ACTUARIAL MATHEMATICS II

About

This course has been created for the Lebanese University's Actuarial Mathematics II course as part of their Masters in Actuarial Science postgraduate programme. It is a concentrated course touching on many of the basic actuarial techniques applied in life- and non-life insurance mathematics.

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Chapter 1 Fundamentals of actuarial mathematics

About:

This chapter summarises the fundamentals of actuarial mathematics and introduces standard actuarial notation that will be used throughout the course. This material is not directly examinable, but knowledge of this chapter is vital to understanding the course and being able to answer exam questions on other sections.

Chapter Contents

1 Basic financial mathematics

1.1 Compound interest

1.1.1 *Simplifying assumptions regarding interest rates*

For most of the work in this course we will assume interest rates are constant. This means two things:

1. The term structure of interest rates is flat. The yield curve is flat.
2. Interest rates do not change over time. Volatility of interest rates is zero.

Note that neither of these assumptions is realistic.

1.1.2 *Definitions of interest rate notation*

Let i be an interest rate.

- i will apply for a certain period (a year, half a year, a quarter or a month typically)
- i will be payable (and this compounded) at some interval (a year, a month etc.)

Table 1.1: Types of Interest rates

Type of interest rate	**Applies for**	**Compounds**	**Conversion to Annual Effective**
NACA = Annual Effective Nominal Annual Compounded Annually	year	yearly	i
NACS = $i^{(2)}$ Nominal Annual Compounded Semi-annually	Year	Twice a year	$\left[(1+\frac{i^{(2)}}{2})^{2}-1\right]$
NACQ = $i^{(4)}$ Nominal Annual Compounded Quarterly	Year	Quarterly	$\left[(1+\frac{i^{(4)}}{4})^{4}-1\right]$

Type of interest rate	**Applies for**	**Compounds**	**Conversion to Annual Effective**
NACM = $i^{(12)}$ Nominal Annual Compounded Monthly	Year	Monthly	$\left[(1+\frac{i^{(12)}}{12})^{12}-1\right]$
NMCM Nominal Monthly Compounded Monthly	Month	Monthly	$\left[(1+i)^{12}-1\right]$
NQCM Nominal Quarterly Compounded Monthly	Quarter	Monthly	$\left[(1+\frac{i}{4})^{12}-1\right]$
NACC = δ Nominal Annual Compounded Continuously	Year	Continuously	$e^{\delta}-1$

This last row is called the **force of interest** and is described in the following sections.

1.1.3 *Continuously compounded rates of interest*

δ ("delta") is the symbol commonly used for the continuously compounded rate of interest. It is also known as the **force of interest**. It is not generally used to quote an interest rate in the market. It is used when performing actuarial calculations when the cashflows are paid continuously over time.

Note that you may also have seen (or will see) the force of interest when it comes to financial economics and the Black-Scholes option pricing formula.

1.1.4 *Derivation of* δ

$$\delta t=\lim n\rightarrow\infty \left(1+\frac{i^{(n)}}{n}\right)^{nt}-1$$

$$\delta t=\left[\ln(1+i)\right]^{t}$$

$$\delta t=t\cdot\ln(1+i)$$

$$i=\mathrm{e}^{\delta}-1$$

$\delta(t)$ is the time-dependent force of interest applicable at duration t.

Our usual simplification is that:

$$\delta(t)=\delta\,\forall\,t$$

1.1.5 *Application of force of interest in actuarial mathematics*

Example

Premiums are paid monthly in advance. To calculate the interest applicable to premiums we could use a NMCM interest rate such that the present value of a premium in 1 month's time is:

$$PV(Premium_1)=Premium_1\cdot(1+i)^{-1}$$

Or, we could use an annual effective (NACA) rate to perform the same calculation:

$$PV(Premium_1)=Premium_1\cdot(1+i)^{\frac{-1}{12}}$$

These make sense for premiums since premiums are paid as specified intervals according to the policy terms. Still, we could write the same calculation using a continuously compounded (NACC in this case) force of interest like this:

$$PV(Premium_1)=Premium_1\cdot e^{\frac{-\delta}{12}}$$

However, when are claims paid? Typically, we assume they are paid at the end of the period we are considering. Most of our examples will be annual examples (for simplicity) but we will also work with monthly examples. But is it correct to assume that policyholders will wait until the end of the month for their death benefit?

A more accurate approach would be to assume that claims are paid continuously over time. The PV of benefits paid during a year could be written as follows:

$$PV(Benefits)=SA\cdot\int_{t=0}^{1}\mathrm{e}^{-\delta t}\cdot\mathrm{e}^{-\mu_x(t)}\cdot\mu_x(t)\,dt$$

Where $\mu_x(t)$ is the instantaneous annualised probability of death at age x+t. We won't deal much with continuous-time assurance functions. This is given as an example to better understand why we may need a force of interest or continuously compounded rate of interest.

1.1.6 *Interest, discounts and discount factors*

If i is the annual effective rate of interest, then the following also exist:

Discount factor: $v^t = (1+i)^{-t}$

The **discount factor** is used to take present values of cashflows in a more concise way.

Discount: $d = 1 - v$

The discount is the rate of interest when the interest is "paid in advance". For example, if interest rates are 10%, borrowing USD100 for one year will require repayment of the USD100 and USD10 of interest for a total payment of

$$USD110 = USD100 \cdot (1+i)$$

However, some loans are structured so that the interest payable is deducted off the amount borrowed. If the loan is the discounted value of USD100, then the borrower receives

$$USD100 \cdot (1-d) = USD90.91$$

and the total interest paid is

$$USD100 - USD100 \cdot (1-d) = USD100 \cdot [1-(1-d)] = USD100 \cdot d$$

As an exercise, prove that $d = \frac{i}{(1+i)}$

$$d = 1 - v = 1 - \frac{1}{(1+i)} = \frac{1+i-1}{(1+i)} = \frac{i}{(1+i)}$$

1.2 Actuarial notation for financial mathematics

This section describes some of the basic actuarial notation used for non-contingent financial maths. That is, financial maths that doesn't not depend on **contingent event** such as death.

1.2.1 *Annuity certain payable in arrears*

Consider a regular stream of constant payments of amount 1, payable annual in arrears for n years. With a constant annual effective interest rate of i, the formula for this can be written as:

$$PV = \sum_{t=1}^{n} v^t$$

This is a geometric sum with geometric increases of $\frac{1}{(1+i)}$ or $(1+i)^{-1}$ for n periods. Thus:

$$PV = \frac{1-(1+i)^{-n}}{i}$$

If you need to, show the derivation of the above as an exercise.

The actuarial notation for this formula is:

$$a_{\overline{n}|} = \frac{1-(1+i)^{-n}}{i}$$

Check that you can calculate the following:

- $a_{\overline{10}|}$ = 6.14 for i = 10%
- $a_{\overline{10}|}$ = 7.72 for i = 5%
- $a_{\overline{1}|}$ = 0.95 for i = 5%

1.2.2 *Annuity certain payable in advance*

Consider a regular stream of constant payments of amount 1, payable annual in arrears for N years. With a constant annual effective interest rate of i, the formula for this can be written as:

$$PV = \sum_{t=1}^{n} v^{t-1}$$

$$PV = \frac{1-(1+i)^{-n}}{d}$$

As an exercise, prove that the previous two statements are equivalent

The actuarial notation for this formula is:

$$\ddot{a}_{\overline{n}|} = \frac{1-(1+i)^{-n}}{d}$$

Check that you can calculate the following:

- $\ddot{a}_{\overline{10}|}$ = 6.76 for i = 10%
- $\ddot{a}_{\overline{10}|}$ = 8.11 for i = 5%
- $\ddot{a}_{\overline{1}|}$ = 1.00 for i = 5%

Now show that $\ddot{a}_{\overline{n}|} = (1+i) \cdot a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$

Explain in words what this relationship means.

1.2.3 *Accumulation factors*

For each symbol "a" there is an equivalent symbol "s". a is the present value function for a constant series of cashflows. s is the future value function for a constant series of cashflows.

Payments in arrears:

$$FV = \sum_{t=1}^{n} C \cdot v^{-(t-1)}$$

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

Payments in advance:

$$FV = \sum_{t=1}^{n} C \cdot v^{-t}$$

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

Prove the following:

- $v^n \cdot s_{\overline{n|}} = a_{\overline{n|}}$
- $v^n \cdot \ddot{s}_{\overline{n|}} = \ddot{a}_{\overline{n|}}$

2 Introduction to concepts of life contingencies

2.1 Survival probabilities

The probability of survival for a life aged x for t years is shown as ${}_t p_x$.

If the t subscript is omitted, it is assumed to be 1. Thus,

$$p_x = {}_1 p_x$$

Mortality in each year is independent. Thus, the following equation holds:

$${}_2 p_x = {}_1 p_x \cdot {}_1 p_{x+1}$$

and further

$${}_n p_x = {}_1 p_x \cdot {}_1 p_{x+1} \cdot \ldots \cdot {}_1 p_{x+n-1}$$

In terms of standard life table notation, ${}_t p_x = \frac{l_{x+t}}{l_x}$ where l_x is the number of lives in the population aged x .

2.2 Death probabilities

The probability of death for a life aged x within t years is shown as ${}_t q_x$.

$$q_x = 1 - p_x$$
$${}_t q_x = 1 - {}_t p_x$$

Prove that

$${}_2 q_x \neq q_x \cdot q_{x+1}$$

Proof:

$${}_2q_x = 1 - {}_2p_x$$
$${}_2q_x = 1 - p_x \cdot p_{x+1}$$
$${}_2q_x = 1 - (1 - q_x) \cdot (1 - q_{x+1})$$
$${}_2q_x = 1 - \left[1 - q_x - q_{x+1} + q_x \cdot q_{x+1}\right]$$
$${}_2q_x = q_x + q_{x+1} - q_x \cdot q_{x+1}$$

Intuitively, the second last line can be interpreted as **The probability of dying within two years is the sum of the probability of dying in the first year and the probability of dying in the second year, less the probability of dying in both years.** Take a moment to ensure that you understand why this (mathematically) true

If $q_x = 0.01$ and $q_{x+1} = 0.02$ what is ${}_2q_x$?

$${}_2q_x = 1 - ({}_1p_x) \cdot ({}_1p_{x+1}) = 1 - (0.99)(0.98) = 0.02980$$

2.2.1 *Force of mortality*

In the same way that for interest we have a force of interest that reflects the continuous compounding of interest over time, we have a **force of mortality** which can be interpreted as the annualised instantaneous probability of death at a particular point.

${}_tp_x = e^{-\int_{m=x}^{x+t} \mu(m)\,dm}$ Where $\mu(t)$ is the force of mortality at time t .

If $\mu(t) = \mu_x \; \forall t \in [x, x+1)$ then ${}_1p_x = e^{-\int_{m=x}^{x+1} \mu(m)\,dm}$ simplifies to become ${}_1p_x = e^{-\mu_x}$

Thus ${}_1q_x = 1 - e^{-\mu_x}$ and so $\mu_x = -\ln(1 - {}_1q_x)$ which is also $\mu_x = -\ln({}_1p_x)$

So we have assumed a **constant force of mortality** within an age bracket in order to calculate the force of mortality operating over that age bracket. This is a common method of calculating monthly mortality rates from annual life tables.

$${}_{\frac{1}{12}}q_x = 1 - e^{-\mu_x/12}$$

■ Prove the above equation.

- If q_{45} = 0.01, calculate $\frac{q_{45}}{12}$
- Then, calculate ${}_{\frac{1}{12}}q_{45}$ assuming a constant force of mortality for lives aged 45. Show you result to at least 5 decimal places.
- Is this the same as $\frac{q_{45}}{12}$?
- If they are different, explain why this is.

We have used the assumption of a constant force of mortality in deriving these results. Another possible assumption is the **uniform distribution of deaths** or UDD.

Under UDD:

$${}_{\frac{1}{12}}q_x=\frac{q_x}{12}$$

or more generally:

${}_t q_x = t\, q_x$ for any integer x and 0 < t < 1

2.3 Deterministic modelling

In this chapter, we consider deterministic models. A deterministic model is one where we consider only a single path of a random variable or time series. We do not consider the potential variation around this sample path.

If we are going to use a single path, the obvious choice is the **Expected Outcome**. For all our calculations, we will be calculating expected future cashflows, or equating expected present values.

The expected value of a cashflow at time t for a poisson distribution is:

$$\mathrm{E}\left[CF_t\right]=\sum_{x=0}^{\infty}\mathrm{pmf}_{CF_t}(x)\cdot x=\sum_{x=0}^{\infty}\frac{\mathrm{e}^{-\lambda}\lambda^x}{x!}\cdot x$$

Fortunately, our modelling is a lot simpler. We are considering a single life which is either alive or dead in any particular period. Thus, the random variable follows a Bernoulli distribution. If the cashflow in question is a premium, then a premium on 1 is payable if the life is alive, and 0 is paid if the life is dead.

$$\mathrm{E}\left[CF_t\right]=\sum_{x=0}^{1}\mathrm{pmf}_{CF_t}(x)\cdot x=1\cdot{}_tp_x+0\cdot(1-{}_tp_x)={}_tp_x$$

The **Expected Present Value** of that same premium payable as described above is:

$$\mathrm{EPV}\left[CF_t\right]=v^t\sum_{x=0}^{1}\mathrm{pmf}_{CF_t}(x)\cdot x=v^t\left[1\cdot{}_tp_x+0\cdot(1-{}_tp_x)\right]=v^n\cdot{}_tp_x$$

2.4 Omega age

Life tables have a maximum age called **the Omega Age.** The probability of death at this age is 1. Thus, nobody can live to be older than age Ω .

We will use this concept in later sections.

3 INTRODUCTION TO ACTUARIAL MATHEMATICS

3.1 Life-contingent annuity factors

3.1.1 *Immediate Annuity*

An **immediate annuity** is an annuity that begins paying 1 immediately (not deferred until some point in the future) but **is paid annual in arrears for the lifetime** of the life. Thus, it is not really paid *immediately* but rather after 1 year. The first cashflow takes place at the end of the period.

An immediate annuity factor is the **E**xpected **P**resent **V**alue of a stream of payments of 1 as long as the life aged x is still alive.

$$\text{EPV} = v^1 \cdot {}_1p_x + v^2 \cdot {}_2p_x + v^3 \cdot {}_3p_x + \ldots$$

In practice, because our life table has a cut-off maximum age, we might express this as:

$$\text{EPV} = v^1 \cdot {}_1p_x + v^2 \cdot {}_2p_x + v^3 \cdot {}_3p_x + \ldots + v^{\Omega-x} \cdot {}_{\Omega-x}p_x$$

The standard actuarial notation for this is:

$$a_x = v^1 \cdot {}_1p_x + v^2 \cdot {}_2p_x + v^3 \cdot {}_3p_x + \ldots + v^{\Omega-x} \cdot {}_{\Omega-x}p_x$$

$$a_x = \sum_{i=1}^{\Omega-x} v^i \cdot {}_ip_x$$

3.1.2 *Immediate Annuity in advance*

As the name suggests, this is the expected present value of an annual payment of 1 for every year that the life currently aged x is alive, with payments occurring at the start of the year.

$$\ddot{a}_x = v^0 \cdot {}_0p_x + v^1 \cdot {}_1p_x + v^2 \cdot {}_2p_x + \ldots + v^{\Omega-x} \cdot {}_{\Omega-x}p_x$$

$$\ddot{a}_x = \sum_{i=0}^{\Omega-x} v^i \cdot {}_ip_x$$

Show that $\ddot{a}_x = a_x + 1$

3.1.3 *Temporary annuities*

A temporary annuity is a contingent annuity that is payable annually until the sooner of death, or a specified term.

