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ABSTRACTS

This book is a reissue of the second edition which appeared in 1940. It has the distinction of being the firstr vintage mathematical work published in the NCTM series "Classics in Mathematics Education." The text include a biography of Pythagoras and an account of historical data pertaining to his proposition. The remainder of the book shows 370 different proofs, whose origins range from 900 B.C. to 1940A.D. They are grouped into the four categories of possible proofs: Algebric (109 proofs): Geometric (255); Quaternionic (4); and those based on mass and velocity, Dynamic (2). Also – included are five Pythagorean magic squares; the formulas of Pythagoras, Plato, Euclid, Maseres, Dickson, and Martin for producing Pythagorean triples; and a bibliography with 123 entries.

THE PYTHAGOREAN PROPOSITION

PREFACE

Some mathematical works of considerable vintage have a timeless quality about them. Like classics in any field, they still bring joy and guidance to the reader. Substantial works of this kind , when they concern fundamental principles and properties of school mathematics, are being sought out by the Supplementary Publications Committee. Those that are no longer readily available will be reissued by the National Council of Teachers of Mathematics. This book is the first such classic deemed worthy of once again being made available to the mathematics education community.

The initial manuscript for The Pythagorean Proposition **was prepared in** 1907 and first published in 1927. With permission of the Luumis family, it is presented here exactly as the second edition appeared in 1940. Except for such necessary changes as providing new title and copyright pages and adding this Preface by way of explanation, no attempt has been made to modernize the book in any way. To do so would surely detract from, rather than add to, its value.

“ In Mathematics the man who is ignorant of what Pythagoras said in Croton in 500 B.C. about the square on the longest side of a right-anbled tringle, or who forgets what someone in Czechoslovakia proved last week about inequalities, is likely to be lost. The whole terrific mass of well-established Mathematics, from the ancient Babylonians to the modern Japanese, is as good today as it ever was.”

E.T.Bell, Ph.d.,1931

FOREWORD

According to Hume , (England's thinker who interrupted Kant's "dpgmatic slumbers"), arguments may be divided into: (a) dmonastrations; (b) proofs: (c) probabilities.

By a demonstration, (demonstro, to cause to see), we mean a reasoning consisting of one or more categorical propositions "by which some proposition brought into question is shown to be contained in some other proposition assumed, whose truth andcertainty being evident and acknowledged, the proposition in question must also be admitted certain. The result is science, knowledge, certailty." The knowledge which cemonstration gives is fixed and unalterable. It denotes necessary consequence, and is synonymous with proof from first principles.

By proof, (probo, to make credible, ot demonstrate), we mean 'such an argument from experience as kleaves no room for doubt of opposition, and adequate to establish it.

The object of this work is to present to the future investigator, simply and concisely, what is known relative to the so-called Pythagorean Proposition, (known as the 47th proposition of Euclid and as Carpenter's Theorem"), and to set forth certain metric proofs and the geometric figures pertaining thereto.

It establishes that:

First, that there are but four kinds of demonstration s for the Pythagorean proposition, viz.:

I. Those based upon Linear Relations. (implying the Time Consept) the Algebraic Proofs.

II. Those based upon Comparison of Areas (implying the Sace Cocept) – The Geometric Proofs.

III. Those based upon Vector Operation (implying the Direction Concept) – The Quaternionic Proofs.

IV. Those based upon Mass and Velocity (implying the Force Concept)—The Dynamic Proofs.

Second, that the number of Algebraic proofs is limitless.

Third, That there are only ten types of geometric figures from which a Geometric Proof can be deduced.

This third fact is not mentioned nor implied by any work consulted by the author of this treatise, but which, once established, becomes the basis for the classification of all possible geometric proofs.

Fourth, that the number of geometric proofs is limitless.

Fifth, that no trigonomeatric proof is possible.

By consulting the Table of Contents any investigator can determine in what field his proof falls, and then, by reference to the test, he can find out wherein it differs from what has already been established.

With the hope that this simple exposition of this historically renowned and mathematically fundamental proposition, without which the science of Trigonometry and all that it implies would be impossible, may interest many minds and prove helpful and suggestive to the student, the teacher and the future original investigator, to each and to all who are seeking more light, the author, sends it forth.

CONTENTS

PORTRAITS

ACKNOWKEDGEMENTS

Every man builds upon his own predecessors.

My predecessors made this work possible, and may those who make further investigations relative to this renowned proposition do better then their predecessors have done.

The author herewith expresses his obligations:

To many who have receded him in this field, and whose text and proof he has acknowledged herein on the page where such proof is found;

To those who, upon request, courteously granted him permission to make use of such proof, or refer to the same;

To the following Journals and Magazines whose owners so kindly extended to him permission to use proofs found therein, viz:

The American Mathematical Monthly;
Heath's Mathematical Monographs;
The Journal of Education;
The School Visitor;
The Scientific American Supplement;
Science and Matehmatics; and Science.

To Theodore H.Johnston, Ph,D, formerly Principal of the West High School, Cleveland, Ohio, for his valuable stylistic suggestions after reading the original manuscript in 1907.

To Professor Oscar Lee Dustheimer, Professor of Mathematics and Astronomy, Baldwin-Wallace Collage, Berea, Ohio for his professional assistance and advice; and to Devid P. Simpson, 330, former Principal of West High School, Cleveland, Ohio, for his brotherly encouragement, and helpful suggestions,1926

To Dr. Jehuthiel Ginsburg, publisher of Scripta Mathematica, New York City, for the right to reproduce the t photo plates of ten of his "Porttaits of Eminent Mathematicians."

To Elatus G. Loomi for his assistance in drawing the 366 figures which appear in this Second Edition.

And to "The Masters and Wardens Association of The 22nd Masonic District of the Most Worshipful Grand Lodge of Free and accepted Masnons of Ohio, " owner of the Copyright Granted to it in 1927, for its generous permission to publish this Second Edition of The Pythagorean Proposition,

the author agreeing that a complimentary copy of it shall be sent to the known Mathematical Libraries of the World, for private research work, and also to such Masonic Bodies as it shall select. (April 27, 1940)

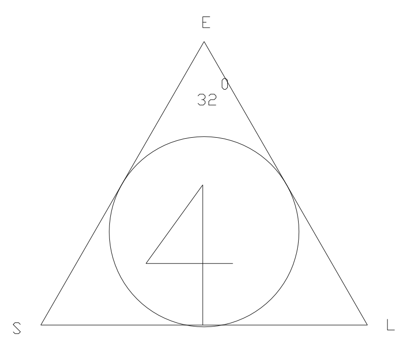
ABBREVIATIONS AND CONTRACTIONS

Am. Math. Mo. = The American Mathematical Monthly, 100 proofs, 1894
a-square = square upon the shorter leg.
b-square = " " " longer leg.
Colbrun = Arthur R. Colbrun LL.M. Dist of Columbia Bar.
Const. = construct.
Const'd = constructed.
Dem. = demonstrated, or demonstration.
Edw. Geom. = Edward's Elements of Geometry, 1895
_ eq. =equation
Eq's = equations
Fig. or fig = figure.
Fourrey = E. Fourrey's Curiosities Geometrique.
Heath = Heath's Mathematical Monongraphs, 1900 Part I and II—26 proofs.
h-square= square upon the hypotenuse.
Jour. Ed'n = Journal of Education.
Legendre = Davies Legendre, Geomotry, 1858
Math. = mathematics.
Math.Mo. = Mathemetical Monthly, 1858-9
Mo. = Monthly.
No. or no. = number
Olney's Geom. = Olney's Elements of Geometry, University Edition.
Outw'ly = outwardly.
Par. = parallel.
p. = page
pt. = point.
Quad. = quadrilateral.
Resp'y = respectively.

THE PYTHAGOREAN PROPOSITION

Richardson = John M. Richardson –28 proofs
Rt.= right.
Rt. Tri. = right tringle
Rect. = rightangle.
Sci. Ame. Supt. = Scientific American Supplement. 1910 Vol. 70.
Sec = secant.
Sin = sine
Sq. = square.
Sq's = squares.
Tang = tangent.
$\therefore$ = therefore
tri. = Triangle.
tri's = tringles
Trap. = trapezoid
V or v =Volume
Versluys = Zes en Negentic (96) Beweijzen Voor Het.
Theorems Van Phythagoras, by J. Versluys, 1914
Wipper = Jury Wipper's "46 Beweise der Pythagoraischen Lehrsatzes," 1880

HE^2 = or any like symbole = the square of, or upon , the line HE , or like Symbole.
AC | AF or like symbole= AC+AF, or AC/AF. See Proof 17



GOD GEPMETROZES

CONTINUALLY~*PLATO*

THE PYTHAGOREAN PROPOSITION 1

This celebrated proposition is one of the most important theorems in the whole realm of geometry and is known in history as the 47th proposition, that being its number in the first book of Euclid's Elements.

