential consumers in the market. A fraction *n* of the consumers are "high-demand" consumers. They have a utility function $U_H = x + B_H(q)$, with $B'_H(q) > 0$ and $B'{}^\wedge(q) <$ 0. The variable *x* represents all other goods. Each consumer chooses at most one variety of the commodity to maximize utility subject to the budget constraint, $P\{q)+x = M$, where *M* is income and $P(q)$ is the price of the product variety with quality *q)* Given this utility function, $B_H\{q)$ can be interpreted as the consumer's willingness to pay for product variety *q,* and utility maximization is equivalent to maximizing $B_H(q) — P(q)$. The other type of consumers are the "low-demand" consumers. These consumers have a utility function $U_L = x + B_L(q)$, with $B_L\{q) > 0$ and $B'[(q) < 0$. We assume $B_H(q) > B_L\{q)$ for all *q;* that is, high-demand consumers are willing to pay more for any variety than are low-demand consumers. We further assume

$$B'_H\{q) > B'_L\{q) \tag{16-15}$$

^Instead of interpreting *q* as product variety or product quality, we can interpret *q* as the quantity consumed. In this alternative interpretation of the model, the producer does not charge a uniform per-unit price and let consumers choose the amount purchased. Rather, he sets a price schedule />(•), where consumers are required to pay a total of $P(q)$ dollars if they choose to purchase *q* units of the good. In general, this price schedule $P(q)$ is a nonlinear function of *q,* which is why the theory of price discrimination under imperfect information is also known as the theory of nonlinear pricing.

for all q. Condition (16-15) amounts to the single-crossing property, which states that the *marginal* value for quality is greater for high-demand consumers than for low-demand consumers. In a commodity space with q on the horizontal axis and x on the vertical axis, the indifference curve of a high-demand consumer [with slope — $B^f_H(q)$] is always steeper than the indifference curve of a low-demand consumer [with slope —$B'_L(q)$]. Thus they intersect only once.

Since there are only two types of consumers in the market, there is no point in making more than two varieties of the product. Let q_H and q_L be the product varieties produced, and let p_H and p_L be the corresponding prices of these two varieties. If there is no hidden information (that is, if the producer can distinguish the high-demand consumers from the low-demand consumers), the producer can sell product variety q_H at price p_H to the fraction of the consumers who are of high-demand type, and sell product variety q_L at price p_L to the remaining fraction of consumers who are of low-demand type. The producer chooses prices and qualities to maximize total profits

$$X[PH \sim C(q_H)] + (1 - n)[p_L \sim C(q_L)] \qquad (16\text{-}16)$$

subject to

$$- PH > 0$$

The solution values, denoted $(q^\circ_H, q^\circ_L, p^\circ_H, p^\circ_L)$, satisfy

$$C'(q^\circ_H) - B'_H(q^\circ_H) = 0$$
$$C'(q^\circ_L) - B>_L(_ql) = 0$$
$$BH(q^\circ_H) - p^\circ_H = 0$$
$$B_L\{q^\circ_L) - p^\circ_L = 0$$

In this full information solution, there is perfect price discrimination. The choice of product varieties is efficient: The marginal cost of quality is equal to the marginal value for each type of consumers. Moreover, the producer completely extracts the surplus from all consumers.

When the producer cannot separate the high-demand consumers from the low-demand consumers, however, the full information solution is not attainable. The implicit assumption behind the objective function (16-16) is that high-demand customers would purchase the variety q_H, while low-demand customers would purchase the variety q_L. However, consider the incentives for a high-demand consumer. If she chooses to purchase $q_H = q^\circ_H$ at a price of p°_H, her consumer surplus is zero. If she purchases the other variety q°_L instead at a price of p°_L, her surplus is greater than zero because

Since the producer cannot identify who the high-demand customers are, he cannot force them to purchase the variety $q°_H$. Indeed, all consumers will purchase the variety $q°_L$ at price $p°_L)$ Perfect price discrimination is not feasible.

With hidden information about consumer types, price discrimination can only be achieved by a price-quality schedule that induces the consumers to sort themselves into different market segments. Formally, the producer's problem is

maximize

$$TIVPH \sim C(q_H)] + (1 - n)[p_L -$$

$C(q_L)]$ subject to

$$B_H(<1H)-PH>0 \tag{16-17}$$

$$B_L(qL)-PL>0 \tag{16-18}$$

$$B_H(q_H) \sim PH> B_H\{q_L) - p_L \tag{16-19}$$

$$B_L(q_L) - P L > B_L(q_H) - PH \tag{16-20}$$

Conditions (16-17) and (16-18) are the participation constraints, which require that each type of consumer prefers buying the good to not buying it. Conditions (16-19) and (16-20) are the incentive compatibility constraints, also known as self-selection constraints in the context of screening models. Because consumer types are unknown to the seller, a feasible price discrimination scheme requires that consumers find it in their self-interests to choose the price-quality combination allocated to them. For example, (16-19) states that high-demand consumers obtain a higher surplus from choosing the combination designed for them (q_H, p_H) than from the other available combination (q_L, p_L). When this constraint is satisfied, they have no incentive to buy the other combination by disguising themselves as low-demand customers.

Two observations about the constraints (16-17) to (16-20) can be made. First, adding inequalities (16-18) and (16-19), we have

$$B_H(qH) \sim PH> B_H(q_L) - B_L(q_L) > 0$$

Therefore constraints (16-18) and (16-19) imply that constraint (16-17) holds as a strict inequality. The participation constraint for the high-demand consumers is not binding. Second, the discussion about the full information solution suggests that high-demand consumers have an incentive to purchase the variety designed for low-demand consumers, but low-demand consumers have no incentive to choose the variety designed for high-demand consumers. It is expected that the incentive compatibility constraint (16-20) for the low-demand customers will not be binding.

low-demand customers have no incentive to buy $q°_H$. If they did, their consumer surplus would be negative because $B_L\{q°_H) - p°_H < B_H(q°_H) - p''_H = 0$.

To solve the maximization problem, we therefore drop constraint (16-20) and then verify that this constraint is indeed not binding at the solution.

Once constraints (16-17) and (16-20) are dropped, the Lagrangian for the profit maximization problem is

$$SB = 7T[p_H \sim C(q_H)] + (1 - Tt)[p_L - C(q_L)]$$
$$+ X_x[B_L\{q_L) - p_L] + X_2[B_H(q_H) - p_H - B_H\{q_L) + p_L]$$

The first-order conditions are

$$n - k_2 = 0 \qquad (16\text{-}21)$$

$$(1 - n) - \mathrm{A.i} + X_2 = 0 \qquad (16\text{-}22)$$

$$-7TC'(q_H) + X_2B'_H\{q_H) = 0 \qquad (16\text{-}23)$$

$$-(1 - 7T)C'(q_L) + X_xB'_L(q_L) - X_2B'_H(q_L) = 0 \qquad (16\text{-}24)$$

From Eqs. (16-21) and (16-22), we get A* = 1 and $X\backslash = TT$. Both multipliers are strictly positive. Therefore the corresponding constraints (16-18) and (16-19) are both binding. Substituting these values of A.j and $X\backslash$ into Eqs. (16-23) and (16-24), the first-order conditions become

$$B'_H(q^*_H)-C'(q^*_H)=0 \qquad (16\text{-}25)$$

$$B'_L\{q^*_L) - C\{ql) - -?—[B'_H\{ql) \sim B'_L(q^*_L)] = 0 \qquad (16\text{-}26)$$
$$Y — 71$$

Furthermore, because constraints (16-18) and (16-19) hold as

equalities, we have

$$Pi = B_L\{ql) \qquad (16\text{-}27)$$

$$PH = B_H(q^*_H) - B_H\{ql) + B_L\{qD \qquad (16\text{-}28)$$

Comparing the first-order conditions (16-25) and (16-26) with the full information solution, we can see that $q^*_H = q^\circ_H$ and $q^*_L < q^\circ_L$. To prevent high-demand customers from choosing the price-quality combination designed for low-demand customers, the seller reduces the product quality of the low-demand combination from q°_L to q^*_L, thereby making the low-quality variety unattractive to high-demand customers.

Also note that price discrimination is imperfect under hidden information. While low-demand consumers retain no consumer surplus (their participation constraint is binding), the producer cannot extract all the consumer surplus from high-demand customers (their participation constraint is not binding). The surplus retained by the high-demand customers is called an *informational rent,* because it arises from the producer's lack of information. Since the producer is unable to differentiate the high-demand consumers from the low-demand consumers, if he tries to extract more consumer surplus from the high-demand consumers by charging them a higher price, he cannot prevent the high-demand customers from buying the low-quality variety instead.

Finally, we need to verify that the incentive compatibility constraint (16-20) for low-demand consumers is indeed satisfied. Because low-demand consumers have zero consumer surplus from the low-quality variety, their incentive compatibility constraint requires that

$$0 > B_L(q^*_H) - p^*_H$$

Substituting the value of p^*_H from (16-28) into this inequality, the condition is equivalent to

$$B_H(q^*_H) \sim B_H\{q^*_L) > B_L\{q^*_H) \sim B_L(q^*_L)$$ This

condition is implied by the single-crossing property,

because

$$= B_H(q^*_H)-B_u(q^*_L)> \qquad B'_L\{q)dq = B_L\{q^*_H) - B_L\{q^*_L)$$

Thus constraint (16-20) indeed holds as a strict inequality. Dropping this constraint makes no difference to the solution of the maximization problem.

Screening models have found many applications in economics besides price discrimination. In a principal-agent setting, screening models are also known as *hidden information* models. The action of the agent is assumed to be observable, so the usual moral hazard problem due to hidden action does not arise, However, the principal does not know the agent's cost structure. The analysis of this problem is similar to the analysis of the price discrimination model, with the principal interpreted as a buyer with monopsony power. In addition, this model is an important building block in optimal income tax models and the theory of auctions.

PROBLEMS

1.401 A producer maximizes $E[pf(x_{\iota}, X2) — W\backslash Xi — w_2x_2]$ without knowing the value of $W\backslash$.
If the distribution of $w\backslash$ becomes more risky in the sense of a mean-preserving spread,
show that the amount she is willing to pay for accurate information about the value of $W\backslash$
increases.

1.402 Suppose the prices different sellers charge are uniformly distributed between 0 and 1.
A buyer plans to purchase /? units of the good, and the cost of search is *nc* if she visits
n sellers. Assume *n* is continuous rather than discrete. Derive the first-order condition
for cost minimization if the buyer follows the fixed sample size rule. Check whether the
second-order condition holds. Derive the comparative statics for the parameters *c* and /?.

1.403 Let prices be uniformly distributed between 0 and *d.* A buyer buys one unit of the good,
and the cost of searching an additional seller is *c.*

1.404 If the buyer follows the fixed sample size search rule, how does the optimal sample
size vary with *dl* Interpret your result.

1.405 If the buyer follows the sequential search rule, what is the optimal reservation price?

1.406 Derive an expression for the expected number of sellers that the buyer visits before she stops searching. How does this number change with the parameter *dl*

4. In the model of equilibrium price dispersion, comparative statics results can be obtained by differentiating Equations (16-5) and (16-6) with respect to parameters and then solving the system of equations using Cramer's rule. Use this method to verify that $dx^*/dp < 0$ and $dy^*/dp < 0$ (that is, the fraction of low-price firms and the fraction of informed consumers both fall when the competitive price p rises).

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maximized. Otherwise, some individual could gain by transferring a resource from a lower-valued use to a higher-valued use. In the words of Adam Smith[†]

> As every individual... endeavors as much as he can both to employ his capital in the support of domestic industry, and so to direct that industry that its produce may be of greatest value, every individual necessarily labors to render the annual revenue of the society as great as he can.... [He] is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention.

The assertion of exhaustion of gains from exchange subject to fixed resource constraints is stated mathematically as

maximize

$$Z = \sum^{n} p_j y_j = \sum^{n} p_j f^j(x_{1j}, \ldots, x_{nj})$$

subject to

$$\sum_j x_{ij} \le x_i \qquad i = 1, \ldots, m \tag{17-1}$$

We shall investigate this very general model in several stages of simplification. Consider first the model reduced to only *two* goods and *two* factors. Let us denote the factors labor L and capital K. Let the labor and capital allocated to industry j be denoted L_j and K_j, respectively. Then we can write this reduced model as follows:

maximize

$$p_1 f^1(L_1, K_1) + p_2 f^2(L_2, K_2)$$

subject to

$$L_1 + L_2 \le L \qquad K_1 + K_2 \le K \qquad L_1, L_2, K_1, K_2 \ge 0 \tag{17-2}$$

Here, L and K are, respectively, the parametrically "fixed" total resource endowments of labor and capital.

Of perhaps greater significance is another simplification commonly introduced into the analysis, viz., the assumption that $f^1(L_1, K_1)$ and $f^2(L_2, K_2)$ exhibit *constant returns to scale.* That is,

$$f^j(tL_j, tK_j) \equiv t f^j(L_j, K_j) = t y_j \qquad j = 1, 2$$

Since this relation holds for *all* t, let $t = 1/y_j$. Then the production relation becomes

†Adam Smith, *Wealth of Nations,* Bk. IV, Chap. 2, 1776. Reprinted by Modern Library, 1937 (New York).

If we let $a_L j = Lj/yj$ and $a_{Kj} = Kj/yj$, the production functions become

$$\boldsymbol{f^j(a_{Lj},a_{Kj}) = 1}$$

This construction indicates that constant-returns-to-scale production functions are completely described by the knowledge *of one* isoquant, here, the unit isoquant, i.e., the isoquant representing 1 unit of output. This is as should be expected, since the level curves of linear homogeneous functions are all radial blowups of each other.

The quantities a,-;- represent the amount of resource *i* used to produce 1 unit of output *j*. They are often known as *input-output coefficients.* They are in general considered to be *variable,* changing continuously over a wide range (0 to +00 in the case of isoquants that are asymptotic to each axis).

When this formulation of the production functions is used, the model expressed in Eqs. (17-2) can be transformed into

maximize

$$z = Piyi +$$

Piyi subject to

$$a_L \backslash y \backslash + a_{L2} y_2 < L$$

$$f \backslash a_{Ll}, a_{Kl}) = l \qquad \text{(17-3)}$$

$$f^2(a_{L2},a_{K2}) = 1$$

$$y \backslash , yi > 0$$

In this form, the theorems of international trade theory (the factor price equalization, Stolper-Samuelson, and Rybczynski theorems) are derivable. We shall derive these theorems at this level of generality in the next chapter. In this chapter, however, we shall consider this model in a still simpler framework, that of fixed-coefficient technology. In so doing, we shall develop the body of analysis known as *linear programming* and illustrate its empirical usefulness. By fixed coefficients we mean the very special case where the a,;'s are constants —fixed, as it were, by nature at preassigned values. In this case, the model described by Eqs. (17-3) becomes a *linear programming* problem:

maximize

$$z — P \backslash y \backslash + piyi$$

subject to

$$a_L \backslash y \backslash + a_{L2} y_2 < L$$

$$a_K \backslash y \backslash + a_{K2} y_2 < K \qquad y \backslash , y_2 > 0$$

Here, $p \backslash$, p_2, L, K, and all the a,-/s are constants. The problem becomes one of maximizing a linear function subject to linear inequality constraints, hence, a *linear*

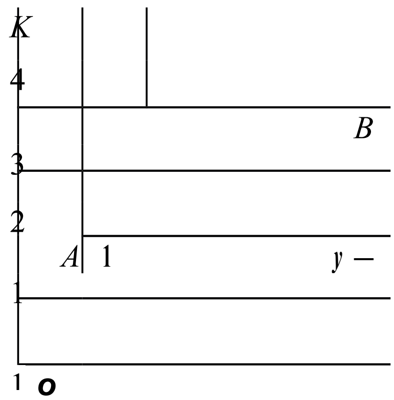

FIGURE 17-1
Fixed-Coefficient Production Functions. The assumption that input-output coefficients of production are fixed leads to L-shaped isoquants. Production of 1 unit of output *requires* a certain amount of each factor. In the figure production of 1 unit of output requires 1 unit of labor and 2 units of capital. This input combination is labeled as point *A*. With 2 units of *K* available, no additional output occurs if more than 1 unit of labor is added. Thus, the isoquant is horizontal to the right of A, and $MP_L = 0$. Similarly, the isoquant is vertical above *A,* since $MP_K = 0$ for $L = 1$, $K > 2$. If both labor *and* capital are increased in the same proportion, then (since the input-output coefficients are constants) output will rise by the same proportion. Hence, this function is a special case of constant-returns-to-scale production functions. For example, in the figure when $L = 2$, $K = 4$, output is $y = 2$.

programming problem. (In general, one could deal with several goods and factors.) Even in this highly restrictive form, the model is capable of yielding insights into the general equilibrium economy. This will be the object of this chapter. Let us first examine more closely the nature of fixed-coefficient technology.

Production functions of this type were discussed briefly in Chap. 9. They can be represented as

$$y_j = \min\left(\frac{L_j}{a_{Lj}}, \frac{K_j}{a_{Kj}}\right) \tag{17-4}$$

The isoquants of this production function (the term is applied loosely) are L-shaped, as depicted in Fig. 17-1. For example, consider the function $y = \min(L/1, K/2)$; 1 unit of output can be produced using 1 unit of labor and 2 units of capital. Thus, $a_{L1} = 1$, $a_{K1} = 2$. If either factor is increased, holding the other factor constant, output remains the same. For example, if L is increased to 2 units, holding K at 2, then

$$y = \min\left(\frac{2}{1}, \frac{2}{2}\right) = 1 \text{ unit of } y$$

The marginal products of labor and capital are never simultaneously nonzero. For example, to the right of point *A,* $MP_L = 0$, since additional amounts of labor yield no increases in output. However, increases in capital will yield increases in output there, and hence $MP_K > 0$. At point *A,* the marginal products of labor and capital are undefined. Any movement at all to the right of A, no matter how small, will yield MP^ > 0. To the left of A, MP^ = 0; however, $MP_L > 0$. Thus, this production function yields discontinuous marginal product functions. The points of discontinuity are the corners of the isoquants, where the marginal technical rate of substitution, MP_L/MP_K is undefined, since no unique slope of the isoquant exists there.

Constancy of the a^-'s is clearly a highly restrictive assumption about productive processes. It can be generalized somewhat by assuming that the firm is faced with not one but several (though finite) distinct production coefficient possibilities, called *activities.* That is, suppose, in addition to the input-output coefficients (a_L\, a_K\) = (1, 2) as in the example in Fig. 17-1, the firm could produce 1 unit of output with the coefficients (b_{Li}, b_Ki) = (3, 1), or (c_{L1}, c_K\) = (2, 3). This situation is depicted in Fig. 17-2. The three activities are denoted A, *B,* and C, respectively. Let us now assume, in addition, that *the firm can choose to use one or more processes, or activities, simultaneously.* That is, assume that these activities are not mutually exclusive but can be used simultaneously side by side.

Suppose, for example, 1 unit of output was to be produced by producing | unit using activity A and \ unit by activity *B.* To produce \ unit of output by A

o

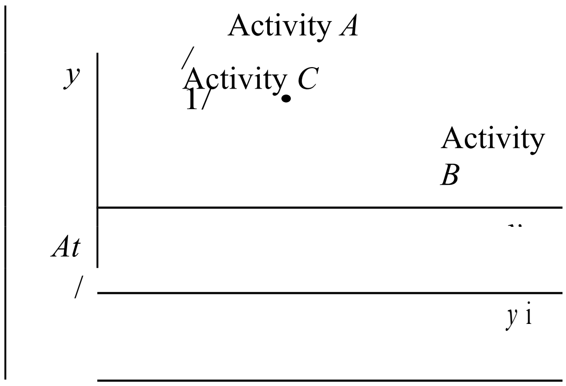

FIGURE 17-2
Several Production Activities. The firm has three distinct production activities, or technologies, available to it. The first, represented by point A, is the technology discussed in Fig. 17-1, wherein 1 unit of output requires 1 unit of labor L and 2 units of capital K. However, the firm can also produce 1 unit of output using 3 units of Land 1 unit of A" (point 5), or 2 units ofL and 3 units of K (point C). In addition, the firm can use any *convex combination* of these processes. That is, any input combination whose coordinates lie along the straight line joining any two processes is also feasible. Thus, 1 unit of output can be produced by using 2 units of L and 11 units of K (point D). This is accomplished by using processes A and B simultaneously, at half the unit output rate each. If process A is used k percent and B is used 1 — k percent, the process $kA + (1 — k)B$ is generated, represented geometrically by the line segment AB.

It is clear that process C will never be used at positive factor prices. There is a process E utilizing activities A and B in some proportion (what proportions?) which yields 1 unit of output using less of *both* labor and capital than activity C. The unit isoquant for this firm is therefore a vertical line above A, the segment AB, and a horizontal line to the right of B. These points represent minimum combinations of L and K needed to produce 1 unit of output. Constant returns to scale imply that all other isoquants will have corners along the rays OA, OB.

will require ^ unit of labor and 1 unit of capital (halfway along the ray from the origin to point A). Similarly, using the *B* activity coefficients, producing | unit of output will require 1 ^ units of labor and \ unit of capital. Together, then, production of 1 unit of output using activities A and *B* together at equal (half) intensities will require 2 (= \ + 1^) units of labor and \\ (= 1 + 0 units of capital. Geometrically, this new composite activity $D = (d_L\backslash, d_K\backslash) = (2, 1\text{^})$ lies midway on the straight line joining points A and *B*. In fact, if each original activity can be used in any proportion with another, then 1 unit of output can be produced by the activities defined by the coordinates of all points lying on the straight line joining the original activity levels. Producing 1 unit by using A to produce \ unit and *B* | unit will be represented by the point three-fourths of the way toward *B* on the line segment *AB*.

Algebraically, this is represented as follows. Suppose A = (L^A, K^A) and B — (L^B, K^B) are any two processes that yield 1 unit of output. Points A and *B* are two points on the unit isoquant. Under the assumptions of constant returns to scale and complete divisibility of these processes, 1 unit of output can be produced using any *weighted average* of processes A and *B* as long as the weights sum to unity. Thus, 1 unit of output can be produced using x = £A + (1 — *k)B,* where $0 < k < 1$. That is, $L^x = kL^A + (l -k)L^B, K^x = kK^A + (l -k)K^B$. As Ovaries from 0 to 1, x traces out the points on the straight line joining A and *B*.

More generally, suppose that $x^1, ..., x^m$ represent *m* points in *n* space. The set of points x = XX=i^x' sucn that $k_t > 0$, ^fc,- = 1, is called a *convex combination* of $x^1, ..., x^m$. The convex combination of points A, *B,* and C in Fig. 17-2 would be represented by all points within and on the boundary of a triangle formed by joining points A, *B,* and *C.* In linear models of this type the assumption is generally made that the convex combinations of unit processes are all alternative processes for the firm to consider.

It is clear from the geometry in Fig. 17-2, however, the activity C will never be used by a cost-minimizing firm. Point *E* on the line *AB,* representing some mix of the activities A and *B,* leads to production of 1 unit of output using less of both labor and capital than point C. Activities A and *B* dominate activity C. Activity *C* becomes irrelevant because it will never be observed.

The unit isoquant for a firm endowed with activities A, *B,* and *C* is therefore the kinked line consisting of (1) the vertical segment emanating from point A (designated *Ayi,* though this line proceeds to infinity), (2) the line *AB,* and (3) a line horizontal from *B,* denoted *By[.* Since these production activities are linear homogeneous, isoquants for other levels of production will be radial blowups of this unit isoquant, with the kinks or corners along extensions of the rays *OA, OB.* Most important, an isoquant map that is convex to the origin has been obtained.

Consider now the nature of a cost-minimizing solution, say for y_t — 1 (perfectly representative, due to homotheticity), as depicted in Fig. 17-3. The slope of line segment *AB* is -|. Thus, along this segment the ratio of the marginal product of labor to that of capital is constant at *MPJ* MP^ = \. Let

w be the wage rate of labor and *r* be the "rental" rate on capital

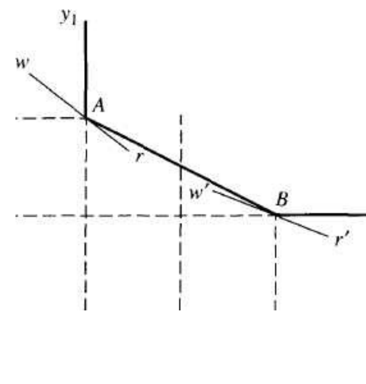

FIGURE 17-3 *Cost Minimization with Linear Segment Isoquants.* When isoquants consist of straight line segments

with corners, as generated by constant-coefficient production functions with more than one technology, the cost-minimizing solution will not in general be the tangency condition $MP_LIMV_K = w/r$. This occurs simply because the isoquant will have only a finite number of slopes (above, only three: —oo, — |, 0), whereas w/r can vary continuously from +00 to 0. For finite factor prices, only if w/r — j will an actual tangency occur. If w/r > ^, the cost-minimizing solution will be at point A. If w/r < ^, the solution will be at point *B*. If w/r = ^, all points along the line segment *AB* will be cost-minimizing solutions. Notice that changes in the factor-price ratio will often produce no change at all in the input combination. Only if in changing, w/r passes through the value of \ (even ever so slightly), will the input combination change. And any such small change around the value \ will produce a *discontinuous* change in the factor mix.

then the ordinary tangency condition for cost minimization is that $MP_L/MPK = w/r$. Here, however, if the wage rate is ever so slightly more than half the rental rate, so w/r > ^, the cost-minimizing solution will occur at point *A*. This will be where an isocost line, *wr* in Fig. 17-3, just touches the isoquant *y\ABy[*.

Only the relatively capital-intensive activity will be used. If the wage rate falls so that w/r < |, as depicted by the isocost segm ent *w'r',* the cost-minimizing solution jumps to point *B.* With this shift in factor prices, only the labor-intensive activity, or process, will be used. We see that small changes in factor prices, from \ + *e* to ~ — e, where *e* > 0 can be as small as one likes, can produce a substantial change in the factor mix. When *w/r* = ^, the isocost line will be tangent to the isoquant along the whole segment *AB,* producing an infinite number of solutions.

Let us formulate algebraically the problem of minimizing the cost of achieving some parametric level of output $v°$, give n the three possible activities above. The objective function is to minimize $z = wL + rK$, where w and rare parametric factor prices and L and K are the total amounts of labor and capital used. Labor and capital can be used in any of the three processes. Denote the amounts of labor allocated to processes A, B, and C as L_A, L_B, and L_c, respectively, and likewise K_A, K_B, and K_c for capital. Thus, $L = L_A + L_B + L_c$, and $K = K_A + K_B + K_c$.

If process A is used, with coefficients (1, 2), then $L_A = K_A/2$ and $y_A = L_A$. Likewise for process B, $L_B = 3K_B$ and $y_B = L_B/3$. For process C, $L_c/2 = K_c/3$

and y_c — $L_c/2$. The problem can thus be posed as minimize

$$w(L_A + L_B + L_c) + r \quad L \quad 3 \quad ,$$

subject to

$$\text{^ \_1\_\_?. \_i\_£.} > \text{-}y^0 \quad £, \quad \text{^o } L_r > o \qquad (17\text{-}5)$$

Although the problem posed in (17-5) involves only simple linear equations, it is not at all trivial to solve. The Lagrangian is

$$X = w(L_A + L_B + L_c) + r\backslash 2L_A \text{ -\textbackslash-------1—} L_c) + x(y^\circ — L_A \text{-----} $$

producing the first-order relations

w + 2r $X > 0$ i $> L$ =
$r\ w$ $X > 0$ i $> L$ =
3 X
2

(17-6)

$$y^\circ - L_A - \text{^} - \text{^} < O \quad \text{if} <, A. = O \qquad (17\text{-}7)$$

The Lagrange multiplier A is interpretable as marginal cost as in the neoclassical case. Since MC = w + 2r if process A is used, MC = 3w> + r if process B is used, and MC = 2vv + 3r if process C is used, relations (17-6) say that at the cost minimum point the only process that will be used is the process for which MC is minimized (unless several are equally minimal). Of course, for linear homogeneous production functions, of which this is a special case, MC = AC, and thus this procedure minimizes total cost. But this does not help us much in actually *finding* the solution, i.e., finding which process or processes to use. Problems of this type require solution by *algorithm.* That is, some iterative procedure is required to approach the solution in a finite number of steps. This algorithm must be able to reveal which first-order conditions are in fact binding and which are to be ignored. This is not usually possible without some search procedure, in which changes in the variables that move the program closer to solution are revealed as the algorithm, or routine, is carried out.

The assumption of constant coefficients in any model of economic behavior can generally be counted on to be in violation of the facts over some finite time period. The assumption may nonetheless be useful, however, if it enlarges the tractability of the model. As pointed out in Chap. 1, assumptions are always simplifications of reality by definition and are incorporated into the analysis to improve the manageability of the model or theory. In the case of linear models, the benefit of this assumption is that a well-established, easy-to-use algorithm exists for finding the solution to the model. Reality of assumptions is traded off for tractability—in this case, actually obtaining solutions. We shall now investigate this class of models and their solution.

17.2 THE LINEAR ACTIVITY ANALYSIS MODEL: A SPECIFIC EXAMPLE

In this section we shall investigate a particular linear programming problem and use it to illustrate the general nature of such models. In the next section, the general theorems and methodology will be presented.

Consider a firm (or an economy made up of many such identical firms) that can produce two goods, food *yi* and clothing y_2. Let us now suppose that *three* inputs are used to produce these outputs: *H,* land; L, labor; and *K,* capital (to use an historically important but misleading taxonomy). These inputs must be combined in fixed proportions to produce 1 unit of either food or clothing. In particular, assume that to produce 1 unit of food requires 3 acres, 2 worker-hours, and 1 "machine" (unit of capital). To produce 1 unit of clothing requires 2 acres, 2 worker-hours, and 2 machines (the same machines as for food). This technology, or state of the art, is representable by the following input-output matrix A:

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 2 \end{pmatrix} \qquad (17\text{-}8)$$

where *a^* = amount of factor / used to produce 1 unit of output *j*

$$i = H, L, K \qquad 7 = 1, 2$$

Since three factors and two outputs are involved, the resulting input-output matrix contains three rows and two columns. These coefficients are *fixed,* i.e., constant at their given values. No other processes or activities are available to this firm or economy, t

Let us assume now that this firm is endowed with 54 acres of land, 40 worker-hours, and 35 machines, representing the resource constraints. Further, assume that food sells for \$40 per unit and clothing for \$30 per unit; that is, *p* = \$40, p_2 = \$30.

tMore general models, called *Leontief input-output models,* after their inventor, Wassily Leontief, allow variables to be *both* the objects of final consumption (outputs) *and* inputs. For example, some food will likely be used in the production of clothing, and vice versa. The matrix of coefficients would have to be suitably expanded. If B = *(btj)* represents the amount of outputs *i* used to produce 1 unit of output;, if C = *(c\, ... , cn)* represents a vector of final consumption of these outputs, and if X = *{x\, ..., xn)* represents total production of these goods, then, by arithmetic, ^"_j *bijXj* + c, =*,-,/ = 1,...,«. In matrix form, BX + C = X, or C = (I — B)X, where I represents the *n* x *n* identity matrix. The production X needed to sustain consumption C is X = $(I - B)^{-1}$C. It can be shown that the inverse (I - B)"[1] exists only if (I — B)"[1] =I + B + B^2 + - - - i s a convergent series, analogous to the sum of an infinite geometric series in ordinary algebra.

We now assert that this firm or an economy made up of many such firms maximizes the total value of output of food and clothing, subject to the constraints imposed by the scarcity of resources (factors) and the nonnegativity of factors. The mathematical statement of this model is

maximize

z = 40jj +

$30y_2$ subject to

$3yi + 2y_2 < 54$ land constraint

$2yi + 2y_2 < 40$ labor constraint (17-9)

$yi + 2y_2 < 35$ capital constraint $y\backslash, y_2 > 0$

Mathematically, the problem is to maximize a linear function subject to linear inequality constraints and nonnegativity of the decision variables. The constraints in (17-9) say that no more than 54 acres, 40 worker-hours, and 35 machines may be used by this firm. However, it is possible to use *less* than these amounts. That is, the firm is not bound to use all its resources. And we can quickly see that it cannot be the case that all three factors will be fully utilized; in that case, the constraints in (17-9) would represent three independent equations in two unknowns, yielding no solution. How shall we discover which resources to utilize fully?

A graphical solution. The solution to this linear programming problem can be obtained graphically, since only two decision variables, $y\backslash$ and y_2, are present. In Fig. 17-4, coordinate axes have been drawn, with y^ the abscissa and y_2 the ordi-nate. Since outputs are constrained to be nonnegative, only the first (nonnegative) quadrant of Euclidean space is relevant. If all land were to be used, the combination of food and clothing that could be produced would satisfy 3 ji + $2y_2$ = 54, a strict equality. This is the line denoted "land" in Fig. 17-4, having horizontal intercept 18, vertical intercept 27. However, the constraint says merely that *not more than 54* units of land may be used, and possibly less. Hence, the set of all output combinations possible under the land constraint is the triangular area bound by this line and the positive axes. It is not possible to obtain output combinations outside this triangle.

However, the preceding is only one of three constraints (aside from nonega-tivity). In addition, no more than 40 worker-hours are available. This implies that no output combinations are attainable which lie outside the line $2y\backslash + 2y_2 = 40$ (denoted "labor" in Fig. 17-4). Output is further constrained to lie within (boundaries included) the triangle defined by this line and the nonnegative axes. Lastly, since only 35 machines are available, all output combinations must lie within the triangle whose boundaries are the nonnegative axes and the line $y_x + 2y_2$ — 35.

The set of all output combinations that satisfy all the constraints (including nonnegativity) is called the *feasible region* for the problem. The feasible region for this problem is the shaded polygonal area *OABCD* in Fig. 17-4. It is called the feasible

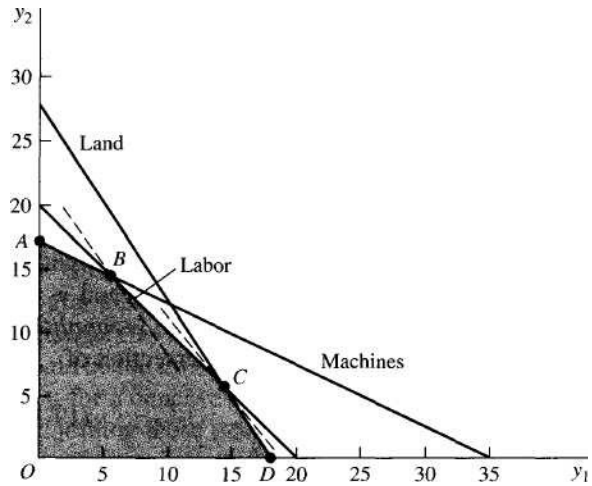

FIGURE 17-4
Graphical Solution of a Linear Programming Problem. The feasible region of a linear programming problem is the set of points that satisfy all the constraints (including nonnegativity) simultaneously. In the above model, this is represented by the region *OABCD.* Along *AB,* the capital, or machine, constraint $y_1 + 2y_2 = 35$ is binding, but the land and labor constraints are nonbinding. Those two constraints lie above and to the right (except at point *B)* of the capital constraint there. At point *B,* both the capital and labor constraints are binding, while along *BC* (excluding point C), only the labor constraint is binding, i.e., holds as an *equality.* Similarly, along *CD* (excluding *C)* only the land constraint holds as an equality. The feasible region is a *convex set.* If *any* two points in *OABCD* are selected, all points on the straight line joining those two points also lie in *OABCD.*

The (maximum) solution of the linear programming problem occurs where the isorevenue line $z = 40y_1 + 30y_2$ is shifted as far from the origin as is consistent with its remaining in contact with the feasible region *OABCD.* This occurs, for this problem, at point C, where the labor and land constraints are binding and the capital constraint is nonbinding. The slope of the isorevenue line is $p_1/p_2 = 40/30 =$ |. This is *greater* than the slope of the production possibilities frontier along *BC* (1) and less than the slope along *CD* (|). No tangency (in the sense of equal slopes) occurs. Instead, the inequality conditions, that for each good, MR < MC for increases in output and MR > MC for decreases in output, hold at point C. The maximum is global because *OABCD* is a convex set. There are no corners jutting out at points removed from C.

region simply because this set of points represents all possible outputs, given the constraints imposed by scarcity of resources.

The feasible regions of all linear programming problems have the important property of being *convex sets.* Recall from Chap. 14 that a convex set is one in which the straight line joining any two points in the set lies entirely in the set. The feasible region is convex here because the constraints, being linear, are therefore *concave* functions (though weakly so). As we saw in Chap. 14, if $g^j(x_1, \ldots, x_n)$ is a concave function, the area defined by $g^j > 0$, $j = 1, \ldots, m$ is a convex set. We shall show this again, in more detail, for linear functions in the next section. The objective function, also linear, is therefore also concave. Therefore, we know that if a *local* maximum of this function is achieved over the (convex) feasible region, it is also a *global* maximum. And if the maximum is not unique, there are an infinite number of local maxima, all equal, along the straight line joining any two maxima.

Let us now move on to the solution of the problem. The objective is to maximize the linear function z = 40ji + $30y_2$ such that the point (ji, y_2) lies in the feasible region *OABCD.* This isorevenue line (hyperplane, in more than three dimensions) achieves higher values the farther it is from the origin. This isorevenue line has a slope = — |. One such line is the dashed line through point *B.* It clearly is not the maximum z, which is in fact obtained when z = 40_y! + $30y_2$ passes through point *C.* At point *C,* only the land and labor constraints are binding. Point *C* lies wholly within the machine constraint, that is, *y* + $2y_2 < 35$. The solution values of *yi* and y_2 are thus obtained by solving simultaneously the land and labor constraints, *as equalities,* ignoring the machine constraint entirely. That is, point *C* is the intersection of

$$3_{yi} + 2y_2 = 54 \quad \text{and} \quad 2y_x + 2y_2 = 40 \qquad (17\text{-}10)$$

Solving simultaneously quickly yields the solution values $y^* = 14$, *y* % = 6, with $_z{}^*$ = 40(14) + 30(6) = $740.

We can also determine the allocation of factor resources to each industry or final good. Since *yi* = 14, the amounts of land, labor, and capital used to produce food, *yi,* are *auyi,*a_2*\y\,* and *a^yi,* respectively, or 42 acres, 28 worker-hours, and 14 machines. For clothing, y_2, the resource requirements are $a_{i2}y_2$, $a_{22}y_2$, and «32j2, respectively, or, since $y_2 = 6$, 12 acres, 12 worker-hours, and 12 machines. All together, 42 + 12 = 54 acres, 28 + 12 = 40 worker-hours, and 14 + 12 = 26 machines are used. Land and labor are fully employed and machines are only partially employed, as indicated by the observation that the machine constraint is the only nonbinding constraint.

Geometrically, it is clear that point C provides the maximum value of *z.* It is also visually obvious for this problem that *C* is the *global* maximum of *z.* This latter statement is a consequence of *OABCD* being a convex set and z = 40_yi + $30y_2$ being a quasi-concave function. The impact of the theorem on maximization of such functions over convex sets is geometrically clear in this example. (Remember, of course, that one example proves little!)

That point *C* is the solution is clear from economic reasoning as well. The boundary of the feasible region, the broken line *ABCD,* is the production possibilities frontier for this firm or economy. The slope of this frontier represents the marginal cost of obtaining more *y*. Along line segment *AB,* this marginal cost is | unit of y_2. The 5 units of *y* produced at *B* are achieved at the expense of 2^ units of y_2 (17^ at A minus 15 at *B).* However, the marginal *benefit* of producing y_x is given by the slope of the isorevenue line, or | unit of y_2. Since MR > MC for y_x, it pays to increase production of *y*. When point *B* is reached, the marginal cost changes discontinuously. Along the segment *BC,* marginal cost = 1. This is still less than marginal revenue, hence it pays to move all the way along *BC* to point *C.* However, it does not pay to move beyond point C Along *CD,* MC of *y* is |, which is greater than MR = |. There is no point on this frontier where MC = MR. However, there is a point *C* for which, in terms of *yi*, MC < MR to the left of *C* and MC > MR to the right of *C*. That is, at point *C, y* = 14. For y_ℓ < 14, MC < MR, whereas for *yi* > 14, MC > MR. (Marginal cost is undefined at y_ℓ = 14, since the frontier has a corner there; i.e., the frontier is nondifferentiable there.) We can thus expect no marginal *equalities* as defining the extreme values of the objective functions, but we can expect a series of marginal *inequalities,* indicating that a move in any direction will leave the firm or economy in a lower-valued option.

There is one instance in which marginal equalities do occur. Suppose *p* had been \$30 instead of \$40. Then the MR of *y* would be unity. This is precisely MC all along segment *BC.* In this case there would not be one solution to the linear programming problem but an infinite number of solutions. The isorevenue line would be tangent to the feasible region at all points along *BC;* all those points therefore would be the solution of the problem. This is the situation where if two local maxima exist for a concave function defined over a convex set, all points along the straight line joining those maxima are also maxima.

To economists, the interest in this problem goes beyond the mere attainment of a solution. We have seen that in constrained maximization models, new variables, the Lagrange multipliers, imputed some sort of value to the constraint, e.g., the marginal utility of money income, or marginal cost of production with resources. These imputed values are present here also, though we have not yet expressly introduced them as Lagrange multipliers in the analysis.

Consider that since land is scarce, output is not as large as it otherwise would be. In particular, suppose this firm or economy possessed 55 acres of land instead of just 54. How would this affect the value of output? The maximum value of output would now occur at a new point *C,* the intersection of the constraint boundaries

3^ + $2y_2$ = 55

land $2y_x + 2y_2 = 40$

labor

Solving, we get *y** = 15, *y%* = 5. The value oftotal income, *z*,* is now *z** = 40(15) + 30(5) = \$750, a gain of \$10.

The fact that income would grow by \$10 if one additional acre of land were available imputes a value, a *shadow price,* or *imputed rent* as it is called, to land. Clearly, the marginal value product of land is \$10. In a competitive economy, this land would rent for \$10 per acre. Moreover, the marginal value of land will remain constant as long as the maximum value of income is determined by the intersection of the land and labor constraints only.

Suppose land is increased to 54 + *AH.* Then the value of total income is determined using the solution of

$3y_\ell + 2y_2 = 54 + AH \qquad 2y_x + 2y_2 = 40 \qquad$ (17-11)

Thus

y\ = 14 + AH   y* = 6 - AH

Formerly, $z^* = 740$. The new value of z^* is

$$z^* + \Delta z^* = 40(14 + \Delta H) + 30(6 - \Delta H) = 740 + 10\,\Delta H$$

Thus, $\Delta z^* = 10\,\Delta H$, or, taking limits,

$$\frac{\partial z^*}{\partial H} = 10 \qquad (17\text{-}12)$$

This is precisely the envelope theorem, which says that the rate of change of the objective function (here z^*) with respect to a parameter representing a resource constraint is the imputed value of that resource.

We shall denote this imputed value, or shadow price, of the first factor, land, as

$$u_1 = \frac{\partial z^*}{\partial H} = 10 \qquad (17\text{-}13a)$$

Note that u_1 is *constant* at $10. It does not depend on the parametric values of either the land or labor resource constraints, 54 acres and 40 worker-hours, respectively. This result is the basis of what is known as the Stolper-Samuelson and factor price equalization theorems. In the next chapter we shall show that for the general case of linear homogeneous production functions (of which fixed coefficients are a special case), factor prices are functions of output prices only. Hence, the preceding result, that the factor price of land is constant (subject to the qualification below), independent of the resource endowments in some neighborhood of the initial solution.

We shall see more explicitly how the factor prices become Lagrange multipliers (called *dual variables* there) in the next section. Note, however, in the solution to Eq. (17-11), that ΔH must be less than 6; otherwise y_2 would become negative. The algebraic algorithm for solving these problems is vitally dependent on these changes. When ΔH becomes greater than 6, the solution will move to a new corner of the feasible region and the marginal valuation of resources will change. (In fact, the marginal value of land will fall to zero. Why?)

In a similar manner, we can determine the imputed wage rate in this "economy." If the amount of labor is increased by an amount ΔL, the new value of output is determined by

$$3y_1 + 2y_2 = 54 \qquad 2y_1 + 2y_2 = 40 + \Delta L$$

Thus

$$y_1^* = 14 - \Delta L \qquad y_2^* = 6 + \tfrac{3}{2}\Delta L$$

Therefore,

$$z^* + \Delta z^* = 40(14 - \Delta L) + 30(6 + \tfrac{3}{2}\Delta L) = 740 + 5\,\Delta L$$

Subtracting $z^* = 740$, we have $\Delta z^*/\Delta L = 5$, or, taking limits,

$$u_2 = \frac{\partial z^*}{\partial L} = 5 \qquad (17\text{-}$$

13/7)

oL

The marginal value of labor is \$5. A competitive economy would result in labor receiving this wage. Here, since $y^* = 14 — AL$, AL < 14. If labor increased by more than 14 worker-hours, the marginal values of the factors would change since a new corner of the feasible region would be reached. (In fact, the marginal value of labor would fall to 0 if AL > 4.5. Why?)

Lastly, consider what the effects on z^* would be if more machines AK were available. The machine constraint is already nonbinding. Only 26 machines are used, in spite of 35 being available. An additional machine would add nothing to income; its marginal value is 0. Machines, though limited, are not scarce. They are a free good, being available in greater supply than the quantity demanded at zero price. Thus,

$$\hat{}3 = \% = 0 \qquad (17\text{-}13c)$$

Do not assume that since the *marginal* evaluation of capital (machines) is 0 capital is redundant in this economy. In fact, 26 machines are used. Moreover, with fixed coefficients, production, by definition, is impossible without certain amounts of each factor. The *marginal* product of capital is 0 because the nine *extra* machines are incapable of being combined productively with the available land and labor. This is a feature of fixed-coefficient technology. The *total* product of capital is certainly not zero. Capital is merely redundant at the particular margin in question, where $y_1 = 14$, $y_2 = 6$.

The preceding phenomena are special cases of the factor price equalization theorem. With unchanging output prices, factor prices remain the same when endowments are changed. This holds until endowments change sufficiently to cause new factors to be brought in and one or more formerly positively used factors to fall from use.

17.3 THE RYBCZYNSKI THEOREM

Let us now consider the effects on output of changes in the land H and labor L endowments. We found $y^* = 14 + AH$, $y^* = 6 — AH$, and y* = 14 — AL, v| = 6+| AL. Combining these into one total differential expression gives

$$dy^* = dH - dL + - dL \qquad dy^* = -\overset{3}{d}H$$

These results are examples of the *Rybczynski theorem.* This important theorem says that if the endowment of some resource increases, the industry that uses that resource most intensively will increase its output while the other industry will decrease its output. The relative factor intensity is measured by the ratio of factor use in each industry. For example, $Li/K\backslash$, L_2/K_2, the labor-capital ratios in industry 1 and industry 2, are compared. The industry for which this ratio is higher is relatively labor-intensive; the other is relatively capital-intensive. However, note that

$$a_{Kj}$$

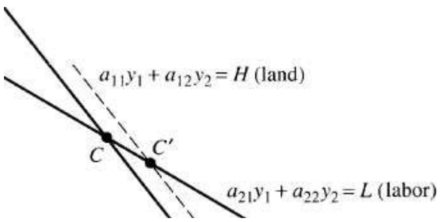

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FIGURE 17-5
The Rybczynski Theorem in a Linear Model, Showing the Land and Labor Constraints. The capital constraint, being nonbinding, is omitted. Maximum income occurs at point *C*. The slope of the land constraint is —*a\\ /an* — — \ in our model. The slope of the labor constraint is — *a₂\ /a-n* = — 1 ■ Since *a\\la\2 > a2\Ia-n, a\\Iai\ > a^laxi,* or industry 1, food, is /and-intensive. Industry 2, clothing, is labor-intensive. Notice how the solution to the problem changes when the amount of land //is increased. The land constraint shifts to the right, producing a new output mix designated by point *C*. At *C,* output of >'i, the land-intensive good, is increased, while the output of the labor-intensive good decreases.

Hence, the factor intensities can also be determined by the ratios of the a_{ij} 's, in the appropriate manner.