In arrears:

$$a_{x:\overline{n}|}=v^1\cdot{}_1p_x+v^2\cdot{}_2p_x+v^3\cdot{}_3p_x+\ldots+v^n\cdot{}_np_x$$

$$a_{x:\overline{n}|}=\sum_{i=1}^{n}v^i\cdot{}_ip_x$$

In advance:

$$\ddot{a}_{x:\overline{n}|}=v^0\cdot{}_0p_x+v^1\cdot{}_1p_x+v^2\cdot{}_2p_x+\ldots+v^{n-1}\cdot{}_{n-1}p_x$$

$$a_{x:\overline{n}|}=\sum_{i=0}^{n-1}v^i\cdot{}_ip_x$$

- Show that $\ddot{a}_{x:\overline{n}|}=1+a_{x:\overline{n-1}|}$
- Show that $a_{x:\overline{n}|}=a_x-v^n\cdot{}_np_x\cdot a_{x+n}$

3.2 Premium payable for an annuity

A policyholder usually purchases an immediate annuity by paying **a single premium**. The insurer invests this money to earn interest (i) and pays regular payments every year to the policyholder as long as he or she is alive. **The policyholder is insured against the risk of living too long.** The insurer has taken on the risk that the policyholder lives longer than expected.

The premium charged by the insurer for the annuity depends on the interest rates available in the market, and the level of mortality expected.

Explain whether (and why) the insurer will have to charge a higher or lower premium if:

- interest rates rise

- mortality rates decrease

If interest rates rise, the discount rate will increase. If the discount rate increases, the present value of future cashflows will decrease. Thus, a lower premium can be charged. Another way of looking at this is to consider that with higher interest rates, the premium received will earn higher investment return (interest) and will thus grow faster to be able to pay higher benefits. For the same benefit, a lower premium can be charged.

If mortality rates decrease, policyholders are expected to survive for longer. Annuity payments are made as long as the policyholder survives. Thus, lower mortality implies more payments. The present value of these higher payments will be higher and the premium charged will have to increase.

To calculate the premium for any type of policy in the traditional actuarial way, we equate the expected present value of income to the expected present value of outgo. In this case we are only considering premiums and benefits.

Premiums are paid by the policyholder to the insurer. Benefits are paid by the insurer to the policyholder. Unless told otherwise, the premiums will be constant and denoted as P

The first step should always be:

$$\text{EPV}(\textit{Premiums}) = \text{EPV}(\textit{Benefits})$$

Now, we fill in the LHS and RHS separately with more detailed formulae.

$$LHS = \text{EPV}(\textit{Premiums}) = P$$

Since there is only one premium paid, and it is paid at the start of the contract, the EPV is simply P.

Now if the annual benefit payment (in arrears) is B,

$$RHS = \text{EPV}(\textit{Benefits}) = B \cdot a_x$$

Make sure you understand the above equation. Why is $B \cdot a_x$ *the expected present value of benefits?*

Next step is to complete the original equation:

$$\text{EPV}(\textit{Premiums}) = \text{EPV}(\textit{Benefits})$$
$$P = B \cdot a_x$$

Therefore the premium that the insurer should charge is simply $B \cdot a_x$.

What is the premium payable for a temporary annuity paid annually in advance for up to 25 years? Define all symbols used. Will this be smaller or greater than the premium for an immediate annuity paid in advance?

3.3 Assurance factors

3.3.1 *Assurance factor*

An **assurance factor** is the expected present value of a sum assured of 1 for a life aged x paid at the end of the year.

An assurance factor is the **E**xpected **P**resent **V**alue of a stream of payments of 1 in the year of death of the policyholder.

$$\text{EPV} = v^1 \cdot {}_0p_x \cdot q_x + v^2 \cdot {}_1p_x \cdot q_{x+1} + v^3 \cdot {}_2p_x \cdot q_{x+2} + \ldots$$

In practice, because our life table has a cut-off maximum age, we might express this as:

$$\text{EPV} = v^1 \cdot {}_0p_x \cdot q_x + v^2 \cdot {}_1p_x \cdot q_{x+1} + v^3 \cdot {}_2p_x \cdot q_{x+2} + \ldots + v^{\Omega - x + 1} \cdot {}_{\Omega - x}p_x \cdot q_\Omega$$

The standard actuarial notation for this is:

$$A_x = v^1 \cdot {}_0p_x \cdot q_x + v^2 \cdot {}_1p_x \cdot q_{x+1} + v^3 \cdot {}_2p_x \cdot q_{x+2} + \ldots + v^{\Omega - x + 1} \cdot {}_{\Omega - x}p_x \cdot q_\Omega$$

$$A_x = \sum_{i=0}^{\Omega - x} v^{i+1} \cdot {}_ip_x \cdot q_{x+i}$$

3.3.2 *Term Assurance factor*

$$A^{\,1}_{x:\overline{n}|} = \sum_{i=0}^{n-1} v^{i+1} \cdot {}_ip_x \cdot q_{x+i} = \sum_{i=1}^{n} v^i \cdot {}_{i-1}p_x \cdot q_{x+i-1}$$

Explain the above equation in words. What does the 1 over the x signify?

Confirm that you understand why both definitions (from i = 0 and from i = 1) are correct.

3.3.3 *Pure Endowment factor*

A pure endowment is a benefit that pays only only survival to time n. The EPV of the benefit is given by the following equation:

$$A_{x:\overline{n|}}^{\ \ \ \ 1} = v^n \cdot {}_n p_x$$

Who would want to buy such a policy?

Why might it be unpopular?

A policyholder with a specified financial need in future might purchase the policy as a savings vehicle, provided the financial need exists only for that policyholder and not for any dependants. An example might be

- Saving for retirement (with no family). If the policyholder dies, no savings are needed.
- Saving for university education. The education will not be needed if the policyholder is dead.

It is often unpopular because no benefit is paid on death. On death occurring shortly before maturity, a large number of premiums will have been paid but no benefit is received. Problems arise through poor selling processes where the policy may not be fully explained. Further, the family of the deceased policyholder may be unhappy that there is no benefit to be paid to them. As a result, these policies are usually not sold by insurers any more.

3.3.4 *Endowment Assurance factor*

An Endowment Assurance (often called just an “endowment” - for this course use the full name to prevent confusion) is a policy that pays a benefit at the end of year of death of the life assured, or at the end of year of maturity given survival to that period.

The EPV of benefits is given as:

$A_{x:\overline{n|}} = A_{\overset{1}{x}:\overline{n|}} + A_{x:\overline{n|}}^{\ \ \ \ 1}$ since it combines the benefit of the Term Assurance with the benefit of a pure endowment. One can also write:

$$A_{x:\overline{n|}} = \sum_{i=1}^{n} v^i \cdot {}_{i-1}p_x \cdot q_{x+i-1} + v^n \cdot {}_n p_x$$

Show that this can further be written as:

$$A_{x:\overline{n}|}=\sum_{i=1}^{n-1} v^i \cdot {}_{i-1}p_x \cdot q_{x+i-1} + v^n \cdot {}_{n-1}p_x$$

An n-year endowment assurance pays a benefit at the end of year of death for n-1 years. On survival to n-1 years, a benefit is paid on death or survival at the end of year n.

3.4 Guaranteed Endowments

A guaranteed endowment is not really an insurance policy. It pays a benefit at maturity on death or survival to maturity. The "guaranteed endowment factor" is simply v^n

Show that v^n is equal to $v^n \cdot {}_np_x + \sum_{i=0}^{n-1} {}_ip_x \cdot q_{x+i} \cdot v^n$

A guaranteed endowment pays a benefit on death before time n at the end of year n, and also on survival to time n.

3.5 Premium calculations for assurance benefits

Given an expression for the regular annual premium (paid in advance) for an endowment assurance policy.

Remember, start with $\text{EPV}(\textit{Premiums})=\text{EPV}(\textit{Benefits})$.

Give similar expressions for:

- regular annual premium (paid in advance) for a term assurance policy
- regular annual premium (paid in advance) for a pure endowment
- regular annual premium (paid in advance) for an endowment assurance policy
- regular annual premium (paid in advance) for a term assurance policy
- regular annual premium (paid in advance) for a pure endowment
- How does the single premium for a pure endowment policy compare with the amount you would need to invest in a bank account earning the same amount of interest, over the same period? Why is this?

3.6 Fractional year factors

All the above factors can be considered with payments more frequent than once per year. This will not be covered in this course to make room for other concepts.

3.7 Joint-life factors

Many of the above factors also exist in joint-life versions. These depend on the survival or death of one or both of two lives (typically a married couple). These are not frequently used in practice and have also been excluded from this course to allow other concepts to be covered in more detail.

Chapter 2 Introduction to reserves and liabilities

About:

This chapter introduces the concepts of actuarial reserving and explores the differences between reserving principles for solvency, and reserving / liability principles for financial reporting. The Actuarial Control Cycle will be explored as a framework for solving actuarial problems (including reserve calculations).

Chapter Contents

1 The Actuarial Control Cycle

1.1 Overview

The Actuarial Control Cycle ("ACC") is a problem solving tool used in actuarial work. There are control cycles in many professions and industries that follow a similar form. The key idea is that it is a **cycle** that controls how to ensure a systematic approach to ongoing problem solving.

It is a useful tool to solve actuarial problems in practice. It is also an extremely useful tool for developing answers for actuarial exams. It encourages thinking around the problem and following the problem through to its logical, long-term result.

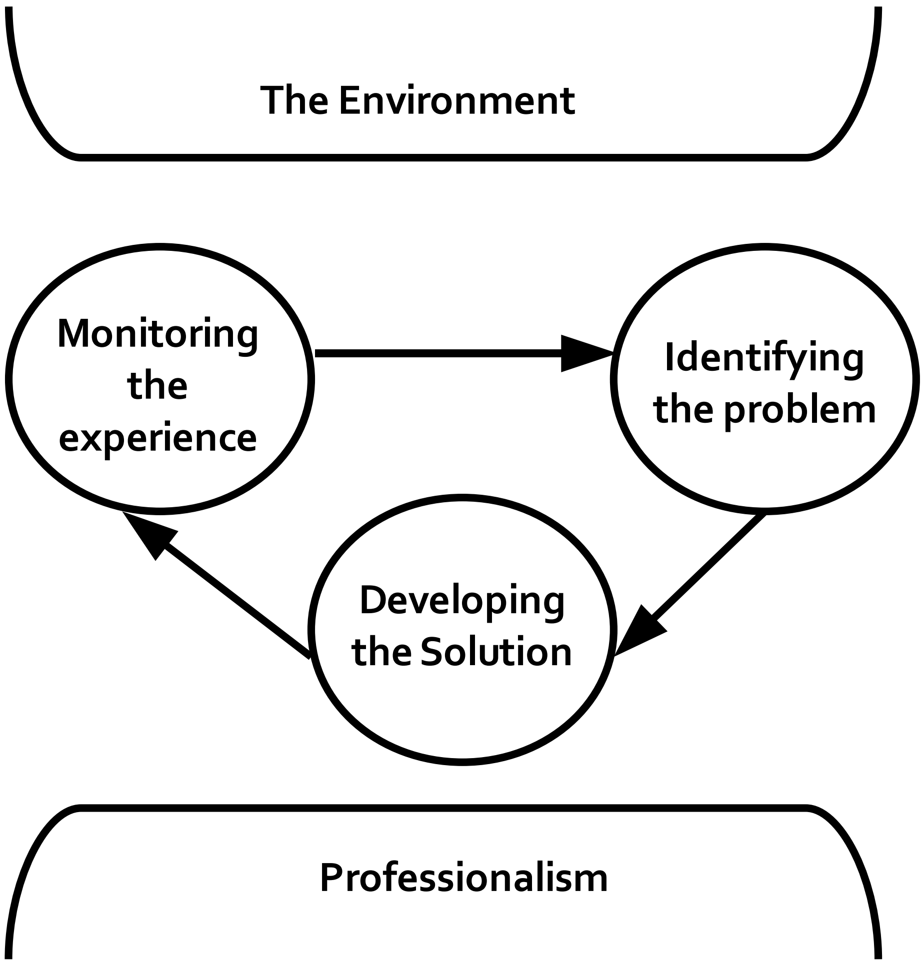


Illustration 2.1: The Actuarial Control Cycle

1.2 Identifying the problem (or "defining the problem")

The first step is to identify the problem. Sometimes this will be apparent from the real-life (or exam) problem, but sometimes significant work is required to understand what the problem is. This is an active step (rather than a passive step) that requires the actuary to consider the drivers of the problems, risks inherent in the problem, different stakeholders who may have an interest in the problem.

> Tip – think about why the problem isn't trivial to solve. What makes it difficult, why hasn't this problem already been solved? What solutions have already been tried? Who is happy with the current solution, and who is unhappy?

The problem may be:

- Our Term Assurance policies are not profitable
- What will future mortality rates be for our policyholders?
- Our business is exposed to several risk

This last problem will probably need to be devolved further by identifying each of the risks to identify specific problems. This identification process is part of the "Identify the Problem" step.

This step is also called "defining the problem" since to accurately identify the problem, it is necessary to properly define the key aspects of the problem.

1.3 Developing the solution

Once the problem has been properly identified and defined we need to work to solve it. This step is called "Developing the Solution". Given the information specified in the previous step, several solutions will be proposed, with flawed solutions being discarded. The problem characteristics identified in the previous step should be kept in mind as an effective solution will have to address these elements.

Solutions may involve:

- We will increase term assurance premium rates
- We will estimate future mortality based on past mortality experience, allowing for apparent trends. We will make use of reinsurers' data to supplement our data where it is not credible.
- We will make use of reinsurance and pooling to reduce risks to our business.

1.4 Monitoring the experience

This step is what make the ACC a "control cycle". Once we have identified the problem and developed a solution, it is critically important to monitor the experience. We monitor the results to ensure that any assumptions we made in developing the solution are reasonable. We look for unintended consequences and gather new information for the next step. The next step is to "Identify the Problem" or redefine the problem based on updated information from our experience, and given the solutions we have already implemented.

- We must monitor sales of term assurances. If our increased premium rates lead to significant lower volumes of business, our unit expenses may increase resulting in lower profitability.
- Future mortality may differ from past mortality. Our past experience may not have been a reliable estimate of future mortality. Mortality improvements (through improved medical care) or worse mortality (HIV/AIDS or pandemics such as bird flu) result in current and future mortality being very different from past mortality.
- Our reinsurance programme may not be sufficient and our business may still be exposed to too much risk. Alternatively, our reinsurance programme may be too extensive, with the result that our profits are too low.

1.5 Environment

This part of the ACC is not a step in the process. Rather, it is a factor that must be considered at each stage of the cycle. The specific elements considered will vary depending on the problem at hand, but this list gives an idea of the typical categories to consider:

- Economic environment (interest rates, inflation, economic growth, currency stability, GDP per capita, income inequality)
- Regularly environment (licensing requirements, asset admissibility rules, policyholder protection, reporting requirements, capital requirements, taxation rules)
- Business environment (number, size and strength of competitors, distribution channels, employee considerations)
- Political environment (political stability, potential for political interference in business, impact on the economic environment)
- Social environment (policyholder activism, treating customers fairly, socially responsible investing, limits on underwriting, limits on exclusions)

1.6 Professionalism

All aspects of actuarial work must be performed with utmost professionalism and ethics. Much of our work relates to individuals long-term savings and other critical events in their life such as the loss of a bread-winner for a family. Trust in placed in insurers companies to look after individuals in their time of need and actuaries are usually the custodians of this trust.

The following list represents just a few of the typical considerations under professionalism:

- Actuaries should consider the impact of their actions on all stakeholders.
- It may often be important to convey this understanding to non-actuaries in a clear manner.
- Actuaries should not perform work for which they are not qualified or in which they do not have sufficient experience.
- Actuaries should avoid conflicts of interest, and if it is necessary to work under conflicts of interest, then this should be communicated to all relevant parties.
- Actuaries must maintain the confidentiality of their clients.

You will receive a copy of the UK's Professional Conduct Standards which governs the professional conduct of members of the Institute and Faculty of Actuaries in the UK. It also applies to Fellows of the Actuarial Society of South Africa. The content is not examinable for this course, but you will be expected to consider aspects of professionalism where relevant in exam questions.

2 Risk management in context of Actuarial Control Cycle

2.1 Steps in Risk Management

- Risk Identification
- Risk Measurement
- Risk Management
- Risk Monitoring

Each of these steps is outlined in the sections below.

2.2 Risk Identification

Risk Identification is commonly performed through maintaining a **risk register**. The risk register is a structured record of all the important risks faced by the organisation. It can be difficult to list all relevant risks in a useful way, as many risks will interact with one another and these links must be made clear in the risk register.

2.3 Risk Measurement

The risks should then be measured. There are three primary methods used to measure risks.