It is also (erroneously) sometimes called the Pons Asinorum. Although the practical application of this theorem was known ling before the time of Pythagoras he, doubtless, generalized it from an Egyptian rule of thumb ($3^2+4^2 = 5^2$) and first demonstrated it about 540B.C., from which fact it is generally known as the Pythagorean Proposition. This famous theorem has always been a favorite with geometricians.

(The statement that Pythagoras was the inventor of the 47th proposition of Euclid has been denied by many students of the subject.)

Many purely geometric demonstrations of this famous theorem are accessible to the teacher, as well as an unlimited number of proof based upon the algebraic method of geometric investigation. Also quaternions and dynamics furnish a few proofs.

No doubt many other proofs than these now known will be resolved by future investigators, for the possibilities of the algebraic and geometric relations implied in the theorem are limitless.

But before proceeding to the methods of demonstration, the following historical account translated from a monograph by Jury Wipper, published in 1880, and entitled "46 Beweise des Pythagouaischen Lehrsatzes," may prove both interesting and profitable.

Wipper acknowledges his indebtedness to F. Graap who translated it out of the Russian. It is as follows: "One of the weightiest propositions in geometry if not the weightiest with reference to its called Pythagorean proposition."

The Latin reads: In rectagulis triagulis quadratum, quod a latere rectum angulum subtendente descrobitur, aequale est eis, quae a lateribus rectum angulum continentibus descrobuntur.

German: in den rechtwinkeligen Dreiecken ist das Quadrat, welches von der dem rechten Winkel gegenuber lienden Seite beschrieben Wird, den Quadraten, welche von den inn umschlieben Seiten beschrieben warden, gleich.

According to the testimony of Proklos the demonstration of this proposition is due to Euclid who adopted it in his elements (I 47). The method of the Pythagorean demonstration remains unknown to us. It is undecided whether Pythagoras himself discovered this characteristic of the right triangle , or learned if from Egyption priest , or took it from Babylon: regarding this opinions vary.

According to that one most widely disseminated Phythagoras learned

from the Egyptian priests the characteristics of a triangle in which one leg = 3(designating Osiris), the second = 4 (designating Isis), and the hypotenuse = 5 (designating Horus): for which reason the triangle itself is also named the Egyptian or Pythagorean,*

The characteristics of such a triangle, however, were known not to the Egyptian priests alone, the Chinese scholars also knew them. "In Chinese history ," says Mr. Skatschlow, "great honor are awarded to the brother of the rurler Uwan, Tschou-Gun, who lived 1100B.C. : he knew the characteristics of the right triangle ,(perfected) made a map of the stars, discovered the compass and determined the length of the meridian and equater.

Another scholar (Cantror) says: this emperor wrote of shared in the composition of a mathematical treatise in which were discovered the fundamental features, ground lines, base lines, of mathematics, in the form of a dialogue between Tschou-Gun and Schau-Gao. The title of the book is: Tschaou pi; i.e., the high of Tschao. Here too are the sides of a triangle already;y named legs as in the Greek, Latin, German and Russan.

Here are some paragraphs of the 1st,chapter of the work. Tschou-Gun once said to Schau-Gao: "I learned , sir, that you numbers and their applications, for which reason I would like to ask how old Fo-chi determined the degrees of the celestial sphere. There are no steps on which one can climb up to the sky, the chain and the bulk of the earth are also inapplicable; I would like for this reason, to know how he determined the numbers,"

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*(Note . the Grand Lodge Bulletin, A,F. and A.M. , of Iowa, Vol.30,No 2, Feb 1927, p.42 has: In an old Egyptian meanuscript, becently discovered at Katan, and supposed to belong to the time of the Twelfth Dynasty, we find following equations: $1^2 + (3/4)^2 = (1\frac{1}{4})^2$; $8^2+6^2 = 10^2$; $2^2 + (1\frac{1}{2})^2 = (2\frac{1}{2})^2$; $16^2+12^2=20^2$; all find that this triangle was to them the symbol of universal nature. The base 4 represented Osiris; the perpendicular 3, Isis; and hypotenuse represented Horus, their son, being the producet of the two principles, male and female.)

Schau-Gao replied: “The art of counting goes back to the circle and squre,”

If one divides a right triangle into its parts the line which unites the ends of the sides when the base =3 the altitude = 4 is 5.

Tschou-Gun cried out: “That is indeed excellent.”

It is to be observed that the relations between China and Babylon more then probably led to the assumption that this characteristic was already known to the Chaldeans. As to the geometrical demonstration it comes doubtless of from Pythgoras himself. In busyin with the addition of the series he could very naturally go from the triangle with sides 3,4,and 5, as a single instance to the general characteristics of the right triangle.

After he observed that addition of the series of odd number (1+3=4,1+3+5=9 etc.) gave a series of squares, Pythagoras formulated the rule for finding, logically, the sides of a right triangle: Take an odd number (say7) which forms the shorter side, square it ($7^2=49$), subtract one (49-1=48)halve the number (48/2=24) this half is the longer side, and this increased by one (24+1=25) is the hypotenuse.

The ancients recognized already the significance of the Pythagorean proposition for which fact may serve among other as proof the account of Diogenes Laertius and Plutarch concerning Pythagoras. The latteris said to have offered (sacrificed) the Gods an ox in gratitude after he learned the notable characteristics of the right triangle. This story is without doubt a fiction, as sacrifice of animals, i.e., blood-shedding, antagonizes the Pythagorean teaching.

During the middle ages this proposition which was also named *inventum hecatombe dignum* (in-as-much as it was even believed that a sacrifice of a hecatomb—100 oxen was offered) won the honor-designation *Magister matheseos,* and the knowledge thereof was some decades ago still the proof of a solid mathematical training (or education). In examinations to obtain the master’s degree this proposition was often given; there was indeed a time, as is maintained, when from every one who submitted himself to the test as master of mathematics a new (original) demonstration was required.

This latter circumstance, or rather the great significance of the proposition under consideration was the reason why numerous demonstrations or it were though out.

The collection of demonstrations which we bring in what follows,*must ,

- Note. There were but 46 different demonstrations in the monograph by Jury Wipper, Which 46 are among the classified collection found in this work.

in our opinion, not merely satisfy the simple thirst for knowledge, but also as important aids in the teachin of geometry. The variety of demonstrations, even when some of them are finical, must demand in the learners the development of rigidly logical thinking, must show them how many sidedly an ofnet can be sonsidered, and spur them on to test their abilities in the discovery of like demonstrations for the one or the other proposition.

Brief Biographical Information Concerning Pythagoras

"The birthplace of Pythagoras was the island of Samos; There the father of Pythagoras , Mnessarch, obtained citizenship for services which he had rendered the inhabitants of Samos during a time of famine. Accomanied by his wife Pithay, Mnessarch frequently traveled in business in business interests; during the year 569C.E. he came to Tyre; jeer Pythagoras was born. At eighteen Pythagoras, secretly, by night, wentfron (left) Samos, Which was in the power of the tyrant Polycrates, to the island Lesbos to his uncle who welcomed him very hospitably. There for two years he received instruction from Ferekid who with Anaksimander and Thales had the reputation of philosopher.

After Pythagoras had made the religious ideas of his teacher his own, he went to Anaksimander and Thales in Miletus (549C.E.). The latter was then already 90 years old. With these men Pythagoras studied chiefly cosmography, i.e., Physics and Mathematics.

Of Thales it is known that he borrowed the solar year from Egypt; he knew how to calculate sun and moon eclipses, and determine the elevation of a pyramid from its shadow; to him also are attributed pyramid from its shadow; to him also are attributed the discorery of geometrical projections of great import; e.g., the characteristic or the angle which is inscribed and rests with its sides on the diameter as well as the charactreistics of the angle at the base of an (equilateral) isosceles triangle.

Of Anaksimander it is known that he knew the use of the dial in the first who taught geography and drew geographical maps on copper. It must be observed too that Anaksimander was the first prose writer, as down to his day all kearned works were written in verse, a procedure which continued longest among the East Indians.

Thales directed the eager youth to Egypt as the land where he could satisfy his thirst for knowledge . The Phoenician priest college in Sidon must in some degree sefve as preparation for this journey. Pythagoras spent an entire year there and arrived in Egypt 547.

Although Ploikrates who had forgiven Pythagoras' nocturnal flight addresses to Amasis a letter in which he commended the young scholar, it cost Pythagoras as a foreigner, as one unclean, the most incredible toil to gain admission to the peiest caste which only unwillingly initiated eventheir own people into their mysteries or knowledge.

The priests in the temple Heliopolis to whom the king in person brought Pythagoras declared it impossible to receive him into their midst, and directed him to the oldest priest college at Memhis, this commended him to Thebes. Here somewhat sever conditions were laid upon Pythagoras for his reception into the priest caste; but nothing could deter him. Pythagoras performed all the rites, and all tests, and his study began under the guidance of the chief priest Sonchis.

During his 21 year stay in Egypt Pyghagoras succeeded not only if fathoming and absorbing all the Egyptian but also became sharer in the highest honors of the priest caste.

In 527 Amasis died; in the following (526) year in the reign of Pasmmenit, son of Amasis, the Persian king Kambis invaded Egypt and loosed all his fury against the priest caste.