In the present example, the only two factors relevant at the solution point are land and labor. The food industry ;yi is relatively *land-intensive,* since 011/021 > 012/022, that is, I > |. Clothing y_2 is relatively *labor-intensive.* When the endowment of land *increases* by *dH,* food production is *increased* by *dH* while clothing production actually *decreases,* also by *dH.* On the other hand, if the endowment of labor were to *increase* by *dL,* the labor-intensive industry y_2 would *increase* by I *dL,* whereas food output would *decline* by *dL.* These results are shown graphically in Fig. 17-5.

Algebraically, the Rybczynski theorem results from the simple solution of simultaneous equations. In our example, the solution values of the model are determined by the land and labor constraints,

$$+ a_{n}y_2 = H \qquad \text{land} \qquad (17\text{-}14a)$$

$$+ a_{22}y_2 = L \qquad \text{labor} \qquad (17\text{-}146)$$

Solving by Cramer's rule yields

$$yi = \frac{a_{22}H - a_{n}L}{011022 - 012021} \qquad y_2 = \frac{-a_{21}H + a_{u}L}{0)1022 - 012021} \qquad (i/\text{-}ijj$$

The factor intensities determine the sign of the denominator. If say, the food (>>i) industry is relatively *land-intensive,* then *H\/L* > H_2/L_2. This is equivalent to 011/021 > 012/022 or0ii0₂2— 012021 > **0. (It is clearly critical that «i** i«22—012021 7^0. Otherwise, with equal factor intensities, the constraints determining the solution would be parallel to each other, i.e., linearly dependent. In that case, no solution to the model in its present form could exist; one constraint would have to be discarded as nonbinding.) If the denominator is positive, then by simple differentiation,

$$\frac{dy_1}{dH} = \frac{a_{22}}{(+)} < 0 \qquad \frac{dy_2}{dL} = \frac{a_{11}}{(+)} > 0 \qquad (17\text{-}16)$$

the Rybczynski results. Moreover, when some factor endowment changes, the output that is intensive in that factor will change in *greater* absolute proportion than the parameter change. For example,

$$y_1 = k_1H +$$

k_2L where

$$k_1 = \text{---------------}$$

$$-a_{12}$$

Thus

$$\frac{\Delta y_1/y_1}{\Delta H/H} = \frac{H}{y_1}\frac{dy_1}{dH} = \frac{H}{y_1} \qquad = \frac{k_1H}{k_1H + k_2L}$$

Since $k_2 < 0$,

$$k_1H > k_1H+k_2L$$

hence

The same result obtains for the change in y_2 with respect to a change in L. Outputs respond *elastically* (absolute elasticity greater than unity) to change in resource endowments in which they are intensive.

17.4 THE STOLPER-SAMUELSON THEOREM

The solution to this linear model will remain at point C in Fig. 17-4 as long as p_1/p_2, the ratio of output prices, is less than | and greater than 1. In our present example, p_1/p_2 = !|j = !. Let us calculate the effect on factor prices produced by an increase in the price of clothing, say, to p_2 = 33, i.e., by 10 percent.

The new shadow factor prices can be calculated as before. However, consider the unit factor cost of y_1 and y_2. The first column of the (a_{ij}) matrix gives the amounts of land, labor, and capital needed to produce 1 unit of y_1. The unit factor cost of y_1 is therefore

$$+ a_{21}u_2 + a_{31}u_3 = 3(10) + 2(5) + 1(0) = \$40 = p_1 \qquad (17\text{-}17a)$$

Similarly, the second column of $a^\wedge$'s gives the amounts of land, labor, and capital needed to produce 1 unit of y_2. The unit factor cost of y_2 is therefore

$$+ \text{tf22"2} + a_{32}u_3 = 2(10) + 2(5) + 2(0) = \$30 = p_2 \quad (17\text{-}176)$$

Unit factor cost equals output price. Equations (17-17) represent zero-profit conditions. Since the production function here exhibits constant returns to scale, zero profits are to be expected. Since in this example *UT,* — 0 (the marginal product of capital is 0) at point C, the zero-profit conditions are

$$p_x$$

$$a_{X2}u_x + a_{22}u_2 = p_2$$

These are two equations in two unknowns, from which we can solve for u_x and u_2 in terms of p_x and p_2, using our data about the a_ifs. [Remember, though, that these equations apply only when the solution is at point *C.* If the solution were at point *B,* u_x would be 0 and Eqs. (17-8) would involve the coefficients a_{3X} and a_{32}.] Most importantly, note that the coefficients of these equations are the transpose of the coefficients in Eqs. (17-14). The only difference is that $a\backslash_2$ and $a_2\backslash$ are interchanged. The algebra of the relations between factor and output prices is therefore virtually identical to the relations between physical outputs and resource endowments. Solving Eqs. (17-18) by Cramer's rule gives

$$u_x = \frac{ai_2p\backslash - a_{21}p_2}{a_na_{22} - a_{x2}a_{2X}} = \frac{2/?, -2p_2}{2} = p_x - p_2 \quad (17\text{-}19a)$$

$$u_2 = \frac{auP2 - a_{12}pi}{a_na_{22} - a_{x2}a_{2x}} = \frac{3p_2 - 2p_x}{2} = \frac{3}{2}p_2 - p\backslash \quad (17\text{-}1\%)$$

Notice the direction of change: if p_x increases (in price of the land-intensive good increases), then the price of land u_x increases while the price of labor decreases. Likewise, if the price of the labor-intensive good y_2 increases, land decreases in price while labor increases in price. With $p_x = 40$, $p_2 = 30$, we derived $u_x = 10$, $u_2 = 5$, in accordance with Eqs. (17-13). With, say, $p\backslash = 40$, $p_2 = 33$, we find $u_x = 7$, $u_2 = 9.5$. With the price of the labor-intensive good rising by 10 percent, the price of land falls by 30 percent, whereas the price of labor nearly doubles. These results are known as the *Stolper-Samuelson theorem.*

The Stolper-Samuelson theorem states that if, say, the price of the labor-intensive good *rises,* the price of labor will not only *rise* but will rise in greater proportion to the output price increase. The price of the other factor *falls,* not necessarily in greater proportion to the rise in output price. The same elastic response that was observed for the physical quantities occurs for prices. This duality is apparent from the similar structures of Eqs. (17-14) and (17-19). Since the algebra is identical (save for interchanging a_{X2} and $a_2\backslash$), we shall not repeat it.

The direction of change of factor prices can be seen geometrically in Fig. 17-6. There, the zero-profit Eqs. (17-17) are plotted in the u_xu_2 plane. The steeper line is Eq. (17-17a). The slope of this line is — a_{xx}/a_{2X} — —|. The less steep line is Eq. (17-17Z?), which has slope —$a\backslash_2/a_{22}$ = — 1. The intersection, point *P,* represents

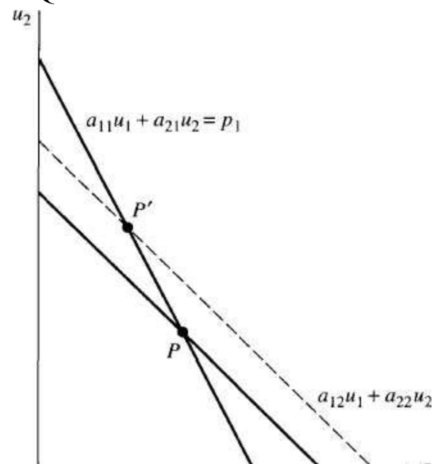

FIGURE 17-6 *The Stolper-Samuelson Theorem in a Linear Model, Showing Plot of the Two Zero-Profit Conditions (17-17).* Capital is nonscarce, with M3 = 0, and is thus omitted from the diagram. The slopes of the two lines depend on the factor intensities. For *y*, the slope of the

zero-profit equation is $-a_{11}/a_{21}$. For y_1, the slope is $-a_{12}/a_{22}$. Given the numbers in our example, $a_{11}/a_{21} = \S$, $a_{12}/a_{22} = 1$, and hence y_1, food, is land-intensive, whereas y_2, clothing, is labor-intensive. Under these conditions, and increase in p_2, which causes a parallel shift in (11-11b), produces a new solution at P'. This lowers u_1 and raises u_2. In words, if the price of say, the labor-intensive good rises, the price of labor will rise while the price of the other factor (here, land) will fall.

5 2

^{2} $y_1, y_1 > 0$

the solution values of u_2 and u_1. Suppose now that p_2 increases. This is represented geometrically by a parallel shift in (11-11b), as shown by the dotted line. The new intersection is at F'. At F, u_2 has increased and u_1 has decreased. Again, an increase in the price of the labor-intensive good will raise the price of labor and lower the price of the other factor, in this instance land.

17.5 THE DUAL PROBLEM

Let us summarize the analysis of the problem just solved. The problem, again, was to maximize

subject to $z = \ldots y$

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The coefficients in the constraints, the a,/s, represent the (constant) amounts of factor *i* used in the production of 1 unit of output *j*. The solution to this particular linear programming problem is $y^* = 14$, y % = 6. At that point, the land and labor constraints are binding, but the capital constraint is nonbinding. That is,

$$3y^* + 2y\backslash = 54$$
$$2y^* + 2v^* = 40$$
$$y^* + 2y^* < 35$$

By considering how much income would change if factor endowments were altered incrementally, marginal evaluations, or shadow prices, were imputed to the three factors. Letting u_1, u_2, and u_3 represent the shadow factor prices of land, labor, and capital, respectively, we found, at point C, where total income was maximized,

$$u_1 = \$10 \qquad u_2 = \$5 \qquad u_3 = 0$$

We also found that the unit factor costs of y_1 and y_2 equaled the output prices, \$40 and \$30, respectively. For food, y_1, the first column of the (a_{ij}) matrix gives the amounts of land, labor, and capital needed to produce a unit of y_1. Thus, the unit cost of y_1 given the preceding shadow factor prices is

$$u_1a_{11} + u_2a_{21} + u_3a_{31} = 10(3) + 5(2) + 0(1) = \$40 = p_1$$

Likewise, for y_2, total unit factor cost is

$$u_1a_{12} + u_2a_{22} + u_3a_{32} = 10(2) + 5(2) + 0(3) = \$30 = p_2$$

The linear model implies the zero-profit conditions:

$$\begin{aligned} a_{11}u_1 + a_{21}u_2 + a_{31}u_3 &= p_1 \\ a_{12}u_1 + a_{22}u_2 + a_{32}u_3 &= p_2 \end{aligned} \tag{17-17}$$

In general, we would expect the unit factor cost to be *greater than or equal to the price* of that good. If unit cost were *less* than the price, no finite solution to the linear programming problem could exist; increasing the output of such a good would lead to ever-increasing total revenues. It might be the case, however, that unit factor cost *exceeded* price in a finite solution. Then we should expect the output of that good to be zero (negative outputs are not admissible).

There is a symmetry, or duality, in the preceding analysis. In the original model, in which income (revenue) was maximized, the constraints were statements of limited resource endowments. When these constraints held as equalities, a nonnegative, generally positive new variable, a shadow factor price, appeared. This "price" was 0 when the constraint was nonbinding, i.e., when a strict inequality appeared, as in the capital constraint above.

However, achieving an actual maximum of revenue also implies that unit costs will be at least as great as output prices,

$$a_{1j}u_1 + a_{2j}u_2 + a_{3j}u_3 \geq p_j \qquad \text{for all } j \tag{17-20}$$

Moreover, revenue maximization implies that if this relation holds as a strict *inequality,* output y_j will be zero. Otherwise, $y_j \geq 0$ (with $y_j > 0$ expected). This is the same "algebra" as when the constraints were the original resource constraints involving y_1 and y_2 and the new, dual variables were the shadow prices u_1, u_2, and u_3. The preceding symmetry, or duality, occurs because the u_i's are the Lagrange multipliers for the original, primal problem, whereas the outputs y_1 and y_2 are Lagrange multipliers for an associated constrained minimization problem. It is an example of the Kuhn-Tucker saddle point theorem. Let us see how this occurs.

Denote the original revenue maximization problem (17-9) as the primal problem. When we let u_1, u_2, and u_3 represent the Lagrange multipliers associated with the three resource constraints, the Lagrangian for this problem is

$$= P\backslash y\backslash + Piyi + u_x(H - a_u y\backslash -$$
$$+ u_2(L - a_{2i}yi - a_{22}y_2) + u_3(K - a_{3l}yi - a_{32}y_2) \quad (17\text{-}21)$$

Among the Kuhn-Tucker first-order conditions for a maximum are that the first partials of ££ with respect to $y_{\{}$ and y_2 be nonpositive, and if 3i£/3j$_7$ < 0, v$_;$- = 0. At an interior solution, *dX/dyj* = 0. If a maximum should occur at *yj* = 0, some *j,* this must happen because the Lagrangian would become larger if *smaller,* i.e., negative, values of _y$_y$ were allowed. Hence, it must be that if a maximum occurs at *yi* — 0, *d!£/dyj* < 0 there. Applying these conditions to the Lagrangian (17-21), we have

$$= p_x - a_n ui - a_2\backslash u_2 - a_{3J}u_3 < 0 \qquad \text{if} <, \quad y_x = 0 \qquad (\backslash l\text{-}22a)$$
$$dy\backslash$$

$$\text{----} = p_2 — a\backslash_2 ux — a_{22}u_2 — a_{32}u_3 < 0 \qquad \text{if} <, \quad y_2 = 0 \qquad (17\text{-}22Z?)$$
$$dy_2$$

These first-order conditions are precisely the nonpositive profit conditions (17-20) just discussed. The remaining first-order conditions are, of course, the inequality constraints, obtained by differentiation with respect to the Lagrange multipliers, here $u\backslash$, u_2, and u_3:

$$\text{----} = H — axxy\backslash — a\backslash_2 y_2 > 0 \qquad \text{if} >, ux = 0$$

$$= L — a_{2l}yx — a_{22}y_2 > 0 \qquad \text{if} >, u_2 = 0$$
$$du_2$$

$$\text{----} = K — a_{3l}yx — a_{32}y_2 > 0 \qquad \text{if} >, u_3 = 0$$
$$du_3$$

These are precisely the resource constraints with the added stipulation that if the resource constraint is nonbinding, the imputed shadow price of that factor is 0, in accordance with our previous reasoning and results.

There is more to the Lagrangian (17-21) than first meets the eye. Let us rearrange the terms of (17-21) as follows:

$$Lu_2 + Ku_3 + y\backslash(pi — ax\backslash U\backslash — a_2xu_2 —$$
$$P2 — anUx — a_{22}u_2 — a_{32}u_3) \quad (17\text{-}23)$$

This functional form can be interpreted as a Lagrangian for an extremum problem with choice variables $u_x, u_2,$ and u_3 whose objective function is w = $Hu\backslash + Lu_2 + Ku_3$, the total value of the resource endowment. The outputs $y_{\{}$ and y_2 appear in the position of Lagrange multipliers for profit constraints.

A competitive economy can be expected to utilize its resources efficiently. We should expect in this model that revenue maximization implies, and is implied by, *minimization of the total value of resources,* subject to the constraints that profits are nonpositive. The Lagrangian (17-23), if *minimized* with respect to *u\,* *u₂,* and W3, yields the following first-order inequality conditions:

----= $H - a_n y\backslash - a_n y_2 > 0$ i f > , « i = 0 (17-24a)

----= $L - a_2 \backslash y \backslash - a_{22} y2 > 0$ if >, w2 = 0 (17-246)

f)SP.

;0 i f > , « 3 = 0 (17-24c)

and

----— $p\backslash — a\backslash\backslash U\backslash — ci2\backslash U2 — ci3\backslash U3 < 0$ if <, _yi = 0 (17-25a)

$Pi— anU\backslash — CI22U2 — CI32U3 < 0$ if <, $\_y_2 = 0$ (17-25Z?)

exactly the same resource and nonpositive profit conditions derived from the primal problem.

Thus, the following two problems yield the same first-order conditions:

1. The primal problem:

   maximize

$$z = piyi + P2y2$$

   subject to

$$+ anyi < H$$
$$+ 022)\text{^}2 < L \ 031 \ yi$$
$$+\text{^}32\text{^}2 < K$$
$$y \backslash , y 2 > 0$$

2. The dual problem:

   minimize

$$w = Hu\backslash + L1A2 +$$

   K113 subject to

$$>Pi$$
$$>Pi \qquad U\backslash,\ U_2,\ U_3 > 0$$

Moreover, the values of the objective functions of these two problems are identical when the solutions are obtained. In this model, this is a statement that the total value of output equals the total value of resource endowment when resources are used efficiently. In the preceding example, the maximum value of output was

$$z^* = {}_{Pl}y^* + p_2 y^* = 40(14) + 30(6) = \$740$$

as computed earlier. The total value of the resource endowment at

that point is $w^* = Hu\backslash + Lu\backslash + Ku^* = 54(10) + 40(5) + 35(0) = \$740$

the same as for the primal problem.

This adding-up or exhaustion-of-product theorem is a consequence of the homogeneity of the objective and constraint functions. As we showed in Chap. 14, when the objective and constraint functions in a maximum problem are all homogeneous of the same degree [the constraints appearing as $g^j(x_{15} \ldots, x_n) < k^j$], then the indirect objective function 0 $(k^1, \ldots, k^m)$ is homogeneous of degree 1 in the k^j 's. Hence, by Euler's theorem,

$$z^* = — H + — L + —K = u^*H + uVL + u^*K = w^*$$

where w^* is the minimum value of the dual objective function (total factor cost) and the w*'s are the M,-'S that achieve that minimum.

This remarkable duality was first noted and explored by Koopmans and others.^ They converted a purely mathematical puzzle into an interesting (albeit highly restrictive) economic model.

In general, consider the linear programming problem

maximize

subject to

$$Xj\blacksquare > 0 \quad j = 1, \ldots, n$$

There is no need for m, the number of constraints, to be less than the number of decision variables, since these are *inequality* constraints. Some of these constraints will in general be nonbinding, though it is not easy to discover which ones will be binding. In fact, finding the solution to this problem consists precisely of discovering which constraints are binding and which are not. (The algorithm for doing so will be presented in the next section.)

^T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation, Cowles Commission Monograph* 13, John Wiley & Sons, Inc., New York, 1951.

The preceding problem can be written using matrix notation. Denote the column vector of JC/S as

the objective coefficient matrix [redacted] s as

and the right-hand-side coefficients, the &,-'s, as

Denote the matrix of technical coefficients, the flj/s, as A:

/an ...

a A= :

\\a_m\\ a_{mn}

Let x > 0 mean *Xj* > 0, *j* = 1, ..., *n.* The prime denotes the transpose of a matrix. The general linear programming problem can then be written

maximize

$z =$ **p**

x subject to

Ax < b x > 0 (17-27)

Associated with the linear programming problem is a dual problem: minimize

$w =$

b'u subject to

A u > p u > 0 (17-28)

where

u =

***Ku**$_r$*

is a new vector of decision variables, the *dual variables.*

Note that the primal problem involves *n* decision variables and *m* linear inequality constraints. The dual problem involves *m* decision variables and *n* linear inequality constraints. The coefficient matrix of the constraints of the dual problem is simply the transpose of the coefficient matrix of the primal problem. The right-hand-side coefficients of one problem appear as the objective function coefficients of the other. *These problems are self-dual;* i.e., the dual problem of the dual problem is the original primal problem. In fact, either problem can be considered the primal problem.

We shall now state and briefly discuss the fundamental theorem of linear programming.

> **Theorem.** Suppose there exists an x° > 0 which satisfies the constraints of the primal problem, that is, Ax° < b (x° is *& feasible* solution) and a u° > 0 which satisfies the constraints of the dual problem, that is, A'u° > p (u° is a solution of the dual problem). *Then both problems possess an optimal solution,* i.e., a finite solution to the problem posed, *and these two solution values are in fact identical.* That is, suppose x* > 0 is that x vector which maximizes *z* = p'x subject to Ax < b. The maximum value of p'x is *z** = p'x*. Similarly, let u* > 0 be the u vector for which *w* = b'u is a minimum, and let w* = b'u*. Then *z** = *w**.

Discussion. It is easy to show that *z** < w*. The constraints of the primal problem are

n
y^aijXj < bj *i* = 1, ..., *m*
7 = 1

Multiply each constraint by w, > 0 and add:

m
^b Ui = w

In matrix notation, this is simply premultiplying **Ax** < b by the vector u', yielding

u'Ax < u'b = *w* (17-29)

(Note that the term **u'Ax** is the product of matrices of respective size 1 x *m,m* x *n,* and n x l . Hence, **u'Ax** has size (1 x 1); that is, it is a simple number, or scalar.) Now consider the constraints of the dual problem,

m
^atjUi^Pj *j = l, . . . , n*
1=1

Multiply each constraint by *x* ,■ > 0 and add:

:—:

Again, in matrix terms, this is simply multiplying, on the right, **u'A** > p' by the vector x, yielding

$$\mathbf{u'Ax} > p'x = z \qquad (17\text{-}30)$$

From (17-29) and (17-30)

$$z < \mathbf{u'Ax} < w \qquad (17\text{-}3.1)$$

Since this holds for *all* feasible **u** and x vectors (including u* and x*),

$$z^* < w^* \qquad (17\text{-}32)$$

Consider now the statement of the preceding fundamental theorem. Suppose x° and **u°** are (finite) feasible solutions to the primal and dual problems, respectively. Then from (17-31),

$$p'x° < \mathbf{u° \, b}$$

This relation implies that a finite maximum exists for the primal problem and a finite minimum exists for the dual problem. For consider that

$$p'x^0 < p\ x^* < u^{*\prime}b < \mathbf{u°'b} \qquad (17\text{-}33)$$

by the definition of the optimality of x* and u* and Eq. (17-31). But since $u^{0\prime}b$ and $p'x^0$ are finite numbers, p'x*, the maximum, or optimal, value of z = p'x, is bounded from above by **u°'b.** Likewise, u*'b is bounded from below by $p'x^0$. Therefore, if *feasible* solutions exist to both the primal and dual problems, finite *optimal* solutions must exist for each problem.

That z* = p'x* = u*'b = w* at this optimum is a consequence of the Kuhn-Tucker saddle point theorem. Since the objective and constraint functions are all linear, they are all concave. Therefore, the existence of a solution x* to the maximum (primal) problem implies and is implied by the Lagrangian of the primal problem having a saddle point.

Write the constraints of the primal problem as

$$b_\iota — \text{^^}atjXj > 0 \qquad i = 1, \ldots, m$$
$$\mathbf{i{=}i}$$

Multiply each constraint by a Lagrange multiplier w, and add. This yields the Lagrangian

$$\sum^{n} p_j x_j + \sum_{i=1}^{m} u_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

7=1

In matrix notation this Lagrangian is

$$££(x, u) = p^r x + u'(b - Ax) \qquad (17\text{-}34)$$

The saddle point theorem says that if x* is the solution to the primal problem, there exists a u* > 0 such that

5£(x, u*) < ££(x*, u*) < ^(x*, u) (17-35)

However, ££(x, u) can be rewritten

$$5£(x, u) = ub + (p' - u'A)x \qquad (17\text{-}36)$$

Define M(u, x) = -5£(x, u) = -ub + (u'A - p')x. Then, from (17-35),

$$M(u, x^*) < M(u^*, x^*) < Af(u^*, x) \qquad (17\text{-}37)$$

Since M(u, x) has a saddle point at (u*, x*), u* maximizes

$$—w = — u'b = — \overset{m}{V\text{^}}\, b[Ui$$

subject to

$$u'A - p' > 0$$

or

This is precisely the dual problem (17-28). (Of course, minimizing ^ &,-w,- is equivalent to maximizing — ^ fc,M,-.)

From the first-order conditions for maximizing *!£* with respect to x,

$$S_x = p'-u'A<0 \qquad if <, x = 0 \qquad (17\text{-}38a)$$

$$i6_u = b-Ax>0 \qquad if >, u = 0 \qquad (17\text{-}38Z?)$$

where $£g_x$ simply means the whole vector of $!£_{Xj}$'s, 7 = 1,... *,n,* etc. Equivalently, from (17-38a),

$$(p' - u'^*A)x^* = 0 \qquad (17\text{-}39a)$$

and from (17-38Z?),

$$u^{*\prime}(b - Ax^*) = 0 \qquad (17\text{-}3\%)$$

Hence, at x*, u*, the Lagrangian has the common value

$$££(x^*, u^*) = p'x^* + u^{*\prime}(b - Ax^*) =$$

p'x* = *z** and

$$\text{^}(x^*, u^*) = u^{*\prime}b + (p' - u^{*\prime}A)x^* = u^{*\prime}b - w^*$$

Therefore, *z** = *w** at the saddle point, which represents the *maximum* in the x directions (the solution to the primal problem) and a *minimum* in the u directions (representing the solution to the dual problem). As a final note, from the envelope theorem, We showed in Chap. 14 that when the objective and constraint functions $f(x)$ and $g(x)$ [the constraints being $g(x) < b$] are all homogeneous of the same degree, then $z^* = \phi(b_1, \ldots, b_m)$ is homogeneous of degree 1 in the b_i's. Therefore, by the converse of Euler's theorem,

$$w = \frac{\partial h}{\partial b} = z$$

Equivalently,

$$x_j = \frac{\partial w^*}{\partial p_j}$$

and therefore

$$= w$$

17.6 THE SIMPLEX ALGORITHM*

In the previous sections we discussed some of the economic aspects of the solutions to linear programming problems. In particular, we have shown that there is often an interesting dual problem associated with any linear maximization model. The primary reason for considering as restrictive a model as that of constant coefficients, however, is the trade-off in terms of ease of empirical solution. In fact, a relatively simple, easily programmable (in the computer sense) algorithm,

Mathematical Prerequisites

Consider the general linear programming problem

maximize

(17-40)

7 = 1

subject to

$a_{ir}x_r < b_x$

$o\text{-}m\backslash \bullet\bullet + < b$ X (17-41)

$\backslash, \ldots, x_r > 0$

Here, we have r decision variables and m linear inequality constraints.

^This section uses concepts developed in the Appendix to Chap. 5.

The first step in solving such a problem is to convert the *inequality* constraints to *equalities* (which are easier to deal with) through the introduction of *slack variables.* The constraints (17-41) are replaced by the set

$$+ a_{1r}x_r + x_{r+1} = b_1$$
$$\cdots + a_{2r}x_r + x_{r+2} = b_2$$

(17-42)

$$, \ldots, x_{r+m} > 0$$

Since the slack variables $x_{r+1}, \ldots, x_{r+m}$ are constrained to be nonnegative, the equalities (17-42) define the same feasible region as the inequalities (17-41). We see, therefore, that no generality is lost by considering a linear programming problem in which a linear function is maximized subject to linear *equality* constraints, plus nonnegativity. If the constraints are of the form $\sum_j a_{ij}x_j \geq b_i$ however, the slack variable must enter with a *negative* sign, to preserve the meaning of the inequality. Thus, if the constraint were $x_1 + x_2 > 10$, the relevant equality constraint would be

$$x_1 + x_2 - x_3 = 10, \quad x_1, x_2, x_3 > 0.$$

Thus, the problem we shall attempt to solve is maximize

subject to

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad i = 1, \ldots, m$$

$$x_j > 0 \qquad j = 1, \ldots, n$$

In matrix notation, this problem can be written maximize

$$\mathbf{p}'\mathbf{x}$$

subject to

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \mathbf{x} > 0$$

where p, x = $n \times 1$ matrices, or *column vectors* A = $m \times n$ matrix of coefficients b = $m \times 1$ column vector

No generality is lost if we assume that these m constraints are all *independent,* i.e., that it is not possible to derive any constraint by combining the remaining $m - 1$. Mathematically, we say that the matrix A has *rank m.*

Example. Consider the constraints

$$x_1 + 2x_2 + x_4 = 2$$
$$2x_1 - 2x_2 + x_3 = 3$$
$$2x_2 + x_3 + 2x_4 = 6$$

If the first equation is multiplied by 2 and added to the second equation, the resulting equation is

$$4x_1 + 2x_2 + x_3 + 2x_4 = 7$$

The left-hand side is identical to the third constraint above. The three constraints are obviously inconsistent with each other, since $6 \neq 7$. If the 6 had been a 7 in the original third constraint, that constraint could have been ignored, since it would be redundant. We could then simply consider this a two-constraint system.

Denote the rows of the matrix A by $A_1, \ldots, A_m$, respectively. The matrix A has rank m if there do not exist scalars $k_1, \ldots, k_m$ such that

That is, A has rank m if no row is a linear combination of the remaining rows. This ensures that any $m \times m$ determinant formed from A will be nonzero. The number of decision variables n must be greater than the number of constraints m to have any meaningful problem. If $m = n$, a unique solution of the constraints exists; the feasible region consists of that one point, and hence, the maximization part of the problem is trivial. If $m > n$, the feasible region is void.

With m independent equations, it is possible to solve for any m x_j's uniquely, in terms of the remaining $n - m$. If these remaining $n - m$ x_j's are set equal to 0, a *basic feasible* solution results (assuming that nonnegativity, as always, holds as well). That is, a *basic feasible* solution is a feasible solution in which $n - m$ of the x_j's equal 0, or the number of positive x_j's is no greater than the number of constraints. The m x_j's in the basic feasible solution are called the *basis*.

Geometrically, the set of basic feasible solutions corresponds to the *corners* of the feasible region, e.g., the origin and points A, *B, C,* and *D* in Fig. 17-4. We can see this as follows. A corner is a point that does *not* lie between two other points in the feasible region. Suppose $y = (y_1, \ldots, y_m, 0, \ldots, 0)$ is a basic feasible solution, that is, $Ay = b$, $y \geq 0$, where the coordinates have been numbered so that the last $n - m$ y_j's are zero. If y is not on a corner of the feasible region, then there exist two other feasible solutions, u and v, such that y lies on the straight line joining u and v, that is,

$$y = ku + (1-k)v \qquad 0 < k < 1$$

For the last $n - m$ components of **u** and **v**,

$$0 = ku_j + (1-k)v_j \qquad j = m+1, \ldots, n$$

Since *Uj* > 0 and *Vj* > 0, this can happen only if *Uj* = *vj* = 0, *j* = *m* + 1, ..., *n*. Thus, u and v are also basic feasible solutions. However, y, u, and v must all be the same point. With the last *n—m* components of each vector equal to zero, the matrix equation of constraints, Ax = b, reduces to

m 7

= 1

$a_j Uj$ —*b[* $\qquad i = 1, \ldots, m$

7 = 1

jVj = *bi* $\qquad i = 1, \ldots, m$

7 = 1

These are the same *m* equations in *m* unknowns. The equations are all linearly independent by assumption. Hence, there is a unique solution; that is, y = u = v. This contradicts the assumption that y lies *between* two other feasible solutions. Hence, the set *of basic feasible* solutions is the corners of the feasible region.

As indicated earlier, the feasible region is always a *convex set* for linear programming problems. This was illustrated earlier. The proof is quite simple and follows from more general considerations since the constraints are all concave functions. Suppose u and v are any two feasible solutions, that is, Au = b, Av = b, u, v > 0. Then any point y = *ku* + (1 — *k)*, 0 < *k* < 1 is also a feasible solution:

Ay = *A(ku* + (1 - *k)\)* = *kAu* + (1 - *k)A* = *kb* + (1 - *k)b* = b

Hence, Ay = b. Clearly, y > 0, since u, v > 0, and the scalar *k* > 0. Hence, y, which represents all points on the straight line joining u and v, is feasible whenever u and v are, and thus by definition the feasible region is convex.

The importance of these results is that they tell us that any local maximum must be the global maximum (though not necessarily unique) of the problem. There can be no "hills further on" with higher maxima than the given one.

It is geometrically obvious from Fig. 17-4 that in general the maximum, or *optimal solution* as it is usually called, *will be at a corner of the feasible region.* We shall not prove this important theorem, as it depends upon more advanced techniques of linear algebra. It may be the case that there are an infinite number of solutions. This occurs when the objective hyperplane is parallel to a flat portion of the feasible region at the maximum position. In the example in Sec. 17.2, if *p* = p_2 so that *P1/P2* = 1, for example, the maximum will occur at all points along the line segment *BC* in Fig. 17-4. However, the *basic* feasible solutions *B* and *C* will still be optimal. *In any linear programming problem in which a finite optimum exists, there exists an optimal solution that is a basic feasible solution. If the optimal solution is unique, it is a basic feasible solution.*

These results drastically reduce the number of points over which we have to search for the optimal solution. One approach to the problem would be simply to program a computer to search all the corners, evaluate z = p'x at each one, and pick the largest. However, vastly more efficient routines are available. The number of corners can be quite large; e.g., a model with 10 equations and 20 variables may contain

$$\binom{20}{10} = \frac{20!}{10!\,10!} = 184{,}756$$

basic feasible solutions.

The Simplex Algorithm: Example

We shall now illustrate the simplex algorithm for solving linear programming models by using the algorithm on the three-factor, two-good model analyzed in Sec. 17.2. The problem, again, is

maximize

$$z = 40y_1 + 30y_2$$

subject to

$$3y_1 + 2y_2 \le 54 \qquad 2y_1 + 2y_2 \le 40 \qquad y_1 + 2y_2 \le 35 \qquad y_1, y_2 \ge 0$$

The first step is to convert the constraints to equalities, by adding slack variables: maximize

$$z = 40y_1 + 30y_2 \tag{17-43}$$

subject to

$$3y_1 + 2y_2 + y_3 = 54 \tag{17-44a}$$

$$2y_1 + 2y_2 + y_4 = 40 \tag{17-44b}$$

$$y_1 + 2y_2 + y_5 = 35 \tag{17-44c}$$

$$y_1, y_2, y_3, y_4, y_5 \ge 0$$

We now have three independent constraints in five variables. A basic feasible solution is a vector $y = (y_1, \ldots, y_5)$ such that any *two* (=5 — 3) y_i's are set equal to 0, and the remaining y_i's are nonnegative and satisfy the preceding constraints. Let us arbitrarily set $y_4 = y_5 = 0$ and see if this yields a basic feasible solution. However, let us keep y_4 and y_5 in the equations and *solve for* y_1, y_2, and y_3 *in terms of* y_4 *and* y_5.

Subtracting (17-44c) from (17-44b) and rearranging terms gives

$$y_1 = 5 - y_4 + y_5 \tag{17-45a}$$

Substituting this into (17-44c) and solving for y_2, we have

$$2y_2 = 35 - y_5 - (5 - y_4 + y_5)$$

or

$$y_2 = 15 + 0.5y_4 - y_5 \qquad (17\text{-}456)$$

Lastly, we substitute (17-45a) and (17-456) into (17-44a) to solve for y_3:

$$y_3 = 54 - 3(5 - y_4 + y_5) - 2(15 + 0.5y_4 - y_5)$$

or

$$y_3 = 9 + 2y_4 - y_5 \qquad (17\text{-}45c)$$

Consider this solution, Eqs. (17-45). Setting $y_4 = y_5 = 0$, we get $y\backslash = 5$, $y_2 = 15$, y3 = 9. This is a basic *feasible* solution since $y_{1?}$ y_2, y_3 > 0. It corresponds to point *B* in Fig. 17-4. Notice that the slack variable y_3 is positive. Hence, the first (land) constraint is nonbinding, but with $y_4 = y_5 = 0$, the labor and capital constraints are binding.

However, is this solution optimal; i.e., does it maximize the objective function? Let us substitute these values of y_{ℓ} and y_2 into the objective function [y_3 does not appear in (17-43)], carrying y_4 and y_5 along:

$$\begin{aligned} z &= 40(5 - y_4 + y_5) + 30(15 + 0.5y_4 - y_5) \\ &= 650 - 25y_4 + 10y_5 \end{aligned} \qquad (17\text{-}46)$$

Equation (17-46) is the indirect objective function for this solution vector. It quickly reveals that the solution $y\backslash = 5$, $y_2 = 15$, $y_3 = 9$ just obtained is *not* optimal. Using the envelope theorem, we get

$$\frac{51}{} = 10 > 0 \qquad (17\text{-}47)$$

This relation says that the value of the objective function z can be increased by increasing the variable y_5. In fact, z will increase by a factor of 10 for each unit increase in y_5.

Thus, we should seek to make y_5 as large as possible. How large is "possible"? We have to make sure that the remaining variables remain nonnegative. Consider Eqs. (17-45) again. We know we have to bring y_5 into the basis, but which variable, yi, y_2, or y_3, shall we take out to leave only three basic variables? There is no reason, from Eq. (17-45a), why y_5 cannot be increased indefinitely (along with y_{ℓ}). However, (17-456) tells us, for example, that y_5 cannot be made larger than 15. Ignoring y_4 (which remains at 0), nonnegativity and Eqs. (17-45) mean that we must satisfy, simultaneously,

$$y\backslash = 5 + y_5 >$$

$0 \ y_2 = 15 - y_5$
$> 0 \ y_3 = 9 -$
$y_5 > 0$

In fact, the last inequality indicates that ys < 9 is required for nonnegativity. This relation tells us both that "as large as possible" for y_5 is in fact $y_5 = 9$ and that y_3 is the variable that comes out of the basis.

The new basis is therefore *y*, *y2,* and V5. We must now solve for these variables and in the same manner check whether this solution is optimal. We know it will be an improvement, since $dz/dy_5 > 0$. In fact, we know (since $dz/dy_5 = 10$ and V5 = 9) that z will increase by 10(9) = 90, yielding $z = 650 + 90 = 740$. Let us proceed.

From (17-45c),

$$y_5 = 9 + 2y_4 - y_3 \qquad (17\text{-}48a)$$

Substituting this into (17-45a) yields

$$yi = 5 - y_4 + (9 + 2y_4 - y_3)$$

or

$$y\backslash = 14 - y_3 + y_4 \qquad (17\text{-}48^*)$$

Substituting (17-48a) into (17-456) gives

$$y_2 = 15 + 0.5y_4 - (9 + 2y_4 - y_3) \text{ or}$$

$$y_2 = 6 + y_3 - l.5y_4 \qquad (17\text{-}48c)$$

Equations (17-48) are the new basic feasible solution. Setting $y_3 = y_4 = 0$, we get $y_i = 14$, $y_2 = 6$, $y_5 = 9$, and $z = 40(14) + 30(6) = 740$, as expected. This is point *C* in Fig. 17-4.

Is this solution optimal? Using (17-486) and (17-48c), we get

$$z = 40(14 - y_3 + y_4) + 30(6 + y_3 - 1.5y_4) \text{ or}$$

$$z = 740 - \backslash 0y_3 - 5y_4 \qquad (17\text{-}49)$$

Equation (17-49), the indirect objective function for this solution, tells us that this is indeed the optimal solution. This relation implies that $dz/dy_3 < 0$, $dz/dy_4 < 0$. Bringing in either of the nonbasic variables will only *reduce* the value of the objective function. Moreover, this solution is *globally* optimal. Convexity of the feasible region and concavity of the objective function (weak concavity here; the objective function is linear) tell us that any local maximum must be a global maximum. We have thus found the maximum solution in one easy iteration, a combination of luck and the efficiency of the simplex algorithm.

To summarize, the simplex algorithm consists of first finding some basic feasible solution by rote, if necessary. The basic variables are solved in terms of the nonbasic variables. These values are then substituted into the objective function, a procedure by which the objective function becomes expressed in terms of the non-basic variables only. If $3z/3y_j > 0$ for any nonbasic variable, the original solution is not optimal. Suppose one or more *yj* is such that *dz/dyj* > 0. Pick one of them (perhaps the one for which *dz/dyj* is largest, though this will not ensure that the optimal solution will be reached the fastest). Use the previously obtained solutions to determine how large that nonbasic variable can be made and which previously basic variable must be set equal to zero, i.e., taken out of the basis. Solve for the new basic feasible solution. Repeat this process until *dz/dyj* < 0 for all nonbasic variables. When this condition holds, the solution is optimal, since bringing new variables into the basis will not increase *z*.

Remark 1. If *dz/dyj* = 0 for some nonbasic variable, bringing that *yj* into the basis will neither increase nor decrease *z*. If this occurs at an optimal solution, multiple optima are indicated.

Remark 2. For a *minimization* problem, the criterion that must be satisfied is *dz/dyj* > 0 for all nonbasic *yj*. For minimization problems, the optimal solution is characterized by having the nonbasic variables *increase* the objective function if they are introduced.

Example. Let us consider the following minimization problem, stripped of any economic content, for the purposes of exhibiting the simplex algorithm once more:

minimize

$$z = !x\backslash + x_2 +$$

$2x_3$ subject to

$$X\backslash - 2x_2 + x_3 > 16$$

$$2x_2 + x_3 < 10$$

$$X\text{-}i < 6 \quad X\backslash, X2, X\text{-}i > 0$$

The first step is to add slack variables to convert the constraints into equalities. Note the direction of the inequalities:

minimize

$$z = 2xi + x_2 +$$

$2x_3$ subject to

$$Xi - 2x_2 + JC_3 - x_4 = 16$$

$$2x_2 + X3 + x_5 = 10$$

$$6 \qquad Xi, x_2, x_3, x_4, x_5, x_6 > 0$$

Let us choose $X\backslash$, x_2, and JC_3 as a basis. Solving for these variables in terms of the remaining ones yields (the student should work through this algebra)

$$x_3 = 6 - x_6$$

$$x_2 = 2 - 0.5x_5 + 0.5x_6$$

$$*i = 14 + x_4 - x_5 + 2x_6$$

Computing *z,* we get

$$z = 2(14 + x_4 - x_5 + 2x_6) + (2 - 0.5x_5 + 0.5x_6) +$$

2(6 - x_6) or

$$z = 42 + 2x_4 - 2.5JC_5 + 2.5x_6$$

We see that $dz/dx_5 < 0$; bringing x_5 into the basis will decrease *z.* Hence, this solution is not optimal. How large can we make x_5? From the preceding solution, to ensure nonnegativity of *Xi,* x_2, and x_3, we must have (ignoring x_4 and x_6, which remain at 0)

$$x_2 = 2 - 0.5JC_5 > 0 \qquad xi = 14 - x_5 > 0$$

Since these must both be satisfied, we cannot increase x_5 beyond 4. Hence, x_2 comes out of the basis. The new basis is *X[, xj,* and x_5. Solving for these variables yields

x_3 — 6 — $x_6 x_5$ — *4* — $2x_2 + x_6 x\backslash = 10 + 2x_2 + x_4 + x_6$ Computing the new *z* [we expect this *z* to be less than the old *z* by 2.5(4) = 10] gives

$$z = 2(10 + 2x_2 + x_4 + x_6) + x_2 + 2(6 -$$

JC_6) or

$$z = 32 + 5x_2 + 2x_4$$

Notice that $dz/dx_2 > 0$, $dz/dx_4 > 0$ but that $3z/3x_6 = 0$. This solution is optimal, since there is no variable to be brought in which would *lower z.* However, this is not the only solution. Bringing *x^* into the basis will keep *z* the same. There are an infinite number of solutions along the line segment between this solution (JCI = 10, $x_5 = 4$, $JC_3 = 6$) and the one which results from bringing *X(,* in. The remainder of this problem is left as an exercise for the student.

PROBLEMS

1. Consider an economy made up of many identical fixed-coefficient firms, each of which produces food y_x and clothing y_2. There are three inputs: land, labor, and capital, inputs 1, 2, and 3, respectively. The matrix of technological coefficients is

Each firm has available to it 30 units of land, 40 units of labor, and 72 units of capital. Prices are $20 per unit for food and $30 per unit for clothing.

1.407 Find the production plan that maximizes the value of output.

1.408 Find the shadow prices of land, labor, and capital.

1.409 Write down the dual problem and interpret it.

1.410 Solve the dual problem and verify that the optimal value of its objective function equals the maximum value of output.

1.411 From the factor intensities of the goods at the optimum, predict the changes in output levels that would occur if an additional unit of labor were available. Check by actual solution.

(f) From the factor intensities, predict the change in factor prices if p_x rises to 21. Check by actual solution.

(g) What effect does a small increase in factor endowments have on factor prices? 2. Answer the same questions as in Prob. 1 for an economy made up of firms with coefficient matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

endowments of

/40

(

and prices $p\backslash = 15$, $p_2 = 10$. For part (/), suppose here that p_2 rises to 11. 3. Solve the following linear programming problem:

minimize

subject to

$$*i + x_2 > 9$$
$$X1+X3 <8 \qquad XJ, x_2, x_3 > 0$$
$$2x_2 — X3 > 5$$

Start with $X\backslash$, x_2, x_3 as a basis.

4. A firm makes banjos $(x\backslash)$, guitars (x_2), and mandolins $(x\text{-}\$)$. It uses three inputs: wood, labor, and brass, inputs 1, 2, and 3, respectively. Let

Gij = amount of *ith* input used in production of 1 unit of product *j* The matrix of these technological coefficients is

(2 2 l)

The firm has available to it 50 units of wood, 60 units of labor, and 55 units of brass. The firm sells banjos for \$200, guitars for \$175, and mandolins for \$125.

1.412 Find the production plan that maximizes the total value of output. Start with x,, x_2, x_3 as your first basis.

1.413 Formulate the dual for this problem and explain its economic interpretation. Solve the dual problem using, if you wish, whatever information about its solution you can glean from the solution of the primal problem. How much would the firm be willing to pay for an additional unit of wood, labor, and brass?

5. *The Diet Problem.* Suppose a consumer has available *n* foods, $X\backslash, \ldots, x_n$. Each food contains a certain amount of nutrients (vitamins, minerals, etc.). Let $a\text{-},j$ = amount of nutrient *i* in food *j*. If the foods JC, cost p, per unit, and the consumer wishes to obtain a minimum daily requirement (MDR) *bj* of nutrient *j,* what diet should be consumed?

1.414 Formulate this problem as a linear programming problem. What assumptions about how nutrients are combined is needed?

1.415 Formulate and interpret the dual problem.

1.416 Suppose there are only three nutrients to consider, *A, B,* and C, with MDRs of 34, 32, and 50, respectively. There are three foods, $x_{\{}$, x_2, and JC$_3$ with prices /?, = !, *P2* = 3, /?$_3$ = 2. The matrix of nutrient contents of the foods is

/I 2
IX A= 2
1 3
V2 3 1/

Find the diet that minimizes the cost of satisfying the MDRs of each nutrient. Check by solving the dual problem also.

6. Complete the discussion of the linear programming problem presented in the text preceding the problems. Bring x_6 into the basis and show that the maximum value of z is unchanged. Formulate and solve the dual problem and verify that its objective function has as its solution the same value as z^*.

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CHAPTER

18

GENERAL EQUILIBRIUM II: NONLINEAR MODELS

18.1 TANGENCY CONDITIONS

In this chapter, the more plausible model of general equilibrium based on *variable* coefficients of production will be investigated. The bulk of the chapter will be concerned with the derivation of the Stolper-Samuelson and Rybczynski theorems in this more general context. It is a rather remarkable feature of these models that when the production functions are assumed merely to be *linear homogeneous,* the comparative statics of the model yield the same implications as in the case of fixed-coefficient technology (indeed, the algebra is identical).

The most general model to be considered here is the one in which n final goods, $y_1, \ldots, y_n$, are produced using m factors of production, $x_1, \ldots, x_m$. The economy faces world output prices, $p_1, \ldots, p_n$. If we let

$$x_{ij} = \text{amount of factor } i \text{ used in production of}$$

good j the production function for y_j is

$$y_j = f^j(x_{1j}, \ldots, x_{mj}) \qquad j = 1, \ldots, n \tag{18-1}$$

We assert that an invisible hand leads the economy to maximize

subject to

$$ij < x_i \qquad (18\text{-}3)$$

$$> 0 \qquad \text{all } i, j$$

The model, at this point, is a general nonlinear programming problem. Without the knowledge of the specific functional forms for the production functions (18-1), no solution algorithm is available (as opposed to the linear model of the previous chapter). We shall therefore concentrate on the *comparative statics* of the model, assuming that the JC(/S are the ones that the economy uses in positive amounts. In that case, the Kuhn-Tucker first-order conditions become the classical Lagrangian techniques.