2.3.1 *Detailed modelled based on actual data*

If reliable, credible data is available from the company's own experience, or through industry data, then the risks can be modelled in detail through fitting probability distributions to the magnitude and frequency of losses, allowing for correlations and dependencies between risks.

This is typically the case for insurance risks, persistency risks and market risks. Credit risks is increasingly being modelled accurately as banks collected relevant default data for Basel II compliance.

2.3.2 *Approximate modelling based on subjective estimates*

Some risks are difficult to model exactly. Credit risk is still difficult to model accurately for some classes of business. Operational risk is another that is typically measured in an approximate manner. This can be based on subjective estimation of distribution functions, or through an arbitrary factor.

2.3.3 *Qualitative assessment*

Some risks (often operational or catastrophe risks) cannot be measured at all with current information and tools. For these, a subjective assessment of the probability (or frequency or likelihood) and severity (or "impact") is made.

For example:

- an earthquake might be very infrequent, but have a severe impact
- Fraud by an employee might be fairly frequent, but low in impact.

2.4 Risk Management

This section is called "Risk Management" which is also the name of the overall process! However, it is in this section that the risks already identified and measured are acted upon. Some of the possible actions include:

- Avoid the risk
- Reduce the risk
- Mitigate the risk
- Transfer the risk
- Pool the risk
- Share the risk
- Hedge the risk
- Insure the risk
- Accept the risk

2.5 Risk Monitoring

As with the Actuarial Control Cycle, it is critically important that risks are monitored on a regular basis. Risks may change in nature over time, and unless the risk registers are regularly updated and the measurements re-performed, the risk management actions may no longer be appropriate.

Additionally, if the risks are regularly monitored, it serves to focus management's attention on risk management as a critical component of their job, rather than as something peripheral to their main function.

3 Cashflows arising from insurance products

3.1 Definition of cashflow

Cashflow is the transfer of cash from one entity to another. It only counts as a cashflow when it has moved or "flowed". Thus, cash at a bank is not a cashflow. Premiums flow from the policyholder to the insurance company and are thus a cashflow.

Profit is not a cashflow since no money flows anywhere. It is an accounting result.

3.2 Operating cashflows

Operating cashflows are those cashflows that occur during the normal operating activities of the company

3.2.1 *Cash in-flows*

- Premiums
- Investment income (interest and dividends, proceeds from sale of financial instruments)
- Reinsurance claims
- Reinsurance commission

3.2.2 *Cash out-flows*

- Claims
- Expenses
- Commission
- Reinsurance premiums
- Profit commission
- Tax

3.3 Cashflows of a capital nature

Some cashflows arise due to the company increasing or decreasing the amount of capital it has. These are not operational cashflows but are capital cashflows.

3.3.1 *Capital Cash in-flows*

- Issue Shares
- Issue Preferred Shares (or "preference shares")
- Issue bonds / debentures (borrow money from investors)

3.3.2 *Capital Cash out-flows*

- Pay dividends
- Pay interest on bonds / debentures
- Repay capital on bonds / debentures
- Buyback shares
- Make a capital distribution (pay out some share capital. Normal dividends are paid from retained earnings)

3.4 Mismatch in timing of cashflows in insurance products

For most insurance products, cash in-flows occur mostly at the start of the policy, and cash out-flows mostly at the end of the policy.

3.4.1 *Immediate annuity example*

- Single premium is received up front at t = 0
- Claims and expenses are paid every month or year from t = 1 for up to 30 or 40 years

3.4.2 *10 year Endowment Assurance example*

- Premiums are paid regularly from t = 0
- Small death claims are paid regularly from t = 1 to 10. These cashflows are much smaller than the premiums
- Expenses are paid regularly from t = 0 to 10. These cashflows are much smaller than the premiums
- At t = 10, a large maturity payment is made to the survivors. This is much larger than the premium received in the last year.

3.4.3 *Term Assurance example*

Term assurance has closer matching of in- and out-flows. The major mismatch is because the premium is level, but the risk increases over time due to increasing mortality rates.

- Level premiums are paid each year
- Expenses are paid each year
- Claims are small at the start of the policy, but increase over time

It is likely that

Premiums > Expenses + Benefits for early durations

Premiums < Expenses + Benefits for later durations

4 Actuarial reserves and liabilities

This section covers the need for actuarial reserves and liabilities, explains the difference between them and the types of reserves and liabilities calculated.

4.1 Actuarial reserves for solvency

As we saw in the previous section, cashflows for insurance company are "mismatched". That is, we receive cash before having to pay it out. Sometimes this difference can be many years. The case of the immediate annuity is an easy example. We receive cash upfront and then have to make payments for many years (sometimes 20 or 30 or even 40 years). It is important that we ensure we have enough money to be able to make these payments when they fall due.

An insurance company is solvent if it can meet is liabilities as they fall due.

Thus, actuaries calculated **the amount of money that must be held in reserve** to ensure that liabilities can be met as they fall due. It is common for this calculation to be **prudent** since we have a responsibility to our policyholders to look after them.

In general, the more prudence in the calculation of the solvency reserves the better.

4.2 Actuarial liabilities for financial reporting

Financial reporting is the disclosure of:

- the company's financial position (balance sheet); and
- financial performance (income statement)

to allow current and potential future shareholders to:

- assess the value of their shareholding;
- estimate how the company will perform in future; and
- make informed investment decisions about whether to buy, hold or sell the company

and to assist management to:

- make good operational and strategic decisions in the management of the company.

4.2.1 *External reporting*

Shareholders and investors need to know the current financial position and past financial performance of the company. They will also be interested in the solvency of the company (if the company is insolvent and cannot pay policyholders, there will be no money left for shareholders and other investors). However, they also need to know accurately what the balance sheet and income statement of the insurer looks like.

Since most insurance products have large cashflows upfront followed by negative cashflows at later durations, without calculating the liabilities due to policyholders and placing this on the balance sheet, insurance companies will show large profits in the early years of a policy, followed by large losses at the end of a policy. This clearly doesn't reflect the realistic scenario for the insurer.

4.2.2 *Internal reporting*

Internal management also need to know the accurate financial position of the company so that they can manage it effectively and make appropriate strategic and operational decisions. They need to know the assets and liabilities on the balance sheet, as well as how much profit the company makes as a whole, and how much profit is made by each business unit or product line.

4.3 Difference in purpose between actuarial reserves and actuarial liabilities

Actuarial reserves for solvency are intended to protect policyholders against the risk that the insurer becomes insolvent. Prudence is a good thing. More prudence is generally better.

Actuarial liabilities for financial reporting are to provide shareholders and other investors with accurate information about the financial position of the company. A small element of prudence is good to ensure that profits are recognised prematurely. However, too much prudence paints an unrealistic picture of the financial position of the insurance company and can be misleading.

This is an important difference in the purpose of solvency calculations and financial reporting.

A secondary difference is that financial reporting generally requires accurate information for individual business units and product lines to enable management to make appropriate decisions on this level. On the solvency side, as long as the entire company is solvent, we are less concerned (although not unconcerned) with individual product lines.

4.4 Actuarial Balance Sheet

The largest elements of an insurers' balance sheet are the actuarial reserves or liabilities. The illustration below gives an example of a life insurance company with total assets = 100 and total liabilities = 75.

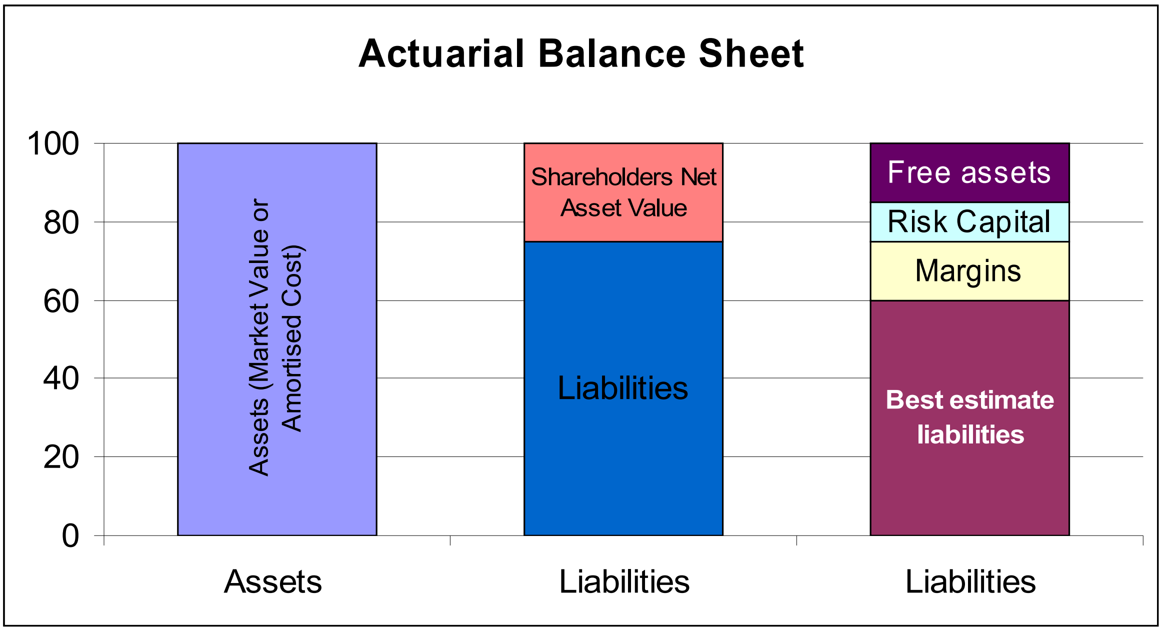


Illustration 2.2: Actuarial balance sheet

The two sides of the balance sheet are the assets and liabilities. The illustration above shows the assets on the left hand side, and two different breakdowns of the liabilities on the right hand side.

4.4.1 *Insurance company assets*

Insurance companies' assets can be valued at market value or some form of **historical cost** or **amortised cost.**

4.4.2 *Liabilities*

The middle column shows the most basic breakdown. 75 of liabilities and 25 of “shareholders' net asset value”, which is also known as “shareholders' equity” or “shareholders' capital”.

The column on the right provides more detail. Total liabilities of 75 consist of best estimate liabilities of 60 and **margins for prudence** of 15. The level of prudence will differ between companies, between countries, and may be different for solvency reserving and financial reporting liabilities. Prudence is usually introduced through conservative assumptions.

4.4.3 *Risk capital and free assets*

Risk capital is capital held as an additional buffer against fluctuations in experience and other risks to the insurer. Regulators may stipulate the minimum level of risk capital that must be held over and above the reserves. Risk capital belongs to shareholders, but may be needed to pay benefits to policies if experience is much worse than expected.

4.5 Typical actuarial reserves

4.5.1 *Prospective reserves*

- Net Premium Valuation
- Gross Premium Valuation or Realistic Discounted Cashflow Valuation

4.5.2 *Retrospective reserves*

- Unit fund reserves
- Unearned Premium reserves

4.5.3 *Approximate reserves*

Some reserves or liabilities are difficult to calculate, or there may not be sufficient information to calculate them accurately. In these cases, approximate methods may be used. This is often the case when the overall size of these reserves / liabilities is small.

Common approximations may include:

- multiple of premium
- multiple of sum assured
- retrospective accumulation of premiums, expenses and benefits

4.5.4 *Non-life reserves*

Non-life insurance generally does not used discounted cashflow reserves. The reserves commonly used for non-life insurance are:

- Unearned Premium Reserves (UPR) or Unearned Premium Provision (UPP)
- Additional Unexpired Risk Reserves (AURR) or Additional Unexpired Risk Provision (AURP) .
  - This is sometimes called the Unexpired Risk Reserve or (URR) but this is not technically correct
- Incurred But Not Reported Reserve
- Outstanding claims reserve.

4.6 Special reserves

In addition to the above reserves, special reserves are sometimes held to meet specific needs.

4.6.1 *Bad data reserves*

If there are concerns about the data on which the reserves are based (number of policies, premium, age etc.) then a **bad data reserve** may be held to reflect the potential additional outgo in future arising from corrections to the data.

A bad data reserve should not be held forever. The data problems should be fixed and the reserves **released.** When a reserve is no longer held it is said to be "released".

4.6.2 *Mismatch reserves*

A mismatch reserve may be held when assets and liabilities are not **matched**.

4.6.3 *Investment Option and Guarantee Guarantee Reserves*

Investment options and guarantees can present significant risks to an insurer. Many insurance company failures have been the result of problems with investment guarantees and options.

The reserves for this risks are usually calculated using **stochastic models** that explicitly model the possible variation in investment returns and interest rates.

4.6.4 *Non-Investment Option & Guarantee Reserves*

Options and guarantees that do not involve investment returns are also significant sources of risk. Some examples include:

- Guaranteed Insurability Benefit
- Optional premium increases
- Implicit option to lapse a policy

4.6.5 Regulatory special reserves

The insurance regulator may require additional special reserves to be held. These could be a percentage of premiums, or a fixed amount for each class of business.

Chapter 3 Actuarial reserving techniques for life insurance

About

This chapter covers the core techniques of actuarial reserving. We will cover the net premium valuation and gross premium valuation.

Chapter Contents

1 Practical Reserving Principles

1.1 Prospective valuation

In a prospective valuation, we take the reserve to be the present value of future cashflows.

If we invest the reserves in assets that earn the discount rate, then the reserve will be sufficient to meet future cashflows.

1.2 Assumptions required (the "basis")

In valuing the expected present value of future cashflows, we need to make certain assumptions regarding what these expected future cashflows will be. These assumptions are collectively called **the basis**.

1.2.1 *Mortality (q_x)*

Expected mortality rates for each age are required. Often, this will be different for males and females. These rates will be based on:

- past mortality experience of the insurance company
- industry data if this is available
- data from reinsurers
- mortality experience from similar countries
- known or expected trends, such as annuitant mortality improvements and deteriorating mortality from HIV/AIDS

1.2.2 *Discount rate (i)*

The discount rate (or valuation **rate**) is used to calculate the present value of cashflows for the reserves. It can be set in two ways:

1. The expected future investment return (or yield) on the assets backing the reserves
2. The market-consistent yield expected on similar instruments in the market.

If the assets backing the reserves closely match the liability cashflows, then approach 1 and 2 will produce the same result.

Note, the discount rate we are considered is the expected **future** investment return or yield. It does not necessarily reflect recent past investment returns and does not directly affect the actual interest received on the investments.

1.2.3 Expenses

Future expenses (commission, salaries, rent, depreciation, electricity etc.) are sometimes modelled explicitly (typically with a Gross Premium Valuation, but not for a Net Premium Valuation). An assumption regarding the expected level of future expenses per policy will be required.

1.2.4 Other decrements

A **decrement** is a change from one state to another. For example, death is a decrement from the state “alive” to the state “dead”. Other decrements that are of interest in insurance are:

- Lapse (the policyholder stops paying premiums so the contract is cancelled)
- Surrender (the policyholder stops paying premiums and receives a “surrender value” and the contract is cancelled)
- Paid-up (the policyholder stops paying premiums, by the policy continues under modified terms, usually with a lower Sum Assured)

None, some, or all of these decrements may be modelled in a Gross Premium Valuation.

1.2.5 Other assumptions

Depending on the product and valuation methodology, a range of other assumptions may be required.

2 Gross Premium Valuation

2.1 Definition of Gross Premium Valuation

A **Gross Premium Valuation** is a reserving technique that discounts the expected value of all future cashflows, including the actual premium paid by the policyholder (the **Gross Premium**) and any expenses expected to be paid in future.

We will use the symbol GP for the Gross Premium. In these examples, we model expenses as E payable annual in advance as a fixed amount per policy.

2.2 Impact of basis changes on gross premium valuation

If the basis is changed (e.g. Mortality, discount rate or expenses) then the calculation of the gross premium reserve must take these new assumptions into account. Thus, the basis change is directly incorporated into the reserve.

2.3 Examples of gross premium valuations

The reserve for a policy **in force** (still active) at time t is calculated as ${}_tV$ as shown for each regular premium product below.