Nearly all members thereof fell into captivity, among them Pythagoras, to whom as abode Babylon was assigned. Here in the center of the world commerce where Bactrians, Indians Chinese, Jews an other folk came together, Pythagoras had during 12 years stay opportunity to acquire those learning in which the Chaldeans were so rich.

A singular accident secured Pythagoras liberty in consequence of which he returned to his native land in his 56^{th} year. After a brief stay on the island Delos where he found his teacher Ferekid still alive, he spent a half year in a visit to Greece for the purpose of making himself familiar with the religious, scientific and social condition thereof.

The opening of the teaching activity of Pythagoras, on the island of Samos, was extraordinarily sad; in order not to remain wholly without pupils he was forced even to pay his sole pupil, who was also maned Pythagoras, a son of Eratokles. This led him to abandon his thankless land and seek a new home in the highly cultivated cities of Magna

Graecia (Italy).

In 510 pythagoras came to Kroton. As is known it was a turbulent year. Tarquin was forced to flee from Rome, Hippias from Athens; in the neighborhood of Kroton, in Sibaris, insurrection broke out.

The first appearance of Pythagoras before the people of Kroton began with an oration to the youth wherein he rigorously but at the same time so convincingly set forth the duties of young man that the eiders of the city entreated him not to leave them without guidance (counsel). In his second oration he called attention to the family. In the two following orations he turned to the matrons and children. The result of the last oration in which he specially condemned luxury was that thousands of costly garments were brought to the temple of Hera, because no matron could make up her mind to appear in them on the street.

Pythagoras spoke captivatingly, and it is for this reason not to be wondered at that his orations brought about a change in the morals of Kroton's inhabitants; crowds of listeners streamed to him. Besides the youth who listened all day ling to his teaching some 600 of the worthiest men of the city, matrons and maidens, came together at his evening entertainments; among them was the young, gifted and beautiful Theana,
who thought it happiness to become the wife of the 60 year old teacher.

The listeners divided accordingly into disciples, who formed a school in the narrower sense of the word, and into auditors, a school in the broader sense. The former, the so-called mathematicians were given the rigorous teaching of Pythagoras as a scientific whole in logical succession from the prime concept of mathematics up the highest abstraction of philosophy; at the same time they learned to regard everything fragmentary in knowledge as more harmful than ignorance even.

From the mathematicians must be distinguished the auditors (university extensioners) out of whom subsequently were formed the Pythagoreans, These took part in the evening lectures only in which nothing rigorously scientific was taught. The chief themes of these lectures were: ethics, immortality of soul, and transmigration—me-tempsycholosy.

About the year 490 when the Pythagorean school reached its highest splendor-brilliancy-a certain Hypasos who had been expelled from the school as unworthy put himself at the head of the democratic party in Kroton and appeared as accuser of his former colleagues. The school was

broken up.the property of Pythagoras was confiscated and he himself exiled

The subsequent.16 years Pythagoras lived in Tarentum, but even here the democratic party gained the upper hand in 474 and Pythagoras a 95-year old man must flee again to Metapontus where he dragged out his poverty-stricken existence 4 years more. Finally democracy triumphed there also; the house in which was the school was burned , many disciples died a death of torture and Pythagoras himself with difficulty having escaped the flames died soon after in his 99thyear.*

Supplementary Historical Date

To the following (Graap's) translation, out of the Russian, relative to the great master Pythagoras, these interesting statements are due.

"Fifteen hundred years before the time of Pythagoras, (549-470 B.C.),** the Egyptians constructed right angles by so placing three pegs that a rope measured off into 3,4 and 5 units would just reach around them, and for this purpose professional 'rope fasteners' were employed.

" Today carpenters and masons make right angles by measuring off 6 and 8 feet in such a manner that a 'ten-foot pole' completes the tringle.

"Out of this simple Nile-compelling problem of these early Egyptian rope-fasteners Pythagoras is said to have generalized and proved this important famous theorem, - the square upon the hypotenuse of a right triangle is equal to the sum of the squares upon its two legs, --- of which the right triangle whose sides are 3,4 and 5 is a simple and particular case; and for having proved the universal truth implied in the 3-4-5 triangle, he made his name immortal --- written indelibly across the ages.

In speaking of him and his philosophy, the Journal of the Royal Society of Canada, Section II, Vol. 10, 1904, p. 239, says: "He was the Newton, the Galileo, perhaps the Edison and Marconi of his Epoch……
'Scholars now go to Oxford, then to Egypt, for fundamentals of the past

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*Note. The above translation is that of Dr, Theodore H. Johnston, Principal (1907) of the West High School, Caeveland,o.

**Note. From recent accredited biographical data as to Pythagoras, the record reads: "Born at Samos, c. 582B.C. Died probably at Metapontum, c. 501,B.C.

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……….The philosophy of Pytagoras is Asiatic --- the best of India --- in origin, in which lore he became proficient; but he committed none of his views to writing and forbid his followers to do so, insisting that they listen and hold their tongues.'"

He was indeed the Sarvonarola of his epoch; he excelled in philosophy, mysticism, geometry, a writer upon music, and in the field of astronomy he anticipated Coperincus by making the sun the center of the cosmos. "His most original mathetical work however, was probably in the Greek Arithmetica, or theory of numbers, his teachins being followd by all subsequent Greek writers on the subject."

Whether his proof of the famous theorem was wholly original no one knows; but we now know that geometers of Hindustan knew this theorem centuries before his name; But he of all the masters or antiquity, carries the honor of its place and importance in our Euclidian Geomety.

On account of its extensive application in the field of trigonometry, surveying, navigation and astronomy, it is one of the mist, if not the most, interesting propositions in elementary plane geometry.

It has been variously denominated as, the Pythagorean Theorem, The Hecatomb Proposition, the Carpenter's Theorem, and the Pons Asinourm because of its supposed difficulty. But the term "Pons Asinorum" also attached to Theorem V, properly, and to Theorem xx erroneously, of Book I of Euclid's Elements of Geometry.

It is regarded as the most fascinating Theorem of all Euclid, so much so, that thinkers from all classes and nationalities, from the aged philosopher in his armchair to the young soldier in the trenches next to no-man's –land, 1917, have whiled away hours seeking a new proof of its truth.

Camerer,* in his notes on the First Six Books of Eculed's Elements gives a collection of 17 different demonstrations of this theorem, and from time to time others have made collections, --- one of 28, another of 33, Wipper of 46, Versluys of 96, the American Mathematicla Monthly has 100, others of lists ranging from a few to over 100, all of which proofs, with credit, appears in this (now, 1940) collection of over 360 different proofs, reaching in time, from 900B.C., to 1940C.E.

Some of these 367 proofs, --- supposed to be new ---are very old; some are short and simple; others are long and complex; but each is a way of proving the same truth.

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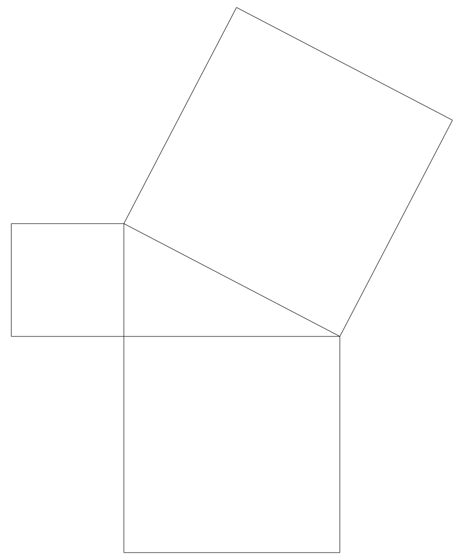
*Note. Perhaps J.G, See Notes and Queries, 1879,Vol,V, No.

Read and take your choice; or better, find a new a different proofs possible, whose figure will be different from any one found herein

10

Come and take choice of all my Library.

---------Titus Acdronicus.



"Mathematics is queen of the sciences and arithmetic is queen of Mathematics. She often condescends to render service to astronomy and other natural sciences, but under all circumstances the first place is her due."

Gauss. (1777-1855)

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THE PYTHAGOREAN THEOREM

From an Arthemetico—Algebric Point of View

Dr, J.W.L, Glashier in his address before Section A of the British Association for the Advancement of Science, 1890, said: "Many of the greatest masters of the Mathematical Science were first attracted to mathematical inquiry by problems concerning numbers, and one can glance at the periodicals of the day which contains questions for solution without noticing how singular a charm such problems continue to exert."

One of these charming problems was the determination of "Triads of Arithmetical Integers" such that the sum of the squares of the two lesser shall equal the square of the greater number.

These triads, groups of three, represent the three sides of a right triangle, and are infinite in number.

Many ancient master mathematicians sought general formulas for finding such groups, among whom worthy of mention were Pythagoras (c.582-c. 501 B.C.), Plato (429-348 B.C.), and Euclid (living 300B.C.), because of their rules for finding such triads.