All the comparative statics results that are forthcoming in this general model can be adequately indicated by reducing the model to two goods and two factors, since we shall not be concerned with which factors are present in the first place. Let us therefore change notation to conform with the previous analysis and consider *two* factors, *L,* labor, and *K,* capital. Let L_j and K_j represent the amounts of labor and capital, respectively, that are used in the production of good *j.* The production functions for each of the two goods are

The thus becomes maxim

$$z \qquad i \qquad (\quad K_2) \qquad (18\text{-}4)$$

subject to

$$+ L = \qquad (18\text{-}5a)$$

$$K + K = \qquad (18\text{-}5b)$$

We are assuming that we shall find L_1, L_2, K_1, and K_2 all positive and fully employed at the parametrically fixed levels L and K, respectively. The Lagrangian for this model is

$$K_1) + p_2 f^2(L_2,$$

$$- L_1 - L_2) \qquad (K - K_1 - K_2) \qquad (18\text{-}6)$$

The first-order equations for constrained maximum are obtained by differentiating $\mathcal{L}$ with respect to the four choice variables, L_1, L_2, K_1, K_2, and the two Lagrange multipliers. When we let

$$df^j \; dy_j \qquad , \qquad dp \qquad d_{yj}$$

$$dL. \qquad J\,L,$$

these first-order conditions are

$Pifb\text{-}k_K=0$ (18-7a) (18-76)

$p_2f^2_L \sim X_L=0$ (18-7c) (18-7c?)

$Plfl \sim\hat{} K=0$ (18-8a) (18-86)

L - Li - L_2 = 0 K - K_{ℓ} - K_2

= 0

The second-order conditions consist of restrictions on the border-preserving principal minors of the border Hessian determinant formed by differentiating Eqs. (18-7) and (18-8) again with respect to the L,-'s, K_i's, and A.'s. Letting $f[_L = d^2f/(dLjdLj)$, etc., we have

$$H = \begin{vmatrix} pif & Pi & 0 & 0 & -1 & 0 \\ Pif & pifL & 0 & 0 & 0 & - \\ 0 & 0 & P2fl & Piflu & -1 & 0 \\ 0 & 0 & Pifh & P2fl & 0 & - \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 & 0 & 0 \end{vmatrix} \qquad (18\text{-}9)$$

Specifically, the border-preserving principal minors of order k alternate in sign, the whole determinant H having sign +1.

Assuming the sufficient second-order conditions hold, Eqs. (18-7) and (18-8) can be solved for the explicit choice functions

i = 1,2 $Li=L^*(p_l,p2,L,K)$ (18-10a)

and $K_i = K^*(p_l,p_2,L,K)$ (18-106)

$X_K=X\backslash\{p_up_2,L,K)$ (18-lla) (18-116)

Equations (18-10) show the quantities of each factor that will be used by each industry at given output prices and total resource constraints. They are in fact neither factor supply nor factor demand equations, since they are not functions of factor prices. The factor supply curves to the whole economy are vertical lines at L and K, respectively. Equations (18-10) represent the solutions to the allocation problem wherein each factor is demanded by two industries.

As in the linear model of the previous chapter, the role of factor prices is filled by the Lagrange multipliers X_L and X_K. Substituting the

L^*'s and K^*'s into the

objective function z yields

$$z^* = 4>(_Pi, Pi, L, K) = _{Px}f\backslash L\backslash, K^*) + p_2f^2(L^*_2, K^*) \qquad (18\text{-}12)$$

Using the envelope theorem, we get

8L *dL*
dL

$$= k^*_K(p_{uP2}, L, K) \qquad (18\text{-}13^*)$$

dK *dK*

That is, k^*_L is the rate of change of maximum NNP with respect to a change in the resource endowment of labor, with a similar interpretation for k^*_K. These multipliers are the incremental increases in income that would result if an additional increment of labor or capital were available. In a competitive economy, these are the marginal revenue products of labor and capital, respectively. They indicate what labor and capital would be paid in a competitive economy. Hence, the Lagrange multipliers represent the imputed values, or shadow prices, of resources in this model. They are not exogenous, as in the previous partial equilibrium treatments of the firm, but endogenous, appearing as part of the solution to the model.

Let us in fact designate the wage rate w as $w = k_L$ and the flow price of capital as $r = k_K$. The symbol r does *not* represent an interest rate. It is the rate at which capital is *rented,* analogous to the rate w at which labor is rented. Capital is treated as a service flow, as is labor, and not as a stock that is purchased outright, with r the wage of capital.

Industry supply curves can be defined in this model by substituting the L*'s and *K*'s* into the production function, yielding

$$yi = f\backslash L\backslash, K\backslash) = y^*(p_{lt}\, p_2, L, K) \qquad (18\text{-}14a)$$

$$y_2 = f(L^*_2, K^*) = y^*(j,_u\, p_2, L, K) \qquad (18\text{-}14^*)$$

These are the industry supply curves because they indicate how much output will be produced for a given price of output, price of the other good, and resource constraints. The demand curves for each industry are the horizontal price lines at the levels $p\backslash$ and p_2, respectively, reflecting competitive output markets.

The supply curves (18-14) are homogeneous of degree 0 in output prices. Increasing both prices by the same proportion leaves output unchanged, or

$$y^*j(t_{Pl}, tp_2, L, K) = y^*(_{Pl}, p_2, L, K) \qquad 7 = 1,2 \qquad (18\text{-}15)$$

This is easily seen from the objective function (18-4), $z = piy\backslash + p_2y_2$. If both prices are increased by the factor t, $z = t(piyi + p_2y2)$, a simple monotonic (linear, in fact) transformation of the original function. The values of the factors that maximize $p\backslash y\backslash + p_2y_2$ also maximize $tp\backslash y\backslash + tp_2y_2$. The solutions (18-10), $L_t = L^*(pi, p_2, L, K)$, etc., are thus homogeneous of degree 0 in $p\backslash$ and p_2, and thus so must be $y^* = f^1(L^*, K^*)$ and $y^* = f^2(L^*_2, K^*)$.

The production possibilities frontier for this economy is defined as the locus of points (vi, y_2) such that for any given y_2, the maximum vi is obtained, or vice versa. It can be obtained by eliminating the prices from Eqs. (18-14a) and (18-14&). In Eq. (18-15), let *t* = \/*pi*, *P* = *P\lPi*- Th e variable p represents the *relative* price of output y_1 (in terms of units of y_2). Then we obtain

$$y_1^* = y_1^*(p_1, p_2, L, K) = Y_1^*(p, L, K) \qquad (18\text{-}16a)$$

and

$$y_2^* = y_2^*(p_1, p_2, L, K) = Y_2^*(p, L, K) \qquad (18\text{-}16b)$$

Assuming $\partial Y_j^*/\partial p \neq 0$, $j = 1, 2$, the variable p, relative output price, can be eliminated from these two equations, leaving

$$G(y_1^*, y_2^*, L, K) = \qquad (18\text{-}17)$$

or, in explicit form,

$$y_2^* = g(y_1^*, L, K) \qquad (18\text{-}18)$$

The typical assumed shape of this function is concave to the origin, as depicted in Fig. 18-1. We shall see presently how this shape is implied by the sufficient second-order equations for this model. As drawn, $dy_2^*/dy_1^* < 0$, and $\partial^2 y_2^*/\partial y_1^{*2} < 0$. Before analyzing the comparative statics of this model, let us investigate the fir st-order marginal relations (18-7). As we have indicated, w (= k_L) and r (= k_K) are the imputed marginal revenue products of labor and capital, respectively. From Eqs. (18-7a) and (18-7c), we see that the marginal revenue product of labor must be the same in both industries. Likewise, from (18-7fr) and (18-7d), the marginal revenue product of capital must be the same in both industries. One would expect this on the basis of intuitive reasoning. If labor, say, were more productive, i.e., yielded more output at the margin, in industry 1 than in industry 2, the owners of labor would

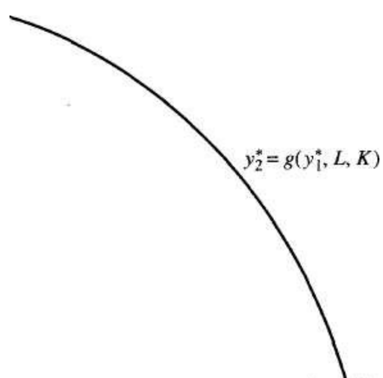

FIGURE 18-1
The production possibilities frontier indicates the maximum amount of one output that is attainable for given amounts of the other. In order for the economy to achieve maximum NNP for given resource constraints, a point on this frontier must be reached. Otherwise, increasing both *y* and j2 will increase NNP = *p\y* + *piyi.* The second-order conditions for the maximization of NNP model imply that *dyl/dy* < 0, *d²y*/dyf* < 0, as drawn above. However, this is not implied by simply maximizing j2 for given *y*, *L,* and *K* unless additional restrictions are placed on the production functions. (See Prob. 5.)

find competitive bids for their services more attractive in the first industry. Labor will leave industry 2 and enter industry 1. In so doing, the marginal product of labor will rise in industry 2 and fall in industry 1. This process will continue until the bids for labor (or capital) are the same in both industries. It is this process of competitive bidding, with owners of labor and capital seeking their highest valued employment, that is the essence of Adam Smith's invisible hand mechanism, in which the value of total output is maximized.

Combining Eqs. (18-7a) and (18-7*b)* gives

4 = — (18-19a)

Likewise, from (18-7c) and (18-7d) we have

4 = - (18-19/))

JK r

These last two equations say that the ratio of the marginal products of labor and capital, for each industry, is equal to the ratio of the (imputed) factor prices. This is analogous to the partial equilibrium tangency condition for profit maximization or cost minimization, in which the isoquants are tangent to the isocost line, with common slope equal to the ratio of wage rates.

In the present model, however, this imputed ratio of wage rates or relative factor costs is common to both industries. From Eq. (18-19),

4 = 7 = 4 (18-20)

JK ' ***JK***

At the wealth-maximizing input combination, the slopes of the isoquants in each industry are the same. Again, if they were different, one factor would be more productive (at the margin) in one industry than the other (the reverse holding for the other factor). In that case, factors would move from the relatively low-valued use to the high-valued use, increasing both the return to that factor and the NNP of the economy.

This situation is commonly depicted in an Edgeworth-Bowley box diagram. In Fig. 18-2, industry 1 is depicted in the usual manner, with origin *O* at the lower left, or southwest, corner of the box. The isoquants of industry 1 are the curves convex to that origin. The axes are finite, however, and extend only to the limits of resource endowments. In the horizontal direction, labor is plotted up to the parametric value *L*. Likewise, in the vertical direction, units of capital are plotted until the parametric value *K* is reached. At these limits, a rectangle is formed, yielding another origin O_2. The production function $f^2(L_2, K_2)$ is plotted upside down, starting at O_2, with increased labor plotted in a westerly direction, increased capital in the southerly direction. Various isoquants of $f^2(L_2, K_2)$ are plotted in Fig. 18-2. These curves are concave to *O* but convex to O_2, the origin from which they are plotted.

Any point in the box represents an allocation of factors to each industry. For example, at point A, *L* units of labor are allocated to industry 1 and L_2 to industry 2. As at all interior points, *L* + L_2 = *L,* the horizontal dimension of the box. Likewise

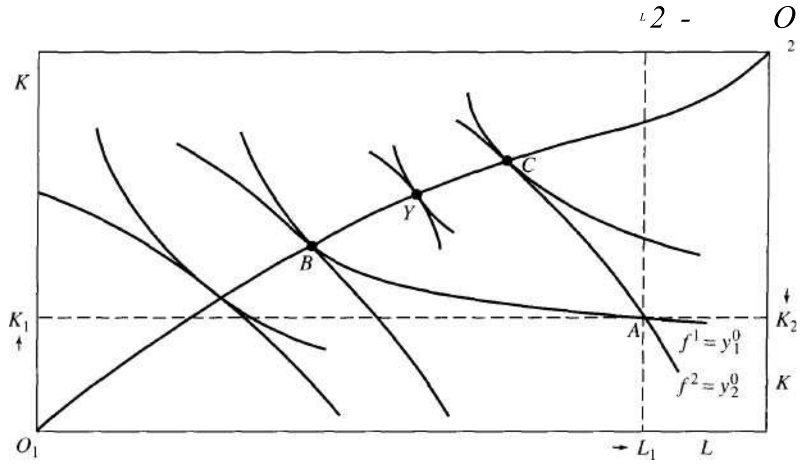

FIGURE 18-2
Edgeworth-Bowley Box Diagram for Factor Utilization. This famous diagram, attributed to F. Y. Edge-worth and A. L. Bowley, depicts the set of factor allocations in an economy in which all gains from trade are exhausted. The dimensions of the box are the total endowments of labor (plotted horizontally) and capital (plotted vertically). The labor *L\,* used by industry 1, at some point, say *A,* is the horizontal distance from A to the vertical axis emanating from origin *O*. Likewise, the vertical distance from the labor axis to *A* is the capital *K* used by industry 1. Continuing to the right from *A* to the right-hand extremity of the box is the remaining labor *Li* in the economy, which is used by industry 2. Likewise, the vertical distance above *A* to the top horizontal axis is the amount of capital *Ki* used by industry 2. Since *L* + *Li* = *L* and *K* + *Ki* = *K,* where *L* and *K* are constants, a rectangle is depicted such that any point inside it defines an allocation of factors to each industry.

With this in mind, the production function /'(Li, *K{)* can be plotted with respect to the origin denned by the southwest corner of the box, and /2(L2, *Ki)* can be plotted with respect to the origin denned by the northeast corner of the box. Industry 2's production function /2(L2, #2), however, is plotted negatively. *Increasing Li* is a leftward movement since it represents *decreasing L*. Likewise, *increasing Ki* is a downward movement since it represents *decreasing K*. Hence, while the isoquants of /' appear normally, the isoquants of *f*2 appear concave to *O*. Actually, though, they are to be interpreted as convex to *Oi,* the appropriate origin from which *f*2*{Li, Ki)* is plotted.

The efficient factor allocations, i.e., those which result in output levels on the production frontier, are those input combinations at which the slopes of the isoquants are equal, in accordance with Eq. (18-20). At such points, each industry values the inputs identically. If industry 1 were willing to give up 3 units of capital for 1 unit of labor and industry 2 were willing to give up 1 unit of labor for 1 unit of capital, both industries could experience an increase in output by, say, industry 2 exchanging only 2 units of capital and getting 1 unit of labor from industry 1 in return. Such mutually advantageous reallocations are possible as long as the slopes of the isoquants for the two industries are different. The locus of points at which those slopes are identical, indicating exhaustion of gains from exchange of factors, is the curve line between *O* and *Oi,* known as the *contract curve* or *efficiency locus.* In the absence of transactions costs, some point on the contract curve must be achieved. If the initial point is A, say, in this diagram, then *voluntary* exchange will lead to an allocation of factors on the contract curve.

at A, $K\backslash$ units of capital are allocated to industry 1 and K_2 to industry 2. Since $K\backslash + K_2 = K,$ the vertical height of the box at all values of $L\backslash,$ the entire amount of capital available to the economy is allocated between the two industries.

Point A, however, cannot be an efficient allocation, i.e., one in which NNP is maximized. At point A, the slope of industry l's isoquant is flatter than that of industry 2. Industry 1 will be bidding relatively low for labor and high for capital. Likewise, industry 2, which at A employs relatively large amounts of capital (though this is not necessary), will bid higher amounts.for labor and lower amounts for capital than industry 1. The slopes of each industry's isoquants at A indicate that labor is valued more to industry 2 than to industry 1 while capital is relatively high-valued to industry 1. The competitive mechanism will induce labor to move to industry 2 and capital to industry 1. In terms of the Edgeworth diagram, the allocation point will move to the contract curve. Eventually, some point will be reached, such as point *Y,* at which the slopes of industry l's and industry 2's isoquants are identical, in accordance with Eq. (18-20). At such a point, the isoquants for the two industries will be tangent to each other. The locus of all such points of tangency within the Edgeworth box is called the *contract curve.* In the absence of transactions costs, a competitive economy must always be at some point on the contract curve. To be otherwise would deny the postulate that more is preferred to less. Individual self-seeking will force the economy to some point along the contract curve at which all gains from trade are exhausted.

One more tangency, which is not depicted but nonetheless is implied, is the response of consumers to this situation. All consumers in this economy face the prices p_1 and p_2 for the output of these industries. Utility-maximizing consumers will consume these goods where their subjective marginal valuation of the goods equals the price ratio, i.e., the relative costs to the consumers. The slope of the consumers' indifference curves, their marginal rate of substitution, will equal the price ratio, or the slope of the production possibility frontier at the final output point.

Any point on the production possibility frontier depicted in Fig. 18-1 corresponds to some point on the contract curve of Fig. 18-2, since only at efficient factor utilization can the maximum output of one good be obtained for a given amount of the other. Any factor allocation not on the contract curve of Fig. 18-2 will result in an output point inside the production frontier depicted in Fig. 18-1. In the literature, two kinds of efficiency are usually defined. If the production point achieved is perceived to be on the production possibilities frontier, then the economy is said to be *efficient in production.* However, because of, for example, monopolies in the economy, the marginal rates of substitution of consumers may not equal the relative cost of production. In that case, gains from exchange could occur, using the reasoning just described, that would allow all consumers to gain. When the MRS of all consumers equals the relative marginal costs of production, the economy is said to be *efficient in consumption.* These welfare-type considerations were first enunciated by V. Pareto. An economy in which all the gains from exchange are exhausted is called *Pareto-optimal* or *Pareto-efficient.* We shall delve into this in more detail in the next chapter.

The student should be warned that efficiency is an essentially unobservable condition. In the absence of transactions costs, all gains from exchange must at all times be exhausted, and hence efficiency follows tautologically. If an economy is asserted to be at an inefficient, or non-Pareto, point, the implied losses to consumers must be reconciled with the consumers' preferences for more rather than less. That is, some observable cost of trading, perhaps embodied in some institutional restriction

on open markets, must be identified to make the theory internally consistent. (A deeper analysis would in fact have to explain why consumers sometimes get together and enact laws resulting in lost gains from trade.)

18.2 GENERAL COMPARATIVE STATICS RESULTS

As we have repeatedly emphasized, the potentially observable, refutable hypotheses generated by a model are the goals of theory. The first-order conditions (18-7) and (18-8) are nonobservable in the absence of knowledge of the specific functional forms. The refutable hypotheses of this model are the restrictions in sign of the various partial derivatives of the explicit choice functions, Eqs. (18-10a) and (18-10b), that are implied by the maximization hypothesis. Additional refutable hypotheses, derived from the preceding, are the possible restrictions in sign of the output supply functions (18-14). Let us investigate these comparative statics relations using the envelope analysis of Chap. 7.

The indirect national income function, defined in Eq. (18-12), is the maximum value of NNP for given output prices and resource endowments. It is found by substituting the explicit choice functions (18-10) into the objective function defining NNP, or

$$z^* = \mathrm{NNP}^* = \phi(p_1, p_2, L, K) = p_1 f^1(L_1^*, K_1^*) + p_2 f^2(L_2^*, K_2^*)$$

Since this is the maximum value of NNP for given parameter values, $\mathrm{NNP} - \mathrm{NNP}^* \le 0$, with $\mathrm{NNP} - \mathrm{NNP}^* = 0$ when $L_i = L_i^*$, $K_i = K_i^*$, $i = 1, 2$. Thus, the function

$$\Phi(L_1, L_2, K_1, K_2, p_1, p_2, L, K) = \mathrm{NNP}(L_1, L_2, K_1, K_2, p_1, p_2, L, K)$$

has a constrained maximum (of 0) at $L_i = L_i^*$, $K_i = K_i^*$, $i = 1, 2$. The Lagrangian for this primal-dual problem is

$$\mathcal{L} = \mathrm{NNP} - \mathrm{NNP}^* + w(L - L_1 - L_2) + r(K - K_1 - K_2)$$

The comparative statics sign restrictions are derived from the bordered Hessian of second partials of $\mathcal{L}$ with respect to the parameters p_1, p_2, L, and K. The first partials of $\mathcal{L}$ with respect to these parameters are the envelope relations

$$\frac{\partial \mathcal{L}}{\partial p_1} = y_1 - y_1^* = 0 \tag{18-21a}$$

$$\frac{\partial \mathcal{L}}{\partial p_2} = y_2 - y_2^* = 0 \tag{18-21b}$$

$$\frac{\partial \mathcal{L}}{\partial L} = -w^* + w = 0 \tag{18-21c}$$

$$\frac{\partial \mathcal{L}}{\partial K} = -r^* + r = 0 \tag{18-21d}$$

The constraints are the first partials of $\mathcal{L}$ with respect to w and r, set equal to 0:

$$\frac{\partial \mathcal{L}}{\partial w} = L - L_1 - L_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} = K - K_1 - K_2 = 0$$

The relevant bordered Hessian is formed from the second partials of $\mathcal{L}$ with respect to p_1, p_2, L, K, w, and *r*. These second par[t]

s are the first partials of Eqs. (18-21) and (18-22) with respect to those variables, since those equations are the first partials of *SE* with respect to those variables. The bordered Hessian is thus

d	***dy***	***dy****	***d***	***d***
L	*dpi*	9/?2	***d***	***d***
1	*dy*	9}>	***L***	***K***
	d	***dp***	***dL***	***d***
	pi	***dw***	***dw***	***K***
	d	***dp***	***d***	***d***
	p		***L***	***K***
	d	***dp***		
	K	0		
	0	0		

0 0

0 0

1 0 (18-23)

Also The first

0 1 / =

1,2 /

(18-

25a)

comparative statics relations that appear are the reciprocity conditions, derived from the symmetry of X_{aot}. We note

d

dp2 *d*

dy* **d**

dL *d*

dy* **d**

dK *d*

n

dw **d**

***** ***r***

dK **dL**

Equation (18-24) says that if the output price of *yi* is raised, say, the effect on industry 2's output is exactly the same as the effect on industry l's output of an increase in p_2. Equations (18-25) indicate, for example, that if the

r e s o u r c e

e n d o w m e n t

of

The reciprocity relations (18-26) are exact analogs of (18-24), with the prices of the goods or factors replacing the respective physical quantities, and vice versa.

Equations (18-24) to (18-26) generalize in a straightforward manner to the case of *n* goods and *m* factors. When we let Ai, ..., X_m represent the shadow factor prices of factors *x\, ...,* x_m, these equations become, respectively,

$$i = 1, \ldots, m \quad j = \backslash, \ldots, n$$

Thus far, however, we have not placed any sign restrictions on these partial derivatives. At this level of generality, it is not possible to say whether an increase in the endowment of labor will increase or decrease the output of either industry. (It can be shown, as one would expect with positive marginal products, that an increase in labor or capital cannot lead to a decrease in output of *both* industries.) The effect on the imputed wages of changes in output prices is likewise indeterminate. The only comparative statics sign restrictions that are derivable at this level of generality relate to the supply of output functions y_i = *y*(pi, Pi, L, K).* These curves must be upward-sloping in their own prices, assuming the sufficient second-order conditions. The parameters in this model are partitionable into two distinct sets, *p* and p_2, which appear only in the objective function, and *L* and *K,* which appear only in the constraints. No signed comparative statics relations can appear for these latter parameters, as the analysis in Chap. 7 makes clear. In the case of *p* and *P2,* the associated decision variables *y** and *y%* behave the same as in an unconstrained model. Since this is a maximum problem, the diagonal elements *—dy*/dpi* and must be negative. Thus,

$$\frac{\wedge}{dpi} > 0 \qquad Z = 1, 2 \tag{18-27}$$

The output supply curves are thus upward-sloping. Again, the analogous assertions for *-dw*/dL, —dr*/dK* are *not* valid, since *L* and *K* are parameters that appear in the constraints. It is *not* possible to say at this level of generality that, for example, an increase in the amount of labor in the economy will depress the wage rate of labor. Plausible as that result sounds, it is not implied by the preceding model. However, if the production functions are concave, for example, if they are homogeneous of degree $s < 1$, then the entire objective function (18-2) or (18-4) is concave. By the theorem at the end of Sec. 7.4, it must then be the case that $dw^*/dL < 0$ and $dr^*/dK < 0$, with the strict inequality holding if the production functions are strictly concave, e.g., $s < 1$. If the production functions exhibit constant returns to scale, the wage rates are *independent* of the resource endowments. We shall explore these issues in the next section.

Let us now consider the implied properties of the production transformation frontier, defined in Eq. (18-17) and depicted in Fig. 18-1. The factor choice equations are, again,

$$L_t = L^*(p_1, p_2, L, K) \qquad K_t = K?(p_1, ft, L, K)$$

Since these equations are homogeneous of degree 0 in $p\backslash$, p_2, they can be expressed in terms of the price ratio $p = p\backslash/p_2$- We shall suppress the parameters L and K in the following discussion for notational ease; these parameters are not relevant to this discussion. Writing $L_t = L^*(p)$, $K_t = K^*(p)$, we define Eq. (18-14) again as

t l (18-28fl)

and

VHP) = f\L*₂{p\ K*₂{p)) (18-286)

Assuming these functions are invertible, the functional dependence

$$yi = yl(p(y^*i)) = yJW) \qquad (18\text{-}29)$$

is valid. From the chain rule,

M m^l (18-30)

When Eqs. (18-28) are used, Eq. (18-30) becomes

Using the first-order conditions pif[= pift = w' Pifk = P^/K = r, we have

dy*x {\/p,)[w{dL\/dp) +

However, since $L\{p) + L^*(p) = L$, $K\{(p) + K^*(p) = K$

dp dp dp dp

The numerator (excluding the price term) is thus exactly the negative of the denominator, or

^f = -(-!) = -/> =

Equation (18-31) asserts that when NNP is maximized, the production possibilities frontier will be tangent to an *isorevenue* (same revenue) line. NNP is given as a linear function of output levels,

$$\text{NNP} = /?iyi + p_2y_2$$

The values of $y\backslash$ and y_2 which maximize NNP, that is, those values that allow this line to move farthest from the origin in the output space, are those values where

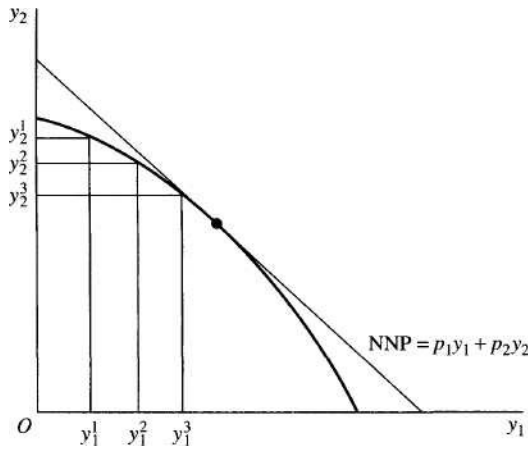

FIGURE 18-3
Maximization ofNNP — p\y\ + piy$_2$. An interior maximization of NNP subject to resource constraints implies a production possibilities frontier that is concave to the origin. This feature is visually obvious in the above diagram. If the frontier were convex to the origin, maximum NNP, where the line NNP — *P*1 *y* + *Pi yi*^{was} ^{as} far from the origin as possible, would occur at a corner, implying production of one good only. The interior maximum above occurs at a point where the consumers' marginal evaluations of the two goods (measured by the slope *—pxlpi* of the isorevenue line NNP = *p*$_x$*y*$_x$ + *p*$_2$*yi)* equals the marginal production trade-off of *y*$_2$ for *y*. This latter trade-off is the marginal cost of *y* measured by the ordinary slope of the production frontier. The concavity of this frontier is indicative of increasing marginal costs of production. As output of *y* increases from *y* to *y*, the *cost* of this increased output is measured by the *decrease* in *yi* available, *y* — *y*. Increasing *y* by the same increment, to *y*, leads to the *larger* sacrifice of *y*$_2$, measured by the vertical distance *y* — *y*. In the limit, for small changes in *y*, the marginal cost of *y* in terms of the rate at which *y*$_2$ must be forgone is the slope of the production possibilities frontier. The increasing (in absolute value) slope as *y* increases indicates increasing marginal cost of *y*. The same situation obtains for *y*$_2$*:* the reciprocal slope measures the marginal cost of *yi* and increases with increasing *yi.*

the production frontier has the same slope as the isorevenue line, *—p* = — This situation is depicted in Fig. 18-3. As drawn there, the production possibilities frontier is concave to the origin. This is verified mathematically by differentiating both sides of the identity (18-31) with respect to y*:

^* 9^2* -1 *-Pi* **<0** (18-32)

Equations (18-31) and (18-32) assert that this economy is characterized by positive and increasing marginal costs of production in the neighborhood of the wealth-maximizing output choices. The fact that *dy^/dy** is negative means that if more *yi* is desired, some j2 will have to be forgone. The partial derivative *dy^/dy** measures the rate at which *y%* must be sacrificed in order to get more *y*. Therefore, *dy%/dy** is the (negative) marginal opportunity cost of obtaining more *y**. Since 3(3v2/3v*)/3y$_1$* = *d*^{2}*y£/dy**^{2} < 0, the marginal cost of yi must be increasing. As *yi* increases, the slope, or marginal cost *dy^/dy*,* of yi becomes more negative.

Hence, the more $y_{(}$ the economy produces the more y_2 must be forgone to obtain additional units of *y*. This situation is depicted in Fig. 18-3. As $y_{(}$ increases from *y* to *y* to *y*, the marginal cost of these increments, measured by the vertical declines in y_2 produced *{y\ — y* and *y\ — y\),* increases.

Given the interpretation of *dy^/dy** as the marginal cost of y_x, Eq. (18-31) becomes the familiar condition for maximizing behavior so that marginal costs equal marginal benefits, i.e., here, marginal revenue. The world market is willing to exchange y_2 for additional units of y_x at the rate *p = p\/p*$_2$. NNP will be maximized when, at the margin, the cost of *y* in terms of y_2 forgone is just equal to the additional revenue produced by the increased *yi* and decreased y_2 production, measured by the price ratio *p = p\/p*$_2$. Equation (18-32) is the statement that at a *maximum* position, marginal costs must be rising. Of course, all this analysis can be done with the axes interchanged, i.e., in terms of marginal costs and benefits for y_2 rather than _yi. The model is perfectly symmetric in *yi* and y_2*;* valid results are obtainable by interchanging $y_{(}$ and y_2, and likewise labor and capital.

We note in passing that in the two-good case, since *dy*/dp*$_x$, *dy\ldp*$_2$ > 0, it must be that *dy^/dpi = dy\/dp*$_2$ < 0. This follows from the homogeneity of the output supply functions and is left as a problem for the student. In the n-good case, however, no sign is implied for the off-diagonal terms *dy*/dpj, i =fc j.*

18.3 THE FACTOR PRICE EQUALIZATION AND RELATED THEOREMS

Let us now move on to the analysis of the classic theorems of international trade: the factor price equalization, Stolper-Samuelson, and Rybczynski theorems. These results were presented for the special case of fixed-coefficient technology in the previous chapter. To derive these results for the case of variable proportions, the assumption of linear homogeneous industry production functions must be added.

We have not been specific about the nature of the firms in this model, as it was not germane to the results. Let us now assume, however, that each industry is composed of many "identical" firms. They may differ in scale, but the underlying production function for each firm in a given industry must be the same. Under these conditions, an industry production function can be well defined. That is, if all firms are identical in the preceding sense, the aggregate output of the industry is expressible as a well-defined (single-valued) function of the total labor and capital inputs. Moreover, this aggregate production function will be linear homogeneous. Consider a 10 percent increase in the demand for the industry's output. With constant factor prices, the long-run effects will be merely to increase the number of firms in the industry by 10 percent (if the scale of each firm is the same). These new firms, being identical to the preexisting firms in the industry, will hire like proportions of labor and capital. Hence, 10 percent more labor and 10 percent more capital will be used to produce the 10 percent increase in output. Since this will occur for any initial input combination, the industry, or aggregate production, function can be characterized as having constant returns to scale.

Example. Consider two firms *in the same industry,* with production functions

$$y_x =$$

$$,2/3\wedge 1/3$$
$$= L_2 \quad K_2$$

The industry output _y by definition is $y = y_l + y_2$. (Remember, this is for *one good,* not the previous model for two goods.) Total factor usage for these two firms is $L = L\backslash + L_2$, $K = K\backslash + K_2$. Is it possible to define a function y — $f(L,K)$, that is, aggregate output, as a function of aggregate factor inputs? The answer is readily seen to be in the negative. Students can convince themselves of this by trying to do so. For example,

$$y = LfK?^3 + L^2{}_2{}^{/3}K_2{}^{1/3} = L\backslash^3K?^3 + (L - \mathrm{L},)^{2/3}(* - \mathrm{AT},)"^3$$

There is no way to eliminate $L_{\{}$ and $K\backslash$ from this equation. Output y will always depend on the allocation of labor and capital between the two firms. That is, if a unit of labor and /or capital is moved from firm 1 to firm 2, then even though aggregate labor and capital remain the same, total output will change, since the firms' production functions are different. Hence, *industry* output, under these circumstances, is *not* a (single-valued) function of *total* resource utilization.

We shall therefore make the assumption that there are two industries, each consisting of many identical "small" firms. Then, although factor prices are endoge-nously determined in the model, each firm can be perceived as taking factor prices as given. That is, the actions of any one firm cannot affect in any substantial way either factor or output prices. The simultaneous (and identical, since all firms are identical) actions of all firms together do affect prices, but these effects are beyond the control of any given firm. Most importantly, the *industry* production functions can be assumed to be linear homogeneous, i.e., exhibiting constant returns to scale. Let us characterize this linear homogeneity as follows. By definition,

$$f^J(tLj,\ tKj) = tf\backslash Lj,\ Kj) = tyj \qquad 7 =$$

1, 2 Since this holds for all t, let $t = \backslash/yj$. Then

$$\wedge,\ \wedge\text{-}) = f^j(a_{Lj},\ a_{Kj}) = 1 \qquad j = 1, 2 \qquad (18\text{-}33)$$
$$y \qquad y)$$

Equation (18-33) defines the production function in terms of the input-output coefficients. These a,/s were utilized in the linear model of the previous chapter. There, they were considered to be constants. Here, they are variable, changing continuously along the firm's *unit isoquant.* Equation (18-33) says that for constant-returns-to-scale production functions, the function is completely described by the unit isoquant. This occurs since *all other isoquants* are linear *radial blowups or contractions of the* unit (or any other) isoquant.

We shall use Eqs. (18-33) explicitly as constraints in the problem. In inequality form, we shall presume that inputs are combined such that

$$f^j(a_{Lj}, a_{Kj}) > 1$$

The resource constraints can also be expressed in terms of the a,-/s. By simple arithmetic,

$$L_x + L_2 = \text{—}y\backslash + \text{—}y_2 = L$$
$$y\backslash \qquad yi$$

or

$$a_L\backslash y\backslash + a_{L2}y_2 = L \qquad \text{(18-34a)}$$

Similarly, the capital constraint is

$$a_K iyi + a_{K2}y_2 = K \qquad \text{(18-346)}$$

The model of revenue (income) maximization subject to resource constraints can therefore be written

maximize

$$z = piyi +$$

Piy_2 subject to

$$a_L\backslash \mathrm{vi} + a_{L2}y_2 < L \qquad a_{Ki}y_x + a_{K2}y_2 < K \qquad \text{(18-35)}$$
$$f\backslash a_{LX}, a_{Kl}) > 1 \qquad f^2(a_{L2}, a_{K2}) > 1$$

The decision variables are the *six* variables Vi, y_2, a_Li, $a_K\backslash$, a_{L2}, a_{K2}. Remember, the *ciij* 's are not constants, as in the linear programming model. They represent input combinations that are jointly determined with outputs. The Lagrangian for the problem posed in (18-35) is

$$p_2y_2 + w\{L - a_{Li}yi - a_{L2}y_2) + r\{K - a_K iy_x - a_{K2}y_2)$$
$$\mathrm{W\ Vi},\ a_{Kl}) - 1) + k_2(f^2(a_{L2},\ a_{K2}) - 1) \qquad \text{(18-36)}$$

The Lagrange multipliers *w* and *r* represent the imputed wages, or rental values, of labor and capital, as before. The Kuhn-Tucker first-order conditions are thus

$$\text{----} = pi - a_{LX}w - a_K ir < 0 \qquad \text{if} <,\ y_x = 0 \qquad \text{(18-37a)}$$

$$\frac{fiSP}{dy_2} = p_2 - a_{L2}w - a_{K2}r < 0 \qquad \text{if} <,\ y_2 = 0 \qquad \text{(18-37/?)}$$

$$\frac{d}{oa_Li} = -_w yi + k_{\{}\frac{\wedge}{da_L\backslash} < 0 \qquad \text{if} <, a_{L1} = 0 \qquad \text{(18-38fl)}$$

$$\frac{}{3a_K\backslash} \qquad \frac{}{da_K i} < 0 \qquad \text{if} <, a_{Kl} = 0 \qquad \text{(18-386)}$$

$$dX \qquad df^2$$

da_{L2} da_{L2}

∂SP $_{ry2} + X_2$ $\partial f^{\wedge}$

da_{K2} da_{K2}

< 0 if $<, a_{L2} = 0$ (18-38c)

< 0 if $<,{}^{\wedge}_2 = 0.$ (18-38J) and the constraints

$$\frac{\partial}{\partial w} = L - a_{L1}y_1 - a_{L2}y_2 \geq 0 \qquad \text{if} >, \ w = 0 \qquad (18\text{-}39a)$$

$$\frac{\partial}{\partial r} = K - a_{K1}y_1 - a_{K2}y_2 \geq 0 \qquad \text{if} >, \ r = 0 \qquad (18\text{-}39b)$$

$$\frac{\partial}{\partial k_1} = f^1(a_{L1}, a_{K1}) - 1 \geq 0 \qquad \text{if} >, \ k_1 = 0 \qquad (18\text{-}40a)$$

$$\frac{\partial}{\partial k_2} = f^2(a_{L2}, a_{K2}) - 1 \geq 0 \qquad \text{if} >, \ k_2 = 0 \qquad (18\text{-}40b)$$

Notice the relations (18-37). These are precisely the nonpositive profit conditions that formed the constraints of the dual linear programming problem of the previous chapter. They say that if the maximum position occurs such that "profits" are negative for either good, then the output of that good will be 0.

The relations (18-38) are the marginal conditions for factor utilization, with the units adjusted to reflect the constraint of being on the unit isoquant. Since $f^j(L_j, K_j)$ is homogeneous of degree 1, f_L^j is homogeneous of degree 0. Thus

$$f_L^j(L_j, K_j) = f_L^j\left(\frac{L_j}{y_j}, \frac{K_j}{y_j}\right) = f_L^j(a_{Lj}, a_{Kj})$$

Dividing Eqs. (18-38) by y_j gives

etc., or

$$\frac{k_j}{y_j}\frac{\partial f^j}{\partial L_j} - w \leq 0 \qquad \text{if} <, \ L_j = 0 \qquad (18\text{-}41a)$$

with a similar relation with respect to capital:

$$\frac{k_j}{y_j}\frac{\partial f^j}{\partial K_j} - r \leq 0 \qquad \text{if} <, \ K_j = 0 \qquad (18\text{-}41b)$$

Equations (18-41) are equivalent to Eqs. (18-38). They are the usual marginal conditions for factor utilization if one interprets (k_j/y_j) as marginal cost of y_j. Equations (18-38) or (18-41) say that if the maximum position occurs where the value of the marginal product of any factor is less than its wage (or rental price), it will not be used. Otherwise, the value of the marginal product equals the wage.

Consider again the maximum problem posed in (18-35) and the associated Lagrangian (18-36). This maximization takes place at six margins, i.e., for six choice variables, $y_1, y_2, a_{L1}, a_{K1}, a_{L2}$, and a_{K2}. It is possible to conceive of this maximization as taking place in two stages. Recall that the assertion that profits are maximized carries with it the implication that the total cost of that level of output must be minimized. The maximization can be achieved by first minimizing cost for *any* output level; then, with costs minimized, that output level which maximizes profits can be determined as the second part of a two-stage maximization procedure.

In the present model, the hypothesis that total revenue is maximized (subject to resource constraints) can be regarded as occurring in two stages also. First, the "correct" input combinations *along the unit isoquant* can be found. That is, holding $yj = 1$, $j = 1, 2$, we select the a,-/s so as to minimize total factor cost for each industry. This results in having the marginal technical rate of substitution between the factors equal the factor price ratio. Then, given this tangency condition, outputs are varied along the expansion path [in this case, along a ray through the point $(a^*_{L}j,\ a^*_{K}j)$ on the unit isoquant] until the _y*'s are found.

This process can be seen algebraically by rearranging the terms in the Lagrangian (18-36) as follows:

$$56 = P\backslash y\backslash + P2y2 + wL + rK - [y_x(a_{LX}w + a_{KI}r) + A,(1 - f\backslash a_{LX},\ a_{KX}))]$$

$$a_{K2}r) + A._2(1 - f^2(a_{L2},\ a_{K2}))] \qquad (18\text{-}42)$$

The maximum value of 56 is NNP* $= p_xy\backslash + p_2y\backslash$. Let us maximize 56 by first *minimizing* the two square-bracketed terms (which enter negatively) with respect to the a,/s, treating y_x and y_2 as parametric. This is equivalent to two separate *minimizations:*

minimize

$$y_x(a_{LX}w + a_{KX}r)$$

subject to

$$f\backslash a_{LX}, a_{KX}) = l \qquad (18\text{-}43fl)$$

and minimize

$$y_2(a_{L2}w + a_{K2}r)$$

subject to

$$f\backslash a_{L2}, a_{K2}) = \backslash \qquad (18\text{-}43*)$$

The Lagrangians for these two problems are exactly the square-bracketed terms in the Lagrangian (18-42). Moreover, these minimization problems are equivalent to the standard cost minimization formats: Multiply the objective functions through by y_x or y_2 as indicated in (18-43). Since $y_xa_{LX} = L_x$, etc., and $yjf^j(a_Lj,\ a_{Kj}) = f^j(yjaLj,\ yjaxj) = f^j(Lj,\ Kj)$ by linear homogeneity, these square-bracketed terms are, respectively, equivalent to

$$(wL_x + rK_x) + X_x(y_x - f^1(L_x,\ K_x)) \qquad (18\text{-}44a)$$

and

$$(wL_2 + rK_2) + \hat{}_2(y_2 \sim f(L_2,\ K_2)) \qquad (18\text{-}$$

44*)

[Again, from homogeneity, $f^{l}\{a_{L}\backslash, a_{K}\backslash)$ = 1 is equivalent to $f^{l}\{L\backslash, K_{l})$ = $v_{[5}$ etc.] The Lagrangian expressions (18-44) are exactly those which result from the problems

minimize

$$wL\backslash + rK\backslash =$$

$C\backslash$ subject to

$$f^{l}(L_{u}K_{l}) = y_{l} \qquad \text{(18-45a)}$$

and minimize

$$wL_{2} + rK_{2} =$$

C_{2} subject to

$$f(L_{2}, K_{2}) = y_{2} \qquad \text{(18-456)}$$

where $y\backslash$ and v_{2} are at this point parametric. Problems (18-43) and (18-45) are thus equivalent. The setup (18-45) is the classic problem of minimizing the cost of achieving some output *yj*. The objective function *Cj* is the total cost of achieving that output level, *and Xj is the marginal cost of that output.* The *Xj*'s are *not* the same, however, in (18-45) and (18-43), because the units of the constraints are different.

The Lagrangian associated with the submodel (18-43a) is

$$SB = y\backslash\{wa_{Ll} + ra_{Kl}) + A_{x}(1 - f\backslash a_{LX}, a_{K}i)) \qquad \text{(18-46a)}$$

Note that since $y\pm$ is treated as a constant here, minimizing $y\backslash(wa_{L}\backslash + ra_{K}\backslash)$ yields the same solution and comparative statics results as minimizing $wa_{L}\backslash + ra_{K}\backslash$. The first-order conditions obtained from (18-46a) are

$$\hat{}- = yw - A\text{-}i\hat{}- = 0 \qquad \text{(18-47a)}$$

$$\text{-----} = {}_{yi}r - A.i -\hat{}— = 0 \qquad \text{(18-476)}$$

$$-\sim = l\text{-}f^{l}(a_{u}, a_{Kl}) = 0 \qquad \text{(18-47c)}$$

$$dki$$

Equations (18-47) are precisely the relations (18-38a), (18-386), and (18-40a), respectively (ignoring the possibility of corner solutions).

A similar set of results follows from the submodel (18-436). The Lagrangian for industry 2 is

$$\% = y2(a_{L2}w + a_{K2}r) + X_2(l - f^2(a_{L2}, a_{K2})) \qquad (18\text{-}466)$$

[Again, from homogeneity, $f^x(a_L\backslash, a_K\backslash) = 1$ is equivalent to $f^x(L\backslash, K\backslash) = y\backslash$, etc.] The Lagrangian expressions (18-44) are exactly those which result from the problems

minimize

$$wLy + rK\backslash = C\backslash$$

subject to

$$\boldsymbol{f^l(L_l, K_l) = y_l} \qquad \textbf{(18-45*)}$$

and minimize

$$wL_2 + rK_2 = C_2$$

subject to

$$/^2(\mathrm{L}_2, K_2) = y_2 \qquad (18\text{-}45\&)$$

where $y\backslash$ and y_2 are at this point parametric. Problems (18-43) and (18-45) are thus equivalent. The setup (18-45) is the classic problem of minimizing the cost of achieving some output *yj.* The objective function *Cj* is the total cost of achieving **that output level,** *and kj is the marginal cost of that output.* The Ay's are *not* **the** same, however, in (18-45) and (18-43), because the units of the constraints are different.

The Lagrangian associated with the submodel (18-43a) is

$$\% = ydwa_{Ll} + \boldsymbol{ra_{Kl})} + \mathrm{Ml} - \boldsymbol{f^l(a_{Ll}, a_K i))} \qquad \textbf{(18-46a)}$$

Note that since ji is treated as a constant here, minimizing $yi(wa_{L[} + ra_K\backslash)$ yields the same solution and comparative statics results as minimizing $wa_L\backslash + ra_K\backslash$. The first-order conditions obtained from (18-46a) are

$$\boldsymbol{k_l\wedge\text{——}=O} \qquad (18\text{-}47a)$$

$$\text{------}=_{yi}r\ -X_x\text{-}l\text{—}\ =0 \qquad (18\text{-}47/7)$$

$$da_{Kl} \qquad da_{Kl}$$

$$\text{--}0 \qquad (18\text{-}47c)$$

Equations (18-47) are precisely the relations (18-38a), (18-38Z?), and (18-40a), respectively (ignoring the possibility of corner solutions).

A similar set of results follows from the submodel (18-43/?). The

Lagrangian for industry 2 is

$$\& - yi(a_{L2}w + a_{K2}r) + A._2(1 - f(a_{L2},\ a_{K2}))$$ (18-46/?)

producing the first-order conditions

$$a/^2$$
$$\hat{}\text{——}=O \qquad (18\text{-}48.)$$
$$9$$

$$(18\text{-}48^*)$$

$$l\text{-}f(a_{L2}, a_{K2})=0 \qquad (18\text{-}48c)$$

These equations are, respectively, the same as relations (18-38c), (18-38d), and *(\S-40b). Hence, the two suboptimizations account for six of the ten first-order equations of the whole model.* These six equations determine the cost-minimizing input combinations *(a^*_{Ll}, a^*_{Kl}), (a^*_{L2}, a^*_{K2})* along the unit isoquants of each industry and the two industry marginal cost functions *(k\/y*)* and *(Xyy_2)*. The remaining four variables to be determined are y^*, y_2, *w,* and *r*.