2.3.1 *Whole of Life Assurance*

$${}_tV = SA \cdot A_{x+t} - GP \cdot \ddot{a}_{x+t} + E \cdot \ddot{a}_{x+t}$$

2.3.2 *Term Assurance*

$${}_tV = SA \cdot A^{1}_{x+t:\overline{n-t}|} - GP \cdot \ddot{a}_{x+t:\overline{n-t}|} + E \cdot \ddot{a}_{x+t:\overline{n-t}|}$$

2.3.3 *Endowment Assurance*

$${}_tV = SA \cdot A_{x+t:\overline{n-t}|} - GP \cdot \ddot{a}_{x+t:\overline{n-t}|} + E \cdot \ddot{a}_{x+t:\overline{n-t}|}$$

2.3.4 *Annuity*

For this annuity, we assume payments annually in arrears = Benefit, and expenses also incurred annually in arrears.

$${}_tV = (Benefit + E) \cdot a_{x+t}$$

2.4 Example of impact of basis changes on the gross premium valuation

Table 3.1.Basis change impact on gross premium valuation for Term Assurance

Basis change	Impact of basic change on GP	Impact of basis change on Reserve
Increase mortality	No change	Increase
Increase discount rate	No change	Decrease
Increase expenses	No change	Increase

3 Net Premium Valuation

3.1 Definition of Net Premium Valuation

A **Net Premium Valuation** is a reserving technique that discounts the expected value of future benefit cashflows and future **Net Premium** cashflows. The **Net Premium** is not the same as the Gross Premium actually paid by the insured. This is an important difference with the Net Premium Valuation.

We will use the symbol NP for the Net Premium. The Net Premium is calculated as the premium necessary, at the outset of the policy (i.e. Where t= 0) to meet future benefits.

No expenses are modelled as is the nature of the Net Premium Valuation.

3.2 Implicit allowance for expenses in net premium valuation

The net premium valuation does not explicitly allow for expenses. However, the net premium used is generally lower than the gross premium actually paid by the policyholder. The net premium is calculated without considering expenses, so it is lower by approximately the amount required to cover expenses.

When we use the net premium in the net premium valuation, because it is lower than the gross premium, the reserve is higher by approximately the same amount as the allowance for expenses in the gross premium.

Consider the example where NP = GP – E for a Whole of Life Assurance

Gross Premium Valuation $_tV = SA \cdot A_{x+t} - GP \cdot \ddot{a}_{x+t} + E \cdot \ddot{a}_{x+t}$

Net Premium Valuation $_tV = SA \cdot A_{x+t} - NP \cdot \ddot{a}_{x+t}$

which is also $_tV = SA \cdot A_{x+t} - GP \cdot \ddot{a}_{x+t} + (GP - NP) \cdot \ddot{a}_{x+t}$

which is also $_tV = SA \cdot A_{x+t} - GP \cdot \ddot{a}_{x+t} + E \cdot \ddot{a}_{x+t}$

And so the net premium valuation and gross premium valuation are equal if NP = GP – E. This is not always the case, particularly where the basis has changed since the policy was issued.

3.3 Impact of basis changes on net premium valuation

If the basis is changed (e.g. Mortality, discount rate or expenses) then the calculation of the net premium reserve must take these new assumptions into account. However, there are two impacts for every basis change:

1. The Net Premium is calculated on the new basis, as at the start of the contract using the standard formula. Thus, the Net Premium changes when we change the valuation basis (or the "reserving basis" or the set of assumptions used to calculate the reserve.
2. The Net Premium Valuation reserve is then calculated on the new basis, but also allowing for the recalculated Net Premium.

The impact of a particular basis change on the Net Premium will be opposite of the direct impact of the basis change on the reserve. This means that the overall impact on the net premium valuation will be less than that for a gross premium valuation. We will consider some examples of this later in this section.

3.4 Examples of net premium valuations

The reserve for a policy **in force** (still active) at time t is calculated as ${}_tV$ as shown for each regular premium product below.

3.4.1 *Whole of Life Assurance*

$$ {}_tV = SA \cdot A_{x+t} - NP \cdot \ddot{a}_{x+t} $$

$$ \text{where } NP = \frac{SA \cdot A_x}{\ddot{a}_x} $$

3.4.2 *Term Assurance*

$$ {}_tV = SA \cdot A^{1}_{x+t:\overline{n-t}|} - NP \cdot \ddot{a}_{x+t:\overline{n-t}|} $$

$$ \text{where } NP = \frac{SA \cdot A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} $$

3.4.3 *Endowment Assurance*

$$ {}_tV = SA \cdot A_{x+t:\overline{n-t}|} - NP \cdot \ddot{a}_{x+t:\overline{n-t}|} $$

$$ \text{where } NP = \frac{SA \cdot A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} $$

3.4.4 *Annuity*

For this annuity, we assume payments annually in arrears. However, because there is no premium to be valued, we cannot allow for expenses as the difference between gross and net premiums. Thus it is typical to allow explicitly for expenses even for the net premium valuation. Once method is given below.

$$_tV = (Benefit + E) \cdot a_{x+t}$$

3.5 Single premium policies under the net premium valuation

The net premium valuation allows implicitly for expenses through the difference in the net premium and the gross premium. However, for single premium policies, the premium is not a component of the reserve. Unless we make an adjustment to the net premium valuation when dealing with single premium policies, our reserves will not be sufficient.

3.5.1 *Single Premium Endowment Assurance*

$$_tV = SA \cdot A_{x+t:\overline{n-t}|} + \text{expense reserve}$$

The expense reserve could be calculated in different ways. A common (and sensible) approach would be as follows:

$$\text{expense reserve} = E \cdot \ddot{a}_{x+t:\overline{n-t}|}$$

In this case, the net premium valuation and gross premium valuation are the same.

3.6 Alternative interpretations of net premium valuation

Some practitioners also use a net premium valuation to be a valuation using:

$$\text{net premium} = \text{gross premium} - \text{expenses}$$

The difference from this and a traditional net premium valuation is that the net premium is calculated explicitly as gross premium less expenses, and that this premium is not recalculated if the basis changes.

This is not strictly a net premium valuation in the formal sense. There is also the requirement to test the net premium to ensure there is a sufficient gap between the net and gross premium to cover future expenses

3.7 Example of impact of basis changes on the net premium valuation

Table 3.2.Basis change impact on net premium valuation for Term Assurance

Basis change	Impact of basic change on NP	Impact of basis change (excluding NP change) on Reserve	Impact of NP change on Reserve	Combined Impact of basis change and NP change on Reserve
Increase mortality	Increase	Increase	Decrease	Small Increase
Increase discount rate	Decrease	Decrease	Increase	Small Decrease
Increase expenses	No change	No change	No change	No change

4 Retrospective versus prospective calculation of reserves

Both the Gross Premium Valuation and the Net Premium Valuation are **prospective** reserving techniques. Prospective means that we “look forward” to future cashflows and take the present values of these cashflows to calculate the current reserve.

A retrospective calculation of reserves considers past cashflows to estimate the current reserve. Under certain assumptions, we can show that the prospective and retrospective calculation will give the same result.

4.1 Assumptions required for equality

- Actual cashflows must be equal to expected cashflows under the basis used for the prospective valuation
- Premiums must have been calculated on the same basis as the valuation

4.2 Typical form of relationship

The equation below gives the typical form of the relationship. This version is for a whole of life assurance.

$$_{t+1}V = \frac{\left[\left({}_tV + P\right)\cdot(1+i) - SA\cdot q_{x+t}\right]}{p_{x+t}}$$

The equation below gives the same equation for an annuity with annual benefits paid in arrears.

$$_{t+1}V = \frac{{}_tV\cdot(1+i) - \text{Benefit}\cdot p_{x+t}}{p_{x+t}}$$

4.3 Intuitive interpretation of relationship

The reserves at time t, plus the premium received at the start of the year and investment return on the reserve and premium just received, is sufficient to pay claims during that year and set up the reserve required at the end of the year for all surviving policyholders.

4.4 Derivation of equality for a Whole of Life Assurance

Required to Prove:

$$_{t+1}V = \frac{\left[\left({}_tV + P\right)\cdot(1+i) - SA\cdot q_{x+t}\right]}{p_{x+t}}$$

Proof:

$$_{t+1}V = SA\cdot A_{x+t+1} - P\cdot \ddot{a}_{x+t+1}$$

$$_{t+1}V = SA\cdot\left[A_{x+t} - q_{x+t}/(1+i)\right]\cdot(1+i)/p_{x+t} - P\cdot\left(\ddot{a}_{x+t} - 1\right)\cdot(1+i)/p_{x+t}$$

$$_{t+1}V = \frac{\left[SA\cdot\left\{A_{x+t} - q_{x+t}/(1+i)\right\}\cdot(1+i) - P\cdot\left(\ddot{a}_{x+t} - 1\right)\cdot(1+i)\right]}{p_{x+t}}$$

$$_{t+1}V = \frac{(1+i)\cdot\left[\left(SA\cdot A_{x+t} - P\cdot a_{x+t}\right) + P - SA\cdot q_{x+t}\right]}{p_{x+t}}$$

$$_{t+1}V = \frac{\left[\left({}_tV + P\right)\cdot(1+i) - SA\cdot q_{x+t}\right]}{p_{x+t}}$$

5 Impact of reserves on profit and loss

5.1 New business strain

Reserves are often calculated on more **prudent** or **conservative** assumptions. This is to ensure that

- profits are not recognised prematurely
- the insurance company has sufficient funds to meet its obligations to policyholders as they fall due

If the reserving basis is more prudent that the basis used to calculate the premium, then the reserve at time 0 will be greater than zero. Thus leads to **new business strain**. New business strain is the loss incurred at the start of a policy due to a prudent reserving basis (and also initial expenses, although this is not covered here). This loss is temporary, and profits will emerge over time as the prudent assumptions turn out to be more conservative than actual experience.

Consider the example below:

A policy pays a benefit of 110 in one year's time, regardless of death or survival. The valuation discount rate is assumed to be 5% as this is a prudent assessment of the expected future investment returns available in the market.

However, the premium is calculated using a discount rate of 10% because this is a **best estimate** of the expected future investment returns available in the market. IT should be easy to see that the single premium payable at the start of the policy is:
$GP = 100$

The reserve at time 0 is ${}_0V = 110 \cdot (1.05)^{-1} - 100 \cdot 1 = 4.76 > 0$

Thus, before the first premium is received, we must create a reserve of 4.76. This will cause a loss of 4.76 at t = 0. This is the **new business strain**.

After the first premium is received, the reserve will simply be the present value of the benefit of 110 at 5%. This is $110 \cdot (1.05)^{-1} = 104.76$. This reserve will earn interest at 10% (not 5%, since 10% is our best estimate of actual investment returns).

At t = 1 the assets have grown from 104.76 to 115.24 ($115.24 = 104.76 \cdot 1.10$). We must pay the benefit of 110 at t= 1, which leaves us with 5.24 profit.

Thus, the new business strain was a temporary loss. It has been reversed by the profit of 5.24 at t = 1.

Importantly, the present value of the profit at t = 1 $5.24 \cdot (1.10)^{-1} = 4.76$ **which is the same as the new business strain or loss at t = 0.**

The present value of profits arising from the contract does not depend on the level of prudence within the reserve. It depends on the actual cashflows (premiums, claims and expenses) incurred. The reserves affect only the timing of the profits and losses.

5.2 Zillmerisation

5.2.1 *Impact of initial expenses on the Gross Premium*

Zillmerisation is a technique used to allow for initial expenses incurred at the outset of the policy. This is only necessary for the net premium valuation. For the gross premium valuation, the premium would be calculated as follows:

$$GP \cdot \ddot{a}_{x:\overline{n}|} = SA \cdot A_{x:\overline{n}|} + I + E \cdot \ddot{a}_{x:\overline{n}|}$$

$$GP = \frac{SA \cdot A_{x:\overline{n}|} + I}{\ddot{a}_{x:\overline{n}|}} + E$$

Where E is the ongoing expenses incurred at the start of every year, and I is the initial expenses incurred once at the outset of the policy.

5.2.2 *Zillmerised Net Premium Valuation*

The Zillmerised Net Premium Valuation is given as:

$$_tV = SA \cdot A_{x+t:\overline{n-t}|} - NP \cdot \ddot{a}_{x+t:\overline{n-t}|} - I \cdot \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}$$

$$\text{where } NP = \frac{SA \cdot A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

$$I = \text{initial expenses}$$

At t = 0, the third time in the above reserves equation = -I. As t approaches n, the third term in the reserve equation above approaches zero. **The reserve is decreased by the amount of initial expenses initially to reduce new business strain**. The adjustment reduces over time as t approaches n and the policy approaches maturity.

This can be intuitively understood as increasing the net premium to reflect the component of the Gross Premium that will be used to recover initial expenses is recognised in the Net Premium Valuation. Without this Zillmer Adjustment, the Net Premium will be too much lower than the gross premium and will result in very high new business strain.

5.3 Basis changes

5.3.1 *Gross Premium valuation*

The full impact of the basis changes is reflected in the liability valuation. This change flows directly through the income statement.

5.3.2 *Net Premium valuation*

The impact of the basis changes is reduced because the net premium itself also changes when the basis is changed. The basis change has a smaller impact on the income statement than the gross premium valuation.

5.3.3 *Impact on financial reporting*

If basis changes result in a large change in reserves or liabilities, then basis changes will have a large impact on the profit of the company for a particular year.

For example, if the total liabilities of a life insurer are USD10m and the average annual profit is USD400,000, then a 5% increase in liabilities resulting from a basis change (maybe an increase in mortality) will produce a USD500,000 increase in liabilities, which is a corresponding decrease in profits of USD500,000. This will erase all the profits for the year and make the insurance company make a loss.

1. Do you understand how a basis change can lead to a change in liabilities?
2. Do you understand how a change in liabilities will lead to a profit or loss in the period in which the basis change is made?
3. Do you understand why one needs to be careful not to change the basis unnecessarily?
4. Do you understand how basis changes might be used to manipulate profits?

1. Changing assumptions changes the expected present value of income and outgo. Since liabilities (and reserves) are calculated as the expected present value of outgo less the expected present value of income, the liability figure will change.
2. The "accounting equation" is A = E + L where A is assets, E is equity and L is liabilities. A change in L without a corresponding change in A must lead to a change in E. This change comes about through a profit or a loss.
3. Unnecessary basis changes will lead to inappropriate profits and losses, distorting the financial performance of the company. This is particularly relevant given the subjectivity involved in setting assumptions and issues of credibility of data and random fluctuations.

5.4 Pattern of reserves over time

We have already covered the graphs of ${}_tV$ over time in lectures.

Chapter 4 Unit and non-unit reserves

About

This chapter will not be covered in the course and as a result the notes provide only the outline of the methodology..

Chapter Contents

1 Unit-linked products

Unit-linked products (also known as Variable Universal Life products) are sophisticated life insurance products that combine risk cover and savings benefits in a flexible structure. The flexibility is two-fold:

1. The mix of risk benefits and savings contributions can be specified by the policyholder at inception of the policy, and often changed within limits during the term of the policy.
2. The investment vehicle(s) into which funds are invested can be selected by the policyholder at the outset, and often changed within limits during the term of the policy.

1.1 Charging structure

Premiums are paid by the policyholder. Of this premium, a certain proportion is allocated to the **unit fund**. The unit fund operates much like a bank account and fees / charges are deducted from the unit fund. Investment return is credited to the unit fund. The unit fund is also sometimes known as the **unit account**.

Death benefits are paid first out of the unit fund, and topped up by the insurer if the Sum Assured is greater than the value of the unit funds. Thus, the risk premium charged each month is calculated as $\text{Max}(\text{SA} - \text{Unit Fund}, 0) \cdot q_{x+t}$.

An annual asset management fee is also charged on a monthly basis. Policy fees, administration charges, advice levies, bid-offer spreads and a range of other charges may be levied.

One of the advantages of a unit-linked product is that the charging structure is **transparent** in that the policyholder can see exactly what charges are levied. However, in practice, policyholders often find that the charging structure is extremely complicated and difficult to understand. High charges will inevitably lead to poor value for money for policyholders.

The unallocated premium plus fees and charges plus risk premiums are income for the insurer. Expenses and death benefits in excess of the unit fund are outgo for the insurer.