In our public libraries may be found many publications containing data relating to the sum of two square number whose sum is a square number among which the following two mathematical magazines are especially worthy of notice, the first being "The Mathematical Magazine," 1891, Vol No, 5, in which p. 69, appears an article by that master Mathematical Analyst, Dr, Artemas Martin, of Washington, D.C.; the second being "The American Mathematical Monthly," 1894, Vol. No.1, in which , p. 6, appears an article by Leonard E. Dickson, B.Sc., then Fellow in pure Mathematics, University of Texas.

Those who are interested and desire more data relative to such number then here culled therefrom, the same may be obtained from these two Journals.

From the article by Dr.Martin. “Any number of square numbers whose sum is a square number can be found by various rigorous methods of solution.”

Case I. Let it be required to find two square numbers whose sum is a square number.

First Method. Take the well-known identity

$(x+y)^2 = x^2 \;\; 2xy \;\; + \;\; y^2 = (x - y) \;\; +4xy$.---------------(1)

Now if we can transform 4xy into a square we shall have expressions for two square numbers whose sum is a square number.

Assume $x = mp^2$ and $y = mq^2$, and we have $4xy = 4m^2p^2q^2$, which is a square number for all values of m, p and q; and (1) becomes, by substitution, $(mp^2 + mq^2)^2 = (mp^2 – mq^2)^2 + (2mpq)^2$ or striking out the common square factor m^2, we have $(p^2 + q^2)^2 = (p^2 – q^2) + (2pq)^2$.-------(2)

Dr. Martin follows this by a second and a third method, and discovers that both (second and third) method reduce, by simplification, to formula (2).

Dr. Martin declares, (and supports his declaration by the investigation of Matthew Collins’ “Tract of the Possible and Impossible Cases of Quadratic Duplicate Equalities in the Diophantine Analysis, published at Dublin in 1858), that no expression for square numbers whose sum is a square can be found which are not deducible from this, or reducible to this formula – that $(2pq)^2 + (p^2 - q^2)^2$ is always equal to $(p^2 + q^2)^2$.

His numerical illustrations are:

Example 1. Let p = 2, and q = 1; then $p^2 + q^2 = 5$, $p^2 - q^2 = 3$, 2pq = 4 and we have $3^2 + 4^2 = 5^2$.

Example 2. Let p = 3, and q = 2; then $p^2 + q^2 = 13$, $p^2 + q^2 = 13$, $p^2 - q^2$

= 5, 2pq = 12 , $5^2 + 12^2 = 13^2$, etc,. ad infinitum.

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From the article by Mr. Dickson: ‘ Let the three integers used to express the three sides of a right triangle be prime to each other, and be symbolized by a, b and h. Then these facts follow:

1. They can not all be even numbers, otherwise they would still be divisible by the common divisor 2.

2. They can not all be odd numbers. For $a^2 + b^2 = h^2$. And if a and b are odd, their squares is even; i.e., h^2 is even. But if h^2 is even h must be even.

3. h must always be odd; and ,of the remaining two, one must even and the other odd. So two of the three integers, a b and h, must be odd. (For proof, see p.7 Vol. I of said Am. Math. Monthly.)

4. When the sides of a right triangle are integers, the perimeter of the triangle is always an even number.

Rules for finding integral values for a, b, and h.

1. Rule of Pythagoras: Let n be odd; then n, $(n^2-1)/2$ and $(n^2+1)/2$ are three such numbers. For $n^2+ [(n^2-1)/2]^2 = (4n^2+ n^4 - 2n^2+1)/4 =(n^2+1)/2]^2$.

2. Plato’s Rule: Let m be any even number divisible by 4; then m,

$(m^2/4)-1$, and $(m^2/4) + 1$ are three such numbers. For $m^2 + \{(m^2/4)-1\}^2 = m^2 + \{(m^4/16) - (m^2/2) +1 = (m^4/16) + m^2/2 + 1 = \{(m^2/4) + 1\}^2$.

3. Euclid’s Rule: Let x and y be any two even or odd numbers, such that x and y contain no common factor greater than 2, and xy is a square. Then $\sqrt{xy}$, $(x-y)/2$ and $(x + y)/2$ are three such numbers. For $(\sqrt{xy})^2 + \{ (x + y)/2\}^2 = xy + (x^2 - 2xy + y^2)/4 = \{ (x + y)/2\}^2$.

4. Rule of Maseres (1721- 1824) : Let m and n be any two even or odd , m>n, and $(m^2 + n^2)/2n$ an integer. Then m^2, $(m^2 + n^2)/2n$ and $(m^2- n^2)/2n$ are three such numbers. For $m^2 + (m^2-n^2)/2n = (4m^2n^2 + m^4 - 2m^2 + n^2 + n^4)/2n^2 = \{(m^2 + n^2) /2n \}^2$.

5. Dickson’s Rule: Let m and n be any two prime integers, one even and other odd, m > n and 2mn a square. Then $m + \sqrt{2mn}$, $n + \sqrt{2mn}$ and m

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$+ n + \sqrt{2mn}$ are three such numbers. For $(m + \sqrt{2mn})^2 + (n + \sqrt{2mn})^2$ $+ m^2 + n^2 + 4mn + 2m\sqrt{2mn} + 2n\sqrt{2mn} = (m + N = \sqrt{2mn})^2$.

6. By inspection it is evident that these five rules, --- the formulas of Pythagoras, Plato, Euclid, Maseres, and Dickson,-- each reduces to the formula of Dr. Martin.

In the Rule of Pythagoras: multiply by 4 and square and there results $(2n)^2 + (n^2 – 1)^2 = (n^2 + 1)^2$, in which p = n and q = 1.

In the Rule of Plato: multiply by 4 and square and there results $(2m)^2 + (m^2 - 2^2)^2 = (m^2 + 2^2)^2$, in which p = m and q = 2.

In the Rule of Euclid: multiply by 2 and square there results $(2xy)^2 + (x - y)^2 = (x + y)^2$, in which p = x and q = y.

In the Rule of Maseres : multiply by 2n and square and results are $(2mn)^2 + (m^2 - n^2) = (m^2 + n^2)$, in which p = m and q = n.

In the Rule of Dickson: equating and solving

$p = \sqrt{\{(m + n + 2\sqrt{2mn}) + \sqrt{(m - n)}\}/2}$ and

$q = \sqrt{\{(m + n + 2\sqrt{2mn}) - \sqrt{(m - n)}\}/2}$

Or if desired, the formulas of Martin, Pythagoras, Pluto Euclid and Maseres may be reduced to that of Dickson.

The advantage of Dickson's Rule is this: It gives every possible set of values for a, b and h in their lowest terms, and gives this set but once.

To apply his rule, proceed as follows: Let m be any odd square whatsoever, and n be the double of any square number whatsoever not divisible by m.

Examples. If m = 9 n may be the double of 1,4,16, 25, 49 etc,; thus when m = 9, and n = 2 , then $m + \sqrt{2mn} = 15$, $n + \sqrt{2mn} = 8$, $m + n + \sqrt{2mn} = 17$. So a = 8, b = 15 and h = 17

If m = 25, and n = 8 we get a = 3, b = 4, h = 5.

If m = 25, and n = 8, we get a = 25, b = 45, h = 53, etc., etc, Table of integers for values of a, b and h have been calculated.

Halsted's table (in his "Mensuration") in absolutely as far the 59th set of values.

MEHTODS OF PROOF

Method is the following of one thing through another. Order is the following of one thing after another.

The type and form of a figure necessarily determine the possible argument of a derived proof; hence, as an aid for reference, an order of arrangement of the proofs is of great importance.

In this exposition of some proofs of the Pythagorean theorem the aim has been to classify and arrange them as to method of proof and type of figure used; to give the name, in case it has one, by which the demonstration is known; to give the name and page of the journal, magazine or text wherein the proof may be found, if known; and occasionally to five other interesting data relative to certain proofs.

The order of arrangement herein is, only in part, my own, being formulated after a study of the order found in the several groups of proofs examined, but more especially of the order of arrangement given in The American Mathematical Monthly, Vols. III and IV, 1896-1899.

It is assumed that the person using this work will know the fundamentals of plane geometry, and that, having the figure before him, he will readily supply the "reasons why" for the steps taken as, often from the figure, the proof is obvious; therefore only such statements of construction and demonstration are set forth in the text as is necessary to establish the argument of the particular proof.

The Methods of Proof Are:

I. ALGEBRAIC PROOF THROUGH LINEAR RELATIONS

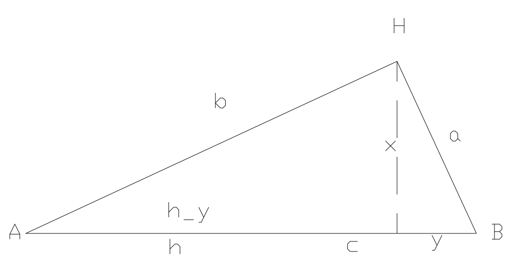
A. Similar Right triangles

From linear relations of similar right triangles it may be proven that. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

And since the algebraic square is the measure of the geometric square, the truth of the proposition as just stated involves the truth of the proposition as stated under Geometric Proofs through comparison of areas. Some algebraic proofs are the following:

<u>O n e</u>

In rt, tri. Fig.1



ABH draw HC perp. To AB. The tri's ABH, ACH and HCB are similar. For convenience, denote BH, AH, AB,HC,CB and AC by a,b,h,x,y,and h-y resp'y. Since, from three similar and related triangles, there are possible nine similar proportions and their resulting equations are:

(1) $a : x = b : h - y$ ∴ $ah - ay = bx$

(2) $a : y = b : x$ ∴ $ax = by$.