The Four-Equation Model

Let us "solve" the systems (18-47), (18-48) for the .*.'s. Dividing (18-47.) by (18-47*) and (18-48.) by (18-48*) yields

$$\frac{dfl/daLl}{df^1/da_{Kl}} \quad \frac{w}{r} \qquad (18\text{-}49.)$$

and

$$\frac{df^2/da_{L2}}{df^2/da_{K2}} \quad \frac{w}{r} \qquad (18\text{-}49^*)$$

Equations (18-49.) and (18-47c) represent two equations in the two unknowns *&L\, <*k* and the variable *w/r*. Likewise, (18-49*) and (18-48c) represent two equations in the two unknowns a_{L2}, a_{K2} and the same variable *w/r*. Thus, we can write the solution of the equation systems (18-47) and (18-48) as

$$J \; v \; (\; j \;) \qquad i = L, K \qquad .7 = 1,2 \qquad (18\text{-}50)$$

and

$$k = \hat{}_{(w,r)} \qquad j = l, 2 \qquad (18\text{-}51)$$
$$yj \quad y^*$$

These equations are a very important feature of this model. They say that the input-output coefficients are functions of the *factor price ratio only.* In particular, the *afj's* are *not* functions of the endowments of either factor. Second, the marginal cost functions (18-51) are not functions of output levels but only of factor prices. This all occurs because of the linear homogeneity of the production functions. The independence of marginal cost from output level was shown in Chap. 9 on cost functions. There, we showed that if $y = f(x_1, x_2)$ was linear homogeneous, the cost function could be written $C^* = yA(w_1, w_2)$. Consequently, $dC^*/dy = A(w_1, w_2)$, which is Eqs. (18-51). Equations (18-50) occur because for linear homogeneous functions, any level curve describes the whole function. Whatever occurs at any output level y° is simply a magnification or contraction of what occurs at $y_{;}$ = 1. It is apparent from general comparative statics theory that

$$\frac{da^{*}_{:}}{dw} < 0 \qquad (18\text{-}52a)$$

and

$$\frac{da_{Kj}}{8r} < 0 \qquad (18\text{-}526)$$

The unit input-output factor levels are downward-sloping in their own price. It is also apparent that *a*j(w/r)* is homogeneous of degree 0 in *w* and r, since *a**- is a function of the ratio *w/r.* From Euler's theorem,

$$\frac{dah}{aw} \quad or \qquad =0$$

Since $da^{*}_{L}\text{-}/dw < 0$,

$$\frac{3a^{*}_{Lj}}{dr} > 0 \qquad (18\text{-}52c)$$

Similarly, it can be shown that

$$\frac{dat}{dw} > 0 \qquad (18\text{-}52d)$$

We shall use these results later.

If we use these solutions to the *six* Eqs. (18-47) and (18-48), the entire 10-equation model (the 10 first-order conditions) can be reduced to four equations in the four unknowns y_u y_2, *w,* and r. The remaining equations of the original 10 are (18-37a), (18-376), (18-39a), and (18-3%). Substituting the solution values (18-50) back into these equations (again, we ignore the possibility of corner solutions) yields

$$a^{*}_{L1}w + a^{*}_{KX}r = p_x \qquad (18\text{-}53«)$$

$$a^{*}_{L2}w + a^{*}_{K2}r = P2 \qquad (18\text{-}536)$$

and

$$a^{*}_{L1}yi + a^{*}_{L2}y_2 = L \qquad (18\text{-}$$

54a)

K (18-546)

The entire model has been compressed to four equations. *Moreover, these are precisely the same four relations as were derived for the linear programming model, two zero-profit conditions Eqs. (18-53) and two resource constraints (18-54).* Here, however, the a^*, 's are not constants. They are functions of the factor price ratio w/r, as indicated by Eqs. (18-50).

Although the preceding are four equations in four unknowns, these equations have a very special structure. The variables *w* and *r* and the parameters *p* and p_2 appear only in the first two Eqs. (18-53). And the variables $y_{\{}$ and y_2 and the parameters *L* and *K* appear only in the second set of Eqs. (18-54). Thus, these equations are actually decomposable, or separable, into two sets of two equations in two unknowns. Just as in the linear model, the coefficient matrix of «*;'s for the first set [Eqs. (18-53a) and (18-53Z?)] is the transpose of the coefficient matrix of the second set [Eqs. (18-54)].

As a consequence of this separability, Eqs. (18-53) can be "solved" independently of (18-54), since the a*;'s are functions of *w/r* only:

$$w=w^*(pi,p_2) \qquad \text{(18-55a)}$$

$$r=r\backslash p_u p_2) \qquad \text{(18-55*)}$$

On the other hand, although Eqs. (18-54) can be solved for *y* and y_2 in terms of *L* and *K,* these solutions will involve the a*/s, which are functions of *w/r* and hence *Px* and p_2 through (18-55). Thus, solving Eqs. (18-54) and using (18-55) leads to the output supply functions

$$y\backslash =y^*_l(pup_2,L,K) \qquad \text{(18-56a)}$$

$$y_2 = y_2(puP2,L,K) \qquad \text{(18-566)}$$

just as in the original model without the homogeneity conditions. As before, the results *dyl/dp\\, dy\fdp*$_2$ > 0 are still valid; the supply curves of each industry are upward-sloping.

The Factor Price Equalization Theorem

Equations (18-55) are the basis of what is known as the *factor price equalization theorem,* a fundamental result in the theory of international trade. Consider the case of two countries, each producing the same two commodities and engaging in trade with one another. In the pretrade, or *autarky,* situation, output prices in the two countries will in general differ, given different marginal costs of production, i.e., different production possibility frontiers, for the two countries. (Of course, consumers' tastes might differ systematically in the two countries, producing different output prices even if the marginal cost functions for the two countries were identical.) However, with the opening up of trade, which will occur precisely because output prices (and hence consumers' marginal evaluations of the goods) are different, the output prices will tend toward equality. With no transportation or other transactions costs of trading, the gains from trade will be exhausted only when output prices are identical in the two countries, i.e., when each country's consumers face the same set of output prices. Given the postulate of "more preferred to less," this outcome is implied.

A less obvious question is the effect on *factor* prices of this tending to equality of output prices. If factors were freely mobile between the two countries at zero cost, clearly, factor prices in the two countries would also have to be identical. Factors would simply move to the higher-paying country, depressing wages or rentals there and raising them in the former location. But what if factors *cannot* move from one country to another? That is, suppose goods can move costlessly from one country to the other but factors can never emigrate. What will happen to factor prices then, when output prices converge?

Equations (18-55) say that under certain conditions, factor prices will also be equal, in the two countries, when output prices are the same for both countries, in spite of factor immobility. This surprising result, known as the *factor price equalization theorem,* depends upon the form of Eqs. (18-55). Those equations indicate that factor prices are functions of *output prices only.* Factor endowments do not enter the right-hand side of these equations and hence are irrelevant in determining factor prices. However, the specific functional form of $w^*(pi, p_2)$ and $r^*(p_{i:}\ p_2)$ will depend upon the underlying production functions in the economy. If the production technology, i.e., the underlying production functions, *is the same in both countries,* i.e., trade is taking place because of different endowments of factors or differences in consumers' tastes (or both) between the two countries, then the functional form of Eqs. (18-55) will be the same for the two countries. In that case, the factor prices will be the same in both countries, since they will depend in identical fashion upon the output prices, which are the same for both countries. An additional qualification, relating to differing relative factor intensities in the two countries, will be explored presently. Notice, too, that the result depends critically on the assumption of linear homogeneous industry production functions. It is that assumption which permits the formulation of the first-order conditions in terms of the factor intensity variables, the a,/s, which, in turn, allows solution of these a_{ij};'s in terms of the relative price ratio *w Ir* alone. It is the dependence of relative factor intensities on factor prices alone which makes Eqs. (18-53), the zero-profit conditions, soluble for factor prices solely in terms of output prices. Without constant returns to scale in each industry, the preceding procedure cannot be carried out.

The Stolper-Samuelson Theorems

Let us now investigate the effects of changes in output prices on factor prices. Since Eqs. (18-53) are the sole determinants of factor prices, the comparative statics of this part of the model is accomplished by differentiating Eqs. (18-53) with respect to output prices. Let us differentiate these equations with respect to *p\,* remembering that the "solutions" $w = w^*(pi, p_2), r = r^*(p_{(}, p_2)$ have been substituted into these equations for w and r, respectively, and that the a^*- 's, being functions of factor prices, are thereby also functions of the output prices. Hence, upon differentiation of(18-53a),

*d ■ w d + *d
Ld * d d

*d ■ a d w
* *

daK\ = j
dpi or
3oh_rJ_aU (]8_57fl)

dpi *dpi*

Similarly differentiating (18-53Z?) gives

d_{Pl} (18-57*)

However, the last two terms on the right-hand side of (18-57a) and (18-57Z?) sum to zero: consider the production function $f^x(a^*_{Ll}, a^*_{KX})$ = 1. Differentiating this identity with respect to $p\backslash$ gives

$$da_{Ll}\ dpi \quad + \quad da_{KJ}\ dp_x \quad = 0 \qquad (18\text{-}58)$$

Using the first-order conditions $df^1/da_L\backslash$ — $y\backslash w/k\backslash$, $df^1/da_K\backslash$ — $y\backslash rfk\backslash$, and eliminating the factor $y\backslash/X\backslash$ in each term, we get

3 f | I ^ 0 (18_59fl)
dpi dpi

A similar procedure shows that

W*_L2 + r*_^2 _ Q (18-5%)
dpi dpi

Therefore, the comparative statics Eqs. (18-57) reduce to the simple form

$$a_{L2}\hat{} + a_{K2}\hat{} = 0$$
dpi dpi

Equations (18-60), which give the changes in factor prices caused by changes in output prices, *have exactly the same structure as the equations that determined these variables in the linear models.* [In the case of constant a_{ij} 's, the differential form (18-60) is directly equivalent to the undifferentiated form (18-53).] Therefore, the analysis of this model is identical, in regard to these variables, to the

+a si (18 60a)

a"£ ^ -£ 0 (18-60/7)

A =
aL2 aK2

Solving for dw^*/dpi, dr^*/dpi by Cramer's rule, we have

$$\frac{3vv^*}{dpi} - -f\frac{at_0}{A} \qquad (1$$

and

$$f = -°-f \qquad (18\text{-}61W$$

d A

In like fashion, if Eqs. (18-53) are differentiated with respect to p_2, one gets

$$\frac{\quad}{dp_2} \quad \frac{a_f}{A} \qquad (1$$

$$\frac{'^l}{dp_2} = \frac{a_k}{A} \qquad \text{(18-6W)}$$

Let us investigate these relationships. In the first place, these solutions are valid only if $A \neq 0$. This is in fact the sufficient condition of the implicit function theorem that the equations $w = w^*(p_1, p_2)$, $r = r^*(p_1, p_2)$ are locally well defined. Hence, this condition is also required for the factor price equalization theorem. The determinant A will be nonzero, in this two-factor, two-good case, if either

$$\frac{4^1}{a^*_{K1}} > \frac{4^1}{a_{K2}} \qquad (\text{1\%-62}a)$$

or

$$aK\backslash \qquad {}^aK2$$

These equations are equivalent to

$$> \qquad \text{(18 63fl)}$$

$$\text{I}^* \quad \text{f}^*$$

or

$$\text{L}^* \qquad L^*$$

$$\text{—}^l\text{-} < \text{-}\pm \qquad \text{(18-63Z?)}$$

$$\text{V} \; {}^*\text{f} \; {}^*$$

$$\text{A1} \quad {}^\wedge 2$$

In other words, if one industry is more labor-intensive than the other, i.e., its capital labor ratio is lower than that ratio in the other industry, then the equations defining factor prices as functions of output prices only will be well defined. Also, the comparative statics relations (18-61) indicating the response of factor prices to changes in output prices will be well defined.

With regard to the comparative statics relations (18-61), the condition that one industry be more labor-intensive is a strictly *local* condition. All comparative statics equations, despite the name that connotes comparing separate equilibria, are in fact simply partial derivatives evaluated at a certain point. The functions defining the choice relations need only be well behaved around that one point; i.e., they must have the various properties of differentiability, nonzero Jacobian determinant, etc., to allow a solution for a choice function at a given point.

For purposes of asserting factor price equalization, however, the *local* condition that $L_1/K_1 \neq L_2/K_2$ is insufficiently strong. The factor price equalization theorem is an essentially *global* assertion. That is, it asserts that, starting at finitely different output prices in two countries, as output prices converge, factor prices will converge also.

But this is supposed to take place over a whole path of prices. Therefore, a strictly local condition on factor intensities cannot be enough to guarantee the convergence of factor prices. If factor prices are to converge *for any initial output* *prices* and for any endowments, then one industry will *always* have to be more labor- (or capital-) intensive than the other. That is, we must have *Lj/Kj* > *Lj/Kj* for *all* output prices. If industry 1 is initially the more labor-intensive industry, as output prices change, that industry must remain the more labor-intensive. Should one industry switch from being relatively labor- to relatively capital-intensive, the direction of movement of factor prices with regard to output price changes will reverse. In Eqs. (18-61), the denominators will all change sign. This means that if, say, industry 1 is labor-intensive at some output prices, wages and rents will move in one direction as output prices converge. However, for different endowments or if output prices are such that industry 1 is c<2/?/ta/-intensive, factor prices will move in the opposite direction as output prices converge in the two countries. What is therefore needed, in order to assert factor price equalization (aside from the other assumptions such as linear homogeneity, etc.) is the *global* condition that *L-JKi* > *Lj/Kj for all* possible output or output price combinations along the production frontier. Strictly local conditions are insufficiently strong.

If the production functions in each sector are homothetic (e.g., linear homogeneous), this "switching" of factor intensities cannot occur. Switching implies that the contract curve depicted in Fig. 18-4 would cross the diagonal line connecting the two origins O_1 and O_2. But then the diagonal would have to be the expansion paths of each production function; the contract curve would have to be the diagonal itself. This situation occurs when the factor intensities are constant and identical, e.g., if the production functions in each industry are identical Cobb-Douglas functions.

Suppose now that industry 1 is the more labor-intensive industry, i.e., that *C^LX/CLKX* > *CLLI/UKI-* (To save notational clutter, the asterisks will now be dropped.) Then $A > 0$ and, from Eqs. (18-6la) and (18-616),

dw

0 (18-

dp and

dpi < 0 (18-646)

These results are the general Stolper-Samuelson theorem. They say, again, that if the price of the labor-intensive industry is increased, nominal wage rates will rise, whereas capital rental rates will fall. If p_1 rises, then we know that y_1 increases and y_2 decreases, that is, $dy_1/dp_1 > 0$, $dy_2/dp_1 < 0$, as the economy moves along the production possibilities frontier. Hence, in this case, the labor-intensive industry is expanding whereas the capital-intensive industry is contracting. This results in a net increase in the aggregate demand for labor and a decrease in aggregate demand for capital. Hence, the factor price of labor rises while that of capital falls. In general, the price of a factor of production will rise if the price of the industry in which that factor is most intensively used rises; it will fall if the industry which is less intensive in that factor experiences an output price increase.

The preceding analysis, however, pertains to *nominal* price changes only. If p_1 and *w both* rise, as in the preceding example, will "real" wages in fact have risen? That is, will the owners of labor be able to purchase more goods at the higher wages

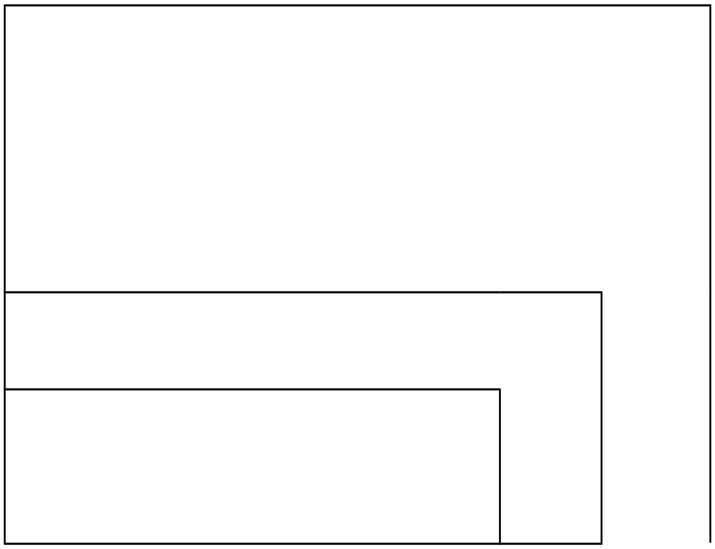

FIGURE 18-4
Diagrammatic Exposition of the Stolper-Samuelson Theorem: Variable Proportions. Consider an Edgeworth-type diagram as in Fig. 18-2, with labor plotted along the horizontal axes and capital plotted vertically. A contract curve *O\ O2* connecting the two origins has been drawn. It has a special shape: It is convex to the labor axis from *O*; that is, the contract curve rises toward *O2* at an increasing rate. It is this shape that guarantees that industry 1 will always be more labor-intensive than industry 2. Consider point F[1] along *O\ O2.* The slope of the chord *O* F[1] is *K\/L*. The slope of the chord connecting F[1] and *O2,* F[1] *O2,* is *K2/L2.* As drawn, *K\/L* < *K2/L2,* or *L\/K* > *L2/K2.* Moreover, this is true along any point of the contract curve. Industry 1 is always more labor-intensive than industry 2.

Consider now the slopes of the isoquants as they cross the contract curve. Near *O\,* where the contract curve is close to the labor axis, the isoquants are quite flat; i.e., they have a low absolute slope. As one moves along *O\C>2* toward *O2,* the isoquants cut the curve at increasing slopes, as depicted at points F' and *Y*[2]. The slope of the isoquants is *w/r,* the ratio of wages to rental rates. Thus, with industry 1 always the more relatively intensive, as output *y* expands, in response to increases in *p\,* wage rates rise relative to rental rates. This is in accordance with Eqs. (18-61) and the subsequent analysis. The increase in *real* wages is not easily depicted geometrically, however. Notice, too, that as *p* and thus *y* increase, *both* industries become less labor-intensive (though industry 1 remains more so than industry 2). As output moves from F[1] to *Y*[2]*,* for example, the capital-labor ratio, measured for industry 1 by the slope of the chord *O\ Y*[2] and for industry 2 by the slope of F[2] *O2,* increases. That is, the labor-capital ratio decreases, or both industries become less labor-intensive. This can be viewed as a response to the increase in real wage rates and the fall of real capital rental rates.

after these two price changes? Clearly, the owners of capital, whose money price has fallen, are worse off in real as well as money terms.

The *real* income of the owners of labor will also rise if the wage rate increases. Wages will increase at a higher percentage rate than the output price, i.e.,

lim *Aw/w* $= \frac{dw\ pi}{dp\backslash\ w} > 1$

(18-65)

If the owners of labor consume the output of industry 1 only, then Eq. (18-65) guarantees greater purchasing power. If the owners of labor consume some y_2 also, then since the price of y_2 hasn't changed, (18-65) represents an even greater increase in real income. (In the limit, if *only* y_2 were consumed, then *any* increase in xv would be an increase in real income, since p_2 is constant here.) From Eq. (18-61a), again,

a_{K2} >0

d

pi A From the zero-profit first-order relation (18-53a),

P

$i = (^L\|^w + \&Ki^r$

or

$$\frac{P\backslash}{xv} = a_{L}\backslash \frac{r}{xv} + a_{Ki}$$

Thus,

p_1 ∂w
a_{K2}
$(r/w)a$
> 1
w ∂p_1
a_{K2}

just as in the case of fixed coefficients. The same procedure shows that

That is, since industry 2 is capital-intensive, an increase in p_2 will not only increase nominal rental rates on capital but real rates also.

Another set of results coming under the heading of the Stolper-Samuelson theorem are the effects on factor intensities of changes in output prices. That is, consider how the labor-capital ratio varies in each industry when, say, p_1 is increased. Assume as before that industry 1 is labor-intensive. The labor-capital ratio in industry j is $Lj/Kj = a_Lj/a_Kj$. Specifically, the a_ij 's are functions of the factor prices w and r, which are in turn functions of output prices, or

$$— = g(w^*(p_1, p_2), r^*(p_1, p_2))$$

Using the quotient rule with the chain rule gives

$d(a_{Kj})$ $d a_{Lj}$ $d w$ r p_1 $d w$ $d p_1$ $d a_{Lj}$ $d r$ d d

- da
dr
q
$^L d$
$_j$

Rearranging terms gives

^$_2d(a_{Lj}/a_{Kj})$ (da_{Lj} da_{Kj}\ dw

(aKj)---------= a_{Kj} ——----a_{Lj} —— -—

dpi \ dw dw J dpi

'J **dr** J **dr**

This result follows from the comparative statics relations derived for the cost minimization submodels (18-43). The comparative statics of those models yielded the results, *for both industries,*

$UL>$ <0 ,- = 1.2

dw

<0 / = 1,2 (18-52&)

dr

dr

da_{Kj}

"$_{Ti}$ > 0 7 = 1,2 (18-52c)

>0 7 = 1,2 (18-52J)

dw

Inserting these sign values and also Eqs. (18-64), that is, $dw/dpi > 0$, $dr/dpi < 0$, into (18-66a) immediately shows that

< 0 $j = l, 2$

dpi

when yi is labor-intensive.

Similarly, with regard to changes in p_2, we have

$d(a_{Lj}/a_{K}j)$ (da_{Lj}

$\boldsymbol{dp_2}$ **V** $\boldsymbol{dw}$ $\boldsymbol{dw}$ $\boldsymbol{J}$ $\boldsymbol{dp_2}$

dr dr

The only differences between (18-66Z?) and (18-66a) are the terms dw/dp_2, dr/dp_2 instead of dw/dpi and dr/dpi. Since these latter two terms have the opposite sign of the first two, respectively,

W*» Q ; = 1,2

Note that Eqs. (18-66) say that if the price of the labor-intensive good (*yi* here) rises, then the labor-capital ratio will fall *in both industries.* With the rise in p_t, more of the labor-intensive good will be produced and less of the capital-intensive good. This results in a net increase in the demand for labor. However, total labor to the economy is fixed. The economy responds to this increase in demand in two ways: The price of labor w rises, and the rental price of capital falls, in accordance with Eqs. (18-64). To economize on the now higher-priced labor, *both* industries reduce the ratio of labor to capital utilized in production. (It may be a surprising piece of arithmetic that this is possible.) This situation is illustrated in Fig. 18-4. These results cannot be observed in the model with fixed-coefficient technology. There, the a_{ij}'s are constant and hence unchanged by output prices.

The Rybczynski Theorem

Let us now turn to the comparative statics of this two-factor, two-good variable-proportions model with respect to changes in endowments. Under the hypotheses of the factor price equalization theorem, which includes the assumption that one industry is always more labor-intensive than the other, a change in the resource endowment of either labor or capital (or both) will have *no effect* on factor prices. Again [Eqs. (18-50)] $a_{ij} = a^*_{ij}(w/r)$, and Eqs. (18-53) imply that factor prices are functions of *output prices only* [Eqs. (18-55)]. Thus, the first result is

$$\frac{\partial w}{\partial L} = \frac{\partial w}{\partial K} = \frac{\partial r}{\partial L} = \frac{\partial r}{\partial K} = 0 \qquad (18\text{-}67)$$

Do not forget that in these relations, *output prices* are being held fixed. Only resource endowments are changing. As endowments shift, the NNP plane $p_1y_1 + p_2y_2$ depicted in Fig. 18-3 shifts *parallel* to itself and becomes tangent to a new production frontier (not depicted) at the same output prices. Since output prices remain the same, factor prices are unchanged, given our assumptions.

Let us now consider the effects of changing the endowment of labor, say, on output levels. Since Eqs. (18-53) involve prices only, the comparative statics relations are derivable from Eqs. (18-54) alone, repeated here:

$$a^*_{L2}y_2 = L \qquad (1$$

$$K \qquad (18\text{-}54/7)$$

These two identities are the original resource constraints of the model, with the important added condition that the linear homogeneity assumption for the production function has been used to express the a_{ij}'s as functions of factor prices w and r (in particular w/r) only. If now either L or K changes, the a_{ij}'s remain constant, since $\partial a^*_{ij}/\partial L = [\partial a^*_{ij}/\partial(w/r)][\partial(w/r)/\partial L] = 0$, since the latter term is 0, from the preceding discussion. Hence, for the comparative statics of this model *with regard to changes in endowments, the a^*_{ij}'s can be treated as constants, even in this variable-proportions model!*

Let us then differentiate Eqs. (18-54), partially of course, with respect to L. (Again, the asterisks will be dropped to save clutter. But do not forget the assumptions needed to perform these operations.) Differentiating gives

$$\frac{\partial y_1}{\partial L} + a \frac{\partial y_2}{\partial L}$$

Thus, using Cramer's rule, we find, as in the linear programming model,

$$\frac{\partial y_1}{\partial L} = \frac{a_{K2}}{A} = \frac{a_{K2}}{a_{L1}a_{K2} - a_{L2}a_{K1}} \qquad (18\text{-}68a)$$

$$\frac{\partial y_2}{\partial L} = \frac{-a_{K1}}{A} = \frac{-a_{K1}}{a_{L1}a_{K2} - a_{L2}a_{K1}} \qquad (18\text{-}68b)$$

Under the assumption that industry 1 is labor-intensive, $A > 0$, and thus $\partial y_1/\partial L > 0$, $\partial y_2/\partial L < 0$. Differentiation of (18-54) with respect to K yields

$$\frac{\partial y_1}{\partial K} = \frac{-a_{L2}}{A} = \frac{-a_{L2}}{a_{L1}a_{K2} - a_{L2}a_{K1}} \qquad (18\text{-}68c)$$

$$\frac{\partial y_2}{\partial K} = \frac{a_{L1}}{A} = \frac{a_{L1}}{a_{L1}a_{K2} - a_{L2}a_{K1}} \qquad (18\text{-}68d)$$

Again, assuming industry 1 is labor-intensive, (18-68c) says that $\partial y_1/\partial K < 0$, and (18-68d) shows that $\partial y_2/\partial K > 0$.

These results, known as the *Rybczynski theorem,* state that under the hypotheses of the factor price equalization theorem, an increase, say, in the endowment of labor (holding output prices constant) will increase the output of the labor-intensive industry and decrease the output of the capital-intensive industry. Likewise, an increase in the endowment of capital, ceteris paribus, will increase the output of the capital-intensive industry and decrease the output of the labor-intensive industry. Again, under our strong assumptions, all these repercussions will leave factor *prices* unchanged. These results were illustrated for the linear models in Fig. 17-5.

Equations (18-68) are in fact derivable from earlier results. Recall the reciprocity conditions (18-25a) and (18-25b), which were derived from the general model, without the homogeneity restrictions:

$$\frac{\partial y_1}{\partial L} = \frac{\partial w}{\partial p_1} \qquad \frac{\partial y_2}{\partial L} = \frac{\partial w}{\partial p_2} \qquad (18\text{-}25a)$$

and

$$\frac{\partial y_1}{\partial K} = \frac{\partial r}{\partial p_1} \qquad \frac{\partial y_2}{\partial K} = \frac{\partial r}{\partial p_2} \qquad (18\text{-}25b)$$

Inspection of Eqs. (18-61) and (18-68) confirms these reciprocity conditions. For example, from (18-61a) and (18-68a),

$$\frac{\partial w}{\partial p_1} = \frac{\partial y_1}{\partial L} = \frac{a_{K2}}{A}$$

The Rybczynski theorems are in fact merely the dual relationships of the Stolper-Samuelson theorems. The relations between factor and output *prices* are identical to the relations between *physical* factors and outputs. All the results for factor prices have exact analogs for the factors themselves, and vice versa.

In particular, the elasticity relationships (18-65) for real factor price changes have corresponding results for the factors themselves. The algebra is identical, since the comparative statics formulas (18-61) and (18-68) are identical. For example, suppose the endowment of labor increases, again assuming that industry 1 is labor-intensive. Then the output of industry 1 will not only increase but will increase at a faster rate than the increase in labor, i.e.,

$y\backslash\ dL$

As before, since $L = a_{L1}yi + a_{L2}y_2$, $L/y_x = a_{LX} +$ a$_L$2(>2/}i)-

(18-69a) Thus

$$\frac{L\ dy_x}{y_x\ dL} = \frac{a_K2CiL\backslash + amaLiiyi/yx)}{a_{K2}a_Li - a_{KX}a_{L2}}$$

A similar procedure shows

$$\frac{K\ dy_2}{y_2 dK} > 1 \qquad (18\text{-}6\%)$$

in perfect analogy with Eqs. (18-65).

18.4 APPLICATIONS OF THE TWO-GOOD, TWO-FACTOR MODEL

The Stolper-Samuelson theorem can be used to determine the effects of tax policies on income distribution. Consider an economy characterized by all the assumptions underlying the theorem. Assume the economy is trading freely with the rest of the world, i.e., with no policy restrictions on the flow of commodities. Assume the size of the economy is small so that it is a price taker, and that transport costs are small. Denote the exogenously given world prices of goods 1 and 2 by *p*\ and /?2, respectively. Assume that under free trade, good 1, the labor-intensive good, is imported, and the capital-intensive good 2 is exported by the economy.

Suppose now the government imposes a tariff on the imported foreign good. Denote the ad valorem tariff rate by *t*. The domestic prices become

$$P1 = (1+0/>? \qquad (18\text{-}70fl)$$

$$Pi = p^\circ_2 \qquad (18\text{-}706)$$

These two equations show that as a result of the tariff, the price of good 1 increases relative to that of good 2. Since good 1 is labor-intensive, by the Stolper-Samuelson theorem, the real wage rate (in terms of either good) increases while the real rental rate decreases. Thus, we have the following result: A tariff will benefit the factor that is used more intensively in the importable sector and will hurt the other factor. Two simple applications of the Rybczynski theorem will now be introduced. First, consider again the small open economy just discussed. The production possibility frontier of the economy is represented by the curve *AB* in Fig. 18-5. Firms in the economy face exogenously given world prices of *p*® and *p*\\ *MN* represents the world price line whose slope equals —*p*®/p_2- As shown in Fig. 18-3, the production point occurs at the point of tangency, point *E,* between curve *AB* and price

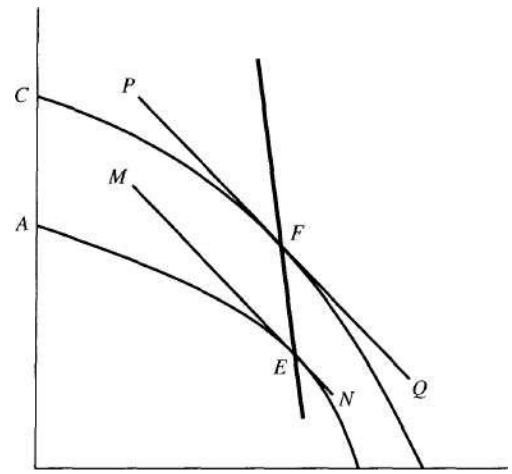

FIGURE 18-5 As a country accumulates capital, its produ

line *MN.* Suppose now that through accumulation, the capital endowment of the economy increases while the labor endowment remains unchanged. With more capital endowment and factor substitution in both industries, more of either good can be produced when the production of the other good is fixed. The production possibility frontier with more capital endowment therefore shifts out to curve *CD* in Fig. 18-5. The new production point occurs at the point of tangency, point *F,* between curve *CD* and a new world price line, *PQ,* whose slope is —*p*®/p_2. Because the economy is small, world prices are not disturbed by the capital accumulation in the economy, meaning line *PQ* is parallel to line *MN.*

By the Rybczynski theorem, production of the capital-intensive good 2 must thereby increase while that of labor-intensive good 1 must decrease. This means that point *F* is to the northwest of point *E.* Line *EF* is the locus of the production points as the capital stock grows. It is, in fact, linear. T

ction frontier shifts out. Letting good 1 be labor-intensive, at constant world prices (indicated by the slope *of MN* and *PQ),* output of good 1 expands, while output of good 2 contracts. The locus of production points *EF* is linear, since the input-output coefficients are functions of output prices only.

o see this, note that the rate of change of good 2 output with respect to good 1 output is given as dy_2/dy_1. Using Eqs. (18-68c) and (18-68*d*),

$$dy_2 \qquad dy_1/dK \qquad (18\text{-}71a)$$

Since the a_{ij} 's are functions of the output prices only, the slope of the locus is constant (and negative). Since point *F* must be above the price line *MN*, *EF* must be steeper than *MN*. The locus *EF* is sometimes called the Rybczynski line (for capital).

The preceding analysis can be similarly used to show the production effects of an increase in labor endowment. The Rybczynski line for labor has a slope of

$$dy_2 \; \_ \; dy_2/dL \qquad dy_x \; d \qquad (18\text{-}71Z?)$$

Because industry 2 is capital-intensive,

$$\frac{a_L}{a_K} > \text{- -}$$

This implies that the Rybczynski line for capital is steeper than the Rybczynski line for labor. In fact, the Rybczynski line for labor is less steep than the price line *MN*. The Rybczynski theorem can also be used to prove the Heckscher-Ohlin theorem, which is widely used in the theory of international trade to explain the patterns of trade of two trading partners. Consider two countries, *A* and *B,* and denote the endowments of capital and labor in the two countries by *K'* and *U, i* = *A, B*. Country *B* is said to be capital-abundant (or labor-scarce) relative to country *A* if and only if

$$\frac{K^B}{L^B} \qquad \frac{K^A}{L^A}$$

The Heckscher-Ohlin theorem is as follows.

Assume:

a. There are two tradeable goods, 1 and 2, and two factors, labor and capital.
b. The technologies are identical across countries in the sense that the production function of a sector is the same in both countries.
c. The countries have identical and homothetic preferences, which are represented by a quasi-concave, increasing (social) utility function.
d. The production function of each sector exhibits constant returns to scale.
e. The factors are perfectly mobile across sectors but immobile across countries.
f. All markets are perfectly competitive.
g. There is no factor intensity reversal in the sense that sector 1 is labor-intensive relative to sector 2 at all factor prices.

Then each country will export the good that uses its abundant factor more intensively.

The theorem is proved as follows. Assume for the moment that the countries are exactly identical, with identical technologies, preferences, and factor endowments. Then the production possibility frontiers of both countries are identical and can be represented by curve *AA'* in Fig. 18-6. The self-sufficient, or autarky, equilibrium point *P* of each country is depicted as the point of tangency between the production frontier *AA'* and an indifference curve. The slope of the tangent to curve *AA'* at *P* equals the autarky relative price of good 1. Obviously, under these conditions, the two countries have no incentive to trade.

Suppose now that country *B* has more capital, implying that country *B* is capital-abundant and country *A* is labor-abundant. Suppose further that the relative price in country *B* remains unchanged. Then by the Rybczynski theorem, the production point will shift to point *Q,* which is above and to the left of point *P.* Country fi's production possibility frontier is represented by curve *BB′*. As explained previously, curve *BB′* is entirely beyond curve *AA′*. Because of homothetic preferences, the consumption point will shift to point C, the point of intersection between the tangent to curve *BB′* at point *Q* and a ray from the origin through point *P.* As a result, at the original

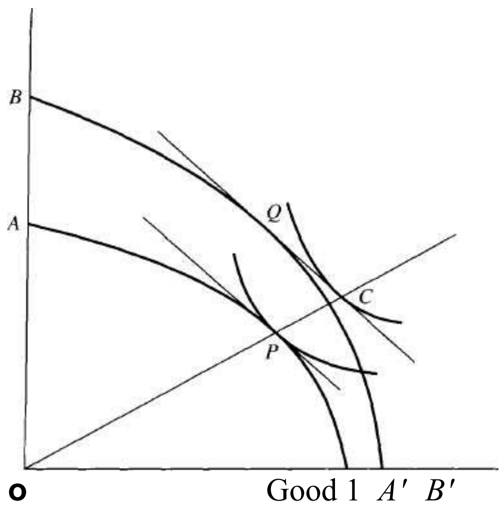

FIGURE 18-6
The Heckscher-Ohlin Theorem. Assume two initially identical countries, *A* and *B,* and let *B* then accumulate capital. Country A's production frontier is *AA'\ B's* is *BB'.* Under autarky, the relative price of the labor-intensive good, *y\,* would rise in country *B* and fall in *A.* With trade, each country would export the good with the lowest (internal) relative price; thus, *B* would export the capital intensive good *yj* . Thus (under the assumptions of the theorem), each country will export the good that uses its abundant factor more intensively.

price ratio, an excess demand for good 1 and an excess supply of good 2 are created in country *B.* This means that under autarky, the relative price of good 1 is higher in country *B,* or the relative price of good 2 is higher in country *A.* Because of the difference in the autarky prices, it is sometimes said that country *A* has a comparative advantage in good 1 and a comparative disadvantage in good 2 relative to country *B.*

Now allow trade between the countries. Each country will export the good that is cheaper under autarky. This means that country *A* will export good 1 while country *B* will export good 2. Thus, we have proved the theorem: The capital-abundant country exports the capital-intensive good, while the labor-abundant country exports the labor-intensive good. Although the proof assumes that country *B* has more capital but the same amount of labor as country *A,* this assumption is not necessary for the theorem. As long as country *B* is capital-abundant, the production point *Q* must be to the left of the consumption point C, and an excess demand for good 1 and an excess supply of good 2 will be created under autarky and at the original price.

As our last application of this model, consider an economy that uses labor *L* and capital *K* to produce a universal consumption good *C, and more capital, K.* The production functions are, respectively, $C = f^c(L_c, K_c)$ and $K = f^K(L_K, K_K)$. Assuming exogenously determined prices p_c and p_K for the consumption good and capital, the first-order conditions for maximization of the value of output include

$$X_K = \text{Rental rate on capital} = p_K \frac{df^K}{\quad} = \frac{df^c}{\quad} = Pcirzr$$

Assuming the capital lasts forever, the interest rate equals the rental rate of capital divided by the price of capital:

$$\cdot = \frac{p_K\, df^K/dK_K}{PK} = \frac{df«}{dK_K}$$

That is, the interest rate in this model is the marginal product of capital in producing more capital. (The letter "r" is commonly used in some models to mean the real interest rate; in others, it means the rental rate on capital. Do not confuse these different concepts!) In this model, the interest rate is determined by (among other things, perhaps, depending on the other assumptions in the model) the relative price of capital vs. the consumption good. Assume for the moment that the capital goods industry is relatively capital-intensive. Then an exogenous increase in the price of capital increases the rental rate on capital by a greater proportion. In order to maintain the first-order condition, therefore, the marginal product of capital must rise, producing a higher real interest rate. However, the interest rate will fall if it is the consumption good industry that is capital-intensive. Thus we see that in such a two-sector model, the determination of the interest rate is a somewhat complicated process, depending in part on the relative capital intensities of the capital and consumer goods industries.

18.5 SUMMARY AND CONCLUSIONS

Let us now briefly summarize the results and the underlying assumptions of the two-good, two-factor model. The fundamental hypothesis is that in a competitive economy, the owners of factors will contract with each other in such a way as to maximize the value of national income. This invisible hand process is not the intention of any person in the economy. Self-seeking owners of resources, in trying to maximize the return of such ownership, can be expected if transactions costs are zero to combine in a way that all gains from trade are exhausted. This must place the economy on the production frontier and at that point on that frontier where the marginal evaluations by consumers of each good, in terms of forgone consumption of the other good, equals the marginal cost of production of each good, measured in terms of forgone production of the other good. This occurs at a point of tangency of the line, or plane, defining NNP, *z = p\y\ + piyi,* and the production-possibilities frontier.

Since factors are completely mobile between the two industries, factor prices must be the same in both industries. Factor prices emerge as the Lagrange multipliers associated with the resource constraints. Although the wage and capital rental rates are determined endogenously by the model, these factor prices are taken ex-ogenously by the relatively "small," identical firms that make up each industry. It is the simultaneous actions of each firm that change factor prices and aggregate output levels.

Under these general conditions, it is possible to show that the supply-of-output curves, ***y?j*** *= y*(pi, P2),* are upward-sloping in their own price. This is a direct consequence of the concavity of the production possibilities frontier with respect to the origin. This shape of the production frontier is indicative of increasing marginal costs (hence, upward-sloping supply curves) for each industry. These matters are discussed in Sec. 18.2.

Lastly, for the general model, certain reciprocity conditions appear, Eqs. (18-24) to (18-26). These relations indicate a duality between physical quantities and their respective prices. The relation of outputs to resource endowments is the same as the relation of resource *prices* to output *prices.* These results are independent of any homogeneity assumptions concerning the production functions.

In Sec. 18.3, the assumption that each industry is characterized by constant returns to scale is added to the model. This permits representation of the production function in terms of the unit isoquant only (since all isoquants are radial blowups or contractions of that or any other isoquant). Mathematically, letting a_{tj} = the amount of input / used to produce 1 unit of output *j,* where *i = L, K, j* = 1,2, the production relations become $f^x(a_L\backslash, a_K i) = 1$ and $f^2(a_{L2}, a_{K2}) = 1$. These a,-/s are the fundamental quantities in this more restricted model. When we use the preceding production relations and the first-order marginal relations (18-38), the *djj's* are shown to depend only on the factor prices, the Lagrange multipliers, *w* and r; in particular, $a_t j = a^*j(w/r)$. This critical step allows the model to be defined by two independent sets of two equations each, Eqs. (18-53) and (18-54). The former indicate that profits are zero in each industry, i.e., that the amount of labor used to produce 1 unit of *y*, times the price of labor plus the amount of capital used to produce 1 unit of *y* times the unit price of capital exactly equals the price of 1 unit of ji. Similarly, the total unit factor cost of y_2 equals the unit price of y_2. The second set of equations constitutes the original resource constraints, with the added feature that the a,-/s are functions of *w/r* only. Because of the dependence of the fl/j's on *w/r* only, the comparative statics of this model is the same as in the case where the a,-/s are technologically fixed. The variable-proportions model (including the assumption of linear homogeneous production functions) yields the same comparative statics results as the fixed-proportions linear programming model.

Equations (18-53), dealing with prices, yield as solutions Eqs. (18-55), factor prices expressed as functions of *output prices only.* This result yields the factor price equalization theorem, under the global assumption that in each country, one industry is always more labor-intensive than the other. This theorem then indicates that if two countries trade with each other, then as output prices converge in the two countries, the factor prices in each country will be functions of output prices only and not dependent in any way on resource endowments. If the production functions are the same in the two countries, the functional relationship $w — w^*(p\backslash, p_2), r = r^*(p_u p_2)$ will be the same for both countries. Then, since with free trade both countries will face the same output prices, factor prices will also equalize in both countries even though factors are immobile between countries.

Differentiation of Eqs. (18-53), dealing with prices only, yields the set of results known as the Stolper-Samuelson theorem. It is shown in Eqs. (18-61) and (18-65) that if the price of the, say, labor-intensive industry rises (inducing an expansion of that industry), nominal *and* real wages will rise and capital rental rates will fall. Likewise, if the price of the capital-intensive industry rises, the capital goods industry expands, the labor-intensive industry contracts, and thus rental rates rise, in real as well as nominal terms, and the wage rate falls. Also, if the price of the labor-intensive industry rises, *both* industries become less labor-intensive, with similar results holding if the price of the capital-intensive good rises.

Analogous results for the physical quantities are known as the Rybczynski theorem. By using the reciprocity conditions (18-24) to (18-26), the algebra of the Rybczynski theorem is shown to be the same as that used in the Stolper-Samuelson theorem. Alternatively, these results are derivable from the second set of two equations defining the model [Eqs. (18-54)], the resource constraints. In Eqs. (18-68) and (18-69), it is shown that if, say, the amount of labor available to the whole economy increases, the output of the labor-intensive industry will not only expand but will expand in greater proportion to the increase in labor. Analogous results hold for an autonomous increase in capital.

This completes our discussion of the two-good, two-factor model. Let us briefly comment on the many-good, many-factor generalization of this model. This generalization is in fact exceedingly complex and beyond the scope of this book. The general model of maximization of NNP subject to resource constraints proceeds in an obvious way, with no difficulty. One derives the upward slope of the supply functions and the reciprocity conditions analogous to (18-24) to (18-26) in the same manner. The difficulty begins with trying to generalize the factor price equalization, Stolper-Samuelson, and Rybczynski-type theorems. In general, if the number of goods exceeds the number of factors, certain goods, not determinable without an algorithmic process, will not be produced. Similarly, if the number of factors exceeds the number of goods in these models, certain factors will not be used and their associated Lagrange multiplier shadow prices will equal zero. The relation of factor price to output price changes is much more complex than a simple dependence upon factor intensities, since higher-order determinants are involved. Under restricted conditions, however, the factor price equalization theorem is valid. However, no easy or intuitive factor intensity rules can be stated to give the results analogous to those derived in Sec. 18.3.

PROBLEMS

1.417 In the «-good, m-factor model, with *y** the output of the /th industry, show that *Yl]=\ ∈U =* 0,wheree,$_{y}$ = *(pj/y*)(dy*/dpj)*. Show therefore in the two-good model that 3 _y*/9^2 >0.

1.418 Show that with linear homogeneous production functions, the model with *n* goods and *m* factors has the property that the (maximum) total value of output equals total factor cost.

1.419 Explain why it is critical, from the standpoint of deriving the Stolper-Samuelson and Rybczynski theorems, for the technological coefficients *a^* to be dependent on factor prices only and not the factor endowments. Explain what assumptions in the model produce this result.

1.420 Explain the assumptions needed to yield the result that factor prices are dependent on output prices only. Does it follow from this alone that if two countries engage in costless trade, factor prices will be the same in both countries? Why or why not?

1.421 The production possibilities frontier is derivable by treating one output level, say y_x, as fixed (parametric) and then using the resources to maximize the output level of y_2. As $y\backslash$ is varied parametrically, the production possibilities locus will be traced out.

(a) Set up this problem for two goods and two factors and interpret the (three) Lagrange multipliers.

1.422 Show that the production frontier is not necessarily concave in this formulation. What
distinguishes these assumptions from the ones used in the text, in which concavity
of the production frontier is implied?
1.423 Show that if both production functions are concave, the production possibilities
frontier is concave.

6. Consider the maximization of NNP model with $y\ — L\ K/\sim$, $y_2 — L_2$ K_2 .
1.424 Show that the capital-labor ratio in industry 1 will always be four times the capital-
labor ratio in industry 2.
1.425 Derive Eqs. (18-50) for this specification; i.e., verify for this model that each a* is
a function of the factor price ratio only.
1.426 Show that $da^*_{Lj}/dWj < 0$, $da^*_{Kj}/dr < 0$ directly from the equations for a^*_r
1.427 On the basis of the factor intensities in each industry, which factor price would you
expect to increase and which to decrease when p_j increases?
1.428 Find the explicit functions $w — w^*(p\, p_2)$ and $r = r^*(pi, p_2)$ and verify the
predictions in part *(d)*.
1.429 On the basis of factor intensities, which industry will increase output and which will
decrease output when the endowment of labor increases?
1.430 Verify this result by applying Eqs. (18-54) for this model.

1.431 Derive the Rybczynski theorem from the Stolper-Samuelson theorem using the reci
procity relations present in the two-good, two-factor model.
1.432 Suppose that the NNP function of an economy is given as

$$f- b—— I\ ioi_LL + a_KK) + I\ c_LL + c_KK + d\ \ I\ \{fi\backslash P\backslash + PiPi)-$$
$$2p_2J \qquad \backslash \qquad 2K)$$

1.433 Show that the NNP function is linear homogeneous in p_x and p_2 and linear homo
geneous in L and K.
1.434 If it is required that the NNP function is a convex function of p, and p_2 and a concave
function of L and K, what restrictions on the sign of b and d are needed?
1.435 Derive the supply functions of goods 1 and 2. Show that these functions are homo
geneous of degree 0 in prices and linear homogenous in factor endowments.
1.436 Derive the shadow prices of L and K. Show that these functions are linear homoge
neous in prices and homogeneous of degree 0 in factor endowments.

9. Define the unit cost function of sector i as

$$c'(w,r) = \min\{wa_{Li} + ra_{Ki}\blacksquare: f'(a_{Li},a_{Ki}) > 1\}$$

where a_{Li} and a_{Ki} are the labor and capital inputs, respectively. Show that the following function is equivalent to the NNP function defined in the text.

$$g(P\backslash, P2, L, K) = \min\{wL + rK: c^1(w, r) > p_x \quad \text{and} \quad c^2(w,r)>p_2\}$$

10. By using the NNP function defined in Prob. 9, show that

$$W = D_1 \frac{\partial y_1}{\partial L} + D_2 \frac{dy_2}{dL}$$

11. "International trade necessarily lowers the real wage of the relatively scarce factor expressed in terms of any good." Comment.

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CHAPTER

19

WELFARE ECONOMICS

19.1 SOCIAL WELFARE FUNCTIONS

Throughout this book it has been stressed repeatedly that the goal of any empirical science is the development of refutable propositions about some set of observable phenomena. Refutable propositions that survive repeated testing form the important principles on which the science is based. (It is easy, of course, to state refutable hypotheses that are in fact refuted.)