1.2 Benefit flexibility

1.2.1 *Risk cover*

A wide range of benefits can be offered. Each benefit is charged with a separate risk premium. As long as the total premium paid by the policyholder is greater than the risk premiums (required for the risk benefits) and expenses, the remaining premium is added to the unit fund and will grow the unit fund over time.

1.2.2 *Maturity benefits*

At maturity, unless there is a guaranteed minimum maturity value (see investment guarantees below) the unit fund will be paid out as a maturity value.

1.2.3 *Surrender benefits*

Surrender values are usually paid as the unit fund less some deduction for

- unrecouped initial expenses incurred
- some amount of the future profit foregone by the insurer

Poor surrender values have been a major criticism of unit-linked policies around the world. This is exacerbated by complicated charging structures and poor communication of surrender terms.

1.3 Investment flexibility

The investment return credited to the policyholder is the actual investment return earned on the underlying funds as selected by the policyholder. Usually several different funds are offered such as:

- Balanced portfolio (equities, bonds, property and cash)
- Aggressive portfolio (mostly equity)
- Defensive portfolio (greater proportion of cash and bonds)
- Cash-only portfolio (investing in money market instruments up to 90 days)
- International portfolios (offering exposure to international currencies, multinational companies and the domestic companies of other countries)
- Sector fund (based on a particular sector of a market, e.g. financials or industrials or commodities)
- Absolute Return Funds / Hedge Funds

The funds can either be own-branded (and managed) funds or funds offered by other financial service providers. Some companies make use of “white labelled” where they repackage another company's fund under their own brand.

1.4 Investment Guarantees

Investment Guarantees can be offered, ensuring that at least a minimum return is offered. These will generally be charged for explicitly and require a great deal of care to ensure that the fee charged is appropriate given the risks involved. The risks will vary depending on the nature of the underlying fund selected by the policyholder so the fee charged must also depend on the underlying fund or combination of funds selected.

2 Unit Fund or Unit Reserves

2.1 Unit pricing

The unit account or unit fund is calculated as a **price per unit** or **unit price** and a number of units. The unit fund for a particular policyholder is:

$$UF_t = UP_t \cdot N_t$$

$$\begin{aligned} \text{where } UF_t &= \text{Unit Fund at time t} \\ UP_t &= \text{Unit Price at tme t} \\ N \quad &= \text{Number of units at time t} \end{aligned}$$

The Unit Price will be the same for all policyholders, but the number of units will differ so the unit fund will differ. Allocated premiums and charges are converted to numbers of units before being added or subtracted from the unit fund. Investment return increases the unit price of each unit.

This is the basic approach. The actual detail of unit pricing is complex and beyond the scope of this course.

2.2 Unit reserve

The unit fund or **unit reserve** is taken as the market value of the units.

$$UF_{t+1} = UF_t + AP_t - \text{charges}_t - RP_t \cdot (1 + \text{investment return})_t$$
$$\begin{aligned} \text{where } AP &= \text{Allocated Premium} \\ RP &= \text{Risk Premiums} \end{aligned}$$

Allocated Premiums are the proportion of the total Gross Premium paid by policyholders that is allocated to the unit fund. The unallocated premium is income for the company as is effectively another form of charge. The Risk Premiums charged are the actual risk premiums deducted from the unit fund to provide the risk benefits.

This formula is a very simplistic version based on a very simple charging structure. In practice, the calculation can be more complex.

3 Non-unit reserves

3.1 Introduction

Non-unit reserves are reserves held in respect of unit-linked business separate from the unit fund or unit reserve. It is calculated as the expected present value of non-unit cashflows such as

- unallocated premiums
- fees and charges
- expenses
- non-unit risks benefits paid.

Since some of the charges and benefits paid on unit-linked policies depend on the size of the unit fund, we need to project the unit fund into the future in order to calculate the expected future non-unit cashflows.

The total reserve for a unit-linked product is the sum of the unit reserve and the non-unit reserve.

Non-unit reserves can be positive or negative, but are usually negative. A negative non-unit reserve decreases the total reserve for a policy.

3.2 Market practice around the use of non-unit reserves

Since non-unit reserves are usually negative, it is not uncommon for a company to set the non-unit reserve to zero. There are two primary reasons for this:

1. Prudence. By not holding a "negative reserve" the overall reserves are higher and thus more prudent
2. Simplicity. The calculation of non-unit reserves is fairly complex.

3.3 Names for non-unit reserves

In many countries, non-unit reserve are named according the local currency. So, in the United Kingdom, non-unit reserves are called **Sterling Reserves** and in South Africa they are called **Rand Reserves**. The term **non-unit reserve** is generic and can be used in all markets.

3.4 Calculation principles

1. Using a set of assumptions, project the unit fund forward allowing for allocated premiums, charges, risk premiums and investment returns and taking decrements into account.

2. Calculate the expected future level of non-unit cashflows for all policyholders.

3. Calculate the present value of these expected future cashflows

- Why are non-unit reserves usually negative?
- What does a positive non-unit reserve imply?

Non-unit reserves are usually negative for two reasons:

- profitability products will have higher future income than outgo. If this difference is greater than any prudential margins included in the valuation basis, the non-unit reserves will be positive. An advanced concept, which is not examinable and will not be explained in detail is that of distortions caused by projecting and discounting the unit funds in a real world (rather than market consistent) manner.
- Initial expenses are usually large. The unit-linked product will have been designed to recoup these costs through future charges. These future charges are capitalised to offset the cost of the high initial expenses.

A positive non-unit reserve is usually an indicator of an unprofitable or very low profit product. This is unusual and the reasons behind positive non-unit reserves should be investigated.

Chapter 5 Actuarial reserving techniques for non-life insurance

About

This chapter covers the core techniques of actuarial reserving for non-life insurance. We will cover the material in less detail than for life insurance

Chapter Contents

Chapter 5 : Actuarial reserving techniques for non-life insurance

Section Chapter 5 : Actuarial reserving techniques for non-life insurance

1 Introduction to reserving for non-life companies

1.1 Differences between life insurance non-life insurance

Life insurance and non-life insurance are different in many respects. The table below outlines some of these differences.

Non-life Insurance	Life Insurance
▪ Policies are short-term (often one year)	▪ Policies are long-term, often ten years or more
▪ Claim frequencies are high	▪ Claim frequencies are very low
▪ More than one claim can occur on a policy	▪ Usually only one claim per policy
▪ Claim amounts are variable	▪ Claim amounts are usually fixed
▪ Policyholders can easily change insurer	▪ Policyholders can change insurer, but given the long-term nature of the policies it doesn't happen as often
▪ The insured person is usually the policy beneficiary	▪ The insured person is usually not the beneficiary
▪ Claims are usually for indemnity against loss	▪ Claims meet a general financial need
▪ Claim amounts are affected by inflation	▪ Claim amounts are generally fixed
▪ Policies do not provide a savings benefit	▪ Policies frequently combine risk cover and a savings element

Table 5.1.Comparison of life and non-life insurance

Group Life Assurance (GLA) policies have many of the same characteristics as non-life insurance. Thus, many of the techniques used for non-life reserving and also used for group life business.

1.2 Need for reserves / liabilities for non-life insurance

1.2.1 *Solvency*

Non-life insurance companies must hold reserves to ensure that they can meet policyholder liabilities (claim payments and associated expenses) as they fall due.

1.2.2 *Financial reporting*

As with life insurance companies, accurate assessment of the company's financial position and financial performance requires an assessment of the outstanding liabilities.

1.2.3 *Monitoring experience and re-pricing*

Accurate pricing is critically important for non-life insurance. Premium rates are regularly re-assessed based on claims experience. To measure recent claims experience accurately, the insurer needs an assessment of claims that have been incurred but have not yet been paid. If only paid claims are considered, the total claims incurred will be underestimated and premiums rates may be set too low.

1.3 Explicit versus implicit prudence

Prudence is usually introduced to non-life reserves in an implicit manner through:

- not discounting future cashflows
- subjective increases in case estimates of claims
- arbitrary increases in the size of reserves held (e.g. 10% higher)

2 Unearned Premium Reserves / Provisions

2.1 Introduction

Premiums are typically received in advance of paying claims. These premiums when received should not be immediately counted as profit as future claims would then lead to losses.

At any point in time, a certain proportion of premiums received will relate to risk cover to be provided in future. We consider these premiums as **unearned** because the insurance company has not yet **earned** the premiums. Once the risk cover is provided to the policyholders in future, and claims have been incurred, then the premiums will be considered to be earned.

In the simplest sense, if premiums are paid annually in advance on 1 January each year, half the premiums will have been earned by 30 June. The other half of the premium will be unearned.

2.2 Distribution of risk over policy term

For most non-life insurance policies, risks are uniform over the lifetime of the policy. This means that the probability of a claim, and the expected size of claims, is constant over the term of the policy.

Some exceptions include:

- Crop insurance.
  Crop (wheat, grapes, fruit) insurance is highly dependent on weather conditions which are seasonal.

- Motor vehicle accident insurance
  In some countries, hail, snow and other adverse weather conditions lead to higher claims in certain seasons (typically winter).

Some examples where claims are uniform throughout the year:

- Motor vehicle theft insurance
  The risk of theft of a motor vehicle will usually be uniform over the term of a policy.

- Public liability insurance

Will buildings and household contents insurance (perils of fire, flood, water damage, subsidence etc.) depend on the time of year? As an exercise, give some examples of claim types that will depend on time of year, and some that are uniform.

Flood and to some extent subsidence claims are usually season-dependent. Some fires may be created from heating appliances used in the house during winter. Theft and most other fires should be uniform throughout the year.

If houses are frequently left unoccupied for extended periods (summer houses, or houses during extended holiday periods when families may vacate the house for several weeks at a time) may be correlated with an increase in theft claims.

2.3 UPR under uniform distribution of risk

When risk is distributed uniformly over the term of the policy, the unearned premium is taken as the proportion of the premium payment interval that lies in the future for the most recently paid premium.

2.3.1 365^{ths} method

The 365^{ths} method consider the proportion of premium unearned measured in days (a year divided into 365 days).

$$\text{UPR} = \left[\frac{\text{days until next premium}}{\text{days from last premium to next premium}} \right] \cdot \text{Premium}$$

2.3.2 24^{ths} method

The 24^{ths} method consider the proportion of premium unearned measured in half-months (a year divided into 24 half-months).

This method is an approximation to the 365^{ths} method. Premiums are usually assumed to be received halfway through a month rather that on the exact date of premium receipt.

2.4 UPR under non-uniform distribution of risk

If risk is not uniformly distributed over the term of the policy, the UPR needs to be adjusted to reflect not only the proportion of time remaining for the most recently paid premium, but the proportion of risk.

Example:

A policy has annual premiums (USD1,000) paid in advance on the 1st of January each year. Risk increases linearly over the lifetime of the policy. What is the UPR at 30 June of each year?

$\text{Risk}\,(t) = c \cdot t$ Where c is the gradient and t is the time in years

Total risk for the policy year is:

$$\begin{aligned}\text{Total Risk} &= \int_0^1 \text{risk}\,(t)\,dt \\ &= \int_0^1 c \cdot t\,dt \\ &= \left[\frac{c \cdot t^2}{2}\right]_0^1 \\ &= \frac{c}{2}\end{aligned}$$

$$\begin{aligned}\text{Risk remaining @ t = 0.5} &= \int_{0.5}^1 \text{risk}\,(t)\,dt \\ &= \left[\frac{c \cdot t^2}{2}\right]_{0.5}^1 \\ &= \frac{c}{2} - \frac{c}{8} = \frac{3}{8} \cdot c\end{aligned}$$

$$\text{Percentage risk remaining @ t = 0.5} = \frac{c \cdot 3/8}{c/2} = \frac{6}{8} = 75\,\%$$

So the UPR required is USD750

This can be expressed graphically as the area under the function of risk as a function of time as shown below:

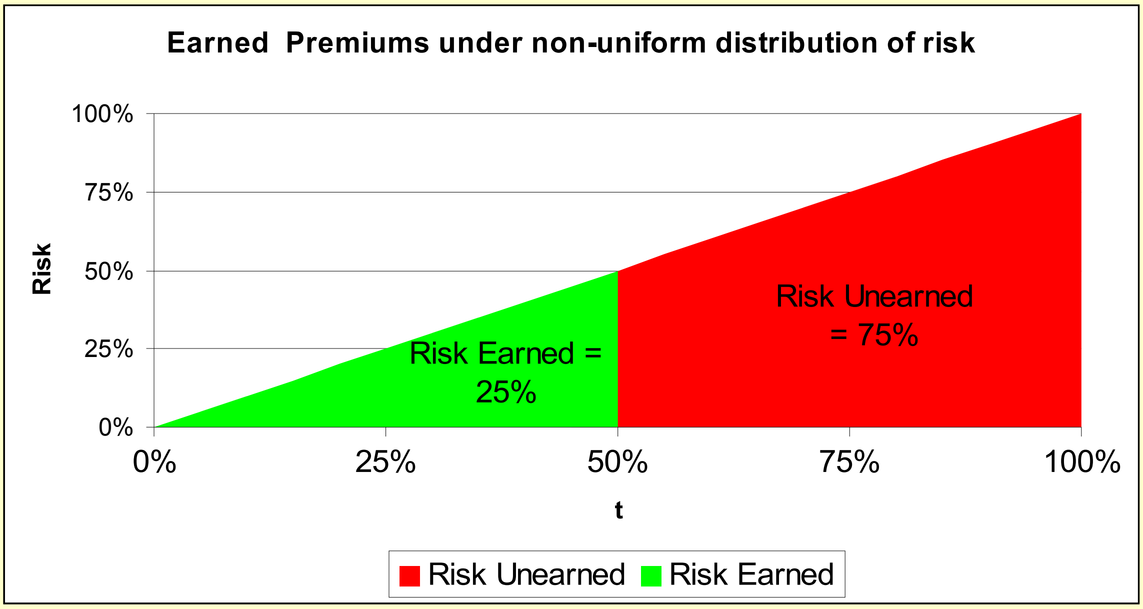


Illustration 5.1: Graphical representation of unearned risk under non-uniform distribution of risk

As an exercise, calculate the proportion of premiums unearned at 30 September if:

- $\text{Risk}(t) = e^{c \cdot t}$
- $\text{Risk}(t) = -t^2 + t$

- $\text{Risk}(t) = e^{c \cdot t}$ therefore $\text{UPR} = \dfrac{e^{0.75c} - 1}{e^c - 1} \cdot \text{premium}$
- $\text{Risk}(t) = -t^2 + t$ therefore $\text{UPR} = \left(-2 \cdot 0.75^3 + 3 \cdot 0.75^2\right) \cdot \text{premium}$

2.5 UPR for monthly premiums

For monthly premium policies, the UPR at the end of the month will always be zero.

Check that you understand why this is true.

3 Additional Unexpired Risk Reserves / Provisions

3.1 Need for an AURR

If the future expected claims are higher than the UPR calculated as described in the previous section, the insurance company will need to hold additional reserves.

This usually occurs for one of two reasons:

1. Premium rates are too low and will not be sufficient to meet normal levels of future claims
2. Significant additional risks have arisen since the premiums were set. For example, court rulings may create a new precedent that will expose the insurer to higher claims in future. Alternatively, an unexpected increase in vehicle replacement parts may mean that future motor accident claims will be more costly to pay.

3.2 Calculation of the Unexpired Risk Reserve

The Unexpired Risk Reserve or URR is the reserve for unexpired risks. Unexpired risks are risks arising from current policies. This is an estimate of future claim payments for which the insurance company has a legal obligation to pay. This will reflect the remaining term on existing policies as the insurer is not obligated to renew policies and may increase the premium rates on policy anniversary in most cases.

Typically, as is the case with most non-life insurance reserves, future claims are not discounted in this calculation as an implicit form of prudence. Although it might be considered more accurate to discount claims and use an explicit amount of prudence, this is not market practice.

3.3 Calculation of the AURR

The calculation of the AURR is straightforward:

$$\text{AURR} = \text{Max}\left(\text{URR} - \text{UPR}\,,\,0\right)$$

If the URR is less than the UPR, no AURR is held.

3.4 Alternative naming conventions

Some insurance companies use the term URR to mean AURR.

4 Outstanding Claims Reserves / Provisions

4.1 Introduction

The process to pay a claim begins once a claim has been reported to the insurance company. However, there may be a delay of a few days to several years from the date the claim is reported to the date the claim is finally settled.