(3) $x : y = h - y : x$ ∴ $x^2 = hy - y^2$.

(4) $a : x = h : b$ ∴ $ab = hx$.

(5) $a : y = h : a$ ∴ $a^2 = hy$.

(6) $x : y = b : a$ ∴ $ax = by$.

(7) $b : h - y = h : b$ ∴ $b^2 = h^2 - hy$.

(8) $b : x = h : a$ ∴ $ab = hx$.

(9) $h - y : x = b : a$ ∴ $ah - ay = bx$. See Versluys, p. 86, fig. 97, Wm. W. Rupert.

Since equation (1) and (9) are identical, also (2) and (6), and (4) and (8), there remain but six different equations, and the problem becomes, how may these six equations be combined so as to give the desired relation $h^2 = a^2 + b^2$, which geometrically interpested is is $AB^2 = BH^2 + HA^2$.

In this proof *one*, and in every case hereafter, as in proof *Sixteen*, p. 41 the symbol AB^2 or a like symbol, signifies AB^2.

Every rational solution of $h^2 = a^2 + b^2$ after fords a Pythagorean triangle. See "Mathematical Monograph, No. 16, Diopha tine Analysis," (1915) , by R.D. Carmichael.

1st. Legendre's Solution

a. From no single equation of the above nine can the desired relation be determined, and there is but one combination of two equations which will give it; viz., (5) $a^2 = hy$; (7) $b^2 = h^2 - hy$; adding these gives $h^2 = a^2 + b^2$.

This is the shortest proof possible of the Pythagorean Proposition.

b. Since equations (5) and (7) are implied in the principal that homologous sides of similar triangles are proportional it follows that the truth of this important proposition but a corollary to the more general truth—the law of similarity.

c. See Davis Legendre, 1858, p. 112 Journal of Education, 1888, V. XXV, p. 404, fig.v.

Heath's Math. Monograph, 1900 No. 1, p. 19, proof III, or any late text on geometry.

d. W. W. Rouse Ball, of Trinity Collage, Cambridge, England seems to think Pythagoras knew of this proof.

2nd. Other Solutions

a. By the law of combinations there are possible 20 sets of three equations out of the six different equations. Rejecting all sets containing (5) and (7) and all sets containing dependent equations, there are remaining 13 stes from which the elimination of x and y may be accomplished in 44 different ways each giving a distinct proof for the relation $h^2 = a^2 + b^2$.

b. See the American Math. Monthly, 1896, V. III p. 66 or Edward's Geometry, p, 157, fig. 15.

Two

Produce AH to C so that CB will be perpendicular to AB at B.

Denote a, b, h, x and y resp'y.

The triangle ABH , CAB and BCH are similar.

From the continued proportion b: h : a =a: x:y =h :b+y: x nine different simple proportions are possible, viz,

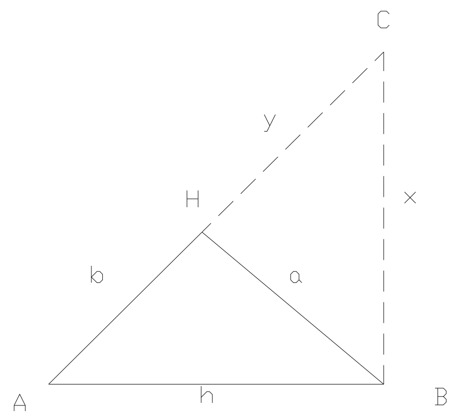


Fig.2

a. possible 20 sets of three equations out of the six different equations. Rejecting all sets containing (5) and (7) and all sets containing dependent equations, there are remaining 13 stes from which the elimination of x and y may Since equations (5) and (7) are implied in the principal that homologous sides of similar triangles are proportional it follows that the truth of this important proposition but a corollary to the more general truth—the law of similarity.

b. See Davis Legendre, 1858, p. 112 Journal of Education, 1888, V. XXV, p. 404, fig.v.

Heath's Math. Monograph, 1900 No. 1, p. 19, proof III, or any late text on geometry.

d. W. W. Rouse Ball, of Trinity Collage, Cambridge, England seems to think Pythagoras knew of this proof.

2nd. Other Solutions

a. By the law of combinations there are be accomplished in 44 different ways each giving a distinct proof for the relation $h^2 = a^2 + b^2$.

b. See the American Math. Monthly, 1896, V. III p. 66 or Edward's Geometry, p, 157, fig. 15.

(1) b : h = a : x. (7) a : x = h : b + y.

(2) b : a = a : y (8) a : y = h : x.

(3) h : a = x : y (9) x : b + y = y : x , from which six different equations are possible as in one above

(4) b : h = h : b + y.

(5) b : a = h : x.

(6) h : a = b +y : x.

1st. ---Solutions From Sets of Two Equations.

a. As in *one*, there is but one set of two equations, which will give the relation $h^2 = a^2 + b^2$.

b. See Am. Math. Mo. V. III, p. 66.

2nd .---Solution Form Sets of Three Equations.

a. As in 2nd under proof *one*, *fig.* 1, there are 13 sets of three eq's, gives 44 distinct proofs that give $h^2 = a^2 + b^2$.

b. See Am. Math. Mo., V. III p. 66.

c. Therefore from three similar. rt. tri's so related that any two have one side in common there are 90 ways of proving that $h^2 = a^2 + b^2$.

Three

Take BD = BH and at D draw CD perp. to AB forming the two similar tri's. ABH and CAD.

a. From the continued proportion a : x = b : h = h : b –x the simple proportions and their resulting eq's are:

(1) $a : x = b : h - a \therefore ah - a^2 = bx$.

(2) $a : x = h : b - x \therefore ab - ax = hx$.

(3) $b : h - a = h : b - x \therefore b^2 - bx = h^2 - ah$.

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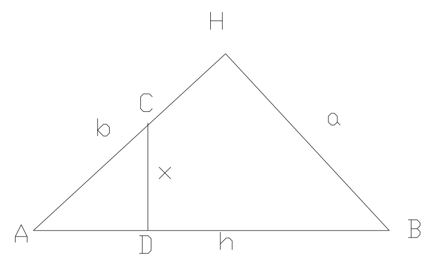

Fig. 3

As there are but three equations and as each equation contains the unknown x in the 1st. degree, there are possible but three solutions giving $h^2 = a^2 + b^2$.

b. See Am. Math. Mo., V. III p. 66 and Math. Mo., 1859, V. II, No. 2, Dem. Fig. 3, on p. 45 by Richardson.

FOUR

In Fig. 4 extend AB to C making BC = BH, and draw CD perp. to AC. Produce AH to D, forming the two similar tri's ABH and ADC.

From the continued proportion $b : h + a = a : x = h : b + x$ three equations are possible giving, as in fig. 3, three proof.

a. See Am. Math. Mo., V. III, p. 67.

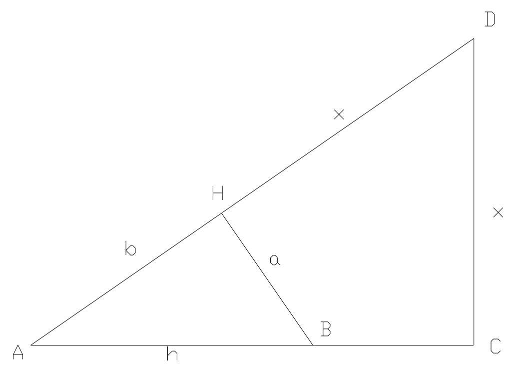

Fig 4.

FIVE

Draw AC the bisector of the angle HAB, and CD prep. to AB, forming the similar tri's ABH and BCD. Then CB = a – x and DB = h – b.

From the continued proportion h : a – x = a : h – b = b : x three equations are possible giving, as in fig. 3, three proofs for $h^2 = a^2 + b^2$.

a. Original with the author, Feb. 23, 1926.

Six

Through D, any pt. in either leg of the rt. Triangle ABH, draw DC perp. to AB and extend it to E a pt. in the other leg produced, thus forming the four similar rt. tri's ABH, BEC, ACD and EHD. From the continued proportion (AB = h) : (BE = a + x) : (ED =v) : (DA = b – y) = (BH = a) : (BC = h – z) : (DH = y) : (DC = w) = (AH = b) : (CE = v + w) : (HE = x) : (CA = z), eighteen simple proportions and eighteen different equations are possible.