Parallel to the development of economics along the preceding lines has arisen a discipline called *welfare economics,* which seeks not to explain observable events but to *evaluate* the desirability of alternative institutions and the supposed resulting economic choices. For example, it is commonly alleged that "too many" fish are being caught in the oceans, that tariffs and other specific excise taxes cause an "inefficient" allocation of resources ("too little" production of the taxed item), that "too much" pollution and congestion occur in metropolitan areas, and the like. In this chapter we shall investigate the basis of these assertions and comment on the empirical content of such pronouncements.

It was common for classical economists to speak of "the benefits to society," the interest of the "working class," and other such phrases that implied a sufficient harmony of interests between members of the relevant class to permit speaking of them as a group. Today, we often hear of individuals representing "the interests of consumers" or of someone taking the position of "big business."

A difficulty in the concept of group preferences, or interests, was pointed out by Kenneth Arrow in his classic paper, "A Difficulty in the Concept of Social Welfare."^ The use of such phrases implies that there is a well-defined function of individual preferences, or utility functions, representing the utility, or "welfare," of the group. Such a function was first posed explicitly by A. Bergson in 1938.* The social welfare function posited by Bergson had the form

$$W = f(U^1, \ldots, U^m) \tag{19-1}$$

where $U^1, \ldots, U^m$ were the utility functions of the m individuals in the group being considered, perhaps the whole economy. Bergson considered various first-order marginal conditions for the maximization of W subject to the resource constraints of the economy.

Arrow's discussion of these matters began with a 200-year-old example of the problem of construction of a group preference function. The example was based upon majority voting. Voting is a very common way for groups to reach decisions. Suppose one were to attempt to define collective preferences on the basis of what a majority of the community would vote for. Suppose there are three alternatives a, b, and **c** and three individuals in the group. Let P represent "is preferred to" so that **aPb** means that **a** is preferred to b.

Suppose now that the three individuals have the following preferences:

Individual 1: **aPb, bPc**
Individual 2: **bPc, cPa**
Individual 3: **cPa, aPb**

Assume, in accordance with ordinary utility theory, that these consumers' preferences are *transitive.* That is, for individual 1, **aPb** and **bPc** means that **aPc,** etc. Then it can be quickly seen that a majority-rule social welfare function will have the unsatisfactory property of being *intransitive.* Consider, for example, alternative a. A majority of voters, namely voters 2 and 3, prefer **c** to a. Likewise, a majority of voters (1 and 3) prefer **a** to b, and another, different majority (1 and 2) prefer **b** to c. Whichever alternative is selected, a majority of voters will prefer some other

Journal of Political Economy, 58:328-346, 1950. This paper was part of a larger study, *Social Choice and Individual Values,* 2d ed., *Cowles Commission Monograph* 12, John Wiley & Sons, Inc., New York, 1963.

^Abram Bergson, "A Reformulation of Certain Aspects of Welfare

Economics," *Quarterly Journal of Economics,* 52:310-334, 1938.

alternative. Thus, the social welfare function based on what the majority wishes will exhibit the properties **aPb, bPc,** *and* cPa.^

Let us now summarize Arrow's theorem about social welfare functions. Arrow uses a weaker form of the preference relation: Let ***a*** *Rib* represent the statement "a is *preferred or indifferent* to b, according to individual /." Suppose there are *n* individuals in this society. Then, by a *social welfare function,* in this terminology, we mean a relation *R* that corresponds to the individual orderings, *Ri, ..., R_n,* of all social states by the *n* individuals in the society. That is, given the preference orderings of all people in the polity, there exists some social ordering *R* which denotes "society's" values and rankings of the alternatives being considered.

Arrow proceeded to list five conditions that he felt almost any reasonable social welfare function ought to contain. The first of these is that the social welfare function is in fact defined for all sets of individual orderings that obey some set of individualistic hypotheses about behavior, e.g., the usual economic postulates of convex indifference curves and the like.

> **Condition 1.** The social welfare function is defined for every admissible pair of individual orderings *R\, R_2.*

Second, the social ordering should describe welfare and not, in Arrow's word, "illfare." The social welfare function should react in the same direction, or at least not oppositely to, alterations in individual values.

> **Condition 2.** If a social state a rises or does not fall in the ordering of each individual without any other change in those orderings, and if **a/?b** before the change, for any other alternative b, then **a/?b** after the change in individual orderings.

tThis voting paradox illustrates one of the outstanding differences between *market* choices and *political* choices. In the former, the consumer has the option of expressing the *intensity* of a preference by the simple act of choosing to purchase differing amounts of goods. In political choice, however, ordinary voters get one and only one vote. The consumer under these circumstances is unable to express intensity of preference. In the above example, the three alternatives were merely *ranked.* The voters were not able to say, for example, that they preferred a a great deal more than b and b only slightly more than c. In legislative bodies, in which there are relatively few voters, the individuals can *trade* votes on successive issues. Suppose, for example, individual 1 has the above-stated intensities of preferences and individual 2 was almost indifferent between a, b, and c. Then voter 1 could make a contract or a deal to vote for some other issue which voter 2 felt strongly about (and which voter 1 had no strong preferences about) in exchange for an agreement from voter 2 to vote for alternative a in the text example. The paradox would be resolved through trade. However, more trade is not necessarily preferred to less trade for individuals, and voter 3 might end up worse off for such political trading. It is for these reasons that many people believe that special-interest legislation is more apt to be enacted by legislative bodies than by referendum vote. But such vote trading also protects minorities who feel intensely about some issue from the "tyranny of the majority." The gains-from-trade aspect of political trading is emphasized in James Buchanan and Gordon Tullock, *The Calculus of*

Consent, University of Michigan Press, Ann Arbor, 1963.

The most controversial of Arrow's conditions is the third, the *independence of irrelevant alternatives.* Consider an election in which three candidates, a, b, and c, are running. Suppose an individual's preferences are **a/?,b/?(c.** Suppose, before the election, candidate b dies. Then we would expect to observe **a/?(c.** In like manner, we expect the social welfare function's ranking of any two alternatives to be unaffected by the addition or removal of some other alternative.

> **Condition 3.** Let R_x, R_2, and $R\backslash$, R'_2 be two sets of individual orderings. Let *S* be the entire set of alternatives. Suppose, for both individuals and all alternatives a, **b** in *S,* that **a/?,b** if and only if a/?-b. Then the social choice made from *S* is the same whether the individual orderings are /?i and R_2 or $R\backslash$ and R'_2.

Conditions 4 and 5 imposed by Arrow amount to assertions that individual preferences *matter.* That is, individual values are to "count" in determining the social welfare function. Conditions 4 and 5 say that the social welfare function is not to be either *imposed* or *dictatorial.* A social welfare function is said to be *imposed* if, for some pair of alternatives a and b, a/?b for any set of individual orderings *R\,R2,* that is, irrespective of the individual orderings *R\,R2,* where *R* is the social ordering corresponding to *R\,R2-* Likewise, a social welfare function is said to be *dictatorial* if there exists an individual *i* such that for all a and b, a/?, b implies ***aRb*** regardless of the orderings of all individuals other than /, where *R* is the social preference ordering corresponding to the R_t*'s.*

> **Condition 4.** The social welfare function is not to
>
> be imposed. **Condition 5.** The social welfare
>
> function is to be nondictatorial.

Arrow succeeded in showing that these five conditions could not all hold simultaneously. In particular, he showed that any social welfare function that satisfied the first three conditions was either imposed or dictatorial. This very strong result is called the *possibility theorem.*^ It says that no matter how complicated a scheme might be constructed for determining a set of social preferences, social ordering *R* cannot meet all conditions 1 to 5. It will be impossible to construct *any* welfare function of the type described in Eq. (19-1), $W = f(U^1,\ldots, U^m)$, that is, some function of individual utility levels, obeying the preceding conditions.

Another interpretation of the possibility theorem is that interpersonal comparisons of social utility are ruled out. It is impossible to say that taking a dollar away from a rich person and giving it to a poor person will make society better off, in some nondictatorial or nonimposed sense. The problem of interpersonal comparisons of utility was a vehicle by which ordinal utility replaced the older cardinal utility idea.

On a less rigorous but more intuitive basis, the reason sensible social welfare functions cannot exist is that they conflict in a fundamental way with the notion that

^The authors would have called it the impossibility theorem.

more is preferred to less. At any given moment, there is a frontier of possibilities for the consumers in any society. Any movement *along* this frontier involves gains for some individuals and losses for others. Without a measure for comparing these gains and losses between individuals, there is no sense to the phrase "social welfare." (We shall explore these matters in more detail in Sec. 19.3.)

A rigorous proof of the possibility theorem is beyond the scope of this book. It can be found in the reference cited. We conclude this section by noting that in spite of this theorem, hundreds, perhaps thousands of articles have been written in economics journals using social welfare functions. Indeed, a whole new area of mathematical theology has arisen. However, to quote Samuelson,* "the theorems enunciated under the heading of welfare economics are not meaningful propositions of hypotheses in the technical sense. For they represent the deductive implications of assumptions which are not themselves meaningful refutable hypotheses about reality."

19.2 THE PARETO CONDITIONS

Faced with the impossibility of constructing a meaningful social welfare function, economists have opted for a weaker criterion by which to evaluate alternative situations. This criterion, known as the *Pareto condition,* after the Italian economist Vilfredo Pareto, states that a social state **a** is to be preferred to b if there is at least one person better off in **a** than in b, and no one is worse off in **a** than in b. This is a weaker value judgment only in the sense that more people would probably accept this judgment over more specific types of social orderings wherein some individuals lose and others gain. A state **a** that is preferred to b in the Paretian sense is said to be *Pareto-superior* to b. One can imagine some sort of frontier of possible states of the economy such that there are no Pareto-superior points. That is, along this frontier, any movement entails a loss for at least one individual. The points for which no Pareto-superior states exist are called *Pareto-optimal.*

In general, we shall find that the set of Pareto-optimal points is quite large. Whether or not these points are a useful guide to policy is debatable. Even so, to say that the economy *ought* to be at a Pareto-optimal state is a value judgment and therefore a part of moral philosophy and not part of the empirical science of economics. We can, however, as economists, investigate the conditions under which various ideal Pareto-optimal states will be obtained. In this section we shall investigate certain famous conditions that achieve Pareto optimality. It is useful, in these discussions, to maintain the perspective indicated in the preceding quotation from Samuelson.

Pure Exchange

Consider an economy containing two individuals who consume two commodities, x and y. Let JC, , y, denote the amounts of x and y consumed by the *ith* person, whose

^*Foundations of Economic Analysis,* Harvard University Press, Cambridge, MA, 1947, pp. 220-221.

utility function is U^{l} $(x_{l}, y_{l}$•). Suppose that the total amounts of x and y are fixed, that is, x\ -\-$x_{2} = x$, $yi + y_{2} = y$, where x and y are constants. Under what circumstances will the allocation of x and y between the two individuals be Pareto-optimal? This problem can be formulated mathematically as follows:

maximize

$$U^{2}(x_{2}, y_{2})$$

subject to

0 (19-2)

$$Xi + X_{2} = X\text{^}1\text{+^}2 = \text{^}$$

It is meaningless to attempt to maximize both individual's utilities simultaneously.t Instead, we first fix either individual's utility at some arbitrary level; then, the other person's utility is maximized. In this way, a position is attained in which neither party can be made better off without lowering the other person's utility. The Lagrangian for the preceding problem is

Differentiating with respect to x\, y\, x_{2}, y_{2} and the Lagrange multipliers yields

$$\text{i£}_{X2} = U^{2}_{x}\text{-}X_{x}=0 \qquad (19\text{-4a})$$

$$\boldsymbol{\text{£}_{n} = U^{2}\text{-}X_{y}=0} \qquad \textbf{(19-46)}$$

$$\text{£g}_{Xi} = \text{-xt/jj - A}_{x} = 0 \qquad (19\text{-4c})$$

$$\boldsymbol{5E_{yi} = \text{-}W^{l}_{y} - X_{y} = 0} \qquad \textbf{(19-4J)}$$

and

$$\text{ig}_{x} = \text{^} - U\backslash x_{u}\ yi) = 0 \qquad (19\text{-5fl})$$

$$X_{Xx} = \text{JC — JCI} - x_{2} = 0 \qquad (19\text{-56})$$

$$\text{^A,} = \text{? - yi} \sim \text{^} = 0 \qquad (19\text{-5c})$$

where $U^{l}_{x} = dU^{l}/dxi$, etc. Combining Eqs. (19-4) gives

^ We leave such constructions to those who aspire to find that economic system which seeks "the greatest good for the greatest number of people."

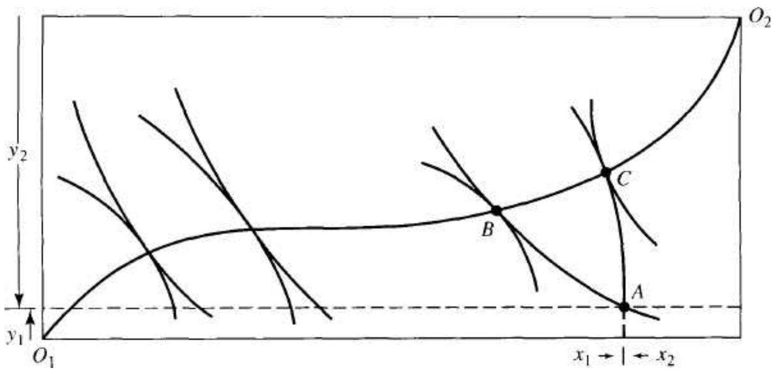

FIGURE 19-1
The Edgeworth box diagram is useful for depicting the set of Pareto-optimal points in a pure trade, zero transaction cost world. The dimensions of the box are the total amounts of each good available, *x* and *y.* Any point, such as *A* in the interior of the box, represents an allocation of *x* and *y* to the two individuals. Individual 1 's utility function is plotted in the usual direction from the origin marked *O*. Individual 2's utility function is plotted opposite (right to left and down) from origin *Oi.* The set of points for which the slopes of U^1 and U^2 are identical at the same point, i.e., a level curve of [/' is tangent to a level curve of U^2, is called the *contract curve,* designated *O\ O2.* This curve represents the set of points for which the gains from trade are exhausted. It is occasionally referred to as the *conflict curve* because movements *along O\ Oj* represent conflicts of interest: one individual gains and the other loses. For that reason, it is the set of Pareto-optimal points in this economy.

Equation (19-6) is the tangency condition that the consumers' indifference curves have the same slope. The marginal rate of substitution of *x* for _y must be the same for both consumers. This is the familiar condition that must hold if the gains from trade are to be exhausted. The set of all points that satisfy (19-6) (and the constraints) is called the *contract curve,* as depicted in Fig. 19-1. This diagram is the Edgeworth box diagram first shown in the chapter on general equilibrium theory. (There, though, the axes were quantities of factors of production, not final goods as is the case here. The mathematics is, of course, formally identical.)

The set of Pareto-optimal points is the set of allocations for which the gains from exchange are exhausted. If the consumers were presented a different allocation, e.g., point A in Fig. 19-1, then with no cost of trading we should expect them to move to some point on the contract curve $O\backslash O_2$. If the trade is voluntary, the final allocation must lie between (or on) the two original indifference curves, i.e., some point on the segment *BC* of the contract curve. Without a further specification of the constraints of the bargaining process, the theory is inadequate to determine the actual final point. But in the absence of transactions costs and coercion, self-seeking maximizers must wind up at *some* point along *BC.*

The problem as posed in (19-2) does not actually start at some particular point such as A and then move to the contract curve. As formulated in (19-2), the indifference level of individual 1 is fixed, say at the level that goes through point A. The resulting solution of the problem, i.e., solution to Eqs. (19-4) and (19-5), would place the economy at point *B,* where person 2 achieves maximum utility, leaving person 1 on the original indifference curve. Hence, the problem posed in (19-2) admits of a unique answer, even if a bargaining process that starts *both* individuals at *A* is unspecified.

The indirect utility function for individual 2 is obtained first by solving Eqs. (19-4) and (19-5) and substituting the chosen values of x_2 and y_2 into $U^2(x_2, y_2)$. Let the solutions to (19-4) and (19-5) be designated

$$Xi = xT(ui,x, y) \qquad _{yi} = y!(Ul,x, y) \qquad (19\text{-}7)$$

and likewise for the Lagrange multipliers:

$$k_x=KM>x>y) \qquad k_y=k^*_y(Ul,x,y) \qquad (19\text{-}8)$$

Then

$$U^2{}^* = U^2(x^*_2, y^*) = f(U^*,x, y) \qquad (19\text{-}9)$$

Holding constant x and $y,$ the total amounts of the goods, one can imagine a utility frontier, defined by Eq. (19-9). Starting with UQ = 0, the maximum level of utility for person 2 is that which is achieved when person 2 consumes all of both goods, i.e.,

Likewise some maximum level of U^1 exists, represented by the indifference curve for person 1 which passes through $O_2,$ for which $U^2 = 0$.

This utility frontier is plotted as the curve *UU* in Fig. 19-2, where the subscript 0 on U^1 has been suppressed. Using the envelope theorem and Eqs. (19-4) leads to

$$du^r \quad u^2 \quad u^2 \quad o \quad k \quad f \qquad (1910)$$

Assuming the tangencies defining the contract curve take place at positive marginal utilities (downward-sloping indifference curves), $dU^2{}^*/dU^1 < 0$, as indicated. The Pareto frontier could not very well exhibit $dU^2{}^*/dU^1 > 0$, since then movements along it in the northeast direction would imply gains for *both* individuals, contradicting the notion of Pareto optimality. It is *not* possible to infer that the Pareto frontier *UU* is concave to the origin; this follows from the ordinal nature of utility. A monotonic transformation of $U^1(x\backslash, yi),$ say, could bend the frontier as desired, though keeping it downward-sloping.

Production

Suppose now we generalize the preceding discussion to the case where JC and y are produced using two (or more) factors of production. In the preceding chapter on general equilibrium, an Edgeworth box diagram was constructed for the two-factor case. In order for consumers to be on the Pareto frontier in consumption,

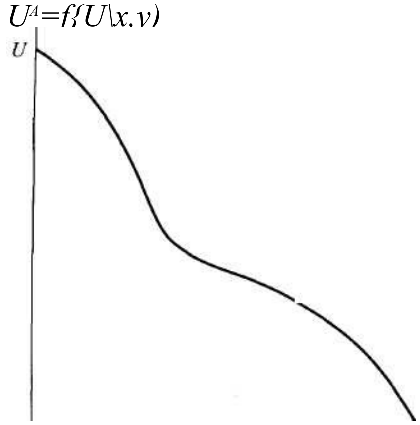

FIGURE 19-2
The Utility Frontier for Given Total Quantities of Goods. For any given amount of the goods x and y there exists a whole set of points for which neither individual can gain without the other person's losing. This Pareto frontier consists of reading off the (ordinal) utility levels for each person at every point along the contract curve $O\backslash$ 02- The frontier is necessarily downward-sloping, by definition of Pareto optimality.

the goods must be produced efficiently. That is, a production point interior to the production possibilities frontier cannot result in a Pareto-optimal state for consumers. The consumers could both (or all, in the n-person case) have more of all goods and hence higher utility if production were moved to the production possibilities frontier in the appropriate manner. Hence, the problem of defining the Pareto frontier for consumers in the case in which x and *y are produced,* and not fixed constants, begins with the problem of defining the production possibilities frontier. Points on the production frontier are called *efficient in production.*

The mathematics for the production case is formally identical to the preceding analysis of final goods. Let there be two factors of production, L and *K,* and let L_x denote the amount of labor used in producing *x,* etc. Then the problem of efficient production can be stated

maximize

$$y = f(L_y, K_y)$$

subject to

$$JL_x \qquad g\{L_x,K_x)=x$$
$$JLy \quad - \quad Li \qquad K.y \quad — \quad K. \tag{19-11}$$

where $f(L_y, K_y)$ and$g(L_x, K_x)$ are the production functions of y and x, respectively. The value x is taken as a parameter; it is *not a* decision variable. The Lagrangian for the problem (19-11) is

$$56 = f(L_y, K_y) + k(x - g(L_x, K_x)) + X_L(L - L_x - L_y) + X_K(K-K_X- K_y) \tag{19-12}$$

The resulting first-order relations are

$$f_K - X_K = 0 \quad (19\text{-}136)$$

$$-Xg_L - k_L = 0 \quad (19\text{-}13c)$$

$$-Xg_K \sim \hat{}K = 0 \quad \textbf{(19-13^)}$$

and the constraints

$$x - g(L_x, K_x) = 0 \quad (19\text{-}14fl)$$

$$L - L_x - L_y = 0 \quad (19\text{-}146)$$

$$K - K_x - K_y = 0 \quad (19\text{-}14c)$$

From Eqs. (19-13),

$$\frac{A}{IK} = \frac{hL}{\hat{}\text{-}K} = \frac{EL}{8K} \quad (19\text{-}15)$$

The ratio of marginal products must be equal for both goods along the production contract curve. This is the tangency condition illustrated in Fig. 18-2. Solving Eqs. (19-13) and (19-14) simultaneously gives

$$L_y = L^*(x,L,K) \quad (19\text{-}16a)$$

$$K_y = K^*(x,L,K) \quad (\backslash 9\text{-}l6b)$$

$$L_x = L^*(x,L,K) \quad (19\text{-}16c)$$

$$K_x = K^*(x,L,K) \quad (19\text{-}16a^1)$$

and

$$X. = X.^*(x,L,K) \quad \textbf{(19-17a)}$$

$$X_L = k^*_L(x,L,K) \quad (19\text{-}176)$$

$$\backslash_K = k^*_K(x,L,K) \quad (19\text{-}llc)$$

Equations (19-16) give the chosen values of labor and capital in both industries. Substituting these values into the objective function gives the maximum *y*, *y** for any value of JC:

$$y^* = f(L^*_y, K;) = y^*(x,L,K) \quad (19\text{-}18)$$

Using the envelope theorem, we have

$$\frac{dJl}{dx} = \frac{™}{dx} = r \qquad (19\text{-}19)$$

Hence, A* has the interpretation of the *marginal cost of x,* since it shows how much/ must be given up in order to get an additional unit of JC. The multiplier *X** is the slope of the production possibility frontier by definition, since A.* = *dy*/dx.* Assuming the marginal products of the factors are positive, A.* < 0; i.e., the production frontier is negatively sloped. As before, from Eqs. (19-13),

r = -^ = -^ = -^ = -^L < 0 (19-20)

8L gL gK gK

This equation has the interesting interpretation that the marginal cost of JC is the same if only labor is varied (the ratio /i/gO or if only capital is varied (/W/g*:) or if both are varied. In the partial equilibrium framework this phenomenon was encountered in the formula

W r W K

MC = — = —- (19-21)

A *IK*

where the w's were the respective factor prices. Here, of course, X_L and X_K are the factor prices, *measured in terms of the physical output y,* i.e.,

XI = ^ (19-22a)

oL

X = ^ (19-22/7)

This interpretation of X_L and X_K makes (19-21) and (19-20) equivalent except for units.

The production possibilities curve yields the set of "efficient" production plans. A necessary condition for overall Pareto optimality is to be on this frontier. However, that in itself is not sufficient. To exhaust all the gains from trade, the goods produced must be allocated to the consumers in an efficient manner. This requires at least that the previous analysis of the *consumer's* Edgeworth box diagram apply, i.e., the consumers must be on their contract curve, for any production levels *(x, y)*. However, one more tangency condition must also apply: For each consumer, the marginal rates of substitution of *x* for *y,* that is, the marginal evaluation of *x* in terms of *y* forgone, must equal the marginal cost of producing *x* (in terms of *y* forgone). This condition implies that the consumers are on their contract curve, since *each* consumer's marginal evaluation of *x* must equal the marginal cost of *x*. Let us see how this last condition is derived.

The only difference between this last, and most general problem, and the first one posed in (19-2) is that instead of *x* and _y being fixed, they are determined by the production possibilities frontier derived in the production model as Eq. (19-18). Thus, the locus of overall efficient (Pareto-optimal) points is defined by

maximize

$$U^2(x_2, y_2)$$

subject to

$$x_2 = x \qquad (19\text{-}23)$$

$$y = y^*(x,L,K)$$

It will simplify the algebra to combine the last three constraints into one. These three equations define the production possibility curve, written in implicit form, as

$$h\{x, y) = h(x\backslash + x_2, yi + y_2) = 0$$

where the parameters L and K have been suppressed because they will not be used. The problem is then simply

maximize

$$U^2(x_2, y_2)$$

subject to

$$h(x\backslash + x_2, y_t + y_2) = 0$$

The Lagrangian for (19-24) is

$$\pounds\pounds = U^2\{x_2,y_2) + k_t(U^* - U^1(x_x,y\backslash)) + kh\{x,y) \quad \text{(19-25)}$$

Noting that $dh/dx_t = (dh/dx)(dx/dxi) = dh/dx$, etc., we see that the first-order conditions are

$$U^2 + Xh_x = 0 \qquad \text{(19-26fl)}$$

$$U^* + kh_y = 0 \qquad (\backslash 9\text{-}26b)$$

$$-\wedge\wedge + \wedge = O \qquad \text{(19-26c)}$$

$$-kiU^* + kh_y = 0 \qquad \text{(19-26J)}$$

and the two constraints

$$UQ — U^1(xi, yi) = 0 \qquad \text{(19-27a)}$$

$$h(x, y) = 0 \qquad \text{(19-27/7)}$$

Eliminating the Lagrange multipliers from Eqs. (19-26), we find

$$-\wedge = -\wedge = — \qquad \text{(19-28)}$$

$$U\} \quad U] \quad h_y$$

The quantity h_x / h_y is the absolute slope of the production possibilities frontier; i.e., in explicit form, by the chain rule,

$$dx \qquad h_y$$

Hence, Eq. (19-28) gives the marginal condition stated above: *For overall (production and consumption) Pareto optimality, the marginal evaluation of each commodity must be the same for all individuals, and that common marginal evaluation*

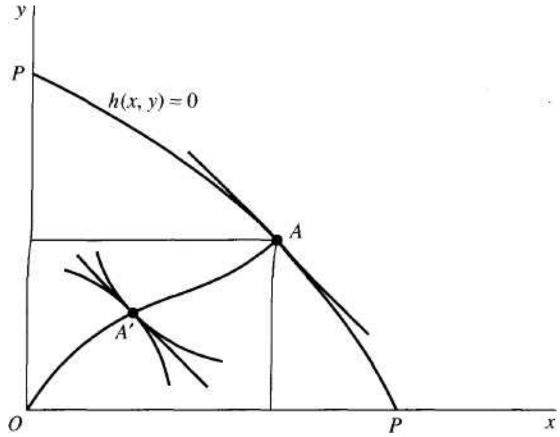

FIGURE 19-3
Overall Pareto Optimality. The curve *PP* represents the *production* possibilities frontier of the economy for given resource endowments. The slope of this frontier is the marginal cost of producing *x,* in terms of *y* forgone. At any point, say *A,* along the frontier, an Edgeworth box can be constructed as shown. The points in the box represent allocations of *x* and *y* to the two consumers. These consumers will presumably trade to the contract curve *OA*. At some point or points on *OA,* the slopes of the indifference curves will equal the slope of the transformation curve at *A*. This is an overall Pareto efficient point, since the MRSs of each consumer are equal and equal to marginal cost.

must equal the marginal cost of producing that good. (The words *all* and *each* have been used instead of *both.* The generalization of these results to *n* goods and m consumers is straightforward.)

The overall utility frontier is found by solving Eqs. (19-26) and (19-27) for $x_t = X^*(UQ)$, y, = y*(£/(]). Substituting these values into the objective function, we derive

$$U^2 = U^2(x2, y_2^*) = U^{2*}(U\hat{}) \qquad (19\text{-}29)$$

This situation is shown geometrically in Fig. 19-3. The curve *PP* represents the production possibilities frontier for given resource endowments. At any point, say A, the slope of this frontier is the marginal cost of*. From this point, which represents a certain total amount of JC and y, an Edgeworth box diagram is constructed. The points in the interior of the box represent the allocations of *x* and y to the two consumers. The curve *OA* represents the implied contract curve for the consumers. At some point (or points) along *OA,* say *A',* the marginal evaluations of *x* (the marginal rates of substitution) will equal the slope of the tangent line at A, the marginal cost of JC. This is an overall Pareto-optimal allocation, i.e., efficient in production and consumption. The point *A'* represents one particular point on the implied *utility* frontier, as depicted in Fig. 19-2. It is a special point, however, in that marginal cost equals marginal benefits there.

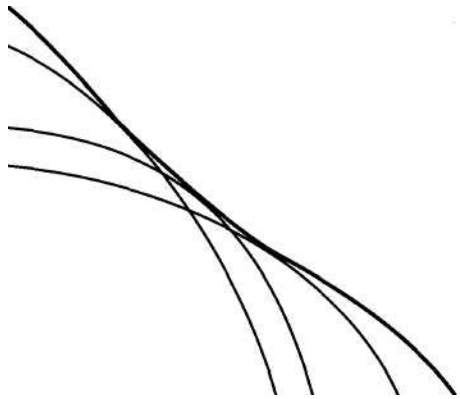

FIGURE 19-4
Partial and Overall Utility Frontiers. For any given *x* and *y,* that is, for some particular point on the production possibilities curve, some *utility* frontier is implied. Several of these are drawn: *U'U', U''U'',* and *U'''U'''*. The *envelope* curve for all these partial frontiers is the overall, or *grand,* utility frontier *UU.* The frontier *UU* represents the maximum utility any one consumer can achieve for given level of the other person's utility. Each point on *UU* represents, in general, a different production point, though there is no reason why some partial frontier could not be tangent to *UU* at more than one point.

At each point along the *production* possibilities frontier, an Edgeworth box can be drawn and the overall efficient allocation(s) can be determined. In Fig. 19-4 the utility frontiers for several production points are drawn. The *envelope* curve for all these partial frontiers is Eq. (19-29), $U^2 = U^{2*}(UQ, X, y)$. The partial frontiers are those for specific values of *x* and *y,* that is, holding *x* and *y* constant. From general envelope considerations

du· _/ dU' 9^0 (19-30)

That is, along the overall frontier, the slope of the frontier at any point is the same if *x* and *y* are held constant or allowed to vary.

The grand utility frontier *UU* represents the complete set of Pareto-optimal, or efficient, productions and distributions of the goods *x* and *y.* The choice of *which* Pareto-optimal point is somehow "best for society" is a value judgment and outside the scope of positive economics. If some social welfare function is posited (social welfare functions can exist, but not with all the properties outlined by Arrow), its indifference curves can be plotted in Fig. 19-4, and some optimal point along the frontier *UU* will be selected. There are some who believe that governments consciously seek some overall optimum as just described. It is difficult to explain political behavior with such a model.

19.3 THE CLASSICAL "THEOREMS" OF WELFARE ECONOMICS

In this section we shall present the classical "theorems" of welfare economics. The quotation marks are *used because the propositions derived in what follows are not* in fact refutable theorems. They represent generally unobservable first-order conditions for maximization, i.e., statements that at an optimum, marginal benefits equal marginal costs. As was indicated in the quotation from Samuelson's *Foundations of Economic Analysis,* these propositions represent the logical implications of propositions that are not themselves refutable.

The first "theorem" is *that perfect competition leads to a Pareto-optimal allocation of goods and services.* This proposition holds only under certain restrictive conditions. Specifically, the formulation of the problems posed in the previous section ruled out two major classes of phenomena: Interdependence of the consumer's utility functions and interdependence of the production functions. In the preceding presentation, there were no *externalities,* or side effects, present between any of the maximizing agents. Such interdependence would be indicated by writing, say,

$$y = f(L_y, K_y, x) \qquad (\backslash 9\text{-}3\backslash a)$$

or

$$U^2 = U^2(x_2, y_2, U^1) \qquad (19\text{-}31/?)$$

In the case of (19-3la), the output of y depends not only on the labor and capital inputs in the production function for y but also the level of x produced. In a later section we shall consider a particular example of this, where the output of a farm depends in part on a neighboring rancher's output of cattle, who trample some of the farmer's output. Similarly, (19-31Z?) indicates that another person's happiness is an influence on one's own utility.

In the absence of occurrences (19-3la) and *(\9-3\b)* and in the absence of monopoly, the prices of goods and services offered in the economy will equal their respective marginal costs of production. The condition for profit maximization under competitive factor and output markets yields, for each industry *h,*

$$P h f - - w_t = 0 \qquad i = 1, \ldots, /i \qquad (19\text{-}32)$$

where

$f^k(x\backslash, \ldots, x_n) = k\backslash h$ firm's production function Wi = wage of Xj

p_h = output price Suppose there are m firms. The supply function of the firms is the solution of

$$Ph - \frac{\mathbf{ac^*}}{T^{JL}} = O \qquad (19\text{-}33)$$

where $Cl(y_k, w\backslash, \ldots, w_n)$ is the firm's total cost function. From (19-32),

— = ^V $k = l, \ldots, m$ (19-34)
"J fj

This is precisely the condition that the economy be on the production possibilities frontier: The ratio of marginal products for all pairs of factors is the same for all firms, equal to the ratio of factor prices.

Moreover, utility-maximizing consumers with utility functions $U^k(y\backslash, \ldots, y_n)$ in the n output goods will set the ratios of marginal utilities equal to the price ratios; i.e.,

^V = — for all *i, /, k* (19-35)
uj PJ

Since all consumers will face the same prices, Eq. (19-35) says that all consumers' marginal evaluations of the good will be identical, the condition for efficient consumption for given outputs. Lastly, using Eq. (19-33),

^ ^ ft* all/,./,* (19-36)
***Uj Pj* MC;**

Hence, not only are all consumers' marginal evaluations equal, they are equal to the ratio of marginal costs of those goods, expressed in money terms. This ratio of money marginal costs is precisely the marginal cost of good *i,* in terms of good *j* forgone. That is, converting to units of good *j* makes $MC_;$ = 1. [Note that the units of MQ/MC/ are ($v,) 4- ($)>_/) = *yj/yi,* the amount of *yj* forgone to produce another increment of $y_{\prime}$, or the *real* marginal cost of v,-.]

Thus, under perfect competition with no side effects (externalities), the Pareto conditions for overall efficiency hold. Therefore, in such a perfectly competitive economy, no individual will be able to improve himself or herself without making someone else worse off.

It does *not* follow from the preceding that it is desirable for the economy to be perfectly competitive. Consider Fig. 19-5, where the grand utility frontier *UU* has been plotted. Suppose, somehow, the economy has situated the two individuals at point *A,* a non-Paretian allocation. Any movement to the right or upward from *A,* resulting in a point on the utility frontier along the segment *BC,* is clearly Pareto-superior to *A*. However, a movement to *D,* a Pareto-optimal point, leaves consumer 2 worse off; it is not an improvement from consumer 2's standpoint. Hence, aside from being a value judgment, a move to the Pareto frontier may involve losses.

The second "theorem" of classical welfare economics is the statement that there is an allocation under perfect competition for any overall Pareto optimum. That is, starting now with a point on the Pareto frontier, there exists a competitive solution which achieves that optimum. The proof of this proposition, for general functional forms of utility and production functions, is a formidable mathematical

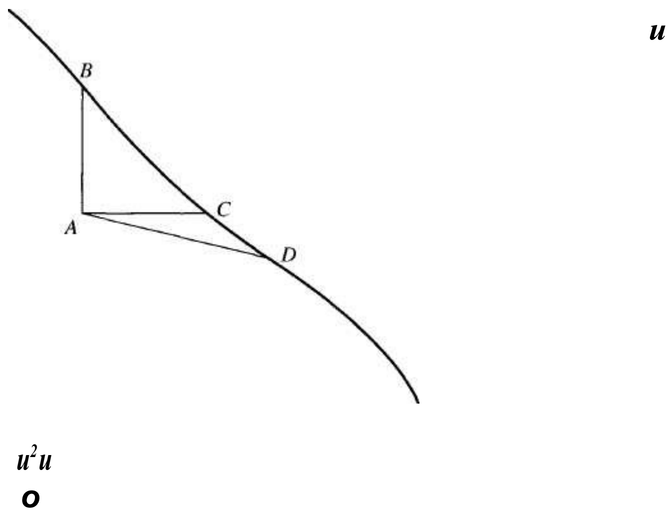

FIGURE 19-5
A Non-Pareto Move. Suppose the economy is at point *A*. Then any move northeast will be to a Pareto-superior position: Each consumer will gain. Any point along the segment *BC* of the Pareto frontier *UU* is Pareto-superior to *A*. However, not every point along *UU* is Pareto-superior to *A*. Point *D*, for example, leaves consumer 1 better off and consumer 2 worse off than at A. Consumer 2 will not advocate economic efficiency if it results in the economy's moving to point *D*. It is not possible to argue, even with the weak Paretian value judgment, that the economy "ought" to be at a Pareto-optimal point.

problem, which has been analyzed by K. Arrow^ G. Debreu,* L. Hurwicz,[§] and others. A rigorous discussion is considerably beyond the scope of this book.

Note what this second "theorem" does *not* say: It does not say that in order to achieve a Pareto position the economy must be competitive. An omniscient dictator could mandate the correct prices and quantities so that the economy would reach the same position as a competitive economy would.

Two of the outstanding reasons why an economy might not be on the overall Pareto frontier are (1) excise taxes and (2) monopolistic raising of price over marginal cost. With regard to the latter, a *perfectly discriminating monopolist,* who extracts all the gains from trade via some sort of all-or-nothing pricing, does *not* disturb the Pareto conditions. The reason, fundamentally, is that all the gains from trade are exhausted. The only difference is that only the perfectly discriminating monopolist

principal investigation of both the above theorems is in K. Arrow, "An Extension of the Basic Theorems of Classical Welfare Economics," in J. Neyman (ed.), *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability,* University of California Press, Berkeley, 1951. ■tGerard Debreu, *Theory of Value,* John Wiley & Sons, Inc., New York, 1959.

§ Leonid Hurwicz, "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences 1959,* Stanford University Press, Stanford, CA, 1960.

gains, whereas with open markets the buyers and sellers both gain. But as long as all the gains from trade are exhausted, there can be no *Pareto-superior* moves.

19.4 A "NONTHEOREM" ABOUT TAXATION

A commonly stated proposition is that to raise any given amount of tax revenue it is best, from the standpoint of consumers' achieving the highest possible indifference curve, to collect those taxes via proportional excise taxes or income taxes. (With no savings in the economy, these taxes are equivalent.) The argument is loosely based on the observation that the Pareto conditions *Pj/pi* = MRS^ = MCy/MC, would not be disturbed if *pj*■■ = (1 + *t)MCj,* where the tax rate *t* is constant across all commodities. This, however, is a logical error, since these first-order marginal conditions for Pareto optimality, while necessary, are not sufficient. Other criteria may lead to the same conditions.

The "theorem" has been criticized on the empirical grounds that not all goods are easily taxed. A person's labor-leisure choice is affected by any tax on income. The price of leisure is the forgone wage; a tax on that wage income is a *subsidy* on leisure. In addition, many commodities, for more or less technological reasons, may be difficult to tax, e.g., services one provides for oneself or family. Under these conditions, a proportional tax on all taxable items is not a proportional tax on all items.

These empirical matters aside, however, a correct theorem is difficult to state. Even if one could tax all goods and services proportionately, this would not in general lead to a Pareto allocation, as we shall presently see. The most famous "proof" of this nontheorem was presented by Harold Hotelling in 1938.^ Hotelling's proof went essentially as follows. Suppose a consumer currently consumes *n* goods, q_h *i* = 1, . . . , «, at prices /?, = MQ. The consumer's *income* is taxed, however, and money income after taxes is *m* = ^ *PiQi-* Since the commodity bundle q = *(q_l, ..., q_n)* was chosen at prices p = *(pi, ..., p_n)* and income *m,* any other bundle of goods *q'* = q + Aq that the consumer *could* have chosen must be inferior. Hotelling was asserting (without using the phraseology, which was not yet invented) that q was *revealed preferred* to q' if

or

$$^PiAqi<0 \qquad (19\text{-}37)$$

^H. Hotelling, "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates," *Econometrica,* 6:242-269, 1938; reprinted in *A.E.A. Readings in Welfare Economics,* Richard D. Irwin, Homewood, IL,1969.

Now suppose prices are changed by amounts A/?,, representing excise taxes, but money income (tax) is also changed so that the consumer can have the same opportunities to purchase goods as before. By definition,

m + Am =]P(Pi + Ap,)(^ +

Aq$_{t}$*)* Subtracting *m* = *Y2 Pili* gives

Am =

Rearranging terms, we

have

^*Pi&qt* = Am - ^Ap,-te + A?,-) (19-38)

Consider this last equation. The term q_t + Ag, represents the q_t 's sold if taxed; hence, the last term represents the total tax revenue from the excise taxes, Ap,, / = 1,..., *n.* The term Am represents the change in income taxes. Therefore, this expression says that if the change in excise taxes results in revenue absolutely greater than or equal to the income tax change, *J2 Pi* ^<7< — 0- In this case, it is argued, that since prices were set at marginal costs, replacing income taxes by excise taxes leads the consumer to purchase some bundle *q'* which was shown to be revealed inferior to *q*. Hence, to quote Hotelling^

> *If government revenue is produced by any system of excise taxes there exists a possible distribution of personal levies among the individuals of the community such that the abolition of the excise taxes and their replacement by these levies will yield the same revenue while leaving each person in a state more satisfactory to himself than before.*

This "proof," however, seems to be merely a theorem about revealed *preferences.* Starting at *any* set of prices whatsoever, making the just stated changes in prices and income will leave the consumer worse off. Nowhere is the condition *Pi* = MC, used in this "proof." That marginal condition is irrelevant to the argument. No assumptions about *production* are contained in the argument; only assumptions concerning *preferences* are used. The same "proof" follows if initially /?, ^ MQ and the *Apt's* and *m* are changed so as to make p_t = MQ in the final position.

19.5 THE THEORY OF THE SECOND BEST*

The problem of optimal excise taxation cannot be handled without considering the ends of this taxation. Suppose there are *three* goods—two private goods, *x* and};, and

^Italics in the original. There is no apparent distinction in Hotelling's paper between income tax, proportional excise tax, and lump-sum or personal-levy tax.

*R. G. Lipsey and K. Lancaster, "The General Theory of the Second Best," *Review of Economic Studies,* 24:11-32, 1956.

government services, z. If these government services are services for which normal pricing is possible, e.g., postal services, the optimal taxes are zero. The government merely sells its services at marginal cost, which, together with selling x and y at their respective marginal costs, will yield a Pareto optimum. The question of optimal taxation makes sense only in the context that some good, say the services of the government, is not, for some reason, to be sold at marginal cost. In some cases, e.g., national defense, it would be difficult to do so. Also, an important class of goods exists, e.g., the so-called public goods discussed in the next section, for which marginal costs are less than average costs—the declining-AC industries. It is impossible to sell these goods at marginal cost without subsidies raised via taxation. The question thus becomes: Suppose some good z is not sold at marginal cost. Is it possible to infer that consumers will be on the highest indifference curves if the remaining goods are sold at prices proportional to their marginal costs, e.g., by proportional excise or income taxes? The answer is no, as the following argument shows.

Consider the simplest case of one consumer. The consumer maximizes utility subject to the production possibilities frontier, or

maximize

$$U(x,y,z)$$

subject to

$$g(x,y,z) = 0$$

The Lagrangian is

producing the first-order conditions

$$U_x+kg_x=0 \qquad U_y+Xg_y=0 \qquad U_z + kg_z = 0$$

or

$$\text{—} = \text{—} \qquad \text{—} = \text{—} \qquad (19\text{-}39)$$

$$\mathbf{Uy} \quad \mathbf{8y} \quad \mathbf{Uy} \quad \mathbf{gy}$$

The marginal rates of substitution equal the respective marginal costs. Suppose now that z is not sold at MC. A simple constraint which expresses this is $U_z = kg_z$, where $k =£ Uy/gy$. Let us now maximize $U(x, y, z)$ subject to this new constraint also, in addition to the resource constraint $g(x, y, z) = 0$. The Lagrangian for this problem is

$$\% = U(x, y, z) + Xg(x, y, z) + /i(U_z - kg_z)$$

The first-order conditions for this maximization are (excluding the

constraints) $£_X = U_X + kg_x + fi(U_{zx} - kg_{zx}) = 0$

$$Xy = Uy + Xgy + fl(U_Zy \sim kg_Zy) = 0$$

$$\%_Z = U_Z + Xg_z + fi(U_{zz} - kg_{zz}) = 0$$

Since the constraint U_z — kg_z is assumed to be binding, /x / 0. Solving for the marginal rates of substitution,

$$\frac{Ux\_}{Uy} = \frac{-*-gx \sim V(Uzx \sim kg_{zx})}{-Xgy \sim /JL(U_{zy} \sim kg_{z}y)} \qquad (19\text{-}40)$$

with a similar expression for U_x/U_z or U_y/U_z.

The left-hand side of Eq. (19-40) is the MRS between *x* and y. It cannot be inferred that this MRS should be equal to MC^/MC^ = g_x/g_y. For arbitrary values of the cross-partials U_{zx}, g_{zx}, U_{zy}, and g_{zy}, *nonproportional* excise taxes on *x* and *y* will in general satisfy (19-40). It might be noted that if these cross-partials are all 0, Hotelling's "theorem" holds, but this is a special case.

In general, therefore, it cannot be argued that if some distortion, that is, *Pj* ^ MC_7, is removed in the economy, consumers will move closer to the Pareto frontier if other distortions are present. If the industries involved are unrelated, a case might be made that the above cross-partials are 0. In that case, a more efficient allocation is implied by removal of the distortion.

Hotelling correctly argued for a nondistorting, or lump-sum, tax. As previously mentioned, an income tax is a subsidy on leisure and hence distorts the labor-leisure choice. A poll tax is cited as an example of a lump-sum tax. More precisely, an existence tax is advocated. Even with this type of tax, however, we shall find, in the long run, less existence, i.e., fewer children, less spent on lifesaving devices, etc. For all practical purposes, it is probably safe to conclude that there is no such thing as a lump-sum tax.

19.6 PUBLIC GOODS

There is an important class of goods that have the characteristic of being *jointly consumed* by more than one individual. These goods, known as *public goods,* are goods for which there is *no congestion.* Ordinary private goods are goods for which congestion is so severe that only one person can consume the good.

The most famous example of a public good is perhaps the service national defense. The protection afforded any individual by the nation's foreign policy and military prowess is substantially unaffected if additional recipients are added to that service flow. Similarly, driving on an uncrowded freeway, watching a movie or play in an uncrowded theater, or watching a television program are services for which the marginal cost, in terms of resources used up, of accommodating an additional consumer is essentially zero. These goods are the polar case of goods for which average costs are forever declining.

The problem such goods raise for welfare economic considerations is that the Pareto frontier is reached only if all goods and services are sold at their marginal cost of production. If public goods are sold at marginal cost, no revenues will be generated to finance the production of those goods. If production of the public good is financed by revenues derived from taxation of other goods, these other goods will be sold to consumers at prices other than marginal cost, thereby moving the economy off the Pareto frontier. The problems of second best, just discussed, apply to these goods.

Matters of financing aside, assuming that the public good is to be sold at marginal cost, that is, zero, what level of the good is to be produced in the first place, i.e., how many uncrowded highways, open-air concerts, etc., are to be produced? The *production* of public goods is not free; these goods are "free" only in the sense that the marginal cost of having an additional individual consume the good, once produced, is zero. In the case of private goods, this problem does not arise (except in the case of declining average costs). The goods are produced by profit-maximizing firms and sold at marginal cost. No private firm, however, could produce a public good and satisfy the Pareto condition p — MC = 0.