Motor accident claims will be paid within a short period of time. The delays here are caused by the need to examine the vehicle for damage, assess culpability and liability, repair the damage and for the final claim amount to be determined.

For product liability claims, it may take several years for the courts to decide what liability is owed by the insured.

Once the claim has been reported to the insurance company, the insurance company must create an **outstanding claims reserve (OCR)** based on the expected claim amount to be paid. There are two primary methods of assessing outstanding claims reserves.

4.2 Case estimates

Specially trained (and experienced) claims assessors or "loss adjusters" consider the details of each claim on an individual basis and estimate the size of the claim. This relies on the expertise of the claims assessor.

This approach is useful when:

- There are a small number of claims so the workload of assessing individual claims is manageable.
- The claims are large or unusual in nature so human judgement is required.
- The claims can be accurately assessed based on the details of the claim. For example, for a specific model vehicle, the cost of replacing a specific part or repairing a specific body panel should be known with high certainty.

4.3 Statistical methods

Statistical methods are used when there is a very large number of claims. Considerable time would be required to estimate claims individually, and the impact of errors in any individual claim estimate should be small given the large number of claims.

The method might allow an average claim amount per claim type, policy type and peril type. More complicated approaches can be adopted if required.

Only very large insurers with very large, homogeneous books use statistical methods.

Example:

Average claim amount per claim estimated over recent years has been USD250. We have 34 reported claims that have not yet been paid. What is the total OCR?

$$\text{OCR} = 34 \cdot 250 = USD\ 8{,}500$$

4.4 Combined methods

In some cases, both methods are used together. For example, claims assessors may make case estimates for each individual claim. The insurance company may then use statistical methods to adjust these estimates for inflation or assessor bias.

5 Incurred But Not Reported Reserves / Provisions

5.1 Introduction

Claims are not always reported to the insurance company immediately. Delays may be:

- Several days for a motor vehicle accident
- Several weeks for a house burglary while the occupants were away on holiday
- Several years for product liability where the product defect is not known for several years.

At any given valuation date (the date at which we calculate reserves) there will be an unknown number of claims that have been incurred (i.e. have happened) but have not been reported. Each claims will have an unknown size. This poses a seemingly difficult question, “How do you reserve for claims that you don't know exist?”

This is the subject of this section. These claims are called **Incurred But Not Reported** claims or simply IBNR claims. The reserves for these claims are known strictly as **IBNR Reserves** but often simply **IBNRs**.

5.2 Percentage of premium approach

The most basic approach is to calculate IBNRs as a fixed percentage of premium. Given that claims will generally increase with increased premium volumes (more policies, larger risks), this is a reasonable simple approach. The problem is that this approach alone does not help us to assess *what percentage of premiums to use.*

5.3 Loss ratio approach

The simplest approach that does help to assess the absolute level of the IBNR reserve is called the **loss ratio approach**. A Loss Ratio (LR) is the ratio of total claims incurred to total premiums earned over a certain period for a certain line of business of other grouping.

$$LR = \frac{\text{Claims Incurred}}{\text{Premiums Earned}}$$

The LR approach assumes that claims (losses) can be expressed as a stable percentage of premiums. This is typically the way policies are priced, and for large portfolios of business the LR may be relatively stable. The LR will usually be estimated in conjunction with the pricing and underwriting department(s).

Year	Earned Premiums	Assumed Loss Ratio	Expected Claims	Reported Claims	Unreported Claims
200x	*EP*	*LR*	*EC= EP * LR*	*RC*	*IBNR = EC - RC*
2004	50,430	60%	30,258	30,100	158
2005	67,800	60%	40,680	38,400	2,280
2006	66,300	70%	46,410	41,310	5,100
2007	75,070	55%	41,289	23,650	17,639
			Total IBNR		**25,177**

Table 5.2: Loss Ratio approach to calculating IBNR reserve

The table above shows the typical way in which the LR approach is applied.

5.4 Basic chain ladder methods

The Basic Chain Ladder (BCL) method is a more sophisticated method of estimating IBNR reserves.

5.4.1 Assumptions required

- The past development pattern of claims will continue in future. (Future reporting delays will be similar to past reporting delays)
- Future claims inflation will be a weighted average of past claims inflation
- The past claims experience on which the reporting delays are based includes at least one year that is fully run-off. (At least one past year of claims must be 100% reported)

5.5 Average cost per claim method

Another method often adopted is called the **average cost per claim**. It is similar to the BCL method in that it assumes past development patterns (reporting delays) will continue in future. However, it assumes the past development pattern **measured in terms of number of claims** will be constant in future and similar to the past. (The BCL assumes past development patterns measured in amounts of claims will be constant in future and similar to the past).

The reserve is calculated in two steps:

1. Using the past development pattern, calculate how many unreported claims have been incurred
2. Separately, calculate an average cost per claim based on past claims information. This figure may be adjusted if necessary to reflect known changes in claim sizes due to inflation, court precedents, changes in policy conditions etc.

More basic methods are sometimes used for step 1 such as subjective assessment of the number of unreported claims, or the average number of unreported claims over similar periods in the past.

5.5.1 *Overview of approach*

The approach can be separated into distinct steps. We will consider the example set at 31 December 2006. Note that in this example we only use three years of claims information. In practice more detail could be (and generally would be) used.

1. Tabulate reported claims year in which claim was incurred, and the number of years delay to the claim being reported.
   $C_{\text{claim year, reporting delay}}$ are shown in their correct positions in the table below

Table 5.3.Tabulation of claims by year of incidence and reporting delay

Reporting Delay	**0**	**1**	**2**
Claim Year			
2004	$C_{2004,0}$	$C_{2004,1}$	$C_{2004,0}$
2005	$C_{2005,0}$	$C_{2005,1}$	Not available
2006	$C_{2006,0}$	Not available	Not available

The "not available" cells are not yet available since (at the end of year 2006) we do not know about claims that will be reported in 2007 and 2008.

2. Then we cumulate the claims in a similar table where each column includes the total claims reported up to that point as shown in the table below.

   Table 5.4.Tabulation of cumulative claims by year of incidence and reporting delay

Reporting Delay	**0**	**1**	**2**
Claim Year			
2004	$C_{2004,0}$	$C_{2004,0}+C_{2004,1}$	$C_{2004,0}+C_{2004,1} + C_{2004,2}$
2005	$C_{2005,0}$	$C_{2005,0}+C_{2005,1}$	Not available
2006	$C_{2006,0}$	Not available	Not available

3. Once we have the cumulative claims reported for each year and each reporting delay, we can calculate "link ratios" that link the claims reported by reporting delay x to the claims reported by reporting year x+1.

$$link_{0,1} = \frac{C_{2004,1} + C_{2005,1}}{C_{2004,0} + C_{2005,0}}$$

$$link_{1,2} = \frac{C_{2004,2}}{C_{2004,1}}$$

   There are fewer data points available for $link_{1,2}$ than for $link_{0,1}$ because of the triangular nature of the information available.

4. We can use the link ratios to estimate the empty cells of the cumulative claims table:

   Table 5.5.Tabulation of cumulative claims by year of incidence and reporting delay

Reporting Delay	**0**	**1**	**2**
Claim Year			
2004	$C_{2004,0}$	$C_{2004,0}+C_{2004,1}$	$C_{2004,0}+C_{2004,1}+C_{2004,2}$
2005	$C_{2005,0}$	$C_{2005,0}+C_{2005,1}$	$(C_{2005,0}+C_{2005,1})$ x $link_{1,2}$
2006	$C_{2006,0}$	$C_{2006,0}$ x $link_{0,1}$	$C_{2006,0}$ x $link_{0,1}$ x $link_{1,2}$

5. The right-most column now includes estimates of total claims incurred (provided the first row is fully "run-off"). If we subtract total reported claims from the total estimated claims incurred, we get an estimate of total IBNR claims.

6. A slight modification of the approach is to calculate **cumulative development factors** (**cdf**)based on the product link ratios. The calculations are algebraically equivalent, but the cumulative development factors are easier to work with and have another useful **interpretation.**

$$\text{cdf}_0 = link_{0,1} \cdot link_{1,2}$$
$$\text{cdf}_1 = link_{1,2}$$

7. $\dfrac{1}{\text{cdf}_t}$ = percentage of claims reported by time t.

5.5.2 *More detailed worked example*

Make sure you receive a copy of the more detailed worked example in an electronic spreadsheet.

5.5.3 *Problems with BCL method*

The BCL method works well if the assumptions underlying it are appropriate.

1. When might the assumption of stable development patterns not hold?
2. When might the assumption relating to inflation not hold?

1. The development pattern will change if any of the factors affecting any of the causes of delays change. For example, different brokers may report claims with different delays, so changes in brokers or the amount of business arising from certain brokers could change the pattern. Administration systems and processes may change resulting in shorter or longer delays (usually longer in the short term, and hopefully shorter in the long-term as the benefits of the system change are recognised). Outsourcing of administration functions can have a dramatic change in the reporting delays.
2. If the economy has moved from a high inflation phase to a low inflation phase or vice versa. This has happened to many developing economies as tighter monetary policy and more careful fiscal policy have led to greater stability and lower inflation.

5.6 Bornhuetter-Ferguson.

More advanced methods exist for estimate IBNR reserves. None of these will be covered in this course. They are generally covered in a specialised non-life course.

The most commonly used advanced method is known as the **Bornhuetter-Ferguson method** or "BF" method. It uses Bayesian techniques to combine estimates using the loss-ratio approach and the basic chain ladder method based on the credibility of the available data.

Chapter 6 Assets and Liability considerations

About

This chapter covers the some basic principles for considering assets and liabilities together. The major aim of this section is to introduce the concepts of matching and immunisation.

Chapter Contents

1 Yield Curves

A yield curve is a function (or graph, if presented graphically) of the **Yield To Maturity** (YTM) for fixed interest securities with term to maturity T.

An example is given below:

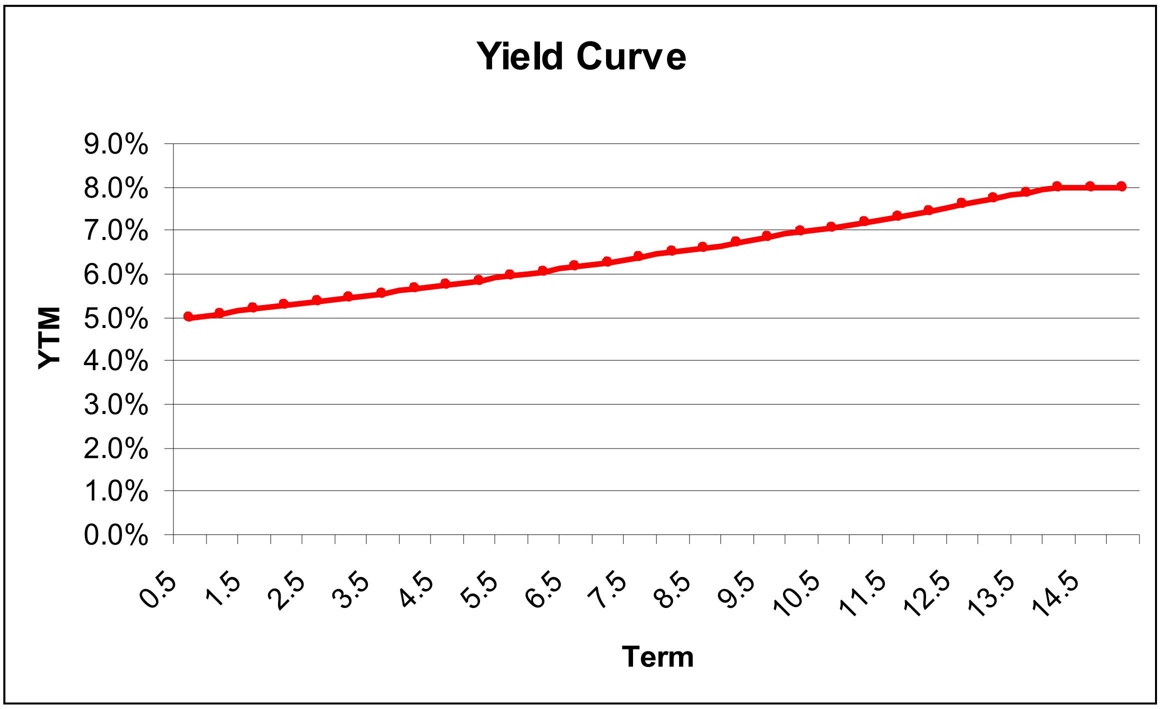


Illustration 6.1: Example of a Yield Curve

1.1 Factors affecting the shape of the yield curve

This have been covered in lectures and are only listed below:

- Expectations of future short-term interest rates
- Preference for liquid (short-term) instruments
- Market Segmentation

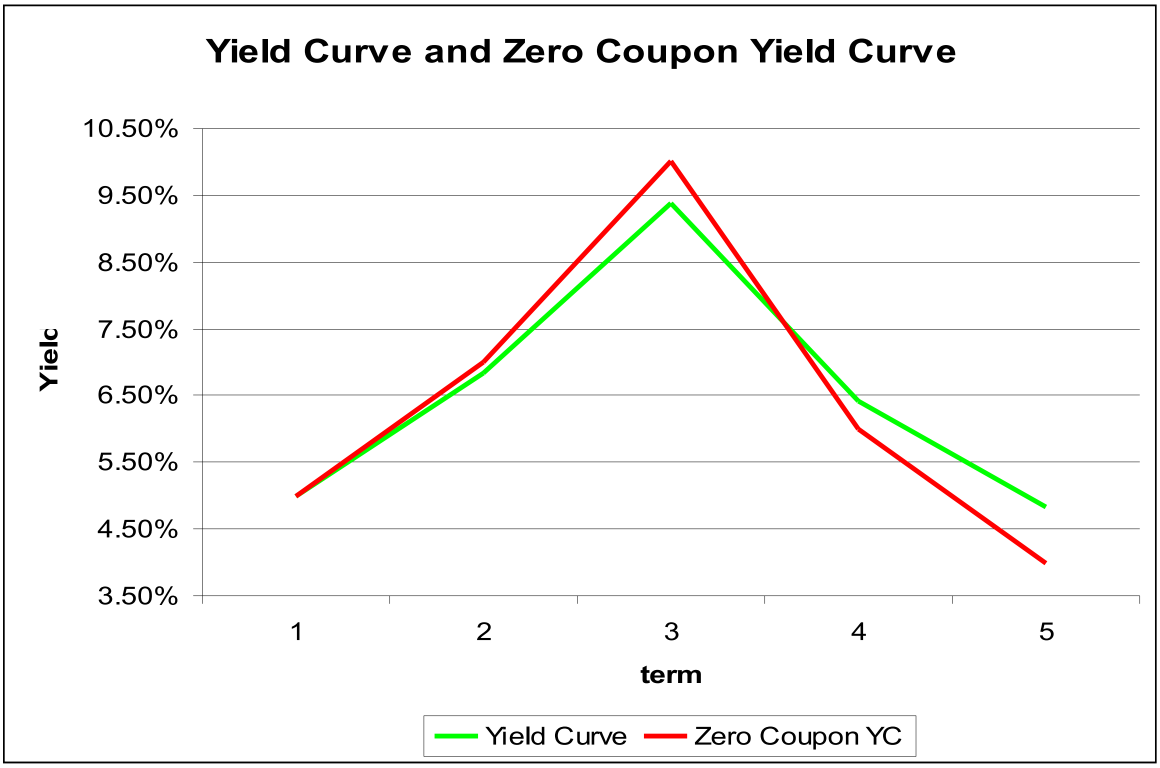
A fourth factor, called Market Preference, is sometimes included. It is a weaker form of the Market Segmentation theory, where different investors in the market have a preference for instruments of a certain term. This contrasts with Market Segmentation where the investors **only** purchase instruments within a particular range of terms.

1.2 Zero Coupon Yield Curve

A normal yield curve shows the YTM for a bond with a given maturity date. However, the YTM is the single discount rate used to discount all the cashflows (coupons and principal) for a **given bond**.