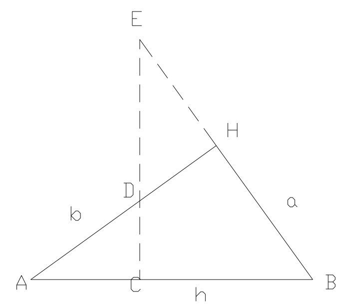

Fig.6

From no single equation nor from any set of two eq's can the relation $h^2 = a^2 + b^2$ be found but from combination of eq's involving three, four or five of the unknown elements u, w, x, y, z, solutions may be obtained.

1st. Proof from sets involving Three Unknown Elements.

a. It has been shown that there is possible but one combination of equations involving but three of the unknown elements, viz., x,y and z which will give $h^2 = a^2 + b^2$.

b. See Am. Math. Mo. , V. III, p. III.

2nd. Proofs From Sets Involving Four Unknown Elements.

a. There are possible 114 combinations involving but four of the unknown elements each of which will give $h^2 = a^2 + b^2$.

b. See Am. Math. Mo., V, III, p. III.

3rd. Proof From Sets Involving All Five Unknown Elements

a. Similarly, there are 4749 combinations involving all five of the un-

- knowns, from each of which $h^2 = a^2 + b^2$ can be obtained.

b. See Am. Math. Mo., V. III, p. 112.

c. Therefore the total no. of proofs from the relations involved in fig. 6 is 4864.

Seven

Produce AB to E, fig, 7, and through E draw, perp. to AE the line CED meeting meeting AH produced in D forming the four similar rt. tir's ABH, DBE, CAE and CDH.

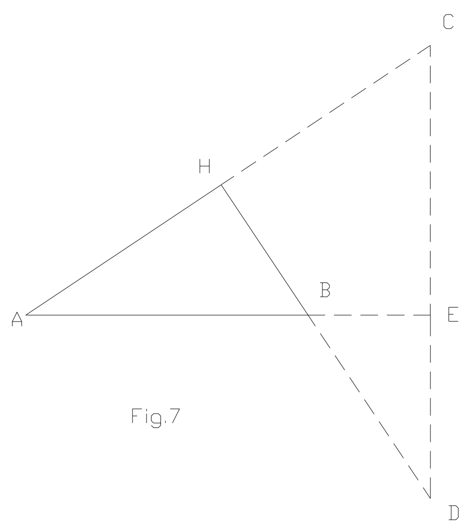

Fig.7

a. As in fig. 6, eighteen different equations are possible from which there are also 4864 proofs.

b. Therefore the total no. of ways of proving that $h^2 = a^2 + b^2$ from 4 similar rt. tri's. related as in fig.6 and 7 is 9728.

c. As the pt. E approaches the pt. B. fig, 7 approached fig. 2, above, and become fig. 2, when E falls on B.

d. Suppose E falls on AB so that CE cuts HB between H and B; then we will have 4 similar rt. tri's involving 6 unknowns. How many proofs

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-                will result?

Eight

In fig. 8 produce BH to D, making BD = BA, and E, the middle pt. of AD, draw EC parallel to AH, and join BE, forming the 7 similar rt. tringales AHD, ECD, BEA, BCE, BFH and AEF, but six of which need consideration, since tri's BED and BEA are congruent and in symnolzation, identical.

See Versluys, p. 87 fig. 98, Hoffmann, 1818.

From these 6 different rt. triangles, stes of 2 tri's may be selected in 15 different ways, sets of 3 tri's may be selected in 20 different ways, sets of 4 tri's may be selected in 15 different way's sets of 5 tri's may be selected in 6 different ways, and sets of 6 tri's may be selected in 1 way, giving, in all, 57 different ways in which the 6 triangles may be combined.

But as all the proofs derivable from the sets of 2, 3, 4, or 5 tri's are also found among the proofs from the sets of 6 triangles, an investigation of this set will suffice for all.

In the 6 similar rt. tri's let AB = h, BH = a, HA = b, DE = EA = x, BE = y, FH = z and BF = v, whence EC = b/2, DH = h – a, DC = h – a /2, EF = y – v, BE = h +a /2, AD = 2x and AF = b – z, and from these data the continued proportion is: b : b/2: y : (h – a) /2 : a : x = h – a : (h – a) /2 : x : b /2 : z : y – v = 2x : x : h : y : v : b – z.

From this continued proportion there result 45 simple proportions which give 28 different equations, and, as groundwork for determining the number of proofs possible, they are here tabulated.

(1) b : b/2 = h – a : (h – a)/ 2, where 1 = 1. Eq. 1.

(2) b : b/2 = 2x : x, whence 1 = 1. Eq. 1.

(3) h – a : (h – a) /2 = 2x : x, whence 1 – 1. Eq. 1^3.

(4) b : y = h – a : x whence bx = (h – a) y. Eq. 2.

(5) b : y = 2x : h, whence 2xy – bh. Eq. 3.

(6) $h – a : x = 2x : h$, whence $2x^2 = h^2 – ah$. Eq. 4

(7) $b : (a + h)/2 = h – a : /2$, whence $b^2 = h^2 - a^2$. Eq. 5.

(8) $b : (h + a) /2 = 2x : y$, whence $(h + a) x = by$ Eq. 6.

(9) $h – a : b/2 = 2x : y$, whence $bx = (h – a) y$. Eq. 2.

(10) $b : a = h – a : z$, whence $bz = (h – a)a$. Eq. 7.

(11) $b : a = 2x : v$, whence $2ax =bv$. Eq. 8.

(12) $h – a : z = 2x : v$, whence $2xz = (h – a) v$. Eq. 9.

(13) $b : x = h – a : y - v$, whence $(h – a)x = b(y – v)$. Eq. 10

(14) $b : x = 2x : b – z$, whence $2x^2 = b^2 - bz$. Eq. 11.

(15) $h – a : y - v = 2x : b - z$, whence $2(y – v) z = (h – a) (b – z)$. Eq. 12

(16) $b/2 : y = (h – a) / 2 : x$, whence $bx = (h – a) y$, Eq. 2.

(17) $b/2 : y = x : h$, whence $2xy = bh$. Eq. 3.

(18) $(h – a)/ 2 : x = x : h$, whence $2x^2 = h^2 – ah$. Eq. 4^2.

(19) $h/2 : (h + a) /2 = (h – a) /2 : b/2$, whence $b^2 = h^2 – a^2$. Eq, 5^2.

(20) $b/2 : (h + a) /2 = x : y$ whence $(h + a) x = by$. Eq. 6.

(21) $(h – a) / 2 : b/2 = x : y$, whence $(h – a) y$. Eq. 2^4.

(22) $b/2 : a = (h – a) /2 : z$, whence $bz = (h –a) a$. Eq. 7^2

(23) $b/2 : a = x : v$, whence $2ax = bv$. Eq. 8^2

(24) $(h – a) /2 : z = x : v$, whence $2xz = (h –a) v$. Eq. 9^2.

(25) $b/2 : x = (h – a) /2 : y – v$, whence $(h – a) x = b (y – v)$. Eq. 10^2.

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(26) $b/2 : x = x : b – z$, whence $2x^2 = b^2 + bz$. Eq. 11^2.

(27) $(h – a) / 2 : y – v = x : b – z$, whence $2(y – v) x = (h – a) (b – z)$. Eq. 12^2.

(28) $y : (h + a) /2 = x : b/2$, whence $(h + a) x = by$. Eq. 6^3.

(29) $y : (h + a) 2 = h : y$, whence $2y^2 = h^2 + ah$. Eq. 13.

(30) $x : b/2 = h : y$, whence $2xy = bh$. Eq. 3^3.

(31) $y : a = x : z$, whence $ax = yz$. Eq. 14.

(32) $y : a = h : v$, whence $vy = ah$. Eq. 15

(33) $x : z = h : v$, whence $vx = hz$. Eq. 16.

(34) $y : x = x : y – v$, whence $x^2 = y(y – v)$. Eq. 17.

(35) $y : x = h : b - z$, whence $hx = y(b – z)$. Eq. 18.

(36) $x : y – v = h : b – z$ whence $(b – z) x = (y – v)$. Eq. 19.

(37) $(h + a) /2 : x = y : v$, whence $2ay = (h + a)z = ab$. Eq. 20.

(38) $(h + a)/2 : x = y : v$, whence $2ay = (h + a)v$. Eq. 21.

(39) $b/2 : z = y : v$, whence $2yz = bv$. Eq. 22.

(40) $(h + a)/2 : x = b/2 : y – v$, whence $bx = (h + a)(y –v)$. Eq.23.

(41) $(h + a) /2 : x = y : b – z$, whence $2xy = (h + a)(b – z)$. Eq. 24.

(42) $b/2 : y – v = y : b – z$. whence $2y(y – v) = b = b^2 – bz$. Eq. 25.

(43) $a : x = z : y – v$, whence $xz = a(y – v)$. Eq, 26.

(44) $a : x = v : b – z$, whence $vx = a(b – z)$. Eq. 27

(45) $z : y – v = v : b – z$. whence $v(y – v) = (b – z) z$. Eq. 28.