Suppose there are two consumers with utility functions $U^1 (x\backslash, y\backslash)$ and $U^2(x_2, y_2)$, where x is the public good and y is the ordinary private good. By definition of a public good, both consumers consume the total amount x of the good produced. Hence,

$$x_x = x_2 = x \qquad \text{(19-41)}$$

For the private good, as before, $y_x + y_2 = y$. Suppose there is a transformation surface $g(x, y)$ defining the production possibilities frontier for the economy. The Pareto optimum is achieved by solving

maximize

$$U^2(x,y_2)$$

subject to

$$U^1(x,_{yi}) = Ul \qquad g(x,y) = 0 \qquad \text{(19-42)}$$

with $y = y\backslash + y_2$. The Lagrangian is

$$\text{Se} = U^2(x, y_2) + X, (Ui - U^1(x, \text{y},)) + {**}(*, y) \qquad \text{(19-43)}$$

Differentiating i£ with respect to $x, y\backslash, y_2$ and the multipliers, noting that $g_{y\cdot} =$
$gy(dy/dyi) = g_y$, i = 1, 2, we have, denoting $U^J_{x\cdot} = U^J{}_X$, etc.,

$$X_x = U^2{}_X - X_x Ul + Xg_x = 0 \qquad \text{(19-44fl)}$$

$$<\text{£}_{yi} = \text{-Aj } U^1{}_y + kg_y = 0 \qquad \text{(19-446)}$$

$$X_{y2} = U^2{}_y + Xg_y = Q) \qquad \text{(19-44c)}$$

with the constraints

$$2*, = Ui - U^1(x,_{yi}) = 0 \qquad \text{(19-45fl)}$$

From (19-44c), $X = -U^2/g_y$. Substituting this in (19-446) gives $X_x = -U^2/U_y{}^l$. Using these two expressions in (19-44a) leads to

$$\wedge + \wedge U_x - \wedge g_x = O \qquad \text{(19-46)}$$

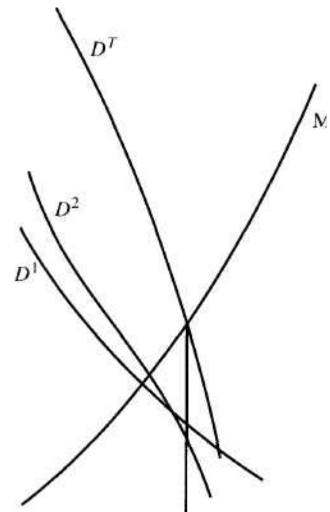

FIGURE 19-6 *Market Demand for Public Good.* If D^1 and D^2 are the two individual demand curves for the public good, the total demand D^T is the *vertical* sum of D^1 and D^2. That is, D^T represents the sum of each con-sumer's margin

al evaluation of the public good. This vertical summation occurs because both consumers consume the total quantity of public good produced. The output x^* at which D^T intersects the marginal cost curve for producing x yields consumption on the Pareto frontier.

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Equation (19-47) admits of an interesting interpretation. U^x_x/U^x_y and U^2/U^2 are, respectively, the marginal rates of substitutions, or the marginal evaluations, of the public good JC. The expression g_x/g_y is the marginal rate of

transformation of _y into x or the marginal cost of the public good in terms of private good forgone. Since both consumers consume the *total* amount of *x* produced, the marginal benefits to society of the public good are the *sum* of each consumer's marginal benefits. Equation (19-47) therefore says that when the Pareto frontier is achieved, the total consumers' marginal benefits equal marginal cost. The usual reasoning of equating benefits and costs at the margin is preserved. The rule is adapted for goods with the characteristic of joint consumption.

Equation (19-47) says that to find the market demand curve for a public good, the individual demand curves are to be added *vertically*, as shown in Fig. 19-6. The market demand for ordinary goods is, of course, the *horizontal* sum of individual demands, because each consumer consumes a part of the total. For public goods, each consumer jointly consumes the total. The height of the individual demand curves, D^1 and D^2 in Fig. 19-6, are the marginal evaluations of the public good *xJ* The curve D^T

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$$i\,l \qquad (19\text{-}39)$$

$$y \quad gy$$

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with $y = yi + y_2$. The Lagrangian is

$$X = U^2(x, y_2) + X_x\,(Ui - U\backslash x, {}_{yi})) + Xg(x, y) \qquad (19\text{-}43)$$

Differentiating X with respect to $x, y\backslash, y_2$ and the multipliers, noting that $g_{y\cdot} = g_y(dy/dyi) = g_y$, $i = 1, 2$, we have, denoting $£// = U^J_X$, etc.,

$$£_x = U^2 - X_lu^l_x + Xg_x = Q \qquad (19\text{-}44a)$$

$$2_{yi} = -\wedge + \wedge = 0 \qquad (19\text{-}446)$$

$$X_{y2} = U^* + kg_y = 0 \qquad (19\text{-}44c)$$

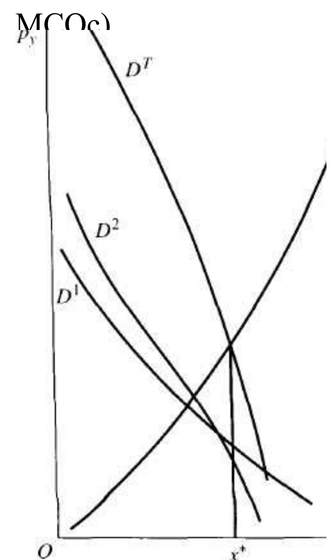
with the constraints

$$5E_{Xl} = Ui - U^1(x,{}_{yi}) = 0 \qquad (\backslash 9\text{-}45a)$$

$$\%x = g(x,y) = 0 \qquad (19\text{-}456)$$

From (19-44c), $X = -U^*/g_y$. Substituting this in (19-446) gives A.j= $-U^\wedge/U^l_v$.
Using these two expressions in (19-44a) leads to

$$U? + 7n\frac{U^2}{U^*} - \frac{U^2}{r}8_x = 0 \qquad (19\text{-}46)$$



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income being held constant in these demand curves is the total value of x and y given by the transformation surface $g(x, y) = 0$.

is the vertical sum of D^1 and D^2, representing the benefits of JC at the margin to both consumers jointly. The quantity JC* where D^T intersects the marginal cost curve of producing JC is the point that satisfies the Pareto conditions for production of a public good.

The preceding analysis generalizes in a straightforward manner to the case of A'consumers. In that case, the Pareto conditions for public good production become

$$\sum_{i=1}^{K} MRS^i = MC \tag{19-48}$$

The problem of private production of public goods is that the ordinary market transactions are not likely to yield the Pareto allocation. In order to arrive at production of JC at the level x^* where $\sum$ MRS = MC, each consumer's differing marginal evaluations would have to be known. However, consumers will have no occasion to reveal these preferences. With private goods, consumers reveal their preferences by their choices in the market, purchasing additional units of a good until the marginal evaluation falls to the market price. There is no comparable mechanism for public goods. Each consumer consumes the total amount produced, and each has in general a different marginal evaluation of that good. Moreover, since the good is to be dispersed in total, it will pay consumers to understate their evaluation of the benefits of the good, lest the government attempt to allocate the good on the basis of fees based on each consumer's personal evaluations of benefits. Lastly, a fee charged for per unit use of the public good will result in "too little" consumption of the good. Consider the case of an uncrowded bridge. When a toll is charged, consumers will not cross the bridge if their marginal evaluation of the benefits is greater than zero but less than the toll. But since the resource cost to society for the consumer's use of the bridge is zero, the ideal Pareto optimum cannot be achieved. Thus, the ordinary contracting in the marketplace for public goods production is not likely to lead to an efficient allocation of resources in terms of the Pareto ideal.

19.7 CONSUMER'S SURPLUS AS A MEASURE OF WELFARE GAINS AND LOSSES

We have previously investigated the problems associated with defining, in units of money income, the gains from trade. One of the most prominent uses of these measures is the evaluation of costs and benefits of alternative tax schemes or the benefits of public good production. Let us briefly recapitulate these issues and apply the analysis to the problem of public good production.

Since the publication of Marshall's *Principles,* economists have attempted to measure the benefits of consumption by some sort of calculation based on the area beneath a consumer's demand curve. In Fig. 19-7, the height of the consumer's demand curve at each point represents the consumer's marginal evaluation of the good in terms of other goods forgone, measured in terms of money. It is therefore tempting to integrate, or add up, these marginal gains to arrive at the total gain received from consuming some positive level of the good rather than none at all.

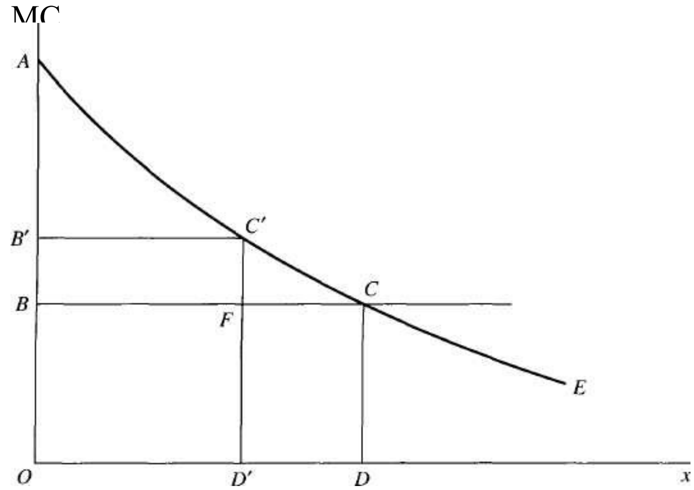


FIGURE 19-7
The Attempt to Measure Welfare Losses by Consumer's Surplus. The analysis of welfare losses is an attempt to have a money measure of the loss in utility incurred from selling commodities at prices other than marginal cost. Let *OB* — MC of *x,* and suppose *OD'* = *B'C* of *x* is sold at *OB'*. The traditional analysis asserts that the benefits from consuming *OD* is the trapezoidal area *OACD.* At price *OB',* total benefits are supposed to be *OACD'*. The difference, *D'C CD,* is partitioned into *FC'C* and *D'FCD*. The latter area is an amount of income spent on other commodities, presumed to be sold at marginal cost. The remaining area *FC'C* is called *deadweight loss,* a money measure of the loss due to the price distortion *BB'*. This distortion is commonly attributed to excise taxation or monopolistic sale of *x*. \fAE is a *real-income,* or *utility-held-constant,* demand curve, then although these areas represent well-defined measures of willingness to pay to face different prices, since these measures hold utility *constant,* they cannot very well measure utility *changes.*

However, we have seen in Chap. 11 that this is not possible. If the demand curve in Fig. 19-7 is a Hicksian, or *utility-held-constant* demand curve, the area *OACD* represents the maximum dollar amount a consumer would pay to have *OD* units of *x* rather than none at all. It likewise follows that for these demand curves, *ABC* represents the maximum amount a consumer would pay for the *right* to consume *x* at unit price *OB*. If the license fee is actually paid, *OD* will be purchased and the consumer will remain on the same indifference level before and after the purchase, by definition of *ABC* as the *maximum* license fee the consumer would pay. This measure has the desirable property of being well defined and at least in principle observable.

If, on the other hand, the demand curve in Fig. 19-7 is a *money-income-held*-constant demand curve, the area *ABC* does *not* represent an observable quantity. The monetary value of gain in utility associated with the terminal prices *OA* and *OB* is generated by a line integral that is generally path-dependent; different adjustments of prices leading to the same initial and final price income vectors will generally lead to different monetary evaluations of the consumer's gain in utility. This is an

inescapable index number problem for nonhomothetic utility functions. Only in the case of homothetic utility functions are changes in utility proportional to changes in income for any set of initial prices.

The quantity *OD* is special in the sense that the marginal benefits to consumers from x exactly equal the consumer's evaluation of the resources used to produce x in producing something else—the marginal opportunity cost of x. If there are no "distortions" of prices from marginal costs elsewhere in the economy, this occurrence is part of the Pareto conditions. However, if there are other goods whose prices differ from MC so that such efficient consumption levels do not occur, then, again, it is not possible to conclude that selling this good x at MC will lead the economy closer to the Pareto frontier. In general, if one good is sold at some price other than MC, say due to an excise tax on that good, then the set of excise taxes $(t\backslash, \ldots, t_n)$ on the n commodities in the economy which will lead to the Pareto frontier will not consist of zero tax rates on the other commodities, nor will they all necessarily be proportional to their respective marginal costs. The specification of such an optimal set of taxes (? *,…, t^*), which leads the economy to the Pareto frontier for given deviations from MC of certain goods or for the purpose of financing government services, is too protracted a discussion to consider here.

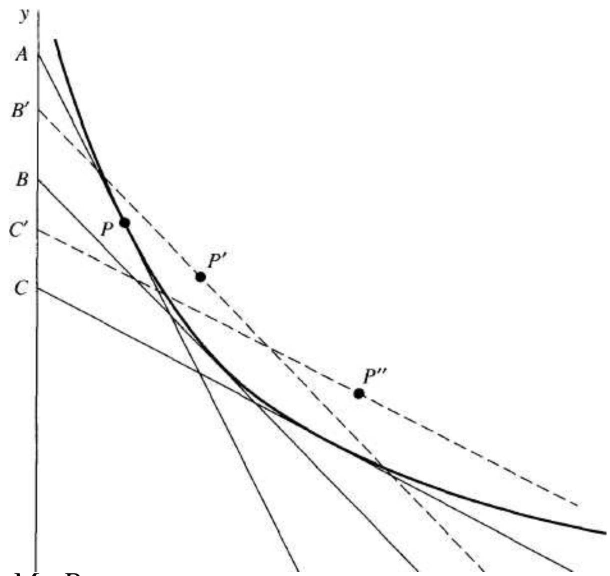
Following the early French economist Dupuit, and stimulated greatly by Marshall's discussion of consumer's surplus, the monetary evaluation of the welfare loss associated with consuming *OD'* instead of *OD* units of x is usually given as the triangular area *FCC* in Fig. 19-7. The total benefits of consuming x are reduced by the trapezoidal area *DC CD*. However, the rectangular area *D'FCD* represents income spent on other goods, *presumably at the marginal cost of those other goods,* eliminating this area as a part of welfare loss. The only remaining deadweight loss of the sale of x at price *OB'* > MC is the area *FCC*. Summing these areas over all commodities is commonly used to measure the welfare loss associated with a set of departures of price from marginal cost.

The compensating variation

$$M^*(p, II) = - (^{Pl} \quad \hat{}\,x/(p, U)\, d_{Pi} \qquad (19\text{-}49)$$

represents the amount of money income *the consumer would be willing to pay* to face the prices p_t instead of /?,+?,, z = 1,…,«. (If some ?, < 0 and $M^* < 0$, M^* represents the amount a consumer *would have to be paid* to accept $p, + tj$ voluntarily instead of /?,-,/ = 1, ..., n.) The problem of using M^* as a measure of the benefits from increased utility is thatM* *depends only on one indifference level.* Utility is held constant in the integral (19-49). This may lead to inappropriate welfare rankings.

In Fig. 19-8, we set $p_y = 1$ arbitrarily. Since the vertical intercepts are $M/p_y = M$ in this case, changes in income can be read directly off the vertical axis. The consumer initially faces price p_x for x, producing the budget line emanating from A, with income *OA*. From the graph, the consumer is willing to pay an amount *AB* to have the price of x reduced to p'_x and willing to pay *AC* to have the price of x reduced to p''_x. Suppose *AB* = \$10 and *AC* — \$20. Suppose the consumer is actually going to have to pay \$5 (*AB'*) to have p_x reduced



M''
$P'x$

$U=U°$

$M\_ Px$ **Ml**

FIGURE 19-8
Measuring Gains from Trade by Compensating Variations in Income. The consumer has income M and faces prices p_x, $p_y = 1$. Money income is therefore measurable as distances along the vertical axis since the budget line intercepts that axis at $M/\,p_y$ — M. At price p_x, the consumer consumes at point P on utility level f/°. The consumer is willing, according to this diagram, to pay amounts AB, AC to face the lower prices p'_x, p''_x, respectively. Suppose the consumer only *has* to pay AB', AC to face those lower prices. Suppose the differences between what the consumer is *willing* to pay and what is actually paid, that is, BB' and CC, are not equal; e.g., suppose $BB' < CC'$. Can one infer that the second situation leaves the consumer on a higher indifference curve? Alas, no. For the first case, the consumer faces price p'_x with income OB', winding up at some point P'. In the second situation, the consumer faces price p'' and income OC, winding up at some point P''. There is no way in general to tell which if either of P' and P'' is on a higher indifference level. The only indifference curve specified is U = f/°; no information is provided (except convexity) about where preferred indifference levels lie. Hence, the differences between compensating variations and actual costs of, say, two mutually exclusive projects may be unreliable measures of their ultimate benefits for consumers.

to p'_x or is actually going to pay \$14 (AC) to have p_x reduced to p''_x. Suppose AB' and AC represent the cost of two alternative, mutually exclusive public works projects. Are these data sufficient to evaluate these projects in terms of answering which will place this consumer on a higher indifference level? Although the gain measured by the compensating variation minus the cost is greater for the second project, one *cannot* conclude that the consumer would be better off with it. With the first project, lowering p_x to p'_x, the consumer will wind up at some point P' on the budget line emanating from B' with slope p'_x. For the second project, the consumer will be at some point P'' on the budget line emanating from C with slope p''_x. Now within a broad range of price changes, there is no way to determine whether P' is more preferred or less preferred than P''. The reason is that nothing has been said of the properties of this consumer's utility function other than the one indifference curve $U = U°$ from which all the compensating variations are derived. One must therefore conclude that integrals of the form (19-49) may not be reliable measures of gains from trade; they hold utility constant throughout.

19.8 PROPERTY RIGHTS AND TRANSACTIONS COSTS

The analysis of the Pareto conditions for economic efficiency has been presented in the absence of any institutional framework. We have assumed that production, exchange, and consumption take place without conflict. In actuality, production and exchange based on mutual benefit is not a universally admired principle; in many parts of the world such activities are severely proscribed by government edict. No society allows literally any mutually advantageous trade; but more importantly, the ability, or *cost* of engaging in trade can vary substantially from good to good, and from nation to nation. The extent to which trade takes place depends on the rights individuals have over the use of resources and the costs of exchange.

Robinson Crusoe will always achieve an efficient outcome given his preferences; he maximizes utility subject to his production constraint. The introduction of another individual, Friday, presents Crusoe (and Friday) with several more "margins" to consider. Gains through specialization are possible, but specialization requires agreement as to the terms of trade, and *enforcement* of the contract. Trade almost always involves "asymmetric information"; one usually knows better what one is giving up than what is about to be received. Crusoe and Friday will have to worry a bit about whether the other individual is living up to the terms of the contract. In modern societies, goods have many dimensions and are difficult to measure completely; production and exchange may involve many individuals, each with their own self-interest, and intruders, who would steal some of the goods, may be present. Whereas it is probably a useful first step to lay out the marginal conditions that must be satisfied in order for all gains from exchange to be exhausted, the empirical realization of such gains is subject to a society's laws and institutions that regulate commerce, and the transactions costs attendant upon production and exchange. Specialization could hardly take place, if, for example, stealing were rampant.

In recent years economists have taken renewed interest in the relationships between property rights and economic activity. While the Pareto conditions are generally unobservable, it is possible to show that certain institutions, or lack thereof, would make the achievement of the Pareto frontier very unlikely. The study of transactions costs, and how the structure of contracts changes to accommodate the realization of gains from trade under varying constraints, is an important new area of economics. Transactions costs are not the same as, say, a tax, which can be analyzed in the usual way by shifting a supply curve by the amount of the tax. Transactions costs are the lost gains from trade, due to imperfect monitoring of exchange, caused by the uncertainty of receiving what is bargained for.

A resource is "private" if it has three essential attributes^

1.437 *Exclusivity*—an individual has the right to exclude others from use.

1.438 *Ownership of income*—an individual may derive (and keep) the income produced by the resource.

1.439 *Transferability*—an individual may transfer the resource to others at some mutually agreed upon price.

In modern societies, these rights may be varyingly enforced (and attenuated) by the government. These rights are almost never complete. Most land in the United States not held by the government is private in the preceding sense, but, for example, use and transferability may be restricted by zoning laws, rights to remove underground minerals may be restricted, etc. When the American west was settled in the nineteenth century, the "homestead acts" gave land title to individuals, but those individuals had to work that land themselves and could not resell the land, usually for about 10 years. Pollution from a factory is an attenuation of our right to breathe fresh air; rowdy neighbors infringe on our ability to enjoy the income produced by our homes, thus reducing the degree to which our homes are "private." Economic activity varies in important ways as the enforcement of private property varies.

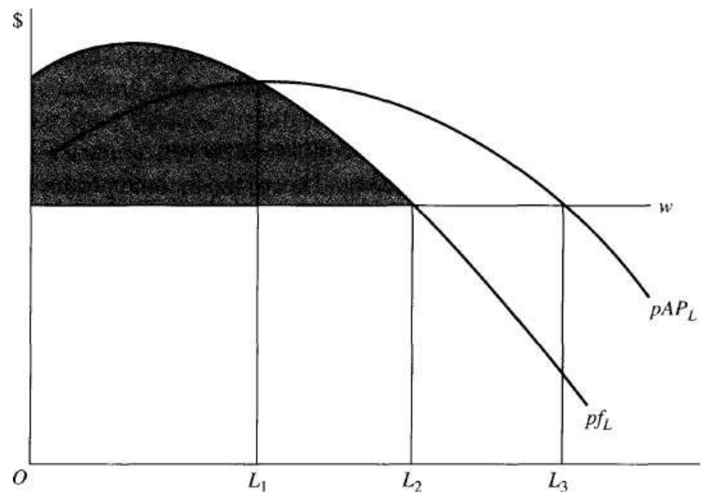
An important polar case occurs when the right to exclude is completely absent. In that case, no one owns the resource; it is called *common property.* A prominent instance is provided by deep-sea fishing. Outside a country's territorial limits, varying from 3 to 200 miles offshore, unless covered by specific treaty, ocean resources, and specifically fish, are often not subject to effective ownership. Even when treaties are present, the ability to police limits on catches offish in, say, the Pacific Ocean may be severely limited. In some countries, private land ownership is severely restricted or forbidden. In such cases we can usually predict the resource will be utilized beyond the level implied by the Pareto conditions.

Suppose the daily production of food takes place by combining labor, L, and land, K, according to some well-behaved production function $y = f(L, K)$; output y is sold competitively at price/?. In Fig. 19-9, the (value of the) marginal and average product curves of labor are shown. Assume workers are available at daily wage w and that this wage represents the opportunity cost of labor in food production. That is, workers could produce nonagricultural output valued at w per day; for each worker in agriculture, nonagricultural output in the amount w is foregone per day.

Consider now two "stylized" systems of property rights.

^This categorization of private property was first presented by Steven N. S. Cheung in "A Theory of Price Control," *Journal of Law and*

Economics, 17(1): 53-72, April 1974. It is similar to the analysis presented in the first edition of Armen Alchian and William Allen's *Exchange and Production,* Wadsworth, Belmont, CA, 1964.



Labor

FIGURE 19-9
Allocation of Resources Under Private and Common Property. Faced with an opportunity cost of labor of w, a private owner of some resource, say land, will hire *L2* workers, where $pf_L = w$. This is an efficient allocation since the marginal value of goods produced in this firm equals the marginal value of labor elsewhere; no reallocation of labor could increase output. Under common property, workers crowd onto the land until *their* own return, which includes some share of the rents on the land, equals their opportunity cost elsewhere. Under this system, L3 workers, where $pAPi = w$, will work the land. This is an inefficient allocation since

workers add only *pfi* on the farm, less than their opportunity cost elsewhere.

PRIVATE PROPERTY. Suppose a fixed plot of land is privately owned by an individual. A private owner maximizes the rents on the land, i.e.,

$$\underset{L}{\text{maximize}}\ R = pf(L, K) - wL$$

The first-order con

ditions for rent maximization are

$$Ph = w$$

or input L_2 in Fig. 19-9. Since the area under the marginal product schedule is total product, the shaded area under pf_L and over the wage line w represents the maximum daily rent on the land.

The important aspect of this outcome is that the Pareto conditions are satisfied: the gains from trade are exhausted. For labor inputs $0 < L < L_2$, $P/L > w$; thus the additional agricultural output generated exceeds the output lost in the other sector of the economy. Beyond L_2, the forgone nonagricultural output exceeds what the economy is getting in the way of food. No further mutual benefits can be realized by applying more workers to the land. The "invisible hand" is working: Private ownership leads to the greatest output gain to society, though the owner of this land neither knows nor intends that outcome.

COMMON PROPERTY. Suppose now that access to the land is unrestricted: Anyone can become a "squatter" on the land. For example, suppose agriculture is organized into "communes," with unrestricted entry. Anyone can join the commune and share equally in the output produced.* Since workers share equally in output, each receives the value of *average product.* In making their choice as to whether to join the commune, workers compare their alternative earnings w with their average product on the farm. At labor input levels less than L_3, workers earn more on the farm. This extra income derives from ownership of the land rents acquired when workers join the commune. With unrestricted entry, however, the rent on the land is *nonexclusive income.* Workers will compete with each other for ownership of this income, until, at the margin, it no longer exists (or exceeds the cost of acquiring it). In this example, workers will continue to join the commune until the marginal gain from joining (the average product) equals their alternative earnings w. When $pAP_L = w$, total product = wL = total factor cost. The rents are *dissipated.*

This outcome is inefficient; i.e., further gains from trade are possible. At labor inputs greater than L_2, the marginal contribution to output when workers engage in farming, pf_L, is less than what workers could produce elsewhere, w. Resources are being directed to activities that lower, rather than increase, total output. If these extra workers could be induced to leave the farm, the resulting increment in output could in principle be shared, making everyone better off.* Nothing in the preceding argument depends on exhaustion of the land, as might especially be the case with ocean fish (though the problem exists with land also). In the case of deep-sea fishing, for example, preservation of the stock of fish for future harvest is an important margin. Increasing the catch this year may reduce the future stock of fish, raising the marginal cost of catching fish in the future. This is a separate and important issue. Under common property, valuable species may be depleted, perhaps to extinction, because no individual owns the right to any *future* income derived from preserving the resource. In that case, wealth maximization leads to shifting consumption to the present to a level where consumers' marginal value of present consumption of that good is less than its opportunity cost, in terms of the present value of future consumption forgone.

^We ignore the problem of "shirking," which is perhaps the main reason this type of firm is not prevalent. ^ Other types of legal ownership can lead to different misallocations. For example, "socialist cooperative" firms, in which workers currently employed decide the labor input, and share, say, equally in the output, will maximize average product (at $L\backslash$ in Fig. 19-9), leading to *too little* agriculture production.

Freeway congestion is another common property problem. As was shown in Chap. 8 (Sec. S.4), with no restrictions on access, cars enter the freeway until the average time cost equals the marginal (and average, if the "bad" roads are never congested) cost on the side streets. However, each car slows all the others; thus the sum of marginal time costs to all drivers will exceed the gain to any one driver who enters the freeway. As drivers compete for the rents received by access to the freeway (in terms of time saved), those rents are dissipated as all traffic slows down. Resources would be saved if some cars took the side streets. Under private ownership, a toll will be charged leading to the efficient outcome; under common property, the freeway is "overutilized."

Price controls typically create nonexclusive income. Suppose the market price of gasoline would be $1.50 per gallon, but, in an attempt to transfer rents to consumers, the government fixes the price at $1.00. If gas tanks hold 10 gallons, say, the price control would grant each driver a gift of $5.00 per fill-up. However, this income is nonexclusive; it can be acquired only by the act of filling up one's gas tank. Car owners will compete for this gift. Though the exact form this competition will take depends upon the additional legal and economic restrictions attendant on the price control, the typical response, such as occurred in the 1970s (apart from some minor violence), is for drivers to compete by waiting in line for purchase. In so doing, the $5.00 gain is at least partially dissipated by having to forego alternative, utility-increasing activities (including, perhaps, leisure). If consumers have identical alternative costs of time, given, say, by a marginal wage rate of $5.00 per hour, the line will be 1 hour long, and the rent will be completely dissipated. The dissipation can be prevented by the issuance of freely tradeable ration coupons; in that case, the price of gasoline would again be $1.50, $1.00 in cash plus $0.50 forgone by not selling the coupon to someone else. By giving exclusive title to the $0.50 gain, the rents can actually be transferred to consumers.^

The Coase Theorem

The first systematic discussion of the role of transaction costs in relation to the allocation of resources was Ronald Coase's pathbreaking article, "The Problem of Social Cost."* The context of the misallocations were various "technological externalities"—the situation where production of one good was, in this case, a negative input in the production of some other good. The example first cited was the historically important case of straying cattle: A rancher-producer raises cattle who invariably trample some of a neighboring farmer's crop.

The classical welfare economic treatment of this problem, in the tradition of A. C. Pigou, took place as follows. Consider Fig. 19-10. The marginal private cost

*A less expensive procedure, however, is to simply tax the gasoline $0.50 per gallon and return the receipts to consumers through some lump-sum tax not related to consumers' own purchases. ^Ronald Coase, "The Problem of Social Cost," *Journal of Law and Economics,*

pp. 1-44, October 1960.

of production is the area OEx^p. At this level of output, resources are misallocated: At output greater than x^s, the marginal opportunity cost of producing cattle is greater than the marginal benefits to consumers, measured by the price OA. Producing x^p instead of x^s results in a deadweight loss in the amount of BEC.

Coase's contribution was to point out that the preceding argument could be valid only if the rancher and the farmer were somehow prevented from further contracting with each other. A misallocation of resources means that some mutual gains from trade or transacting are being lost. *If* the cost of transacting is zero (and no specific mention of transactions costs was presented), it cannot be that individual maximizers would arrive at some point *off the* contract curve. It would be a denial that more is preferred to less for two people to agree to a non-Pareto allocation or misallocation, of resources.

The assignment of legal liability for the wandering cattle constitutes a specification of *endowments* only. Any rancher who does not have to pay damages for trampled crops will be wealthier. It is an expansion of the rancher's property rights and an attenuation of the farmer's property rights. Likewise, a court ruling that the rancher *is* liable for crop damage is a transfer of assets only, from the rancher to the farmer, not a change in production possibilities or preferences. There is no reason why a change in endowments should foreclose a movement to the contract curve, i.e., the Pareto frontier. The classical theorems of welfare economics indicate that individuals will move to the contract curve irrespective of where the endowment point is placed in the Edgeworth box.

The error of assuming a non-Pareto solution hinged upon a failure to consider the range of contracting possibilities available to individuals, e.g., the rancher and farmer in the preceding case. If the rancher is liable for crop damage, no further contracting is necessary; the state enforces the contract that the rancher pay the farmer for damage. If the rancher is *not* liable, however, there are still options to consider. The farmer can contract with the rancher to reduce cattle production for some fee. Consider Fig. 19-10. The damage to the farmer's crops caused by producing x^p instead of x^s is the area $DBEC$. However, the net profit to the rancher derived from this extra production is only part of that area, DBC. Since the damage to the farmer is greater by the amount BEC than the gain to the rancher from producing x^p instead of x^s, the farmer will be able to offer the rancher more than DBC, the rancher's gain, but less than $DBEC$ to induce the rancher to reduce production to x^s. With no transactions cost, this contract is implied, since both the farmer and the rancher are better off. At any level of production beyond x^s, the damages to the farmer exceed the incremental gains to the rancher; both parties will gain by a contract wherein the farmer pays the rancher something in between these two amounts to reduce cattle production to x^s.

If transactions costs are not zero, forgone gains from trade may exist. To point this out, however, is to only begin the problem. The parties involved still have an incentive to consider various contracts to extract some of the mutual benefits. Different contracts have different negotiation and enforcement costs associated with them. Merger or outright purchase of one firm by another can be used to internalize side effects such as trampled crops. With merger or outright purchase, the rancher will produce x^s cattle, since it will now be the rancher's crops that are being trampled. We should expect to see individuals devising contracts that lead to the greatest extraction of mutual gains from exchange. In fact, this hypothesis is the basis for an emerging theory of contracts, based on maximizing behavior^

The Theory of Share Tenancy: An Application of the Coase Theorem

Perhaps the first empirical application of Coase's analysis was the analysis of share-cropping by Steven N. S. Cheung.* Sharecropping is a form of rent payment in agriculture in which the landlord takes some share of the output, specified in advance, instead of a fixed amount, as payment for the use of the land (rent). This form of contract is somehow less enthusiastically regarded by many social reformers than the fixed-rent contract.

Sharecropping as a contractual form of rent payment came under attack by various economists on the grounds that it misallocated resources relative to the fixed-rent contract. In its neoclassical formulation, the rental share paid to the landlord was regarded as equivalent to an excise tax on the sharecropper's efforts, inducing sharecroppers to reduce output below the level where the marginal value product of the sharecropper equaled their alternative wage.

Consider Fig. 19-11. The top curve is the marginal product of labor. Under a fixed-rent or fixed-wage contract, labor input L_2 would be hired, where the marginal product of labor equals its alternative wage *OM*. Suppose, however, the tenant has contracted to pay *r* percent of the output to the landlord as payment for rent. Then the lower curve $(1 - r)MP_L$ represents the tenant's marginal product curve net of rental payments. It is tempting to conclude that the tenant, under these conditions, will produce at input level L_1, an inefficient point since there the true marginal product of labor is higher than its next best use, measured by the wage line w.[§]

The argument is correct up to this point. A *tax* on labor of *r* percent of the tenant's output would indeed lead the tenant to produce at *L*. The mistake is to apply this tax analysis to sharecropping, a situation in which a landlord and tenant voluntarily *contract* with each other. Again, the fundamental issue raised by Coase

^Coase also showed that when transactions costs were not zero, it is not possible to deduce a priori which assignment of liability would reduce misallocation more. Consider the famous case of a railroad that occasionally sets fire to fields adjacent to the tracks because of sparks from the locomotive. If the railroad is made liable for all damage, the farmers lose an incentive to reduce the damage by not planting flammable crops too close to the tracks. The land close to the tracks may have as its highest value use a repository for sparks. On the other hand, if the railroad is not liable, it may run too many trains, i.e., produce beyond where $MC^s = p$. One form of contract which may emerge is for the railroad to purchase land near the tracks, eliminating most, if not all, of the problems.

^Steven N. S. Cheung, *The Theory of Share Tenancy,* University of Chicago Press, Chicago, 1969. ^Curiously enough, much social criticism of sharecropping appears to be based upon the landlord's working his tenants to an undue degree, perhaps, as we shall see, a more astute observation than the above economic argument.

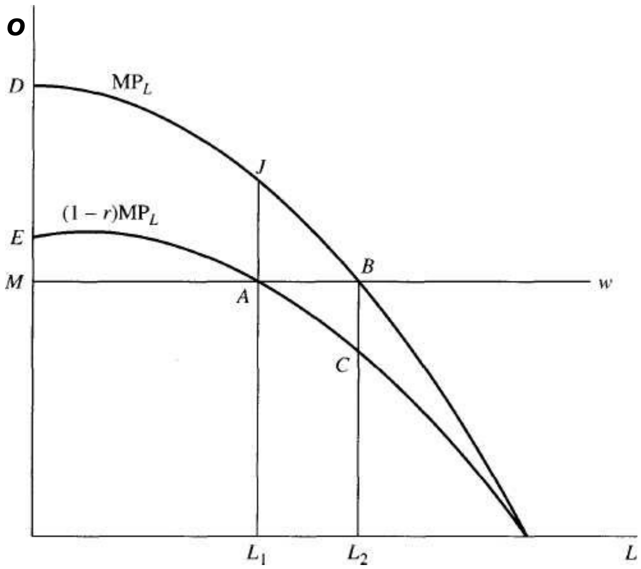


FIGURE 19-11
The Tax-Equivalent Approach to Sharecropping. This diagram has been used to show that sharecropping is an inefficient contract. Using a tax analogy, if MP_L is the marginal product of labor, and if *r* percent of the tenant's output is collected as rent, the net marginal product to the tenant is (1 — *r)MPt.* With such a tax, the tenant would produce at *L\,* where the actual marginal benefits MP^ exceed the opportunity cost of labor measured by its wage *OM.* This argument, while correct with regard to an excise tax on labor, cannot easily be extended to the case of sharecropping. In a share *contract,* many more variables are specified than the share itself. Farm size, nonlabor inputs in general, and labor inputs are negotiable. Under the postulate that the landlord maximizes the rent on the land subject to the constraint of competing for labor at labor's alternative cost, the Pareto condition MP/, = *w* is implied. (From S. N. S. Cheung, *The Theory of Share Tenancy,* University of Chicago Press, Chicago, 1969, p. 43.)

is invoked: Why would utility maximizers get together and *not* exhaust the gains from trade? If *L* instead of L_2 is used, the total output lost is *L\JBL*$_2$, whereas the alternative cost to society of this labor differential, L_2 — *L\,* is *L\ABL*$_2$. Hence, mutual gains *JAB* are lost. Why should the landlord be willing to forgo this additional rental value on the land?

Applying the tax analysis to sharecropping amounts to assuming that the only variable that can be specified is the rental share or the wage rate. A contract, however, need not contain only one clause. It is possible to specify more than one variable in a contract. (Indeed, why else would contracts exist?) Even in the normal wage contract, often an informal agreement between employer and employee, the hourly wage is not the only thing specified. The employer expects the employee to show up on time, work a certain number of hours at some minimum level of intensity, etc. If only the wage were specified, maximizing behavior indicates that workers would show up and do no work at all. Real-world share contracts specify such things as amount of land to be cultivated, nonlabor inputs to be supplied by the tenant, "the droppings [of water buffalo] go to the [landowner's] soil," etc.† Under these conditions, the tax analysis is simply inapplicable. The test conditions of the experiment are entirely different. That sharecropping as a contractual form is consistent with the Pareto conditions is shown by the following argument. Suppose the landlord owns an amount of land (capital) *K*. Labor is available at wage rate *w*, representing the alternative value of labor. The landlord can subdivide his land into *m* tenant farms, *where m is a choice variable*. Similarly, *the rental share r going to the landlord is not fixed but is also a choice variable.* Let the amount of labor supplied to each tenant farm be L. The amount of land supplied to each farm is $k = K/m$. The tenant's production function can therefore be written

m

The landlord will seek to maximize the rent on the land, $R = mry$. However, this is not an unconstrained maximization. Landlords must compete for tenants. Under this constraint of competition, the wage share to the tenant cannot be lower than the tenant's alternative earnings in wage labor. The model thus becomes

maximize
m, *r*, *L*

R

$= mrf(L,k)$ subject to

$$wL = (1-r)f(L,k) \qquad (19\text{-}50)$$

From the constraint,

$$rf(L,k) = f(L,k) - wL$$

Hence, the problem can be posed in the unconstrained form when the variable *r* has been eliminated:

maximize
m, L

$$R = m[f(L,k) - wL] \qquad (19\text{-}51)$$

Differentiating and remembering that *k* — *K/m*, we have

0%∂k(^]\ + t/m(L' *) wL] =

$$\frac{\quad}{dL} = m\frac{\quad}{dL} - mw = 0 \qquad (19\text{-}526)$$

v

† Cheung, op. cit.

From Eq. *(19-52b),* we immediately see that the landlord will contract with the tenant so as to set the (value of the) marginal product of labor equal to the alternative cost of labor. Thus, the labor input in Fig. 19-11 will be L_2, *not* L_1. The Pareto conditions will be satisfied. Substituting $w = \partial f/\partial L$ into Eq. (19-52a) and rearranging leads to

$$\frac{£}{dk} \quad \frac{L}{oL} \quad f(L,k) \qquad (19\text{-}53)$$

Equation (19-53) is a statement of product exhaustion *(not* the Euler expression, which is an *identity).* The imputed value of land (capital) measured by its marginal product times the land input plus the same expression for labor equals the total output of the farm.

The share of output going to the landlord, $rf(L, k)$, from the original constraint in (19-50) and (19-53), is

$$rf(L, k) = f(L, k) - \frac{\partial f}{\partial L}L = \frac{\partial f}{\partial k}k \qquad (19\text{-}54)$$

The landlord's share is precisely the imputed land value of the farm. In Fig. 19-11 this is the area *MDB. When r is chosen so as to maximize the rent of the land,* the landlord's share is also represented by the area *EDBC* = *MDB*. However, *EDBC* is not the landlord's share for *any* arbitrary r, only for the rent-maximizing *r.* This rent-maximizing share, from (19-54), is

y

This share is not determined by custom or tradition; it is a *contracted* amount. It varies with the fertility of the land, the cost of labor, and other variables specified in the share contract.

Showing that sharecropping is consistent with the Pareto conditions, however, is to merely state a normative condition. The interesting question of *positive* economic analysis is why the form of contract varies; i.e., why is it sometimes a fixed rent and other times a share contract? The reader is referred to Cheung for detailed answers to this question. We shall merely indicate here that some answers lie in the area of contracting cost and risk aversion. Share contracting is likely to be a more costly contract to enforce. However, to cite one example from agriculture, if the variance in output, due, say, to weather, is high, the landlord and tenant may *share the risk* of uncertain output by using a share contract. Indeed, empirical evidence from Taiwan indicates that share contracting is more prevalent in wheat than rice farming, wheat having a much higher coefficient of variation of output than rice. Other tests of these hypotheses are available.

It is generally uninteresting merely to pronounce some economic activity inefficient. The normative statements of welfare analysis are perhaps most useful if they are used to investigate why it is that certain ideal marginal conditions are being violated. The analysis then becomes positive rather than normative. Instead of labeling certain actions as irrational or inefficient, one asserts that the participants will seek to contract with each other to further exhaust the mutual gains from trade and one derives refutable propositions therefrom.

PROBLEMS

1.440 Explain why it is nonsense to seek the greatest good for the greatest number of people.

1.441 Suppose two consumers have the utility functions $U^x = x_{\{} \; y/\sim$, $U^2 = x_2 \; y\backslash'$. Suppose $x — x\backslash + x_2$, $y —$ Vi + yi represent the total amount of goods available. Find the equation representing the contract curve for these consumers.

1.442 Suppose there are two goods x and y that are *both* public goods. There are two individuals whose entire consumption is made up of these two goods. There is a production possibilities frontier given by $g(x, y) = 0$. Find the marginal conditions for production levels of x and y that satisfy the Pareto conditions.

1.443 Suppose all firms except one in an economy are perfect competitors, the remaining firm being a perfectly discriminating monopolist. Explain why the Pareto conditions will still be satisfied. What differences in allocation and distribution of income result from that firm's not being a perfect competitor also?

1.444 Two farmers, *A* and *B,* live 8 and 12 miles, respectively, from a river and are separated by 15 miles along the river. The river is their only source of water. Pumphouses cost *P* dollars each and must be located on the river. Laying pipe costs $100 per mile. Once the pipe is laid and pumphouses installed, the water is available at no extra cost.

1.445 Do farmers have an incentive to minimize the total (to both farmers) cost of obtaining water?

1.446 If one pumphouse is used to supply both farmers, show that it will be located 6 miles from the point on the river closest to farm A. (Use either calculus or similar triangles.) What will the cost of water be for each farmer and totally in terms of P?

1.447 Suppose the farmers build their own pumps. What will the cost to each be and the total cost?

1.448 Show that in a certain range of pumphouse costs, one farmer will induce the other to share a pumphouse if transactions costs are low enough. (Assume for simplicity that pumphouse cost is shared equally. Then relax that assumption.)

1.449 Explain why the utility frontier must be downward-sloping and why it is not necessarily concave to the origin on the basis of the elementary properties of utility functions.

1.450 "Interdependences in individuals' utility functions or in production functions will lead to non-Pareto allocations of resources." Evaluate.

1.451 The cases where markets allocate resources less efficiently than the Pareto ideal is often called *market failure.*

1.452 Why isn't the case where governments allocate resources less than the Pareto ideal called *government failure ?*

1.453 Under what conditions will there be market failure?

1.454 Suppose, to cite a famous example, that in a certain region there is apple growing and beekeeping and that bees feed on apple blossoms. If the apple farmers increase their production of apples, they will allegedly increase honey production. The apple farmers acting alone will not, it is said, perceive the true marginal product of apple trees and hence will misallocate resources. Devise a model for this problem. Would the existence of actual contracts between beekeepers and apple farmers affect your conclusions as to whether market failure is a necessary consequence of production externalities, or interdependencies?

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CHAPTER

20

RESOURCE ALLOCATION OVER TIME: OPTIMAL CONTROL THEORY

20.1 THE MEANING OF DYNAMICS

The theory of comparative statics concerns the instantaneous rates of change of choice variables as the parameters (constraints) faced by the decision maker change. In many cases (most, perhaps), this provides an acceptable basis for stating refutable hypotheses by extrapolating these instantaneous changes over a finite interval. Thus, for example, even though the mathematical derivation of the law of demand for compensated price changes is, strictly speaking, a statement about the demand function at a single price and income vector, the additional assumption that the underlying curvature properties hold over an interval of values allows us to state the law of demand in its useful empirical form.

In certain problems, however, the mere statement of the instantaneous movements in choice variables is inadequate. The problem is most apparent in capital theory, where the important quantity—capital—is typically durable, and where an important decision concerns the changing level of service flow to be provided over time. Moreover, it is precisely the changes in the rate of utilization of resources over time that are of interest, rather than the mere specification of the initial direction of change. Such decisions are inherently *dynamic;* the entire future time path of the choice variables changes as decisions are made in the present. The fundamental property of dynamic models is that decisions made in the present affect decisions in the future.

Some of the most important applications of dynamic analysis have been in the area of natural resource utilization. The issue of efficient (long-run wealth maximization) use of some resource, such as fish, which may be depleted through extensive harvest, is a prominent example. For this reason, and because it allows us to illustrate clearly the prominent issues involved in these types of problems, we will use this as our prototype model of resource utilization over time.

In Chap. 12, we briefly analyzed the Fisherian investment problem of maximizing the present value of some resource, say, trees, with growth function *g(t),* where *t* — time. The objective function is *P* — $g(t)e\sim^{rt}$; the problem concerns the length of time the resource should be left to grow. Maximizing *P* with respect to / yields the first-order condition $r = g'/g$; the trees are left to grow until the increase in the value of the stock each year falls to the alternative value of capital, given by the interest rate *r.* Although this is a problem of maximization "over time," it is not really a dynamic problem. There is only one decision to be made, and there is no linkage of that decision with any other choice (there are none, in fact) to be made at some later date.

Even the case of repeated plantings (the so-called Faustmann solution), where an additional opportunity cost is added, that of repeated use of the land through replanting, is essentially a static problem. The solution of this problem, however, is suggestive of the general approach to dynamic problems. After the initial harvest, with value $g\{t)e\sim^{rt}$, the "optimal" policy is to repeat the earlier decision. Thus, the objective function becomes

$$P = g(t)e\text{-}^{rt} + Pe\sim^{rt}$$

or

$$p = \frac{S(Oe-}{1 - e\sim^{rt}}$$

Solving for the wealth-maximizing time of harvest yields a shorter growth period than when the opportunity cost of the land after harvest is zero, as in Fisher's original model. Put somewhat more generally, the policy that maximizes the value (in this case, wealth) today of some extended (perhaps infinite) flow of income must, after the first harvest, be a policy that maximizes wealth from that point on as well. Ignoring, for the moment, exactly how one arrives at the point of the first and succeeding harvests, the entire path cannot be "optimal" (wealth-maximizing, in this case) unless the future after that harvest is optimally timed as well. Otherwise, the entire decision from the initial time forward can be improved simply by replacing the old path after the first harvest with the new. This reasoning was first enunciated by the mathematician Richard Bellman in the 1950s and is known as *the, principle of optimality.* This insight has been used extensively in past decades to analyze problems where decisions are linked, that is, where a decision in one time period affects the level of some relevant variable in the future. In that case, simple replication of past decisions will not be optimal; each decision imposes an "externality" on the future. It is only then that a problem becomes truly dynamic.

To illustrate these issues more concretely, consider a privately owned lake that contains an initial stock offish, x_0. Assume that the only value of this lake is the value of the stock of fish in it. In general, as fish are harvested over time, the size of the stock of fish will change, and the value of the resource will vary correspondingly. Moreover, it is often the case that the greater the number of fish in the lake, the easier it is to catch them; for this reason, harvest decisions in the present may have an additional effect on the marginal cost of fishing in the future and, thus, the present value of the resource. Since the present price of the entire resource is the capitalized value of all future benefits less costs, decisions made "today" that affect the cost of fishing "tomorrow" are reflected in the present value of the resource. An owner who disregarded the future cost of decisions made in the present would not likely use the resource in a wealth-maximizing manner. The situation is directly analogous to Coase's example of the cattle who wander onto a neighboring farmer's land and destroy some crops. Choosing the number of cattle to raise without regard to the cost imposed on farmland will lead to an allocation with less value to all resources (farm and cattle production together) than if those "external" costs are considered. In the instant case, the external costs are those imposed in the future, perhaps on the same owner as in the present.