A **zero coupon yield curve** (sometimes known simply as a **zero curve**) shows the yield to maturity on zero coupon bonds for the given maturity. This YTM is applies only to cashflows of a specific term. Thus, the rates from the zero coupon yield curve can be used to discount specific cashflows of the appropriate term.

Illustration 6.2: Graph comparing a zero coupon yield curve against a yield curve.



Note that the slope (gradient) of the Zero Coupon Yield Curve is greater in magnitude than the Yield Curve. The Yield Curve can be thought of as a form of weighted average of the Zero rates, where the weights are related to the size of the cashflows (coupons and principal).

Coupon bonds can also be priced using the zero coupon yield curve by discounting each cashflow (coupon and principal again) at the appropriate rate from the zero coupon yield curve.

2 Matching of assets and liabilities

2.1 Introduction to matching

Matching assets and liabilities means to choose assets with the same characteristics as the liabilities to reduce risk. Assets and liabilities are both financial instruments and may vary in value independently. If the assets and liabilities have similar characteristics, we would expect them to vary in a similar fashion to changes in external conditions.

The three primary characteristics considered are:

1. Term
2. Currency
3. Nature (fixed or real)

However, there may be other factors such as sensitivity to particular market prices that could be considered.

The concept of matching is similar to that of hedging.

2.2 Perfect cashflow matching and the Law of Once Price

If we have two instruments, A and B, which produce the exact same cashflows as each other under all circumstances, they must have the same price. This is known as the **Law of One Price**.

This should be fairly obvious, but it can be explained further by considering what might happen if A and B weren't priced the same.

Assume $P_A \geq P_B$

Then, we purchase instrument B and sell instrument A. Since $P_A \geq P_B$, this yields profit $\pi = P_A - P_B$. Since all future cashflows from B will perfectly match cashflows required to pay to the owner of A, we cannot make a loss in future.

Thus, we have made **risk-free profit**. The profit we made, π , is made with no risk. This breaks the **no arbitrage assumption**.

Further, we could continue to buy B and sell A to make profit over and over again. However, buying B would increase the price of B, and selling A would decrease the price of A. Eventually, the price of A and B would be equal. Since there is no risk to this activity, anybody who found the price difference would immediately buy as much of B as possible, and sell as much of A as possible. Thus, the pressure to equalise prices A and B would act very quickly and with great force to make the prices equal.

2.3 Difficulties of perfect cashflow matching

In practice, it is often difficult to find two securities with the exact same cashflows under all circumstances. In this case, we make use of more approximate matching techniques. We attempt to match the currency, nature and average term of cashflows. One should also try to match the **convexity** of the instruments as closely as possible.

Convexity is covered in later sections in this Chapter.

1. Why is it important to match annuity cashflows?
2. Why is it less important to match term assurance cashflows?

1. Annuities have a high DMT and large reserves. Changes in interest rates give rise to a large percentage change in the reserves, and a large absolute change since the reserves are large.
2. Term Assurances generally have fairly small DMT and small reserves. A change in interest rates gives rise to a smaller percentage change in reserves than annuities, and as the reserves themselves are usually very small compared with annuities, the absolute financial impact is low.

3 Measures of the term of financial instruments

3.1 Discounted Mean Term

The **Discount Mean Term (DMT)** is the average term of the financial instrument, where the term of each cashflow is weighted by the Present Value of that cashflow.

Below we give the derivation of DMT for an N-year bond with annual coupons C payable in arrears.

$$\mathrm{P}(i) = \text{price of instrument}$$
$$\mathrm{P}(i) = \left(\sum_{t=1}^{N} C \cdot v^t\right) + 100 \cdot v^n$$

$$\mathrm{DMT}(i) = \frac{\sum_{t=1}^{N} t \cdot w_t}{\sum_{t=1}^{N} w_t}$$

$$w_t = C \cdot v^t \text{ for } t \in [1, N-1]$$
$$w_N = C \cdot v^N + 100 \cdot v^N$$

$$\mathrm{DMT}(i) = \frac{\left[\left(\sum_{t=1}^{N} t \cdot C \cdot v^t\right) + n \cdot 100 \cdot v^n\right]}{\left[\left(\sum_{t=1}^{N} C \cdot v^t\right) + 100 \cdot v^n\right]}$$

From the equations above, it should be clear than the DMT of an N-year **zero coupon bond** is N.

A zero coupon bond is a bond that does not pay coupons. It only repays the principal on maturity. Thus, the formula for the price of a zero coupon bond is simply $\mathrm{P} = 100 \cdot v^N$

3.2 Volatility

Volatility in this sense means the sensitivity of price to interest rates. It is related to, but not the same as, volatility in the sense of random variability of market prices of financial instruments.

3.2.1 *General form*

$$\mathrm{P}(i) = \text{price of instrument}$$

$$\frac{\partial \mathrm{P}(i)}{\partial i} / \mathrm{P}(i) = \frac{\mathrm{P}'(i)}{\mathrm{P}(i)} = \text{volatility}$$

3.2.2 *Example for a bond*

Consider an n-year bond with coupons C payable annually in arrears.

$$\mathrm{P}(i) = \left(\sum_{t=1}^{N} C \cdot v^{t}\right) + 100 \cdot v^{n}$$

$$\frac{\partial \mathrm{P}(i)}{\partial i} / \mathrm{P}(i) = \frac{\mathrm{P}'(i)}{\mathrm{P}(i)} = \frac{-\left[\left(\sum_{t=1}^{N} t \cdot C \cdot v^{t+1}\right) + n \cdot 100 \cdot v^{n+1}\right]}{\left[\left(\sum_{t=1}^{N} C \cdot v^{t}\right) + 100 \cdot v^{n}\right]}$$

$$= \frac{-\left[\left(\sum_{t=1}^{N} t \cdot C \cdot v^{t}\right) + n \cdot 100 \cdot v^{n}\right]}{(1+i) \cdot \left[\left(\sum_{t=1}^{N} C \cdot v^{t}\right) + 100 \cdot v^{n}\right]}$$

We obtain the derivative through use of the chain rule.

Taking the derivative requires some care because $v = \frac{1}{(1+i)}$

3.3 Comparison of DMT and volatility

DMT and volatility are similar measures. However, they are defined slightly differently and differ in numerical value for the same instrument.

$$\frac{-\mathrm{DMT}(i)}{(1+i)} = \frac{\mathrm{P}'(i)}{\mathrm{P}(i)}$$

Thus, the DMT for an N-year zero coupon bond is N, but the volatility is $\frac{-N}{(1+i)}$.

4 LINEAR APPROXIMATIONS TO CHANGE IN PRICES

4.1 General form of first-order linear approximation

For a small change in interest rate from i to $i+h$, we can approximate $P_A(i+h)$ as:

$$P_A(i+h) \approx P_A(i) + h \cdot \frac{\partial P_A(i)}{\partial i} = P_A(i) + h \cdot \frac{\partial P_A(i)}{\partial i \cdot P_A(i)} \cdot P_A(i)$$

4.2 Example for a zero coupon bond

Take an N-year zero coupon bond with price $P_A(i)$ at interest rate i and par value = 100. The formula for the price is $P_A(i)=100\,v^N$ and has volatility $\frac{-N}{(1+i)}$ and derivative with respect to i of $\frac{-N}{(1+i)} \cdot 100\,v^N$.

To continue the example, we will assume $N=10$

We calculate the value of the bond at 5% to be: $P_A(5\%)=100 \cdot v^{10}=61.39$

Now, if we need to calculate the value of the bond at 5.1%, we can use the linear approximation as:

$$P_A(5\%+0.1\%)=P_A(5\%+0.1\%)-\frac{0.001 \cdot N}{(1+0.05)} \cdot P_A(5\%)$$
$$=61.39-\frac{0.01}{1.05} \cdot 61.39=61.39-\frac{1}{105}=61.39-0.58=60.81$$

So the approximation yields 60.81. If we now calculate the value directly using the price formula, we get:

$$P_A(5.1\%)=100 \cdot (1.051)^{-10}=60.81$$

In this case, for this instrument, for a 0.1% change in interest rates, the result is the same **up to two decimals places.**

Check for yourself that the results differ from the third decimal place.

4.3 Example for 2 year coupon bond

Consider a 2-year bond, par value 100 that pays a 5% annual coupon in arrears.

1. Calculate the price at i = 5% and at i = 6% directly using the price formula.
2. Calculate the DMT at i = 6%
3. Calculate the volatility at i = 6%
4. Calculate the price at i = 6.2% using the first-order linear approximation
5. Calculate the price at i = 6.2% directly using the price formula.

4.4 Second-order approximation

The first-order linear approximation is only useful for very small changes in i. For larger changes, we need to consider higher order derivatives to more closely match the shape of the price formula curve (the graph of price against interest rates).

The formula for the second-order approximation extends the approach for the first-order linear approximation:

$$\mathrm{P}_A(i+h) \approx \mathrm{P}_A(i) + h \cdot \frac{\partial P_A(i)}{\partial i} + h^2 \cdot \frac{\partial P_A^2(i)}{\partial i^2/2}$$

The graph below shows the blue curved line of the price of a 10-year ZCB for different interest rates. The straight dotted orange line is the first-order linear approximation. The approximation is reasonably close close the point of tangent, but gets progressively worse as we move away from that point.

The curve dotted green line is the second-order approximation. As can be seen from the graph, it is a much better fit at most interest rates.

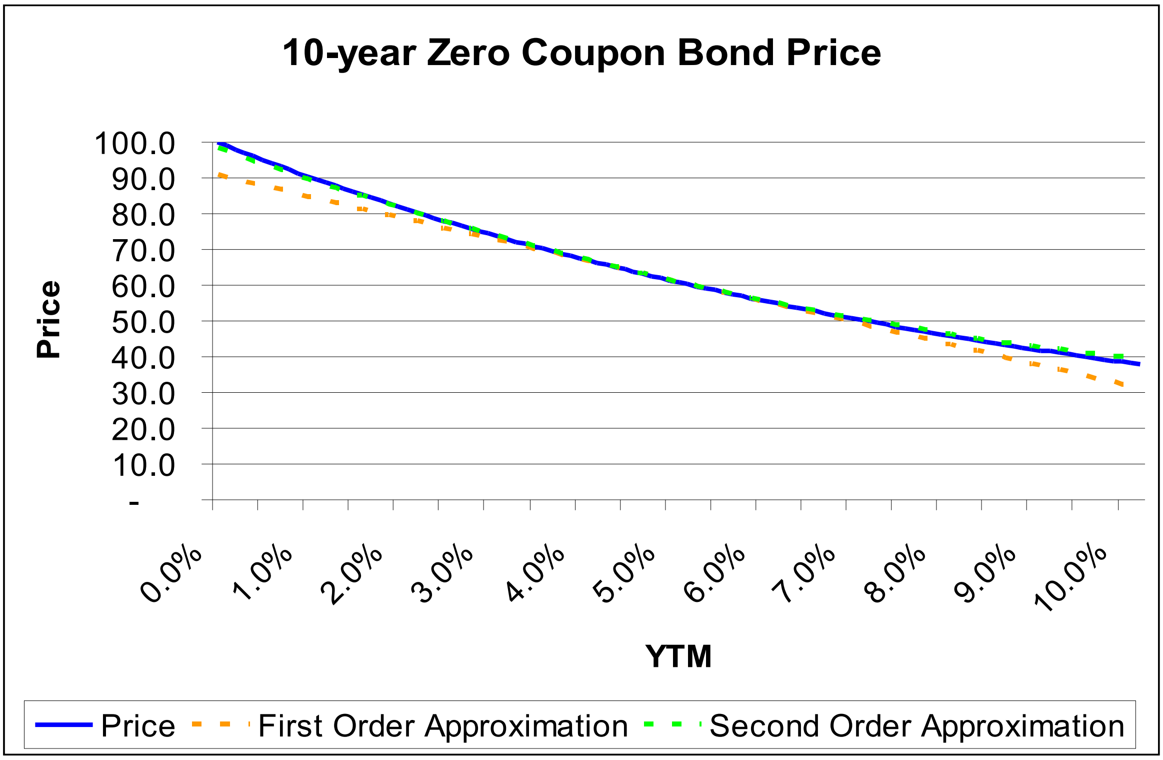


Illustration 6.3: Price sensitivity of a 10-year Zero Coupon Bond

5 Immunisation

5.1 Purpose of immunisation

The purpose of immunisation is to ensure that no losses will be made on changes in interest rates. It is a method of matching that can be used when exact cashflow matching is not possible or too expensive.

Note that immunisation in practice is almost never possible. It requires the assumption of a flat yield curve where changes in interest rates occur only through a parallel shift in the yield curve. This does not happen in practice, since it implies an **arbitrage opportunity** where risk-free profits can be made by investors.

The discussion of arbitrage opportunities is beyond the scope of this course.

5.2 Assumptions

Interest rates are equal for all terms. (The yield curve is flat.)

Changes in interest rates are equal at all durations.

Note that this second point is a necessary result of the first point. If the yield curve is flat (the same interest rate at all durations) then any change in the yield curve at one point must result in the same change at every point on the yield curve.

All changes in interest rates are small. Portfolios can be rebalanced without cost after all small changes in interest rates.

Again, this assumption is not necessarily true in practice. Immunisation only protects against small changes in interest rates. Large changes occur in practice.

Secondly, there are costs to rebalancing portfolios to ensure an immunised position.

5.3 Requirement of immunisation

5.3.1 *Some definitions*

All definitions given for interest rate i

$$\mathrm{f}_A(i) = \text{value of assets}$$

$$\mathrm{f}_L(i) = \text{value of liabilities}$$

$$\frac{\partial \mathrm{f}_A(i)}{\partial i} = \mathrm{f}'_A(i) = \text{first derivate of asset value}$$

$$\frac{\partial \mathrm{f}_L(i)}{\partial i} = \mathrm{f}'_L(i) = \text{first derivate of liability value}$$

$$\frac{\partial^2 \mathrm{f}_A(i)}{\partial i^2} = \mathrm{f}''_A(i) = \text{second derivate of asset value}$$

$$\frac{\partial^2 \mathrm{f}_L(i)}{\partial i^2} = \mathrm{f}''_L(i) = \text{second derivate of liability value}$$

5.3.2 *Requirements for immunisation*

For immunisation, the following three criteria must hold:

- $\mathrm{f}_A(i) = \mathrm{f}_L(i)$ The value of assets and liabilities must be equal
- $\mathrm{f}'_A(i) = \mathrm{f}'_L(i)$ The **volatility** of assets and liabilities must be equal
- $\mathrm{f}''_A(i) \geq \mathrm{f}''_L(i)$ The **convexity** of assets must be greater than convexity of liabilities

Volatility here is defined as the sensitivity of the price of the instrument to changes in interest rates. Similarly, convexity is defined as the sensitivity of volatility to changes in interest rates.

5.3.3 *Practical interpretation*

- Assets and liability values must be the same (i.e. USD100 of Assets = USD100 of Liabilities)
- The volatility or DMT of assets and liabilities must be the same (we call this being **duration matched)**
- The spread of assets (by term) must be greater than that of liabilities

5.4 Example of immunisation

We have an obligation of USD100 in 5 years time. Current interest rates are 5% per annum. We have two assets backing this liability.

Asset X pays 52.5 in year 6

Asset Y pays 47.62 in year 4.

Assets A = X + Y

Check that $f_L(5\%)=78.35$

$$f_X(5\%)=39.18$$

$$f_Y(5\%)=39.18$$

$$f_A(5\%)=f_{X+Y}(5\%)=78.35$$

$$f_L(i)=100\cdot(1+i)^{-5}$$
$$f_L'(i)=-5\cdot100\cdot(1+i)^{-6}$$
$$f_A'(i)=\left(-6\cdot52.5\cdot(1+i)^{-7}\right)+\left(-4\cdot47.62\cdot(1+i)^{-5}\right)$$

$$f_L'(5\%)=f_A'(5\%)$$

Note that volatility, or sensitivity to changes in interest rates is negative. This confirms the principle that bond prices move inversely with changes in interest rates.

$$f_L''(i)=30\cdot100\cdot(1+i)^{-7}$$
$$f_A'(i)=\left(42\cdot52.5\cdot(1+i)^{-8}\right)+\left(20\cdot47.62\cdot(1+i)^{-6}\right)$$
$$f_A''(i)\geq f_L''(i)$$

So it appears that all the requirements have been met. Let's consider what happens the the assets and liabilities if we increase or decrease interest rates. If immunisation is **working, we should make no profit or a small profit (A>L) in both cases.**

For i = 4%

Check that

$$f_L(4\%)=82.19$$
$$f_X(4\%)=41.49$$
$$f_Y(4\%)=40.70$$
$$f_A(4\%)=f_{X+Y}(4\%)=82.20\geq f_L(4\%)=82.19$$

For i = 6%

Check that

$$f_L(6\%)=74.73$$
$$f_X(6\%)=37.01$$
$$f_Y(6\%)=37.72$$
$$f_A(6\%)=f_{X+Y}(4\%)=74.73\geq f_L(6\%)=74.73$$

For both an interest rate increase and decrease, our assets are higher (or not lower) than the liabilities. Thus, we are immune to small changes in interest rates.