The symbol 2^4, see (21), means that equation 2 may be derived from 4 different proportions. Similarly for 6^3, etc.

Since a definite no. of sets of dependent equations, three equations in each set, is derivable from a given continued proportion and since these sets must be know and dealt with in establishing the no. of possible proofs for $h^2 = b^2 + a^2$, it becomes necessary to determine the no. of such sets. In any continued proportion the symbolization for the no. of such sets, three equations in each set, is $\{n^2(n + 1))\} /2$ in which n signifies the no. of simple ratios in a member of the continued porp'n. Hence for the above continued proportion there are derivable 75 such sets of dependent equations. They are:

(1), (2), (3), (4), (5), (6);(7), (8), (9), (10), (11), (12);(13), (14), (15);(16), (17), (18);(19), (20), (21);(22), (23), (24);(25), (26), (27);(28), (29), (30);(31), (32), (33);(34), (35), (36); (37) (38), (39); (40), (41), (42); (43), (44), (45); (1), (4), (16); (1), (7), (19); (1), (10) (22); (1), (13), (25); (4), (7), (28); (4), (10), (31); (4), (13), (34); (7), (13), (40); (10), (13), (43);(16), (19), (20); (16) (22), (31); (16), (25), (34); (19), (22), (37); (19),(25), (40);(22), (25), (43); (28),(310), (37); (28), (34), (40);(31),(34), (43); (37), (40), (43);(2), (5) , (17); (2), (8), (20); (2), (11), (23); (2), (14), (26);(5), (8), (29); (5), (11), (32); (5), (14), (35); (8), (11), (38); (8), (14), (44); (17), (20), (29); (17), (23), (32); (17), (26), (35); (20), (23), (38); (20), (26), (41), (23), (26), (44); (29), (32), (38); (29), (35), (41); (32), (35), (44); (3), (6), (18); (3), (9), (21); (3); (12), (33); (6), (15), (27); (6), (12), (33); (6), (15), (36); (9), (12), (36); (9), (15), (42); (12), (15), (45), (18), (21), (24), (39); (21), (27), (42); (24), (27), (45); (30), (33), (39); (30), (36), (42); (33), (36), (45);(39), (42), (45).

These 75 sets expressed in the symbolization of the 28 eqations give but 49 sets as follows:

1, 1, 1; 2, 3, 4; 2, 5, 6; 7, 8, 9; 10, 11, 12; 6, 13, 3; 14, 15, 16; 17, 18, 19; 20, 21, 22; 23, 24, 25; 26, 27, 28; 1, 2, 2; 1, 5; 1, 7, 7; 1, 10, 10; 1, 6, 6; 2, 7, 14; 2, 10, 17; 5, 7, 20; 5, 10, 23; 7, 10, 26; 6, 14, 20; 6, 17, 23; 14, 17, 26;20, 23, 26; 1, 3, 3; 1, 8, 8; 1, 11, 11; 3, 8, 15; 3, 11, 18; 6, 8, 21; 6, 11, 24; 8, 11, 27; 13, 15, 21; 13, 18, 18, 24; 15, 18, 27; 21, 24. 27; 1, 4, 4; 1, 9, 9; 1, 12, 12; 4, 9, 16; 4, 12, 19; 2, 9, 22; 2, 12, 25; 9, 12,28; 3, 16, 22; 3, 19, 25; 16, 19, 28; 22, 25, 28;

Since eq. 1 is an identity and eq. 5 gives, at once , $h^2 = a^2 + b^2$, there are remaining 26 equations involving the 4 unknowns x, y, z, and v, and proofs may be possible from sets of equations involving x and y, x and z, x and v, y and z, y and v, z and v, x, y and z, x, y and v, x. z and v, y , z and v, and x , y, z and v.

1st. – proofs From Sets Involving Two Unknowns.

a. The two unknowns, x and y, occur in the following five equations, viz., 2, 3, 4, 6, and 13, from which but one set of two, viz., 2, 6, will give $h^2 + a^2 = b^2$, and as eq. 2 may be derived from 4 different proportions, the no. of proofs from this set are 12.

Arrange in sets of three we get,

$2^{4,}$ 3^3, 13 giving 12 other proofs;

(2,3, 4) a dependent set – no proofs;

2^4, 4^2, 13 giving 8 other proofs;

(3, 6, 13) a dependent set – no proofs;

3^3 ,4^2, 6^3 giving 18 other proofs;

4^2, 6^3, 13 giving 6 other proofs;

3^3, 4^2, 13 giving 6 other proofs.

Therefore there are 62 proofs from sets involving x and y.

b. Similarly, from sets involving x and z there are 8 proofs, the equations for which are 4, 7, 11, and 20.

c. Sets involving x and v give no additional proofs.

d. Sets involving y and z gives 2 proofs, but the equations were used in a and b, hence cannot be counted again, they are 7, 13 and 20.

e. Sets involving y and v give on proofs.

f. Sets involving z and v give same results as d.

Therefore the no. of proofs from sets involving two unknowns is 70, making in all 72 proofs so far, since $h^2 = a^2 + b^2$ is obtained directly from two different prop's.

2nd. – Proofs From Sets Involving Three Unknowns

a. The three unknowns x, y and z occur in the following 11 equations, viz., 2, 3, 4, 6, 7, 11, 13, 14, 18, 20 and 24, and from these 11 equations sets of four can be selected in 11x10x9x8/4x3x2x1=330 ways, each of which will gives one or more proofs for $h^2= a^2 + b^2$. But as the 330 sets, of four equations each, include certain sub-sets heretofore used, certain dependent sets of three equations each found among those in the above 75 sets, and certain sets of four dependent equations, all these must be determined and rejected; the proofs from the remaining sets will be proofs additional to the 72 already determined.

Now of 11 consecutive things arranged in sets of 4 each, any one will occur in 10x9x8/2x3 of 120 of the 330 sets, any two in 9x8 / 2 or 36 of the 330 ,and any three in 8/1 or 8 of the 330 sets. There fore any sub-set of two equations will be found in 36, and any of three equations in 8, of the 330 sets.

But some one or more of the 8, may be some one or more of the 36 sets; hence a sub-set of two and a sub-set of three will not necessarily cause a rejection of 36 + 8 = 44 of the 330 sets.

The sub-set which gave the 70 proofs are :

2, 6, for which 36 sets must be rejected;

7, 20, for which 35 sets must be rejected, since

7, 20, is found in one of the 36 sets above;

2, 3, 13, for which 7 other sets must be rejected, since

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2, 3, 13, is found in one of the 36 sets above;

2, 4, 13, for which 6 other sets must be rejected;

3, 4, 6, for which 7 other sets must be rejected;

4, 6, 13, for which 6 other sets must be rejected;

3, 4, 13, for which 6 other sets must be rejected;

4, 7, 11, for which 7 other sets must be rejected; and

4, 11, 20, for which 7 other sets must be rejected; for all of which 117 sets must be rejected.

Similarly the dependent sets of three, which are 2, 3, 4, 6, 13; 2, 7, 14; 6, 14, 20; 3, 11,18; 6, 11, 24; and 13, 18, 24, cause a rejection of 6 + 6 + 6 + 6 + 8 + 7 + 8, or 47 more sets.

Also the dependent sets of four, and not already rejected, which are, 2, 4, 11, 18; 3, 4, 7, 14; 3, 6, 18, 24; 3, 13, 14, 20; 3, 11, 13, 24; 6, 11, 13, 18; and 11, 14, 20, 24, cause a rejection or 7 more sets. The dependent sets of fours are discovered as follows: take any two dependent sets of threes having a common term as 2, 3, 4, and 3, 11, 18; drop the common term 3, and write the set 2, 4, 11, 18; a little study will disclose the 7 sets named, as well as other sets already rejected; e.g., 2, 4, 6, 13. Rejecting the 117 + 49 + 7 = 171 sets there remain 159 sets, each of which will give one or more proofs, determined as follows. Write down the 330 sets a thing easily done, strike out the 171 sets which must be rejected, and, taking the remaining sets one by one, determine how many proofs each will give; e.g., take the set 2, 3, 7, 11; write it thus 2^4, 3^3, 7^2, 11^2, the exponents denoting the different proportions from which the respective equations may be derived; the product of the exponents, 4x3x2x2x = 48, is the number of proofs possible for that set. The set 6^3, 11^2, 18^1, 20^1 gives 6 proofs, the set 14^1, 18^1, 20^1, 24^1 gives 6 proofs, the set 14^1, 18^1, 20^1, 24^1 gives but 1 proof; etc.

b. The three unknowns x, y and v occur in the following tweleve equations, -- 2, 3, 4, 6, 8, 10, 11, 13, 15, 17, 21 and 23, which give 495 different sets of 4 equations each, many of which must be rejected for same reasons as in a. Having established a method in a, we leave details to the one

interested.

c.Similarly for proofs from the eight equations containing x, z and v, and the seven eq's containing y, z and v, and the seven eq's containing y, z and v.