Let us use the variable $x(t)$ to denote the stock of fish in the lake at any time t, where, initially, $x_0 = x(t_0)$. Fish are harvested at some rate $u(t)$, to be chosen by the owner of the resource. The variable $u(t)$ is called the *control;* it is the path of decisions made (with regard to the harvesting of fish, in this case). We make the simplifying assumption that the fish are sold in the world market at price p, assumed constant now and into the future. The amount of fish harvested depends on the stock of fish and the input of labor and other factors. We assume a well-defined cost function with the usual properties; $c = c(x, u, w)$ is defined, where w is a vector of factor prices. For the moment, assume a finite planning horizon so that the owner of the lake maximizes the value of the fish between times t_0 and t_1.

The hypothesis of maximization of the present value of the resource (wealth) is thus

$$\underset{u(t)}{\text{maximize}} \int_{t_0}^{t_1} [pu(t) - c(u(t),x(t),w)]e^{-rt}\, dt \qquad (20\text{-}1a)$$

However, the model is not yet complete because the dependence of the stock of fish in the future on the present rate of harvest has not been specified. Since $x(t)$ is the stock offish at any time, its derivative, $x'(t)$, is the stock's rate of growth or decline at time t. In general, the rate of change of the stock of fish depends on some biological rate of growth of the stock, $G(x)$, and the rate of harvest:

$$x'(t) = G(x(t)) - u(t) \qquad (20\text{-}1b)$$

In addition, restrictions on the values of the control variable must be specified, e.g., $u\{t) > 0$. We say in general that $u(t)$ must belong to some control set U. Lastly, some endpoint conditions must be specified, e.g., the initial stock of capital (the initial stock of fish, in this example), $x(t_0) = xo,$ and perhaps a terminal condition on the stock, $x(t\backslash) = x\backslash$.

Equation (20-1Z?) defines the dynamics of this and similar models. It is called the *state equation; x(t)* is the *state* variable. The variable *u(t)* is called the *control* variable; it is analogous to the decision variables of static theory. This is the variable the decision maker chooses, or controls (e.g., the rate of harvest, the rate of investment in new capital, or any other flow that affects the size of the stock of some resource). The state variable *x(t)* represents the size of the stock of some resource at time *t.* The stock that exists at time *t*\, of course, depends on the initial stock and the path of decisions *u(t)* regarding harvest rates for $t < t\backslash$. (We usually think of choices about *u* affecting the level of *x,* although it is possible, in principle, assuming suitable invertibility of the functions, to imagine choosing the stock *x* at each time and inferring the flow *u* that must be implied to achieve that stock.) Equations (20-1), plus the endpoint conditions and some specified control set *U,* are prototypic of problems in dynamic optimization.

The general form of control theory problems is

maximize

u(t)

$$f^{t} f(x(t),u(t),t)dt \tag{20-2a}$$

subject to

$$\boldsymbol{x'(t)=g(x(t),u(t),t)} \qquad \textbf{(20-26)}$$

with endpoint conditions

$$x(t_0) = x_0 \qquad x(h) = xi \qquad [\text{orx(fi) "free"}]$$

and some specified control set *U(t) e U.* The time period *(to, t*\) is called the planning period. In many important problems, *t*\ -> + oo so that the planning horizon is infinite. Endpoint conditions vary. Typically the initial stock of the state variable is fixed, although the final stock may not be. In addition, there may be restrictions on the variables, such as nonnegativity, and perhaps inequality bounds on the control. We will not cover these more advanced situations.

Problems of the type just outlined are fundamentally different from those encountered in traditional comparative statics analysis. In the so-called static theory, maximizing behavior consists of finding values of the independent variables that maximize functions with specified curvature properties. Although the directions of change of the choice variables with respect to changes in the constraints may sometimes be derived, the empirical properties of the static models do not include specification of the time rates of change of those variables. In dynamic models, the "solution" consists not merely of finding the maximum value of some function, but rather of finding *the actual function* that provides a time path of values of the economic variables so that some value function, specified over an interval of time, is maximized (or minimized). For this reason, the integrand/in *(20-2a)* is often referred to as a *functional,* being a function of *functions x(t)* and *u{t).*

Brief History *

The mathematical problem of finding a function that minimizes or maximizes some integral was first posed by Johann Bernoulli in 1696. Bernoulli challenged his colleagues (and particularly his older brother Jacob, whom he publicly derided as an incompetent) to find the shape of a frictionless wire such that a bead sliding down it would move between two points (not vertically aligned) in the least time. Mathematicians immediately realized the different nature of the problem and set about its solution. Bernoulli's solution was specific to this problem and provided little in the way of generality. (The shape involved is an inverted "cycloid," the path generated by a point on the rim of a coin as it is rolled on a plane.) The first systematic solution was derived in the early eighteenth century by Euler and Lagrange, who provided the general differential equation to be solved for such problems. This result will be discussed shortly. In its original form, this mathematics is called *the calculus of variations.*

In the 1950s, the theory was generalized by L. S. Pontryagin and his colleagues in the Soviet Union and by Richard Bellman and others in the United States. Pontryagin's work was motivated by problems in the physical sciences; Bellman's orientation was generally in the direction of economics and management science. The classical calculus of variations can be considered a special case of control theory; however, the older techniques are still simplest for some problems, although they are usually harder to interpret in terms of economic theory.

20.2 SOLUTION TO THE PROBLEM

We shall exploit the reasoning behind Bellman's principle of optimality to develop a heuristic solution to the control problem. The conceptual "trick" is to divide the entire time period into just two periods: the "present," which lasts only an instant (or some brief time), from some *t* to *t + At, At* > 0; and the "future," consisting of the rest of the planning period, from *t + At* to *t*. Let us interpret the control problem in terms of maximizing the present value of the fish in a lake, as previously described. The integrand in (20-2), *f(x(t), u(t), t),* represents the instantaneous net benefits from fishing at the rate *u(t).* When a decision is made to harvest fish, this produces a flow of net benefits right now, in the present, in the amount of *f At.* In this short period of time, the stock changes little. If the fish in the lake were common property, an individual fishing would maximize short-term profits by setting *df/du* = 0, as in the static framework. (Typically, this would consist of some first-order condition

*Adapted from Richard Courant and Herbert Robbins, *What Is Mathematics,* Oxford University Press, New York, 1941.

such as p = MC.) Under these circumstances, fishers would have no incentive to incorporate the effects of their present actions on the future, e.g., on the stock of fish available and the attendant effect on the marginal cost of fishing. With no ownership of the future stock of fish, there is no personal gain from curtailing present profits to achieve what might in principle be even larger future gains, since these future benefits will likely be captured by someone else.

As fishing proceeds, however, the stock offish and, thus, the value of ownership of the stock begin to change. In a competitive market, assuming the fish in this lake make up only a negligible part of the entire market, the value of the stock in the lake at any time would be $(p$ — MC)x(f), assuming for convenience, constant marginal cost of fishing through time. Even without reference to a market, however, there is an *imputed* value of the stock, given by the product of the stock, x, times the net marginal value of fish. The net marginal value of fish is the increase in the value of the stock if, somehow, an extra fish were placed in the lake. This present increment in the stock might have complicated long-term implications for the future stock, as determined by the biological growth function and harvesting rate.

Let us ignore for the moment exactly how the control problem (20-2) is solved, but assume that a finite interior solution $(u^*(t), x^*(t))$ does indeed exist. The values $(u^*(t), x^*(t))$ represent the "optimal" time paths of the control variables (harvest rate, in this example) and the state variable (the stock of fish). Although we are suppressing it in the notation, x^* and u^* in fact depend on the parameters JC_0, t_0, etc. Denote the resulting value of the objective functional as $V(x_0, to)$, that is,

$$V(x_0, t_0) = f^{t}f(x^*(t), u^*(t), t)\, dt \qquad (20\text{-}3)$$

The function $V(x_0, t_0)$ is directly analogous to the indirect objective functions of comparative statics; it is called the *optimal value function.* Although (20-2) requires us to find an actual path, or function, that maximizes an integral, that function, once found, results in some ordinary maximum value function of the parameters of the model (we suppress $t\backslash$ as not germane to the discussion). The marginal value of an increment in the initial stock of fish is simply 3 $V(x_0, to)/dx_o$. More generally, $V_x(x(t),t)$ represents the marginal value of the resource at time t if the state variable JC is increased exogenously at time t, and the optimal path of values $(x(t), u(t))$ is carried forward from that time until the end of the planning period.

Given x_O, a marginal value of the stock exists for any time t between the initial time to and the terminal time $t\backslash$. Denote this imputed value $k(t) = V_x(x^*(t), t)$. The marginal value of the stock, X (t), is often referred to as the *costate* or *adjoint* variable. The change in the value of the stock of fish caused by fishing is $d[k(t)x(t)]/dt = kx' + xk'$. The true net benefit of fishing at some rate u (t) is the sum of the benefits in the present, $f(x,u,t)$, and the change in the maximum value of the stock caused by taking that action in the present. The optimum (wealth-maximizing, for example) path is obtained by always setting the true (present plus future) marginal net benefits equal to zero along the entire optimal path of values $(u^*(t), x^*(t))$. Thus, we can characterize the solution to the control problem as requiring, at each t, $t_0 < t < t\backslash$,

maximize
u, x

$$f(x,u,t)+kx'+ xX'$$

Using the state Eq. (20-2&), this becomes

maximize
u, x

$$f(x, u, t) + Xg\{x, u, t) + xX'$$

We suppress the dependence on t at this point because we have not yet found the functions *$(u^*(t), x^*(t))$* and expressed them as functions of t. Differentiating with respect to the control u and the state variable x yields

$$f_u +\hat{}g_u=0 \qquad (20\text{-}4)$$

$$f_x + Xg_x + X' = 0 \qquad (20\text{-}5)$$

Equation (20-4) is called the *maximum principle;* (20-5) is called the *costate* or *adjoint* equation. These two conditions plus the state equation

$$x' = g(x,u,t) \qquad (20\text{-}6)$$

are the necessary conditions for an optimal path *$(u^*(t), x^*(t))$* of control and state variables over the planning period. Also determined is the path of marginal values of the stock, $X(t)$.

These equations, however, are not simple equations in x, u, and X, in which case ordinary algebraic or comparative statics techniques would apply. The adjoint Eq. (20-5) and the state equation are *differential equations;* they are, in general, difficult to solve.

Equations (20-4) and (20-5) are generally expressed in terms of the expression $H = f+Xg$, called a *Hamiltonian.* The maximum principle is 3 H/du = 0 (assuming an interior solution to the problem); the adjoint equation is dH/dx = —A/. In the original problem, given the initial value of the stock, x_0, choosing $u(t)$ determines $x'(t)$ and thus $x(t)$, through the state equation. Thus, there is really only one "independent variable," u. However, the introduction of the new variable $X(t)$ adds another degree of freedom; as in static Lagrangian analysis, we "pretend" the problem has one more dimension than it actually has.

Using the maximum principle, Eq. (20-4), which is not a differential equation, and invoking the implicit function theorem, we can "solve" for u: $u = k(x$, A, $t)$. Substituting this into the adjoint and state equations produces two first-order differential equations

$$x' = g(x,k(x,\backslash,t),t) \qquad (20\text{-}7)$$

and

$$X' = -f_x(x, k(x, X, t), t) - Xg_x(x, k(x, X, t), t) \qquad (20\text{-}8)$$

Solving these differential equations (and using the relevant endpoint conditions to evaluate the constants of integration) yields the optimum path of x and *X*. Using the solutions to these equations yields the optimum path of the control variable, *u,* by substituting into *k(x, k, t)*.

Somewhat more formally, consider any point (x_0, *to),* not necessarily the initial point, along the optimum path. The maximum (or minimum, but we proceed in the maximization format) value of the objective integral is some function $V(x_0, t_0)$. As we proceed along some specified path *(x(t*), w(0) for some small interval of time *At,* immediate "net benefits" of *f(x, u, t) At* are realized. At that point, the function *V* is dependent on the new coordinates (JCO + *Ax,* t_0 + *At)* and the path chosen between *to* + *At* and *t\\,* the end of the planning period. Since $V(x_0, to)$ is the value of the objective integral when the optimal path is chosen, for arbitrary initial choices,

$$V(x_0, t_0) > f(x, u, t)\, At + Vix_0 + Ax, t_0 + Af) \qquad (20\text{-}9)$$

Applying the mean value theorem (or, alternatively, a Taylor series expansion) to the last term, we have, approximately,

$$Vix_0 + Ax, t_0 + Af) = V(x_0, t_0) + V_x\, Ax + V,\, At$$

Substituting this expression in the right-hand side of (20-9) and canceling $V(x_0, to)$ from both sides yields

$$fix, u, t)\, At + V_x\, Ax + V_t\, At < 0$$

Dividing by *At* and taking limits, and using the state equation *dx/dt =x'* = g(x, *u, t)* yields

$$fix, u, t) + V_x ix, t) gix, u, t) + V,ix, t) < 0 \qquad (20\text{-}10)$$

Along the *optimal* path, (20-10) holds as an equality; in that form, the equation is known as the *Hamilton- Jacobi* equation:

$$fix^*, u^*, t) + V_x(*\backslash\ t) gix^*, u^*, t) + V_t ix^*, r) = 0 \qquad (20\text{-}11)$$

Recall that A.(f) = V_x(x, f). Making this substitution in (20-10) yields

$$fix, u, t) + kit) gix, u, t) + V_t ix, f) < 0$$

Again, this expression holds as an equality along the optimal path

(x*(f), w*(f)):

$$fix^*, u^*, t) + X^* it) gix^*, u^*, t) + V_t ix^*, 0 = 0 \qquad (20\text{-}12)$$

The last term, V_t, the rate of change of the objective functional with respect to time, is a function only of x and *t*; it is independent of *u.* Therefore, for given x, the optimal path requires maximization of *H*

— *fix, u, t)+kit)gix, u, t)* with respect to *u* along the optimal path. This is the maximum condition (20-4).

The adjoint equation is also derivable from (20-11). This relation is an identity in time and the parameters of the system, in particular x_0, when the optimal paths are substituted back into it, as indicated. Differentiating with respect to x_0 (suppressing the*'s),

fdx\ fdu\ \ fdx\ (du
V3x0/ V9^o/ L \dxoj \dxt

(*L) + vJ*L)=o

\dxoj tx \dxoj

Collecting terms,

—)+[fu + VAx,t)gu](—) = 0 (20-13)
dxoJ \dxoj

However, the last bracketed term is zero, by the maximum condition (20-4), remembering that $k(t) = V_x(x, t)$. Also, differentiating $V_x(x, t)$ with respect to /,

constituting the last two terms in the first set of brackets. Assuming dx/dx_0 ^= 0 (the capital stock is not redundantly abundant, i.e., having more of it would affect the level of the stock later on), Eq. (20-12) thus implies the adjoint equation $f_x + kg_x + k' = 0$. Equation (20-12) yields an interpretation of the Hamiltonian, which is the sum of the first two terms. The last term, V_t, indicates by how much the maximum value of the objective integral will change after an instant of time has passed, holding the stock, x, constant. Therefore, the Hamiltonian equals the (negative) net effect of starting the process a bit later.

Example. Consider the optimal control

problem maximize

— x- -au1 1 dt

subject to

X — U

$x(0) = x_0$

where $a > 0$ is a parameter for this problem. The Hamiltonian for

this problem is

$$H(x, u, \lambda) = -x - au^2 + \lambda u$$

Assuming an interior solution, the necessary conditions are

$$\frac{dH}{du} = -au + 1 = 0$$

d^2H

0

By assumption a > 0, so $d^2H/du^2 < 0$. Solving $dH/du = 0$ for u gives u — X/a. The other necessary conditions are the state and adjoint equations

$$x = \frac{dH}{dX}$$

dH

Using u = A/a in these equations yields

$$x' — X/a \qquad x(0) — x_0$$
$$x(l) = X\backslash\ ),' = 1$$

Integrating $X' = 1$ directly gives $X^*(t) = t + C\backslash$, where $C\backslash$ is an unknown (as of yet) constant of integration. Substitute $X^*\ (t)$ in $x' = X/a$ togetx' = $(t\text{-}\backslash\text{-}C\backslash)/a$. Integrating this equation yields $x^*\{t) = t^2/2a + c_t t/a + c\text{-}i$, where c_2 is another constant of integration. The constants of integration c_t and c_2 are determined by using the initial and terminal conditions x(0) = x_0 and x(l) = $x\backslash$, respectively. Use $x(0) = x_0$ in $x^*(t)$ to get $x^*(Q) = c_2 = x_0$. Now use x(l) = JCI to obtain the value of c_x: $x^*\{\backslash) — 1/2a + C]/a + x_0 = X\backslash\backslash$ thus, $C\backslash — a\{x_t —$ x$_0$) — |. These constants of integration are then substituted in $\{x^*, X^*)$ to yield their optimal paths, and then X^* is substituted into $u = X/a$ to give the control's optimal time path. Doing this gives

$$x\ (t;\ a, x_o, x\backslash)\ \frac{t^2}{2a}\text{--h}\ \backslash(x\backslash - XQ)\frac{11}{2a\ J} \backslash\ t\ + x_0$$

$$k^*(t;\ a,\ x_0,\ X_x) = t + a(xi - x_0) - -$$

$$u\ (t;\ a,\ x_0,\ xi) = \frac{t}{a} - + (x_x - x_0)\ \frac{1}{2a}$$

The optimality conditions (20-4) and (20-5) may also be understood in terms of a discrete-time formulation of the optimal control problem:

maximize

$$t=0$$

subject to

$$x_{t+}i\text{-}x_t = g(x_t, u_t, t) \qquad t = 0, l, \ldots, T$$

In this formulation, the term $x_t+\backslash — x_t$ replaces the term $x'\{t)$ of the state equation (20-2&), and the summation sign replaces the integral in the objective function (20-2a). The choice variables of this problem are JCI ,..., x_T (x_0 and x_T+i are given) and $u_0, \ldots, u_T$. Attaching a multiplier X_t to each of the $T + 1$ constraints produces the Lagrangian function

$$T$$

$$\&t,\ u_t,\ t) + X_t(g(x_t,\ u_t,\ t) - (x_{t+}i - x_t))]$$

The first-order conditions for this maximization are

$$\frac{\partial}{\partial u_t} = f_u(x_t,u_t,t)+\lambda_t g_u(x_t,u_t, t) = 0 \qquad t = 0,\ldots,T \quad (20\text{-}14)$$

$$\frac{\partial}{\partial x_t} = f_x(x_t,u_t,t)+\lambda_t g_x(x_t,u_t,t)+\lambda_t-\lambda_{t-1} = 0 \qquad t = 1,\ldots,T \quad (20\text{-}15)$$

It is clear upon inspection that (20-14) is equivalent to the maximum principle (20-4) of the continuous-time optimal control problem. Similarly, (20-15) is the discrete form of the adjoint equation (20-5), with $\lambda_t - \lambda_{t-1}$ taking the place of $\lambda'(t)$.

The Calculus of Variations

The original formulation of the problem of determining an optimal path is to find some function $x(t)$ that solves

maximize

$$\int_{t_0}^{T} f(x,x',t)dt$$

This is, in fact, a special case of the control problem, where $x' = g(x, u, t) = u$. That is, the time rate of change of the stock is identically the control variable, rather than some more general function that might also include the stock itself and time. Substituting the state equation $u = x'$ into the integrand in (20-2a) yields this specification.

In this case, the necessary conditions for a maximum (or minimum) are as follows. The maximum principle is

However, $g_{x'} = g_u = 1$, so this condition becomes

$$f_{x'} = -\lambda \qquad (20\text{-}16)$$

The adjoint or costate equation is

$$H_x = f_x+\lambda g_x = f_x = -\lambda' \qquad (20\text{-}17)$$

since $g_x = 0$. Since the right-hand side of (20-17) is the time derivative of the right-hand side of (20-16), these equations can be combined into

$$\frac{d}{dt} f_{x'} = \frac{\partial f}{\partial x} \qquad (20\text{-}18)$$

Carrying out the differentiation in (20-18) results in the equivalent expression

$$f_x = f_{x't} + f_{x'x}x' + f_{x'x'}x'' \qquad (20\text{-}$$

18')

Equation (20-18) is the classic Euler-Lagrange relation defining the necessary condition for an optimal path. Application of (20-18) (except for special cases) results in a second-order differential equation, whereas the necessary conditions of control theory result in the simultaneous first-order differential Eqs. (20-7) and (20-8). There is no uniform computational advantage to one approach over the other; however, the Euler-Lagrange equation is difficult to interpret, and the control theory equations often provide useful characterizations of the dynamics of economic models.

The solution to the Euler-Lagrange equation may be obscure. In special cases, however, certain procedures may be of assistance.T In particular, if the objective functional is a function of x and x' only, i.e., not including t explicitly, the Euler-Lagrange equation is

$$Jx - \frac{dfAx,x')}{dt}, - Jx'x^{x} \quad r\, Jx'x'^{x}$$

or

$$Jx \quad Jx'x^{x} \sim\sim Jx'x'X = 0$$

As expected, this is a second-order differential equation. It turns out, however, that x' is an integrating factor for this expression: Multiplying through by x' yields

$$x'(f_x - f_{x,x}x' - f_{x,x,}x'') = \triangle\triangle\triangle = 0$$

Thus, in this case the Euler-Lagrange equation

implies

$$f - x'f_{x,} = k$$

where k is the constant of integration. This equation may (but not always) be easier to solve than the Euler-Lagrange condition in its original form.

Example. Let us prove algebraically a result everyone knows intuitively: The shortest distance between two points on a plane is a straight line. The two points will be designated (t_0, x_0) and $(t_u x\backslash)$. Recalling Pythagoras's theorem, starting at some point and making small movements dt in the t direction and dx in the x direction, the distance traveled is the length of the hypotenuse:

$$ds = [(dt)^2 + \{dxfV^2 = [1 + x'(t)^2]^{1/2}$$

dt We seek to minimize the sum of these little

segments, or

minimize

$$\int_{t_0} [l + x'(t)^2]^{1/2} dt$$

This is a special case: The integrand depends only on x'. Applying the Euler equation in the form (20-18'),

$$f_{x,x,}x'' = 0$$

Thus, either $x'' = 0$ or $f_{x,x,} = 0$. Here / = $[1 + x'(t)^2]^{1/2}$; thus, $f_{x,x,}$ + 0. Therefore, x'' — 0. This simple differential equation has the solution $x = C\backslash t + c_2$, confirming the

. I. Kamien and N. L. Schwartz, *Dynamic Optimization: The*

Calculus of Variations and Optimal Control in Economics and Management, 2d ed. North Holland, New York, 1991, p. 1.5.

result. Using the coordinates of the endpoints to evaluate the constants of integration yields

- Xp) (Xpti -Xjtp)

–

(h -

To gain a better understanding of the Euler-Lagrange condition, consider the discrete-time version of the calculus of variations problem:

maximize

T

T/ \ ^ /V Y t\

v —L J \-*~tl t+l

-If, I) t=0

In this discrete-time formulation, the argument x' in the function $f(x, x', t)$ is replaced by the argument $x_{t+1} - x_t$, and integration is replaced by summation. The choice variables of this problem are $x_1, \ldots, x_T$ (x_0 and x_{T+1} are given), and the necessary first-order conditions for maximization are simply the usual first-order conditions

— - 0

dx_t

The variable x_t appears in the objective function only in the terms $f(x_{t-1}, x_t - x_{t-1}, t - 1)$ and $f(x_t, x_{t+1} - x_t, t)$ for periods $t - 1$ and t. The first-order condition for x_t is therefore

$$f_2(x_{t-1}, x_t - x_{t-1}, t - 1) + f_1(x_t, x_{t+1} - x_t, t) - f_2(x_t, x_{t+1} - x_t, t) = 0$$

where f_1 and f_2 denote the respective partial derivatives with respect to the first and second arguments. This equation can be arranged into

$$f_2(x_t, x_{t+1} - x_t, t) - f_2(x_{t-1}, x_t - x_{t-1}, t - 1) = f_1(x_t, x_{t+1} - x_t, t) \qquad (20\text{-}19)$$

The left-hand side of this equation is the change in the value of f_2 from period $t - 1$ to period t; it is the discrete-time analog of $(d/dt)f_{x'}$. The right-hand side of this equation is simply $\partial f/\partial x$. Equation (20-19) is just Eq. (20-18) in discrete time.

Endpoint (Transversality) Conditions

Up to this point, we have been imprecise as to the effects of assumptions regarding the initial and final values of the optimal path. Endpoint conditions are not generally an issue in comparative statics analysis, since the solutions are assumed to occur at interior points. In dynamic analysis, the path may depend critically on the assumption made regarding initial and final values. The solution to optimal control problems involves solving a second-order differential equation (or, equivalently, two simultaneous first-order equations). In either case, two arbitrary constants of integration appear. For these parameters to be evaluated, additional assumptions must be made about the optimal paths.

Consider the fishing problem. If the model is stated as a maximization problem between time t_0 and finite time $t\backslash$, the model essentially assumes there is "no time" after *t* that is, in essence, the world comes to an end at *t*. (A slightly more general class of models appends a salvage value *S[x(t\), t\]* to the maximization problem.) If, somehow, the stock of fish is simply specified in terms of some initial and final values $x(t_0)$ $= x_0$ and $x(t_i)$ $= x\backslash$, there is no further issue; these values will be used to evaluate the arbitrary constants that appear in the solution to the differential equation defining the optimal path. If, however, a positive stock of fish were to exist at *t* (the end of the world), it surely could have no value. Therefore, necessarily, if *x(t\)* > 0, *X(t\)* *=0.* (With positive salvage values, *X(t\)* = *dS/dxi.)* In many cases, however, the final value of the stock of fish is not specified a priori; it is to be determined by the maximization hypothesis. The same reasoning would then suggest that, since additional stock would have zero value in terms of the objective function/, if *x(ti)* is taken to be "free" (i.e., not specified in advance), then *X(t\)* = 0. Such conditions are known as *transversality conditions;* they are the additional conditions needed in order to evaluate the constants of integration in optimal control problems. For maximization problems, if Jt(*i) > 0 is the constraint on the terminal stock, the transversality conditions can be stated as

Kh) > 0 *x(ti)* > 0 *HhUih)* = 0 (20-20)

In addition, in some problems, the final time itself, *t*, is taken as free. In this case the activity, say, fishing, would cease when prolonging it would have no value, i.e., would add nothing to the value *V(x,t)* of the objective integral. From the Hamilton-Jacobi equation (20-12), **at** /i

$-V_t =$ *fix*, u*, t) + k*it)gix*, u*,* 0 = 0 (20-21)

if *t* is free. These conditions must be modified for more complex models, e.g., those involving inequality constraints and salvage values; the modifications in general resemble the Kuhn-Tucker restrictions in static maximization^

Autonomous Problems

In the general control problem framework, the variable *t* can enter the objective function and the state equation directly. The general specification

maximize

fix,u,t)dt

to

subject to

x' = gix, u, t) *xit*$_0$*)* —

Kamien and Schwartz, op. cit., p. II.7.

where t enters/and g directly, means *the date matters.* That is, the cost or revenue generated by the activity $u(t)$ depends not only on the level of extraction and stock of a resource, or utilization of capital (i.e., on the level of the control and state variables), but also on exactly when this activity is taking place. In many (most, perhaps) economic models, however, such dependence on the date is incorporated only in the term $e\sim^{rt}$, used to discount future income to the present, *to.*

Models in which t is absent from the objective and state equations, i.e.,

maximize

$$\backslash \quad f(x,u)dt$$

subject to

$$x' = g(x,u) \qquad x(t_o)=x_o \tag{20-22}$$

are called *autonomous.* In this case, the maximum condition $H_u = f_u + Xg_u = 0$, the state equation $x' = g\{x,u)$, and the adjoint equation $f_x + Xg_x + X' = 0$ result in differential equations in x' and X' that do not involve t explicitly. These equations are much easier to solve than those in which t appears explicitly. For practical reasons as well, therefore, this modification is important.

Models in which time enters explicitly only as part of the discount factor $e\sim^{rt}$ are generally referred to as autonomous as well, as the time dependence is easily eliminated. That is, consider models of the form

maximize

$$1\, f(x,u)e\sim^{rt}dt$$
$$'to$$

subject to

$$x' = g(x,u) \qquad x(t_o)=x_o \tag{20-23}$$

By replacing time t with the variable $s = e\sim^{rt}$ and defining the initial and terminal times in terms of s, the problem immediately becomes autonomous.

In models of the form (20-23), the costate variable $X\,(t)$ is the *present value* (i.e., at time *to)* of the marginal value of an increment of capital at time /. It is sometimes more convenient to solve these problems using a "current value multiplier," $m(t)$, where

$$e\text{-}^{r}{}'m(t) = X(t) \tag{20-24}$$

The necessary conditions for optimality are, again,

$$H_u = e^{\prime rt}f_u + Xg_u=0 \qquad (20\text{-}$$

25)

and

$$H_x = e^{-rt} f_x + \lambda g_x = -\lambda' \tag{20-26}$$

Using (20-24), however,

$$e^{-rt}m'(t) - re^{-rt}m(t) = \lambda'(t)$$

If we now write the Hamiltonian in current value form as

$$\mathcal{H} = e^{rt}H = f + e^{rt}\lambda g = f + mg$$

the first-order conditions are equivalent to

$$\mathcal{H}_u = f_u + mg_u = 0 \qquad (20\text{-}27)$$

and

$$\mathcal{H}_x = f_x + mg_x = rm - m'(t) \qquad (20\text{-}28)$$

after canceling e^{-rt} from each term. Equations (20-27) and (20-28) are autonomous differential equations; that is, the independent variable t does not enter explicitly as a separate argument. The system is usually more easily solved in this form.

Sufficient Conditions

The Euler-Lagrange equation, or the control theory variant, Eqs. (20-4) and (20-5), plus the state equation (20-6) and transversality conditions are first-order necessary conditions for either a maximum or minimum. Sufficient conditions analogous to those in static theory are as follows:

> If $f(x, u, t)$ and $g(x, u, t)$ are both everywhere concave in x and u for all t, if $g(x,u,t)$ is nonlinear in x or u, if $\lambda(t) > 0$, and if the first-order necessary conditions are satisfied, the solution represents a maximum.
>
> Likewise, if $f(x, u, t)$ and $g(x, u, t)$ are both everywhere convex, then the solution represents a minimum.

Under these conditions, the Hamiltonian will be concave (or convex, for minimum problems). Since the expression

$$H + \lambda' x = f(x, u, t) + \lambda g(x, u, t) + \lambda' x$$

is maximized (minimized) at every point along the optimal path, these conditions are intuitively plausible. Note that if $g(x, u, t)$ is linear in x and u, then concavity of this expression will be independent of g and, thus, the sign of $\lambda(t)$. For classical calculus of variations problems, i.e.,

maximize

$$\int F(x, x', t)\,dt$$

the sufficient condition is that the integrand $F(x, x', t)$ be concave in x and x' for all t. For minimum problems, F must be convex in x and x' for all t. This condition can be applied in control problems if the control variable can be eliminated through substitution, converting the problem to one in the calculus of variations. It is important to note that the preceding sufficient condition requires *global* concavity (or convexity) of/and g (or F); hence, the solution (if one exists) to the first-order necessary conditions yields the global optimum. A weaker condition, $F_{x'x'} < 0$ along the optimal path (>0 for minimum problems), is known as the Legendre condition, and is a *local* curvature property. It is *necessarily* implied by maximization (note the weak inequality). As in static optimization problems, these conditions are often the basis for comparative statics or comparative dynamics results in dynamic problems.

20.3 SOLUTIONS TO DIFFERENTIAL EQUATIONS

Through the use of techniques analogous to comparative statics, the effects of changes in the parameters on the optimal path or on steady state values are sometimes available. However, in order to be more tractable and useful, many models incorporate simplifying assumptions. Many control problems of interest assume specific functional forms in the objective and state equations. The maximum and adjoint equations then result in specific differential equations whose solution is of interest. To that end, we investigate briefly the nature of these solutions.

In general, differential equations are difficult to solve, and some innocent-looking equations are in fact intractable. Certain standard procedures are useful; we review them briefly here.^ Some differential equations can be solved by separation of the variables: To solve $y' = dy/dt = y/t$, we write

$$\frac{dy}{y} \sim \frac{dt}{t}$$

Integrating both sides yields

$$\log 3; = \log t + \log k$$

where the arbitrary constant of integration is denoted $\log k$ for convenience. Thus, the general solution can be written

$$y(t) = kt$$

If it is specified that the curve must pass through some particular point ($?_0$, yo), the constant of integration can be evaluated. Differential equations that can be solved in this manner are the easiest to work with.

Consider now the class of *linear* first-order differential equations

$$y'(t) + b(t)y(t) = c(t) \qquad (20\text{-}29)$$

^The student is cautioned against reinventing the wheel in these procedures, but it is only through practice that skill is acquired.

This equation is called *linear* because there are no terms of the form $(y')^2$, yy', etc. By a *solution* to this equation, we mean a function $y = s(t)$ such that when this function is substituted into this equation, an identity results. The fundamental theorem identifying the nature of these solutions is as follows. Consider Eq. (20-29) without the right-hand side:

$$y' + b(t)y(t) = O \qquad (20\text{-}30)$$

This is called the *reduced* equation. (When there is no right-hand-side function, a differential equation is called *homogeneous.)* This equation is usually much easier to solve than (20-29), assuming a solution exists. The solutions to differential equations, of course, involve arbitrary constants. *However, if any* particular *solution can be found for* (20-29), *the* general *solution to* (20-29) *is the sum of that particular solution plus the general solution to the reduced* Eq. (20-30).

Let us first investigate these equations when b and c are constants, as opposed to being functions of t. Equation (20-29) is then called a first-order differential equation with constant coefficients. In that case, the solution to Eq. (20-30) can always be found by multiplying through by e^{bt}. Note that

$$e^{bt}(y' + by) = \frac{d(e^{bt}y(t))}{dt} = 0$$

Thus, the general solution to the reduced equation is $e^{ht}y(t) = K$, where K is an arbitrary constant, or

$$y(t) = Ke^{-bt}$$

By inspection, *a. particular* solution to (20-29) is $y = c/b$ (note that $y' = 0$). The *general* solution to (20-29) is therefore

$$y(t) = Ke^{-bt} + \frac{c}{b}$$

Substituting this expression into (20-30) confirms that it is indeed

a solution. **Example.** Consider the differential equation

$$y + {}_y = t + i$$

A particular solution of the unreduced equation is $y — t\backslash$ the general solution, since $b = 1$, is therefore

$$y = Ke^{-t} + t$$

In the more general case where $b = bit)$ and $c = c(t)$, finding a particular solution may not be easy. However, by proceeding in a manner similar to the case of constant coefficients, the general solution to the reduced equation is always of the form

Adding a particular solution of (20-29) to this yields the general solution.

The same procedures apply to second-order linear differential equations; however, the algebra is more complex. General solutions of

$$y'' + by' + cy=d \qquad (20\text{-}31)$$

consist of the sum of a particular solution to the entire equation plus the general solution to the reduced (homogeneous) equation

$$y'' + by' + cy = 0 \qquad (20\text{-}32)$$

To solve (20-32), we "try" a solution of the form $y = e^{rx}$. Substituting this into (20-32) yields

This equation will be satisfied for solutions to the quadratic equation, called the *characteristic equation,*

$$r^2 + br + c =$$

0 Using the quadratic formula, the roots

are

$$-b \quad (b^2- \quad \pm \quad \wedge \qquad (20\text{-}33)$$

There are several cases to explore.

1. If $b^2 — Ac > 0$, the roots are real and distinct; in that case the solution to (20-32) is

$$y(t) = cie^{rit} + c_2e^{r2t} \qquad (20\text{-}34)$$

where $C\backslash$ and c_2 are the arbitrary constants of integration. Note that if both roots are negative, $y(t)$ —► 0 as t —► oo. Control theory problems with this type of solution converge asymptotically toward some "steady state"; if one or both roots are positive and the attached constant(s) are not zero, the path will diverge.

2. If $b^2 — Ac = 0$, the roots are identical: $r_{\{} = r_2 = r;$ the solution to (20-32) is then

$$y\{t) = (c_x+c_2t)e^{rt} \qquad (20\text{-}35)$$

3. $lib^2 — Ac < 0$, the roots are imaginary, i.e., they involve i = */—l- The solution is again of the form (20-34) since the roots are distinct; however, (20-34) is not a convenient expression. For this reason, we make use of the well-

known identity

$$e^{lt} = cost + i \text{ sin?}$$

The real and imaginary parts of the roots to the characteristic equation are defined, respectively, by $p = -b/2$, $q = (b^2 - 4c)^{1/2}/2$. The solution to (20-32) is then

$$y(t) = e^{pt}(ci\ cos(qt) + c_2\ sin(qt)) \qquad (20\text{-}36)$$

where, again, c_x and c_2 are arbitrary constants. Note that if the real part of the roots p is negative, the solution $y(t)$ will oscillate around zero, converging to zero as

t -» oo.

Simultaneous Differential Equations

The control theory format results in simultaneous differential equations—typically, one for the state variable $x(t)$ and one for the costate variable, $X.(t)$. These are generally of lower order than the single differential equation resulting from the Euler-Lagrange equation of the calculus of variations. The two, however, are equivalent. We consider only the linear first-order case:

$$x' = a_x x(t) + b_x y(t) +$$
$$f(t) \quad y' = a_2 x(t) + b_2 y(t)$$
$$+ g(t)$$

As in the case of single differential equations, solutions to (20-37) consist of the sum of a particular solution to the complete system and the general solution to the reduced (homogeneous) system

$$x' = a_x x(t) + b_x y(t)$$
$$38)\, f = a_2 x(t) + b_2 y(t) \qquad (20\text{-}$$

We consider only the solution of the homogeneous system, (20-38). Differentiating the first equation with respect to t and substituting for x' and y' in (20-38) yields the equivalent *second-order* differential equation:

$$x'' - (a_x + b_2)x' + (_{ai}b_2 - b_x a_2)x = 0 \qquad (20\text{-}39)$$

This can be solved using the previously discussed methods. However, we can proceed directly with (20-38) and try solutions of the form $x(t) = Ae^{rt}$, $y(t) = Be^{rt}$. Substituting into (20-38) and canceling e^{rt} from each term yields the matrix equation

$$i - r \qquad b_x$$
$$a_2 \qquad b_2 - r)\backslash Bj \qquad \text{V0, This}$$

equation has a nontrivial solution only if its determinant is zero:

$$' - {}^r \qquad *' \qquad = 0 \qquad (20\text{-}40)$$
$$a_2 \qquad b_2 - r$$

Expanding (20-40) yields the characteristic equation

$$r^2 - (a_x + b_2)r + (a_x b_2 - b_x a_2) - 0 \qquad (20\text{-}41)$$

It is apparent that this is the characteristic equation associated with the equivalent second-order Eq. (20-39). The solution thus follows as before. Letting r_x and r_2 be the roots (solutions) of (20-41), if, for example, $r\backslash \,{}^\wedge r_2$, the solutions to the homogeneous simultaneous differential Eqs. (20-38) are

$$
\begin{aligned}
x(t) &= A_x e^{rit} + A_2 e^{r2t} \\
y(t) &= Be^{r}\wedge + Be^{r}{}'' \qquad (2\circ''42)
\end{aligned}
$$

For autonomous infinite-horizon problems, i.e., where the upper limit on the objective functional is infinity, and time t enters directly only in the discount factor, if at all, it is often of concern whether the solution to the control problem converges to some steady state path (usually involving the solution to the nonhomogeneous part of (20-37)). If the roots are negative, or if the real parts of the complex roots are negative, this outcome is assured since $e^{rt} \to 0$ as $t \to \infty$ in those cases. If one root is negative and the other positive, the system is said to have a saddle point. If, for example, $r_1 > 0$, the solution will converge to the steady state if A_1 and $B_1 = 0$.

20.4 INTERPRETATIONS AND SOLUTIONS

Intertemporal Choice

Let us consider first the continuous analog of the model of intertemporal choice investigated in Chap. 12. Assume a consumer has a utility function $U(C(t))$, where $C(t)$ is a flow of consumption. We assume $U' > 0$ and $U'' < 0$, as in static theory. The individual is endowed with an initial stock of capital K_0. The individual's income is the flow iK earned from the capital stock, where i is the market interest rate. In addition, the individual can, by selling capital (we normalize its price to unity), consume the capital stock as well at any time. Finally, assume that the consumer is "impatient," i.e., he or she has a time rate of preference ρ. The model of intertemporal utility maximization can then be stated as

maximize

$$\int_0 U(C)e^{-\rho t}dt$$

subject to

$$K' = iK - C \qquad K(0) = K_0 \qquad K(T) > 0$$

The state variable is the capital stock; the control is the flow of consumption, $C(t)$, the consumer chooses. The state equation (constraint) says that the change in the capital stock ("savings" when positive and "dissavings" when negative) equals the difference between the income earned by the stock, iK, and consumption, C. We assume, of course, that $C(t) > 0$ and $K(t) > 0$ for all t. The Hamiltonian is

$$H = U(C)e^{-\rho t} + \lambda(iK - C)$$

which yields the maximum and adjoint equations

$$H_C = U'(C)e^{-\rho t} - 1 = 0 \tag{20-43}$$

$$H_K = ik = -k' \tag{20-44}$$

Note that the integrand is concave in C, due to the assumption of diminishing marginal utility ($U'' < 0$); the constraint is linear in C and K. Thus, the Hamiltonian is concave in C and K, assuring us that solutions to the first-order equations represent maximum values.

Equation (20-43) says that at every point along the optimal consumption path, the discounted marginal utility of consumption equals the present value (i.e., the value at time t — 0) of an extra unit of capital. Differentiating (20-43) with respect to t yields

$$U''(C)C'e^{-pt} - pU'(C)e^{-pt} = X'$$

From (20-44), $X' = -ik = -iU'(C)e^{\sim pt}$, using (20-43). Using this in the right-hand side of the above equation yields, after canceling the $e^{\sim pt}$ terms,

$$\frac{-U''C}{U'} = i \sim p \qquad (20\text{-}45)$$

The left-hand side of (20-45) is the proportionate change, with respect to time, in the individual's marginal utility of consumption. This represents the marginal benefits of increasing consumption at any point in time. This equation thus says that along the optimal consumption path, these marginal benefits equal the marginal opportunity cost of increasing consumption, the "net" interest rate, i.e., the market (real) yield less the rate of impatience.

Note now the implications of (20-45) for observable behavior. Since $U' > 0$ and $U'' < 0$, C'(r) has the same sign as/— p.Thus, if/ = p, that is, if the rate of interest equals the rate of impatience, then, as in the static models, the consumer chooses constant consumption. Likewise, if the market opportunity cost of consumption exceeds the individual's rate of impatience, consumption rises over time, and vice versa. If the interest rate should rise at some point t, the consumer will accelerate the flow of consumption, thus shifting consumption to the present at a greater rate. Note also that (20-44) is a simple linear homogeneous differential equation; its solution is

$$X(t) = koe^{\prime\prime l} \qquad (20\text{-}46)$$

where $A._o > 0$ is the constant of integration. The present value of the marginal value of capital thus decreases over time; its current value, $e^{lt}k\{t)$ remains constant at $A._o$. Combining this equation with (20-43) yields

$$U'\{C\{t)) = X_o e^{(p-l)t} \qquad (20\text{-}47)$$

Consider now what must happen at the end of the planning period. Recall that the transversality condition (20-20), which is a result of the nonnegativity restriction on the terminal capital stock, requires that $X(T)K(T) = 0$. Either capital must be exhausted, $K(T) = 0$, or its marginal value must fall to zero at the terminal date. The only reason capital would *not* be completely used up is if the additional consumption it afforded had no value, i.e., if the consumer had already been sated so that more income was no longer preferred to less at that margin. If, as we have assumed, $U'(C) > 0$ for any level of consumption, it must be the case that at $t = T$, $K(T) = 0$. Thus, assuming more is always preferred to less, capital will be exhausted at the end of the planning period.

Equations (20-45) through (20-47) characterize the solution to this model. In order to derive actual paths of consumption and capital utilization, we would need to assume a specific functional form for the utility function. To illustrate the solution to control models, let us assume that $U(C) = \log C$. In that case, (20-45) becomes

C

\ \ *d*

Separating variables and integrating,

C(t) = (20-48)

Equation (20-48) is the path of the control variable $C(t)$. To derive the path of the state variable K, recall the state equation (the constraint):

$$K' - iK = -C = -C_o e^{(i-p)t}$$

This can be integrated using the integrating factor e^{-it}:

$d\{e^{-it}K)$

dt $-C_o e$

$e^{-i}\backslash K'$

$-iK) =$ Integrating both sides yields

$e^{-'}K(t) =$ I ^ A (20-49)

where A is the arbitrary constant of integration. At $t = 0$, $K(0) = K_o$; thus,

Likewise, using $K(T) = 0$,

or A —

and thus,

$A =$ $\underline{-K_o e^{-pT}}$ pK_0

The solution to (20-49) is thus

$K\{t) = K_0 e^{it}$ **-pt — 0-**

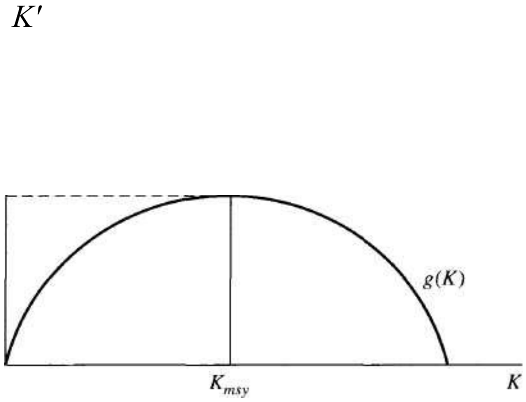

FIGURE 20-1
The function $K' = g(K)$ shows the rate of change of the stock of fish (capital stock) when no harvesting takes place. For $K < K_{msy}$, the stock of fish grows; when $K > K_{msy}$, K shrinks. Left alone, some maximum sustained yield (maximum value of K') occurs at K_{msy}. However, this harvest rate is not in general efficient.

Harvesting a Renewable Resource

A closely related problem to the preceding one has been applied to the problem of harvesting some renewable resource, such as fish.^ We assume the same objective function, except that for this example, we take the time horizon to be infinite. Most importantly, we assume that, left alone, the stock of fish would grow at some rate to some maximum size; and, if the stock were somehow larger, the fish would on net balance die off, reducing the stock to its steady state size. We formulate the model as

maximize
$$\int_0^{\infty} U(C)e^{-\rho t}\,dt$$

subject to
$$K' = g(K) - C \qquad K(0) = K_0 \tag{20-50}$$

where K represents the stock of the renewable resource (in this case, fish). The function $g(K)$ represents the biological growth of the stock; it is depicted in Fig. 20-1. With zero stock ($K = 0$), there is no reproduction, and K remains at zero. Left alone, for positive K, the stock would grow at some rate given by $K' = g(K)$. For many species in given environments, a "maximum sustained yield" K_{msy} exists, where $g'(K) = 0$. We assume that $g'(K) > 0$ for $K < K_{msy}$ and $g'(K) < 0$ for $K > K_{msy}$; thus, $g''(K) < 0$. Therefore, if K_{msy} were the current stock of fish, it would be possible to consume $K' = g(K_{msy})$ forever. It sounds plausible that this is an efficient (utility-maximizing) level of consumption. However, consideration of the dynamic

^See, e.g., C. G. Plourde, "A Simple Model of Replenishable Natural Resource Exploitation," *American Economic Review,* 60:520-522, June 1970.

aspects of the model, in particular the possibilities of time preference and the effect of the size of the fish stock on the marginal cost of fishing, changes that view. The *current-value* Hamiltonian for (20-50) is

$$'K = U(C) + m(g(K) - C) \qquad (20\text{-}51)$$

The maximum and adjoint equations are

$$We = U'(C) \sim m = 0 \qquad (20\text{-}52)$$

and

$$W_K = mg'(K) = pm - m! \qquad (20\text{-}53)$$

Note that the preceding assumptions concerning the shapes of *U(C)* and *g(K),* and the fact that the current marginal value of the stock is positive, guarantee the concavity of the Hamiltonian in *C* and *K;* thus, solutions to the first-order equations are paths that maximize the objective integral. As in the previous model, the maximum equation (20-52) says that along the optimal path, the current marginal imputed value of the stock of capital—in this case fish—equals the marginal utility of consumption of fish. Equation (20-53) can be rewritten

$$g\backslash K) = p - \frac{}{m} \qquad (20\text{-}54)$$

The term *g'{K)* specifies the rate of growth of the stock of fish; it is the benefit of waiting, or delaying consumption an increment of time. In a nondynamic model, wealth or utility maximization would require this marginal benefit to equal the opportunity cost of capital, which in this case is given by the consumer's rate of impatience, *p.* However, decisions in the present affect the future; consumption of fish affects the percent rate of change of the marginal value of fish. This additional cost, the capital loss, *—m'/m,* must be added to the direct cost of waiting. (Of course, *—m'/m* might be negative, thus offsetting the impatience rate.)