6 Mismatch example

Take an immediate annuity for a life aged 50. The DMT for this liability is 20 at an interest rate of 4%. The reserve or liability is USD80,000. We invest the assets backing the reserve into a Zero Coupon Bond with term of ten years.

1. What is the DMT of the ZCB?
2. What is the Volatility of the ZCB?
3. What is the Volatility of the annuity liability?
4. What happens if interest rates increase to 4.5%?
5. What happens if interest rates decrease to 3.5%?
6. What practical implications does this have?
7. What should you do to reduce this risk?
8. Why might it be difficult to perfectly match the liability cashflows arising from annuity products?

1. 10
2. $\frac{-10}{1.04} = -9.62$
3. $\frac{-20}{1.04} = -19.23$
4. Assets decrease to 76,254. Liabilities decrease to 72,683. We make a profit of 3,571.
5. Assets increase to 83,950. Liabilities increase to 88,095. We make a loss of 4,145.
6. The company is exposed to interest rate risk since changes in interest rates have the potential to cause fluctuations in financial performance. Further, we should be particularly concerned about the potential for interest rate decreases since this will lead to a loss.

7. We should try to match our liabilities more closely by selling the 10-year bond and buying longer term fixed interest securities. The first step should be to have the DMT of assets = to that of liabilities. Next steps would be to consider perfect cashflow matching, or attempting immunisation. More complicated hedging strategies such as entering into derivative contracts such as swaps could be considered.

8. Annuity cashflows have a long average term. It might be difficult to find assets in the market with sufficiently long term. This will prevent DMT matching, cashflow matching and immunisation.

   Even if available, the assets may be very expensive due to large demand and limited supply of these long-term instruments. Even if the expected annuity cashflows could be matched perfectly matched, the annuity cashflows depend on mortality risk. If policyholders experience different mortality from assumed (either due to incorrect or basis or simply random fluctuations) then the actual cashflows will not be matched. This could lead to profits or losses, but would also require a rebalancing of the portfolio to match the updated cashflows. This rebalancing can be expensive and the updated assets may not be available.

   Perfect cashflow matching also requires the premiums received from new policies to be invested in exactly matching assets as soon as the funds are received. In practice, new business cashflows are matched monthly in large, fairly sophisticated companies. A few companies match new business on a weekly basis. Again, the costs of this frequent rebalancing can be large.

Chapter 7 Pension fund reserving

About

This chapter covers the basic actuarial techniques used for the calculation of retirement fund reserves.

Chapter Contents

1 The Nature of Pension Funds

1.1 Defined Benefit (DB) structures

1.1.1 *Typical defined benefit structure*

A **defined benefit scheme** is one where the benefit is defined and guaranteed by the employer. A typical benefit definition would be:

$$Pension_{NRA} = Salary_{NRA} \cdot Service \cdot p\%$$

where $Pension_{NRA}$ is the annual pension payable at Normal Retirement Age

$Salary_{NRA}$ is the final salary paid at normal retirement age

Service is the number of years service to the employer

p% is the specified percentage, usually somewhere between 1.5% and 2.5%.

This could also be defined as:

$Pension_0 = Salary_{NRA} \cdot Service \cdot p\%$ where $Pension_0$ is the pension payable 0 years after Normal Retirement Age.

The exact notation is not as important as the concept and the calculation. *You must consider how the terms are defined* and use them appropriately or define your own. No equation is useful unless all terms have been appropriately defined. This is a general comment that applies across all sections (and should apply to your other courses too).

Inflationary increases in the pension payable during retirement are usually at the discretion of the pension fund trustees and will depend on available funds and employee contributions. Some level of increase (either absolute or relative to inflation) may be implied or even guaranteed.

1.1.2 *Interpretation of typical defined benefit structure*

For every year of service (employment), the employer promises to pay p% of the employees final salary as a pension once the employee retires at Normal Retirement Age.

- If p% is 1.5% and the employee works for the company for 40 years, the pension payable is 60% of final salary.

- If p% is 2.5% and the employee works for 30 years, the pension payable will be 75% of final salary.

The factor p% is part of each employer's remuneration package and is usually entirely at the discretion of the company.

1.1.3 *Some variations on typical defined benefit structure*

- Salary_{NRA} may be an average of the last x years of salary (often 3 years). This reduces the potential for manipulation and unequal treatment between equal employees
- Normal Retirement Age might be a band of ages rather than a single specific age, with possibly adjustments to the final benefit depending on actual retirement age
- p% may very outside the usual range specified. p% may differ for different levels of employees.
- Additional years of service may be "purchased" through once-off contributions.

1.2 Defined Contribution (DC) structure

A **defined contribution** pension scheme is one where the **contributions** are defined clearly. The actual benefit paid depends entirely on the investment performance of the underlying assets. This scheme operates as a savings vehicle with no guarantees made in terms of replacement of actual final salary.

At retirement, the value of accumulated funds is used to purchase a retirement income product (often from an insurance company). Depending on the exact legal, tax and regulatory environment, some of the accumulated funds may be taken as cash.

> When might it make sense for a retiring employee to take a portion in cash rather than in a lifetime annuity?
>
> When would it not make sense?

1.3 Movement towards Defined Contribution

DB schemes represent significant risks to the employers. Many prominent companies face severe difficulties in financing their DB liabilities. The employers are exposed to:

- longevity risk
- investment risk

As a result, many companies are moving towards DC schemes, where risk is transferred to the employees.

DC schemes involve little actuarial valuation work. The liabilities are retrospectively accumulated fund values and do not require discounted cashflow valuation techniques.

The rest of this chapter deals with the valuation of DB schemes only.

1.4 Spouse benefits

Many pension schemes provide a **spouse's pension** on death of the employee either during employment, during retirement or both. The spouse's pension will usually be smaller than the employees pension.

1. Why might an employer offer a spouse's pension?
2. Why might the spouse's pension be lower than the employee's own pension?
3. Would the spouse's pension be different before and after the year in which the employee would have reached normal retirement age?

1. The employer might offer the benefit as a way to attract top employees to the company. Also, if the company did not offer the benefit, and an employee with a family died in employment (especially shortly before retirement) and not benefit was paid to the family, the bad publicity might force the company to pay the benefit in any event. Thus, many companies elect to offer the benefit in the first place.
2. The employee's pension is intended to provide for his/her income in retirement, and a proportion for his family. On the death of the employee, he part intended for his/her own income is no longer required, and only the portion intended for his/her family is required.
   Also, by offering a lower spouse's pension, the cost of the benefit is reduced which allows a higher salary or employee's pension to be paid at the same cost to the company.
3. Before NRA, it is more likely that the employee will have a young family with many financial obligations. After retirement, it is more likely that major household debts will be paid off. This is just an example, the exact nature of the benefits would depend on the rules of the pension fund as stipulated by the employer or country-specific law.

2 Pre-funding versus cash-based approaches

Pension funds can be funded (paid for) at different points in time. In this section, we first address two extreme methods ("pay as you go" and complete full funding from inception) before describing the more moderate approaches more often adopted in practice.

2.1 Cash or "pay as you go" system

Under the cash system, pension benefit payments to retired employees are paid when due, and no prior reserves are held. Each payment is an expense to the employer when it is paid.

The risk with this approach to employees is that if the employer can no longer meet these obligations, they have no security and will likely lose their pension.

Under what conditions might the pension be relatively safe?

Under what conditions might the pension be severely at risk?

A state (government-backed) pension might be relatively safe while the population is growing rapidly, such that a large base of active workers are contributing towards the income of pensioners. However, as soon as this population growth slows, the "population pyramid" becomes a population column or even an inverted pyramid and a small number of active workers must support the income of a large number of pensioners. At this point, the sustainability of the system is at risk.

The pension fund of a company depends similarly on the ratio of active employees to pensioners. As long as the company continues to grow in terms of number of employees, the funding requirements are low. However, inevitably this changes and the pension will become onerous to finance.

Pay-as-you-go systems are usually at risk in the long term.

A cash system is used by some companies in practice. It is more often used by the governments of countries rather than individual countries.

Is a cash-based system more safe or less safe if the sponsoring employer is a government?

What is the current problem with the United States Social Security System?

If the government is backing the pension fund, they should be more able to pay the pensions as they can raise additional funds through higher taxes. However, since it is still the working population that are paying the taxes, the burden on the active works can become high if the population growth rate slows down.

The US Social Security System has many problems. One of these is the high growth in the number of people claiming from the system relative to the growth of those contributing to the system.

2.2 Complete full funding from inception

This approach is not adopted in practice.

The employer calculates the full present value of pension payments due when an employee joins the firm. This amount is set aside as a reserve. The full cost of the entire pension is recognised at the date on which the employee joins the company.

Why is this likely to be unpopular with employers?

Are there any risks left to the employer?

Is this an accurate reflection of the cost of employee benefits?

A very high, upfront cost at the point of hiring an employee will decrease profits when employees are hired. This acts as a disincentive to hire employees with full benefits.

There are considerable risks left to the employer. The assets backing the pension fun are exposed to market fluctuations. A decline in asset values will require additional funds from the employer. Further, if employees live longer than expected and require an income after NRA from the company for a longer period than expected. These cashflow risks also give rise to assumption and valuation risks- at the point at which future the company recognises changes in expected mortality or investment returns, the reserves will change without a change in asset values.

No, the cost of employee benefits should be matched with the service provided by the employees to the employer. Under this method, the entire expected cost is recognised upfront.

2.3 Practical pre-funding systems

In practice, many pension funds are gradually funded over the period of an employee's service such that at retirement, the full pension has been reserved for.

$$Reserve_0 = 0$$

$$Reserve_{NRA} = \text{“}Full\ PV\ of\ pension\ payments\text{”}$$

NRA here means **normal retirement age.** The rate and pattern at which $Reserve_0$ grows to $Reserve_{NRA}$ depends on the funding approach chosen.

In the next section we will discuss the **projected unit credit method** which is one of the more common approaches.

3 Projected Unit Credit Method

The Projected Unit Credit Method is one popular method used to reserve for pre-funded pension funds. It is covered here as an example of an appropriate methodology. Other methods are not covered in this course but involve similar concepts and techniques.

3.1 Reserving for benefits in payment

3.1.1 Reserving methodology

The reserving methodology for benefits in payment are the same as those used in valuing an immediate annuity. The expected present value of future cashflows is calculated, allowing for:

- probability of survival
- inflationary increases in pension payments
- discount rates based on available returns in the market and taking into consideration the assets backing the pension liabilities.

3.1.2 Economic assumptions required

- nominal discount rate and pension payment inflation rate

OR

- real discount rate

3.1.3 Demographic assumptions required

- base mortality (by gender, and sometimes class of worker)
- mortality improvements

3.2 Reserving for active employees

3.2.1 Reserving methodology

The reserving methodology involves two key steps:

1. The first step is to calculate the expected present value of the pension payments based on projected final salary.
2. The second step is to determine what proportion of this liability has been accrued based on past service compared with expected total service.

It is step 2 above that defines the Projected Unit Credit Method differently from alternative pension reserving methodologies.

$$Reserve = \text{EPV}(\text{future pension payments}) \cdot \frac{\text{past service}}{\text{expected total service}}$$

More rigorously, we might define this as:

$$Reserve_x = v^{NRA-x} \cdot salary_x \cdot (1+e)^{NRA-x} \cdot p\% \cdot (NRA-EA) \cdot \ddot{a}^{\,j}_{NRA} \cdot \frac{(x-EA)}{(NRA-EA)}$$

$$Reserve_x = salary_x \cdot \left(\frac{1+e}{1+i}\right)^{NRA-x} \cdot p\% \cdot \ddot{a}^{\,j}_{NRA} \cdot (x-EA)$$

where $v = (1+i)^{-1}$

$i =$ nominal discount rate

$e =$ salary inflation

$salary_x =$ current salary for employee aged x

$j = \frac{(1+i)}{(1+g)} - 1 = \frac{(1+i)-(1+g)}{(1+g)} = \frac{i-g}{1+g}$

$g =$ pension increases in retirement

$x =$ current age

$\text{EA} =$ age at start of service or employment

$\ddot{a}^{\,j}_{NRA} =$ immediate annuity in advance at discount rate j

3.2.2 Economic assumptions required

- nominal discount rate and salary inflation rate

AND

- nominal discount rate and pension payment increase rate

OR

- real discount rate (usually different rates for the period before and after retirement

3.2.3 Demographic assumptions required

- base mortality
- mortality improvements

3.3 Change in reserve

From t to t+1, the reserve required will change for both retired employees and active employees.

3.3.1 Change in reserve for pensioners

The change in reserve here operates like an immediate annuity. The factors affecting the change in reserve are:

- unwind of the discount rate
- benefit payments made
- Survival (0 or 1 for each person) to next period (actual versus expected mortality)
- Difference between assumed pension increases and actual pension increases declared (actual versus expected pension increases)
- Changes in mortality basis
- Changes in the economic basis

For each of these factors, make sure you understand:

1. Why it impacts the reserve
2. In which direction it impacts the reserve
3. How large or frequent the changes are likely to be
4. How the company might manage the risks of these factors

The "unwind of the discount rate" is the increase in the reserve as a result of the passage of time since future cashflows are discounted by less time. This is not a risk since it will happen over time with certainty.

When benefit payments are made, they are in the past. Since the reserve takes into account future cashflows, they are no longer part of future cashflows and thus reduce the reserve. This also happens with certainty and so no risk management is required.

Each pensioner has a survival probability. However, each pension with either die or survive. For every pensioner that survives, the reserve will increase (since mortality was lighter than expected) and for every pensioner that dies the reserve will decrease (since mortality was heavier than expected). If the total number of pensioners that survive matches expectations, the overall reserve will not change as a result since actual deaths are the same as expected. By having a large number of pensioners, the random fluctuations will average out and the overall variability of reserves due to actual mortality being different from expected will decrease. Accurate reserving assumptions will help to ensure that overall mortality expectations are accurate. Trends (such as annuitant mortality improvements) should be taken into consideration.

If actual pension increases are higher than assumed, then the reserve will increase since the expected future cashflows will now be higher than previously expected since the base (current pension) is higher than previously assumed. This will generally occur annually when pension increases are decided. The approach followed to setting the pension increases should be taken into consideration when determining the pension increase assumption. Another form of risk management is "automatic" since a key input into increases declared will be the surplus available in the pension fund (assets less liabilities). Thus, high pension increases will result in an increase in pension liabilities (reserves) but this will often correspond with a previous increase in asset values from good investment performance. Thus, although liabilities (reserves) will increase, the surplus will not be inappropriately decreased.

Any change in basis will affect the reserves through a change in expected future cashflows. The basis will usually be considered annually. However, changes may be made less frequently. The impact of economic basis changes can be managed through matching assets and liabilities such that assets and liabilities move in line with each other. Changes to the mortality basis are more difficult to hedge. Care should be exercised when setting the basis in the first place, and spurious changes (unnecessary small changes that might be reversed in future) should not be made.

3.3.2 *Change in reserve for active employees*

The change in reserve for active employees is an important part of the actuarial calculation for DB schemes.

The change can be expressed as:

$$\Delta\, Reserve_t = Reserve_{t+1} - Reserve_t$$

The actual change depends on the approach adopted to move from $Reserve_0 = 0$ to $Reserve_{NRA} = \text{PV of pension benefits}$.

This may be considered along with the Analysis of Profit in AMIII. This will not form part of AMII.