Without specific functions for *U(C)* and *g(K),* an analytical solution of the model is impossible. However, for autonomous models such as these, an analytic device known as a *phase diagram* can be used to characterize the solution and to derive comparative statics results.

From the maximum condition (20-52), *U'(C) = m.* Since *U" <* 0, this is a monotonic function; it can be inverted, using a global version of the implicit function theorem, yielding *C = c(m).* Using this in the state equation (the constraint) yields two differential equations determining the motion of the model:

$$K' = g(K) - c\{m) \qquad K(0) = K_o \qquad (20\text{-}$$

55)

$$m! = pm - mg\backslash K) \qquad (20\text{-}56)$$

A steady, or stationary, state occurs when the values of the variables remain constant over time. These values are therefore determined by setting $K' = m' = 0$ in the

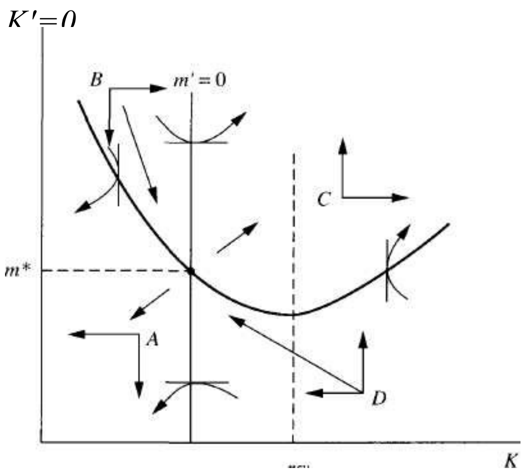

FIGURE 20-2
Phase Diagram Showing Saddle Point Stability. The optimal (utility-maximizing) harvest rate occurs at K^*, where $g'(K) = p > 0$, i.e., to the left of K_{msy}. The steady state occurs at the intersection of the $K' = 0$ and $m' = 0$ loci. Above K' — 0, K increases over time; below it, K decreases. Similarly, to the left of $m' = 0$, the current value of the stock, m, decreases; to the right, m increases. At points such as A and C, the paths diverge from the steady state; at points such as B and D, they converge to the steady state.

preceding equations, resulting in

$$g(K) - c(m) = 0 \qquad (20\text{-}57)$$

$$p - g'(K) = 0 \qquad (20\text{-}58)$$

These two equations are plotted in Fig. 20-2. Denote the steady state values of K and m as K^* and m^*. Consider the locus where $K' = 0$, Eq. (20-57), first. Since $U\backslash C) > 0$ and $U''\{C) < 0$, $c'\{m) = \backslash/U'' < 0$. By assumption, $g(K)$ first rises and then, after K_{msy}, falls. Therefore, $c(m)$ must rise and then fall to maintain the equality in (20-57). Since $c'(m) < 0$, as K increases from the origin, m itself must fall and then rise, reaching its minimum value where $g(K)$ is largest, at K_{msy}. Thus, the locus of (K, m) where $K' = 0$ is the U-shaped curve depicted. On the other hand, for $m' = 0$, Eq. (20-58) is simply a vertical line, at $K = K^*$. But note that since $g'\{K) = 0$ at $K = K_{msy}$ and $g\backslash K^*) = p > 0$, K^* is to the left of K_{msy}. With positive time preference, steady state consumption is shifted toward the present. Moreover, as the rate of time preference (or the market interest rate, in an equivalent model) increases, the steady state capital stock and $m(t)$ fall, since, to the left of K_{msy}, $g'(K) > 0$ wd $g''(K) < 0$.

We have not shown, however, that for some arbitrary K_o, the optimal path will actually tend toward the steady state. Consider how the values of K and m will change in the four areas of the phase plane between the curves $K' = 0$ and $m! = 0$. At all points above the U-shaped locus defined by $K' = 0$, $K' > 0$; below it, $K' < 0$. Likewise, to the left of the vertical line $m' = 0$, $m' < 0$; to the right of this line, $m' > 0$. Thus, if K and m take on values other than (K^*, m^*), they will move in the directions indicated by the signs of K' and m'. These directions are indicated by the arrows in Fig. 20-2. At points A and C, the path moves *away* from the steady state. From points such as B and D, the path converges toward (K^*, m^*). This stationary point is thus a saddle point. The characteristic equation determining the solution to the control problem contains one positive and one negative root. The optimal solution is obtained when the constant of integration attached to the positive root is set equal to zero.

It is important to note that this procedure is valid only because the system is autonomous. If time entered Eq. (20-55) or (20-56) directly, i.e., *if the date mattered,* then merely knowing K and m would not be sufficient to determine the movement of the variables. The date, i.e., if the value of t, would also have to be specified.

That the steady state is the optimal path is often an assertion. It can sometimes be justified by appeal to the curvature properties of the functions in the model. In this case, for example, if we assume that as $C \to 0$, $U'(C) \to \infty$, and that as $C \to \infty$, $U'(C) \to 0$, then paths that converged to zero consumption, for example, could not be optimal; the marginal value of an increment of consumption at small values of C will exceed the full marginal cost of harvest. Likewise, paths that diverged to infinity could not be optimal with positive marginal costs of fishing.

An alternative justification for the interest in a steady state solution is that stable rather than explosive behavior seems to be empirically more relevant. The world does not seem to provide examples of divergence of capital to infinity, and extinction of resources is uncommon with well-defined property rights.

A related model of renewable resource extraction formulates the objective functional in terms of wealth maximization, as introduced earlier in this chapter:

maximize

$$\int_0^\infty [pu - c(K, u)]e^{-rt}dt$$

subject to

$$K' = g(K) - u \qquad K(0) = K_0 \qquad (20\text{-}59)$$

In this model, fish are harvested and sold at the rate $u(t)$ at some constant price p; the cost function depends on the stock as well as the rate of extraction. (Note that the symbol c denotes "cost" in this formulation, not "consumption," as in the previous model.) The current-value Hamiltonian is

$$\tilde{\mathcal{H}} = pu - c(K, u) + m(g(K) - u)$$

We assume the cost function is strictly convex in K and u, and we maintain the concavity assumption concerning $g(K)$. Thus, the Hamiltonian itself is strictly concave, and the first-order necessary conditions are sufficient for a maximum. The maximum and adjoint equations are

$$\tilde{\mathcal{H}}_u = p - c_u - m = 0 \qquad (20\text{-}52')$$

and

$$\tilde{\mathcal{H}}_K = -c_K + mg'(K) = rm - m' \qquad (20\text{-}53')$$

Equation (20-52') is equivalent to (20-52) in the sense of defining the ordinary conditions for maximization; here the marginal current value of the stock, m, equals

the net current benefits offish extraction, price minus marginal cost. In static models, this condition would simply be price equals marginal cost; in dynamic models, the opportunity cost of future events are capitalized into present decisions. Rearranging (20-530, we have

$$g'(K) = r - \frac{m'}{m} + \frac{c_K}{m} \qquad (20\text{-}54')$$

This equation is similar to (20-54) except that an additional term c_K/m is present. Recall that in the earlier analysis, the steady state, derived by setting $m' = K' = 0$, occurred where $g'(K) = r$ (or the rate of impatience, p). Since r (or p) is assumed positive, $K^* < K_{msy}$. However, in Eq. (20-54'), we have the extra term c_K/m, the sign of which is an empirical matter. (We, of course, assume $m > 0$; the capital stock never has negative marginal value.) This term indicates the effect on the cost of fishing of an increase in the stock of fish. It is easy to imagine (and empirically likely) that a larger stock of fish lowers the marginal and total costs of fishing for any level of activity. In that case $c_K < 0$, and it is possible for K^* to occur where $g'(K) < 0$, implying $K^* > K_{msy}$. It is possible that *delaying* harvesting and waiting for the stock to build up can produce a sufficiently large gain in the future to offset the opportunity cost of funds by lowering the cost of harvesting.

Capital Utilization

Let us now consider a model of capital utilization with a somewhat more general objective function. Imagine a stock of capital $x(t)$ at time t that enables a firm (or person—perhaps this is human capital) to earn a stream of rents $R(x)$. Subsumed into this function, for simplicity, is some behavior in which the person or firm combines some other inputs (e.g., labor) with the capital stock in some presumably cost-minimizing manner. Assume that capital depreciates (or "evaporates") at a linear rate bx and that the cost of investing in new capital is given by c(w). The firm wishes to utilize and acquire capital so as to maximize wealth over an infinite horizon. The model is

maximize

$$\int_0^\infty [R(x(t)) - c(u(t))]e^{-rt}\, dt$$

subject to

$$x'(t) = u(t) - bx(t) \qquad x(0) = x_0 > 0 \qquad (20\text{-}60)$$

We assume an interior solution exists, with $u(t) > 0$, and that $x(t) > 0$ throughout. The state equation $x' = u - bx$ defines, as always, the dynamics of the model; it says that the rate of change in the capital stock equals the rate of acquisition of new capital minus the evaporation at time t. The control variable is the acquisition rate of capital, i.e., the investment rate. The objective function is again autonomous (in the sense that time enters only in the discount function), and the state and control variables are functionally separated. These simplifying assumptions, though limiting in terms of their theoretical application, provide substantial increases in tractability. We shall assume that $R(x)$ is concave and $c(u)$ is convex [and, thus, $-c(u)$ is concave] so that a maximum is assured if the first-order necessary conditions are satisfied. Note that the state equation is linear, so that it has no effect on the sufficient conditions. The current-value Hamiltonian is

$$\mathcal{H} = R(x) - c(u) + m(u - bx)$$

producing the maximum and adjoint equations

$$\mathcal{H}_u = -c'(u) + m = 0 \tag{20-61}$$

and

$$\mathcal{H}_x = R'(x) - bm = rm - \dot{m} \tag{20-62}$$

along with the state equation

$$x'(t) = u(t) - bx(t) \tag{20-63}$$

Equation (20-61) says, as in the previous models, that the current marginal value of the capital stock, $m(t)$, equals the current value of marginal costs of investing in new capital, $c'(u)$. The adjoint Eq. (20-62) is easiest to interpret by writing it as

$$R'(x) + \dot{m} = (b + r)m$$

The right-hand side is the opportunity cost of funds, consisting of rate of depreciation of *value* of the capital stock x at time t plus the alternative investment yield, r. Along a wealth-maximizing path, this marginal opportunity cost must equal the marginal rate at which benefits are being produced. These marginal benefits derive from two sources: The instantaneous (marginal) profits from an additional increment of capital, $R'(x)$, plus the capital gain $m'(t)$ (i.e., the rate of change in the marginal value of the capital occurring at time t, which derives from future wealth-maximizing use of the capital stock). As in all dynamic processes (and this is what makes them dynamic), the value of actions taken in the present have *two* components: Some immediate net benefits plus the sum of the future net benefits.

Equation (20-62) can be further interpreted in this manner by multiplying through by $e^{-(b+r)t}$ and writing it as

$$e^{-(r+b)t}[\dot{m} - (b + r)m] = -e^{-(r+b)t}R'(x)$$

Integrating both sides and assuming $R'(x)$ is bounded from above so that the integral function evaluated at the upper limit is zero,

$$e^{-(r+b)t}m = \int^{\infty} e^{-(r+b)s}R'(x(s))\,ds$$

or, multiplying through by $e^{(r+b)t}$,

$$m(t) = \int_{t}^{\infty} e^{-(r+b)(s-t)} R'(x(s))\,ds \tag{20-64}$$

Equation (20-64) says that the current (at time t) marginal value of capital is the future net marginal profits discounted back to time t, where the interest rate used for discounting is the sum of the real interest rate r and the depreciation rate b, reflecting the true opportunity cost of using this particular capital. Lastly, combining this equation with Eq. (20-61) says that marginal costs equal these marginal benefits:

$$c'(u(t)) = \int_t^{\infty} e^{-(r+b)(s-t)} R'(x(s))\, ds \qquad (20\text{-}65)$$

Let us now "solve" this problem, in the sense of investigating whether some steady-state solution exists and what its properties are. We use a diagrammatic analysis similar to that used previously and investigate paths in the (x, m) phase plane that satisfy the first-order conditions.

Since marginal cost is strictly increasing ($c'' > 0$), we eliminate the control variable u by inverting $c'(u) = m$; thus, $u = h(m)$, where $h'(m) = 1/c'' > 0$. Substituting this into the state equation gives

$$x' = h(m) - bx \qquad (20\text{-}66)$$

Equation (20-66) and the adjoint equation (20-62)

$$R'(x) + m' = (b + r)m \qquad (20\text{-}62)$$

constitute two differential equations in x and m. The steady state occurs where $x' = m' = 0$:

$$h(m) = bx \qquad (20\text{-}67)$$

$$R'(x) = (b + r)m \qquad (20\text{-}68)$$

These equations are sketched in Fig. 20-3. Since $c'(0) = 0$ and $c'' > 0$, Eq. (20-67) passes through the origin and is positively sloped. The intersection of the two curves is the steady state and is denoted S; let x^* and m^* be the steady-state values of x and m.

We next ask if paths exist that are consistent with the first-order equations and that approach S asymptotically. Since $h(m)$ is increasing in m, at points above the line, $x' = h(m) - bx > 0$; likewise, $x' < 0$ below the line. Also, since $R''(x) < 0$, $R'(x)$ is decreasing in x; thus, Eq. (20-68) is negatively sloped in the phase plane. Above this curve, $m' = (b + r)m - R'(x) > 0$; thus, m is increasing above the curve and decreasing below the curve.

The combined effect of these movements is indicated by the directions indicated by the arrows in Fig. 20-3. Paths converging to the steady state exist starting either to the "northwest" (but below $m' = 0$) or to the "southeast" (but above $m' = 0$) of S. The other indicated paths are unstable, i.e., they diverge to infinity or zero. The steady state is a saddle point. We reject the divergent paths as empirically unobserved or ruled out by the curvature properties of $R(x)$ and $c(u)$, e.g., $R^{\prime}(x) \longrightarrow \infty$ as $x \to 0$, etc.

A common algebraic procedure used to confirm the properties of the steady state, when the defining equations are nonlinear (as in this example), is to linearly

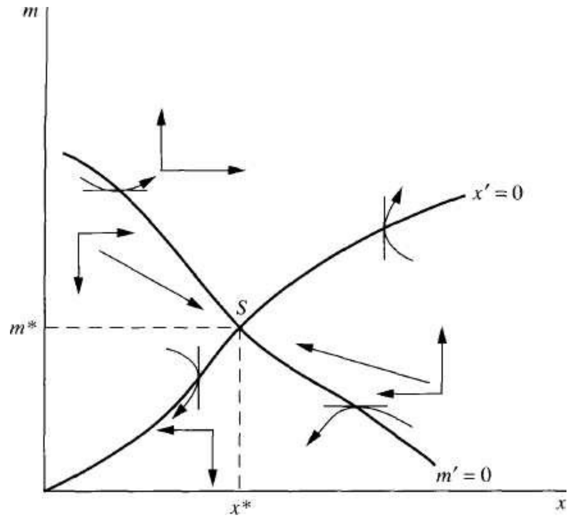

FIGURE 20-3 *Phase Diagram Showing Saddle Point Stability for the Capital Utilization Model.* The steady state 5" occurs at the intersection of the $x' = 0$ and $m' = 0$ loci. Above $x' = 0$, x is increasing; below it, x is decreasing. Also, above $m! = 0$, m is increasing; below it, m is decreasing. The combined effects are shown by the arrows. Some paths starting from the "north-

west" and "southeast" of S converge to the steady state. An increase in r leaves the $x' = 0$ locus unaffected but shifts the $m' = 0$ locus toward the origin. Therefore, increases in the interest rate lower both the steady state capital stock x^* and current marginal value m^*.

approximate the x' and m' differential equations around the steady state, using the first-order terms of a Taylor series. Performing this operation,

$$x' = -b(x - x^*) + h'(m^*)(m - m^*)$$

$$m' = -R''(x^*)(x - x^*) + (r + b)(m - m^*)$$

The characteristic roots are [see Eq. (20-33)]

$$k_1, k_2 = \tfrac{1}{2}\{r \pm [(r + 2b)^2 - 4h'(m^*)R''(x^*)]^{1/2}\}$$

Since $h' > 0$ and $R'' < 0$, the roots are real, and since the term in the radical is larger than r, the roots are of opposite sign, with the absolute smaller root negative. Thus, the steady state is a saddle point within some neighborhood.

It has been stressed throughout this book that the goal of mathematical modeling is to derive refutable hypotheses, and that such propositions generally take the form of statements about the directions of responses of the decision variables to changes in the constraints. In dynamic models, such questions can be posed, for example, about the effect on the steady-state values of the capital stock as various parameters change. These are termed comparative statics questions, as in static models. A more difficult inquiry concerns the effect of a change in a parameter on the entire path of either the control or the state variable; these questions are termed *comparative dynamics.* We briefly illustrate this analysis using the present example.

It is clear from Eqs. (20-67) and (20-68) that an increase in the interest rate leaves the first equation $(x' = 0)$ unaffected but shifts the second $(m' = 0)$

locus toward the origin. As a result, the steady state capital stock is lowered, i.e., $dx^*/dr < 0$. From the positive slope of the $x' = 0$ locus, $dm^*/dr < 0$ as well, and from the maximum condition $c'(u) = m$, $du^*/dr < 0$. We expect these results. If the opportunity cost of funds for use in the present increases, deferring consumption to the future (by investing in new capital) should decrease, resulting in a smaller final capital stock. An increase in the depreciation rate b produces somewhat more ambiguous changes; b enters Eq. (20-68) ($m' = 0$) in the same manner as does r, but it also enters Eq. (20-67) (x^r — 0). Since $h'(m) > 0$, an increase in b shifts the $x' = 0$ locus up. As a result, x^* must clearly fall, but the change in m^* (and, thus, u^*) is ambiguous.

Although general comparative statics theorems of this type are difficult to state, a result is available for the effects of changes in the interest rate (or rate of time preference) on the steady state capital stock, for "autonomous" models.

Caputo's Theorem.^ Consider a general autonomous optimal control model with infinite horizon,

maximize

/"OO

f(x,u,a)e-[rl]*dt*

'to

subject to

X = g(jt, *u)* *x(0) — XQ* *U* €
U

where U is the control set and $a > 0$ is a time-independent parameter. Assume an optimal solution exists that converges to the saddle point steady state of the model. Let $(x^*(a, r), u^*(a, r))$ denote this steady state, with the *'s on the functions used to indicate that they are evaluated at the steady state values.

1. The response of $x^*(a, r)$ and $u^*(a, r)$ to a change in the interest rate, r, is given by

sgn *I -?- j* = sgn(/;g;) (20-69a)

sgn — = -sgn(/;^) (20-6%)
\ *d r J*

2. If the parameter a enters/such that it is attached to x only, that is, $f_{ua} = 0$, then the effect of change in a is given by

SM^T-) =[s]§[n](/,«) (20-70a)

-^) = -sgn(g>;/;[a]) *(20-10b)*

Michael R. Caputo, "The Qualitative Structure of a Class of Infinite Horizon Optimal Control Problems," *Optimal Control Applications and Methods,* 18:195-215, 1997. See corollary l(a).

3. If the parameter a enters / such that it is attached to u only, that is, $f_{xa} = 0$, then the effect of change in a is given by

= Sgn(/ »*«(r ~ 8 *)8 ''} (20-71a)

r - £)£) (20-71/7)

Results 2 and 3 are "conjugate pairs" theorems, analogous to those derived earlier in the static models. It can be shown that g^* appears in all the steady state comparative statics results for $u^*(a, r)$. Thus, if $g^* = 0$ (or if $g_x = 0$), that is, if the state equation is independent of the state variable at the steady state (or globally), then the steady state value of the control variable is independent of a and the discount rate r.

In the preceding example, $f_u = -c'(u) < 0$ and $g_u = 1$; thus, $dx^*/dr < 0$, as derived directly. The theorem has wide applicability in resource extraction models. The signs of these partials are often apparent. Typically, the integrand function $f(x,u)$ measures some sort of net benefits (or negative values of costs) which are increasing in u, so typically $f_u > 0$ and the state equation has the form $x' = h(x) - u$ so that $g_u < 0$ (see, e.g., the models in the section on harvesting a renewable resource). Thus, in those models, we will generally find $dx^*/dr < 0$.

Comparative dynamics concerns the responses of the entire paths $x(t)$, $u(t)$ and $k(t)$ [or $m(t)$] as the parameters of the model change. The procedure is similar to that used in comparative statics in that the "solutions," $x(t, r, b)$, $m(t, rb)$ are substituted into the simultaneous differential equations defining the paths of x and m [(20-62) and (20-66) in this model]. These simultaneous equations are then differentiated with respect to some parameter, producing what is called a *variational differential equation system.* It is sometimes possible to determine the shift in the path, based on the curvature properties of the functions in the model. With more than one state variable, however, two-dimensional graphical analysis is impossible. Such material is beyond the scope of this text; the references at the end of the chapter contain discussions of this problem.

PROBLEMS

1. Solve the optimal control problem maximize

$$\int_0 -u^2\, dt$$

subject to

$$x' = x + u \qquad x(0) = 1$$

$x(1) = 0$ 2. Solve the optimal control problem maximize

2

I

$$(x + tu - u^2)dt$$

subject to

$$x' = u \qquad x(1) = 3 \qquad JC(2)=4$$

3. Solve the optimal control problem

maximize

/'("""" T) *

subject to

$$x' = u - x \qquad x(0)=x_0 \qquad x(1) = 0$$

1.455 In the section entitled "Capital Utilization," let $R(x) = ax — x^2/2$ and $c(u) = cu^2$. Solve the model explicitly, and relate your solution to the analysis in the chapter.

1.456 Consider a mine containing some amount X of some mineral resource. Let x(0 represent the cumulative amount mined at time t, so that $u — x'(t)$ is the rate of extraction. Suppose the current rate of profits of the mine is given by $P(u)$, where $P' > 0$, $P'' < 0$. Assume that the owner of the mine maximizes wealth over the period [0, T] and that there is no salvage value after T. Assume a fixed market interest rate, r;

1.457 Show that the present value of marginal profits is constant over [0, T]. Explain.

1.458 Show that the extraction rate declines over time.

1.459 Suppose the current rate of profits of the mine is given by log u. Find the actual wealth-maximizing path of resource extraction, where x(0) = 0, $x(T) = X$.

1.460 How is the exploitation of the resource affected by changes in the interest rate, r?

1.461 Resolve the renewable resource model, (20-50), using a phase diagram in (K, C) phase space. (*Hint:* Differentiate the maximum condition with respect to time, then use the adjoint and maximum equations to eliminate m and m'.)

1.462 Solve the optimal control problem

maximize

$$\int_0^2 (2x — 3u — au)dt$$

subject to

$$X = X + U$$

$$*(0) = 5 \quad x(2)$$

free SELECTED REFERENCES

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CHAPTER 1

1. The law of demand asserts that the quantity demanded will fall when the price is raised, *holding various other things constant.* In particular, tastes are assumed constant. In this scenario, tastes change as the price changes.
3. Follows from the law of demand.

1.463 Yes; the human mind is incapable of dealing with *all* aspects of a given situation.

1.464 No; not very.

1.465 *(a)* $a, b, k > 0$. *(b)* $x^*(t) = [(a/b) - t]/[(2/b) + 2k]$. (c) $-2/b - 2k<0$; weaker (less restrictive on the values of *a, b, k;* since *b* may be negative, this expression is *not* equivalent to $bk > -1$. *(e)* Differentiate $x^*(t)$ directly.

1.466 Let $R(x) = px$, where *p*, output-price, is parametric.

1.467 Since $dx^*/dt < 0$ is implied here, no amount of data relating to *changes* in quantities sold and *changes* in tax rates will ever distinguish this theory from the others.

1.468 Substitute $y^*(t)$ into the gross and net functions, and differentiate with respect to *t.* Use the first-order conditions to cancel out some terms, and remember this when you get to Chap. 7.

1.469 This problem is most tractable as a cost (physical amount of metal used) minimization problem. When the corners are wasted, the ends use D^2 each, and the sides use πDh. The volume of one can is $(\pi D^2/4)h$. Use this to eliminate *h* and minimize with respect to *D*. The waste per can is $2k(D^2 - \pi D^2/4)$. Subtract this from the original objective function, and see how h^*/D^* varies with *k*.

CHAPTER 3

Section 3.5

1. *(b)* $U_{11} = U_{21} = U$ $V_{12} = V_{21} = 4x_1x_2$, $W_{12} = W_{21} = 0$. (c) MRS $= -x_2/x_1$ for *U, V, W.* *(d)* The MRSs. 3. *(a)* $MP_L = a(K/L)^{1-a}$, $MP_K = (1 - a)(L/K)^a$. *(c)* Yes.

1.470 $dy/dt = [an + (1 - a)m]L_0^aK_0^{1-a}e^{[an+(1-a)m]t}$.

1.471 *(a)* Follows from $V_i = F'(U)U_i$. *(b)* Use product rule on

above; $V_u = F'Uij + F''UiUj$.
Although $F' > 0$ is stipulated. F'' can have either sign.

8. $dU/dp_x = -\{x_2/x_x)^2\wedge(M/p^2) <$ O;yes

$dU/dp_2 = -l(x_l/x_2)\wedge(M/p^2{}_2) < 0$; yes

$dU/dM = \{x_2/x_x)^2'\backslash\backslash/_{Px}) + |(*i/*2)^{1/3}(1/p_2) > 0$;

yes Section 3.6

2. *(a)* $y = \log(xijc_2)$; $x_x x_2$ is homogeneous, (c) $y = F(z) = z^2 - z$, where $z = x_x x_2$, a homogeneous function.

1.472 Note that $f_t = F'hi\backslash$ slopes of level curves are /)//}; result follows.

1.473 Apply Euler's theorem to f_x.

1.474 Follow proof in text.

CHAPTER 4

Section 4.2

1. *(a)* The origin; saddle point; *(b)* (y, y); minimum; (c) (4, 2); maximum. 3. When $a + p = 1$, $L^a K\wedge$ is (weakly) concave.

5. g, = F'/; since at a stationary value / = $g_t = 0$, $g,_7 = F' f_{ih}$ result follows by applying second-order conditions.

Section 4.6

1.475 Since the term f_{x2} enters the expressions for $dx*/dp$ and $dx\backslash ldp$, these partials are indeter
minate in sign. If one assumes, however, that both are negative, then after eliminating the
positive term in the denominators, a contradiction of the second-order conditions occurs
after a little manipulation.

1.476 Since $y* = f(x;,x*)$, $dy*/dw_x = f_x\{dx*Jdw_x) + f_2(dx*/dw_x)$. Applying Eqs. (4-20)
gives the negative of the expression for $dx\backslash/dp$. The same analysis follows for $dy*/dw_2$.

1.477 Assuming $a_x + a_2 < 1$ (otherwise the second-order conditions for profit maximization
are violated), the factor demand for x_x, letting $P = a_x + a_2 — 1$ (note $p < 0$) is

$$x\backslash = a^{t''_2\sim l)lfi}ct^{\sim a2l}t^lp^{\sim llfi}w^{(}_x{}^{\sim a2)lli}w_2{}^{2lli}$$

Since the exponent of w_x is negative, $dx*/dw\backslash <0$. To find the factor demand for x_2, interchange all the Is and 2s.

1.478 (a) Follows from $f_{x2} = f_{2X}$, $dx*/dw_2 = dx\wedge/dwi$. *(b)* Follows from Eqs. (4-20&) and
(4-20c). (c) They aren't; $dx*/d\backslash Vj$ involves more than simply fa. Other second partials
will be present.

1.479 *(a)* $dy*/dt — 7T_{22}/(n_n n_{22} — TT^2{}_2) < 0$; *(b)* nothing; $n_{x2} = C''(y)$ has either sign.

1.480 *(a)* $dy*/dt < 0$; $dy*/dt \wedge 0$, / = 1, 2. *(b)* $(dy_x/dt)_{y2} < 0$.

1.481 There are no observable differences unless the cost and revenue functions can be measured.

Both yield $dx^*/dt < 0$; dx^*/dt ^ 0 (prove).

1.482 The cost of hiring x_2 is now $w_2x_2 + tw_2x_2$ — $(1 + t)w_2x_2$. The factor demands are still homogeneous of degree 0 in all prices. Increasing t clearly has the same effect of the firm as increasing w_2; thus, the qualitative comparative statics results are the same.

1.483 *(a)* The factor demands are not homogeneous of any degree. Doubling both factor prices leaving the demand function unchanged would certainly change factor demand. Show this algebraically by trying to duplicate the proof in the text for the competitive case. *(b)* Proceed as before, differentiating with respect to w_x. *(c)* Define $R^*(wi,w_2)$ as $R(x^*, Xj)$; differentiate with respect to w^. Answer: No.

1.484 *(a)* Nothing, *(b)* dy^$/dt < 0$. (c) No differences except that now $dy^*/dt = 0$. *(d)* Yes. This part is really like two separate firms; there is no interaction term, *(e)* No. Is the price a parameter, or is it endogenously determined by the maximization hypothesis? *if)* No differences, *(g)* Same as earlier exercises.

CHAPTER 5

Text

1. *(a)-I. (b) 2.* (c) -2 (shortcut: add row 3 to row 1). *(d) 2.* 3. Apply Cramer's rule.

Appendix

1. rank A = 1, rank B = 2, rank C = 3; |C| ^ 0.
3. A"[1] (A"[1])"[1] = I by definition. However, A"[1] A = I. Since inverses are unique, A = (A"[1])-[1].

1.485 Let *hi*, *hj = 0, i, j* = 2,..., *n.* Then h'Ah — *a.**h*\ < 0; hence $a_u < 0$. A similar procedure shows $a_{ti} < 0$, *i* — 1,..., *n.*

1.486 Apply the definition of orthogonal matrices.

CHAPTER 6

1.487 Convexity of indifference curves means —$U_u pj$ + $2U_U P\backslash P2$ — $Ui2p\backslash > 0$. This neither implies nor is implied by $U_u < 0$, £/$_2$2 < 0 because of the U_{l2} term.

1.488 *(a)* x^* — 1, *2 = 1; max. *(b)* $x^* = 1$, $x\%$ — 1; min. (c) $x^* = M/2p_f$, x^ $= M/2p_2$; max. *id)* $x^* = (p_2U°/p_xyi\backslash$ $x^*_2 = \{_{Px}U°/p_2yi^2$; min.

1.489 *(a)* For *a,* your right-hand-side column matrix in the comparative statics system should be (—1,0, 0)', yielding dx^*/da = —$H_u/H > 0$. For *fi,* the cofactors are all off-diagonal. *(b)* Find expressions for the component parts of these expressions and combine.

1.490 *(a)* This is really a special case of *6(b)* above, *(b)* The objective function in Prob. 6 produces this result.

1.491 This says firms will hire inputs until wage equals the value of marginal product (VMP);

however, VMP = pf_i = AC*/, (c) The right-hand-side column is *not* (1,0)' or (0, 1)';
don't forget w, is part of AC*. You can multiply through by $y^*\backslash$ terms in the left-hand-side
matrix are then — *AC* fjj.* *(d)* This is an identity in w_i and w_2, *not* in $x\backslash$ and x_2.

1.492 *(a)* By an increase in $k\backslash$. *(b)* Yes; find $dx^*dk\backslash$. *(c)* Can go either way. *(d)* Wages are
not parameters here; one cannot write $x_i = x^*(w_u\ w_2, p)$, as in the competitive case.
(e) Essentially the same analysis as the competitive case.

CHAPTER 7

1. This problem follows the text presentation and is intended as a review. The only difference is the special form of the constraint. Compare your results with those derived by the traditional methodology in Chap. 6, Prob. 7. *if)* This result shows that the marginal increase in the value of the (indirect) objective function when the resource is increased is the Lagrange multiplier *X.* This important concept originated in economics in the theory of linear programming. Of course, *dX*/dk* ^ 0.

1.493 It was this problem that led us to the primal-dual analysis. Compare with the traditional methodology, as outlined in Chap. 6, Prob. 8. *(b)* AC is linear in w]; since AC* is *minimum* AC, it must lie below AC (except at *x*, x%)* and is therefore concave, (c) *x*/y*. (g)* When the output price is continuously adjusted to the minimum average cost of the (identical) firms in an industry, the short-run demand functions become the long-run demands, by definition, *(h)* Differentiate with respect tow,. Use the reciprocity condition for *dx^ /dp* first derived in Chap. 4.

1.494 The Lagrangian for the "short-run" model is i£ = *pf(x\, x₂)* — *W[X₁* — *w₂x₂* + *X(k* — *W]X* — *w₂x₂),* but at the profit maximum, the constraint is just binding so that *X** = 0. However, *dX*/dw* ^0. Develop a reciprocity condition for *dX*/dw\.* Use this with the tangency condition

Alternatively, define the conditional demand for *x*: x*(wᵤ w₂, p)* ~ *x\(w\, w₂, p, k** (w i, *w₂, p));* differentiate with respect to Wi, and use the homogeneity of x* in evaluating *dk*/dwₓ.*

CHAPTER 8

1.495 The factor demands derived in this chapter are functions of factor prices and output level. Previously, they were functions of factor prices and output *price.* They are different functions. They are both, however, downward-sloping in their own price, perhaps the only property useful for deriving refutable hypotheses.

1.496 For two factors, *(i)* and *(ii)* are equivalent (see Prob. 4, Sec. 4.6), whereas by *(Hi),* the factors are always substitutes. For more than two factors, knowledge that two factors are substitutes (or complements) by any one or two definitions provides no information about the sign of the third type of expression.

7. *(a)* Apply Euler's theorem to *f_L, f_K*. *(d)* Follows from *dK/dL* = —/L/ZK- (/) Apply the formulas in *(a)* by multiplying row 1 by L, row 2 by *K,* and adding one row to the other. Repeat for columns. What effects do these manipulations have on *HI*

CHAPTER 9

1.497 Returns to scale is a broader concept than homogeneity.

1.498 C = $W\backslash X\backslash + w_2x_2 — pf\backslash X\backslash + pf_2x_2 — rpy = rTR$. This model does not specify the recipient of these rents. (Indeed, there is no explanation of who it is that is maximizing profits.) Entry will always exist, driving firm size, output price, and profits to 0.

1.499 *(a)y* = Iog4xi*$_2$.

1.500 Suppose *X]* is held fixed. Then from Euler's theorem, Xw=i *f>*^{x}' = ^{r}*y* anc* *Yl/=2* *f$_i$Xi = sy*. Combine and integrate, remembering that the arbitrary constant of integration is a function of the variables held fixed in partial differentiation. Apply to each x, in turn.

6. Since for homothetic functions, C = $J(y)A(w_u\ w_2)$, MC = $J'(y)A(wi, w_2)$ and AC = $[J(y)/y]A(w_l, w_2)$. At min AC, AC = MC, or $J' = J/y$. Integration yields, for min AC outputs only, $J(y)$ — ky, an equation in y only.

CHAPTER 10

1. Diminishing MRS is a two-dimensional concept; quasi-concavity is a much stronger restriction of the curvature of the utility function. 3. None.

1.501 No. If $U\{x_u ..., x_n)$ is a utility function and $V = F(U), F'\{U) > 0$, V_u and U_{tj} need not have the same sign.

1.502 *(a) (i). (b)* Yes. (c) For *(ii)* yes; for (/) no, because of the possibility of asymmetrical income effects.

10. (a) Can change its size and sign, *(b)* No such law. (c) No effect, *(d)* No effect, *(e)* No effect, *if)* Size can change; not the sign, however.

12. *(a)* Not necessarily, *(b)* Intuitively, if a person is a net saver this year, an increase in the interest rate will provide a larger income next year, and vice versa.

18. *(a)* Differentiate the identity with respect to *M,* noting that X^M = dU^*/dM. *(b)* Differentiate the identity with respect to *P2,* using the above and Roy's identity.

20. *(a) Vertically parallel* means 3 $(U_x / U2)$^X2 = 0. Use the quotient rule on this expression; the numerator is proportional to D_{31}, the relevant cofactor in the expression for dx^/dM. *(b)* Follows from part *(a)* and the Slutsky equation, (c) Note that $U\backslash/U_2$ = 1 /x\, a function of X\ only. Hence, $U\backslash/U_2$ is independent of x_2. *(d)* Show that dx^/dpi — 0.

CHAPTER 11

1. The border-preserving principal minors of order 2 are all positive; in the case of separable utility functions, this condition implies —*U-'pj — U'-p]* > 0, all *i,j, i =£j.* Hence, there cannot be *two U'/'s* that are both positive; otherwise one of the above conditions would be violated.

We have *U!(x['')* = $X^M p_h$ Differentiating with respect to *M*gives U'' *(dx^/dM)* = $p_i dX^M/dM$, from which parts *(i)* of (a) and *(b)* follow. For the compensated demands, $X^u U'_i(x_i)$ — p_t. Differentiate with respect to *pj,* noting that dX^u/dpj — dx^/dU^0. Can inferiority or superiority be defined in terms of the sign of dx^/dU^0?

1.503 Use the same hints.

1.504 From envelope considerations, one gets *Roy's equality,* U^* = — Xxf^t. Differentiate with respect to p_k, noting that $U^*_{.pk}$ = 0. Do the same for U^*.. Note that $U^*. = V^*(l/M)$.

1.505 Use Prob. 3 and part *(a)* of Prob. 2.

1.506 $U!(Xi) = X^{M}{}_{Pi}$. Therefore, $U!'(dx^/dpj) = Pi\{dX^{M}/d_{Pj}) = 0$. Therefore, $dxP/dpj = 0$,
$i,j = 1,\ldots, n, j' ■ j = i$. Result follows from budget equation.

1.507 *(a) A* theory, utility maximization, was invented because it implied (under certain re
strictions) downward-sloping demand curves. The theory also implied other things, e.g.,
symmetry of the substitution terms, but those properties do not follow from the assertion

of downward-sloping demand functions. See the 1975 reference by El Hodiri in Chap. 14 for an amusing exposition of this point.

1.508 At the very least, Leo is retrading coats for the people up north.

1.509 *(a)* Consistent, *(b)* inconsistent, (c) consistent.

1.510 I wouldn't touch this one with a 10-foot pole. Strange behavior.

1.511 $U = F(x_2 + \log Xi)$

13. *(a)* At least $4. *(b)* Less than $6. (c) Approximations; bias indicated, *(d)* Not answerable.

CHAPTER 12

1.512 Construct the ratio of marginal utilities in consecutive time periods.

1.513 *(b)* If you had no heirs and you were going to die tomorrow.

1.514 *(a)* Once-and-for-all loss of wealth, *(b)* Price decreases by present value of tax savings for the marginal buyer, (c) Bad news.

1.515 Assuming diminishing marginal value of wool and mutton, no.

1.516 This is all capitalized into the present price.

1.517 *(a)* Lower both, *(b)* With greater inflation, this feature raises the relative value of holding these assets, increasing their price relative to depreciable assets.

1.518 This shifts the real burden of repayment to the present, possibly imposing liquidity constraints.

9. It is interesting that the annual amount saved varies dramatically with the initial mileage assumed. Calculate the present value of these savings.

CHAPTER 13

1.519 Let $v = a + bu$. Then $v' = bu'$ and $v'' = bu''$. The coefficient of absolute risk aversion for v is $-bu''/bu' = -u''/u'$.

1.520 $v' = f'u'$, $v'' = f''u'^2 + f'u''$, $-v''/v' = -(f'u'^2 + f'u'')/f'u' = -u''/u' - f'u'/f >$ $-u''/u'$ since /" < 0.

1.521 *(a)* $u! = W\sim^a$, $u'' = -aW\sim^a$-\ $-Wu''/u' = aW$-$^a/W\sim^a = a$.
(b) $u' = l/W$, $u'' = -l/W^2$, $-Wu''/u' = (\backslash/W)/(l/W) = 1$.

1.522 *(a)*$u^r = a - 2bW, u'' = -2b, -u''/u' - 2b/(a - 2bW)$. As Wincreases, the denominator decreases so that the coefficient of absolute risk aversion rises.
(b) Let x be the amount invested in risky assets. The choice problem is

$$\max E[a(W + xR) - b(W + xR)^2]$$

The first-order condition is

$$E[aR - 2bR(W + x^*R)] = 0$$

That is, $aR - 2bWR - 2bx^*(R + a^2) = 0$. This gives

„ $(a - 2bW)R$

(c) $x^{*\prime}(W) = -R/(R + a\backslash) < 0$.

5. max $E[-e^{-a(W+xR)}]$. First-order condition: $E[aRe-"(*+*«)] = 0$; i.e., $ae^{-aW}E[Re^{-axR}] = 0$; i.e., $E[Re^{-axR}] — 0$. The first-order condition for x does not involve W. Therefore, the amount of investment in risky assets is not a function of initial wealth.

CHAPTER 14

1. The Kuhn-Tucker conditions specify *necessary* conditions for a corner solution, not *sufficient* conditions. At some point, MP, may be greater than w, even if at x, = 0, MP, < w,.
3. $f(x\, x_2)$ has to be concave to achieve a saddle point solution.
1.523 $X\ = 5, x_2 = 5$.
1.524 $k = 5$.
O. $X\ := 5, X_2 = 7$-
10. *(a)* This is the *Fisher separation theorem* (see Chap. 12). If the consumer can borrow and lend, maximizing wealth leads to the largest opportunity set. Consumers can then rearrange consumption in accordance with their preferences by borrowing or lending. But don't take my word for it; read Fisher, *(b)* $X\ — 4.93$, $x_2 = 12.82$; consumer is lender in period 1, present value = 14.78. (c) JCI = 5.15, $x_2 = 12.36$, present value = 15.45.

CHAPTER 15

1.525 The first-order condition for the agent's problem is $f\$M_x + y$ — C" = 0. Comparative statics analysis yields $dx^*/d0 = -M_x/(fiM_{xx} - C")$ and $dx^*/dy = -\/(PM_{xx} - C")$. The principal maximizes $E[B(x^*, e) — C(x^*, e) — u_0]$ by choosing fi and y. Derive the first-order conditions and solve the resulting equations simultaneously by assuming $f3M_{xx} — C"$ is independent of e. This gives $f5^* = \text{cov}[M_x$, fiJ/var[MJ and $y^* = E[B_X]-/3^*E[M_X]$.
1.526 Minimize $w + K\{n)$, where $n — c(x)/w$. Then differentiate the first-order condition with respect to k.
1.527 Substitute $C" — 2c$ into Eq. (15-30) and differentiate with respect to c.
1.528 *(a)* The worker's expected utility is $nf(ew) + (1 — n)f(w) — g(e)$. (c) The worker is risk-loving if /" > 0.

CHAPTER 16

1.529 The indirect profit function is a convex function of $W\$.
1.530 $E[P_{min}(n)] = /Ju - p)^n\, dp = \/\{n + 1)$. Choose n to minimize $pE[P_{min}(n)] + en$.
1.531 *(a)* The expected minimum price is $J_o (1 — p/dY dp — d/(n + l)$. *(b)* $H(x) — /3f^*(p/d) dp = fix^2/2d$. *(c)* Expected number is $l/F(p^*) — dip^*$. Use the value of p^* obtained

from part *(b)*.

CHAPTER 17

1. *(a)* $z^* = 700$. *(b)* $u\backslash = 10$, $u_2 = 10$, $M_3 = 0$. *(e)* Industry 1 is relatively land-intensive. Therefore, if an additional unit of labor were available, industry 1 would expand and

industry 2 would contract. (/) If the price of the land-intensive industry rises, the shadow price of land $u\$ must rise in greater proportion than the rise in p_{l} (5 percent). The shadow price of labor must fall, *(g)* None.

1.532 $z^* = 37$.

1.533 $z^* = \$7000$.

CHAPTER 18

1.534 The output supply functions are homogeneous of degree 0 in output prices. The result follows from the application of Euler's theorem to these functions.

1.535 This is a direct application of the adding-up theorem of Sec. 14.5.

1.536 This allows analysis of the four-equation model [Eqs. (18-53) and (18-54)] consisting of two zero-profit conditions and two resource constraints as two separate parts, with endowments appearing in only the latter two. With respect to endowment changes, the *ciij* 's are constant, and hence this part of the model behaves like the linear models of Chap. 17 for that reason. From cost minimization considerations, the a,-/s behave as though they were constants in the first two equations dealing with output price changes.

5. *(b)* This production frontier is not necessarily concave because no matter what the production functions themselves are—e.g., there may be extreme increasing returns to scale—as long as marginal products are finite and resources are limited, there must be some finite maximum production of either good for fixed amounts of the other good. Thus, the only curvature properties needed for this problem are convex (to the origin) isoquants, i.e., quasi-concavity. The production frontier may therefore be convex to the origin, e.g., if both production functions exhibit rapidly increasing returns to scale, and the maximum value of output may very well occur along either axis, i.e., for positive output of only one good.

8. (c) and *(d):* Use the envelope theorem.

10. With linear homogeneous production functions, total factor cost equals total output, i.e.,

$$w(pi, p_2)L + r(pi,p_2)K = p\backslash y\backslash(p\backslash, p_2, L, K) + p_2y_2(p\backslash, Pi, L, K)$$

Differentiate this identity.

11. The statement is valid if the conditions for the Stolper-Samuelson *and* the Hecksher-Ohlin theorems are valid.

CHAPTER 19

1.537 With finite resources and unlimited wants, a Pareto frontier of allocations exists along which any greater good for one person means lesser good for some other person.

1.538 $4y/y_t - x/x_t = 3$.

1.539 A perfectly discriminating monopolist will product output as long as some consumer will
pay at least MC. Hence, the Pareto condition p = MC will be satisfied, except that the
monopolist will be the sole gainer from the trade. If the monopolist's income elasticities
differ from other consumers, overall production will change due to the redistribution of
income only.

1.540 *(a)* Yes, if transactions costs are low. *(b)* 1000 + \ *P* to A, 1500 + \ *P* to 5, 2500 + *P*
total, (c) 800 + *P* to A, 1200 + *P* to *B,* 2000 + *IP* total, *(d)* If 500 < *P* < 600, A's gain

from sharing a pump will be greater than 5's loss from so doing. With zero contracting costs, *A* and *B* will contract to share the overall gain and will thus share.

1.541 The curvature of the utility frontier is sensitive to the (ordinal) units of utility. Its negative slope is a consequence of scarcity.

1.542 Depends on transactions costs.

1.543 *(a)* Curious, *(b)* Generally, when property rights are costly to define or enforce, (c) See several articles on this subject in the April 1973 issue of the *Journal of Law and Economics.*

CHAPTER 20

1.544 $x^*\{t) = -e^2k\{-t) + k(t); k^*(t) = 4e^2k(-t); u^*(t) = 2e^2k(-t)$, where $k(t) = [e'/(l-e^2)]$.

1.545 $x^*(t) = t + 2; u^*(t)$ - 1; $k^*(t) = -t + 2$.

1.546 $x^*(t, a, XQ) = a(t - 1) + (x_0 + a)k(t) - e^2(x_0 + a)k(-t); k^*(t, a, x_0) = 2(x_0 + a)k(t)$; $u^*(t, a, x_0) = at + 2(x_0 + a)k(t)$, where $k(t) = [e'/(l - e^2)]$.

7. $x^*(t,a) = (e^2/2a)e^{-'} + (1/2a) + [(10a - e^2 - l)/2a]e^l; k^*(t,a) = 2(1 - e^{2''}); u^*(t,a) = (e^{2-'} - 1)/a$.

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