3^{rd}. Proofs From Sets Involving the Four Unknowns x, y, z and v.

a. The four unknowns occur in 26 equations; hence there are 26x25x24x23x22/5x4x3x2x1 = 65780 different sets of 5 equations each. Rejecting all sets containing sets heretofore used and also all remaining sets of five dependent equations of which 2, 3, 9, 19, 28, is a type, the determination of which 2, 3, 9, 19, 28, is a type, the determination of which involves a vast amount of time and labor if the method given in the preceding pages is followed. If there be a shorter method, I am unable, as yet, to discover it; neither am I able to find anything by any other investigator.

4^{th}. – Special Solutions

a. By an inspection of the 45 simple proportions given above, it is found that certain proportions are worthy of special consideration as they give equations from which very simple solutions follow.

b. Hoffman's solution.

Joh. Jos. Ign. Hofmann made a collection of 32 proofs, publishing the same in "Der Pythagoraisch Lehrasatz," 2^{nd}, edition Mainz, 1821, of which the solution from (7) is one. He selects the two triangles, (see fig. 8), AHD and BCE, from which b : (h + a) /2 = h – a : b/2 follows, giving at once $h^2 = a^2 + b^2$.

See Jury Wipper's 46 proofs, 1880, p. 98, credited to Hoffmann, 1818. Also see Math. Mo., Vol. II, No. II, p. 45, as given in Notes and Queries, Vol. 5, No. 43, p. 41.

c. Similarly from the two triangles BCE and ECD b/2 : (h + a)/2 = (h – a)/2 : b/2, $h^2 = a^2 + b^2$.

Also from the three triangles AHD, BEA and BCE proportions (4) and (8) follow, and from the three triangles AHD, BHE and BCE proportions (10) and (37) give at once $h^2 =a^2 + b^2$.

See Am. Math. Mo., V. III, pp. 169-70.

NINE

Produce AB to any pt. D From D draw DE perp. to AH produced, and from E drop the perp. EC, thus forming the 4 similar rt. tri's ABH, AED,ECD and ACE.

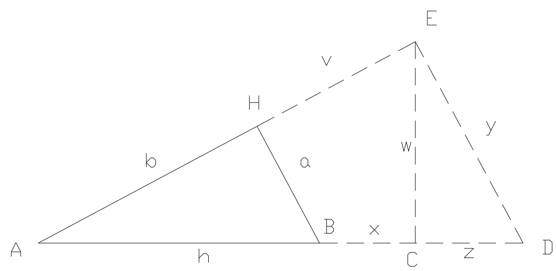

Fig. 9

From the homologous sides of these similar triangles the following continued proportion results:

(AH =b) : (AE = b + v) : (EC = w) : (AC = h + x) = (BH = a) : (DE = y) : (CD = z) : (EC = w) = (AB = h) : (AH = h + x + z) : (DE = y) : (AE = b + v). Note – B and C do not coincide.

a. From this continued prop'n 18 simple proportions are possible, giving, as in fig. 6, several thousand proofs.

b. See Am. Math. Mo., V. III, p. 171.

TEN

In fig. 10 are three similar rt. tri's ABH, EAC and DEF, from which the continued proportion,

(AH = b) : (AC = h + v) : (DF = DC = x) = (HB = a) : (CE = y) : (FE = z) = (AB = h) : (AE = h + v + z) : (DE = y – x).

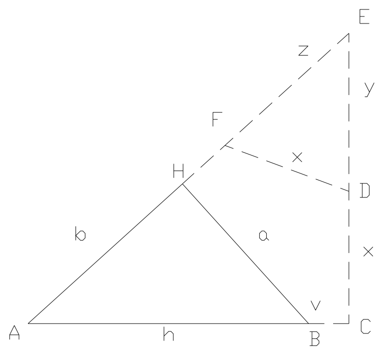

Fig. 10

Follows giving 9 simple proportions from which many more for $h^2 = a^2 + b^2$ may be obtained.

a. See Am, Maths. Mo ., V. III p. 171.

ELEVEN

From D in AH , so that DH = DC, draw DC par. to HB and DE perp. to a AB, forming the 4 similar rt. tri's ABH, ACD, CDE, and DAE, from which the continued proportion (BH = a) : (CD = DH = v) : (EC = y) : (DE = x) = (AH = b) : (DA = b – v) : (DE = x) : (AE = z) = (AB = h) : (AC = z + y) : (CD = v) : (AD = b – v).

Follows; 18 simple proportions are possible from which many more proofs for $h^2 = a^2 + b^2$ result.

By an inspection of the 18 proportions it is evident that they give no simple equations from which easy solutions follow, as was found in the investigation of fig, 8, as in *a* under proof *Eight.*

a. See Am. Math. Mo., V. III, p. 171.

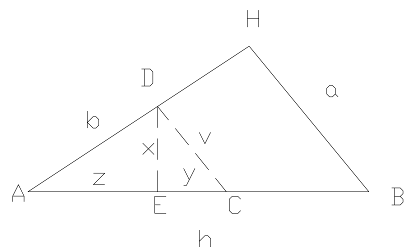

Fig. 11

TWELVE

The construction of fig. 12 gives five similar rt. triangles, which are: ABH, ADH, HBD, ACB and BCH, from which the continued prop'n (BH = a) : (HD = x) : (BD = y) : (CB = a^2/x) : (CH = ay/x) = (HA = b) : (DH = h – y) : (DH = x) : (BA = h) : (HB = a) = (AB = h) : (AH = b) : (HB = a) : (AC = b + ay/x) : (BC = a^2/x) follows, giving 30 simle propotions from which only 12 different equations result. From these 12 equations several proofs for $h^2 = a^2 + b^2$, obtainable.

a. In fig, 9, when C falls on B it is obvious that the graph become that of fig, 12. Therefore, the solution of fig. 12 is only a particular case of 12 are identical with those of case 1, proof *One.*

b. The above is an original method of proof by the author of this work.

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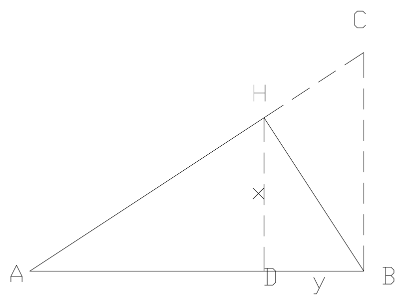

Fig.12

THRTEEN

Complete the paral. and draw HF perp. to, and EF par. with AB resp'ly, forming the 6 similar tri's, BHA, HCA, BCH, AEB, DEF. and DCB, form which 45 simple proportions are obtainable, resulting in several thousand more possible proof for $h^2 = a^2 + b^2$, only one of which we mention.

(1) From tri's DBH and BHA, DB : (BH = a) = (BH = a) : (HA = b); ∴ DB = a^2/b and (2) HD : (AB = h) = (BH = a) : (HA = b) ; ∴

HD = ah/b.

(3) From tri's DEF and BHA,

DF : (EB – DB) = (BH = a) : (AB = h), or DF : $b^2 - a^2/b$: a : h;

∴ DF = a { $(b^2 - a^2)$/bh}.

(4) Tri. ABH = 1/2par. HE =(½)AB xHC = (1/2) ab = (1/2) [AB {AC + CF}/2] = (1/2) [AB(HD + DF)] = (1/4) [h {ah/b + (a(b^2 – a^2)/bh)}] = ah^2/4b + ab/ 4 – a^3/4b :. (5) ½ ab = (ah^2 + ab^2 – a^2)/4b, whence (6) $h^2 = a^2 + b^2$.

a. This particular proof was produced by prof. D. A. Lehman, Prof. of Math. at Baldwin University, Berea, O., Dec. 1899.

b. Also see Am. Math. Mo. , V. VII, No. 10, p. 228.

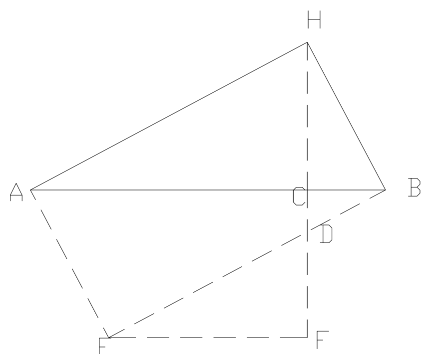

Fig. 13

FOURTEEN

Take AC and AD = AH and draw HC and DH.

Proof. Tri's CAH and HAD are isosceles. Angle CHD is a rt. angle, since A is equidistant from C, D and H.

Angle HDB = angle CHD + angle DCH. = angle AHD + 2 angle CHA = angle CHB.

:. tri's HDB and CHB are similar, having angle DBH in common and angle DHB = angle ACH.

:. CB : BH = BH : DB or h + b : a = a : h – b. Whence $h^2 = a^2 + b^2$.

a. See Math. Teacher, Dec., 1925. Credited to Alvin Knoer, a Milwaukee High School puple; also Versluys, p. 85 fig. 95; also

b. Encyclopadie der Elementar Mathematik, von H. weber and J. Wellsein, Vol. II, p. 242, where, (1905) , it is credited to C.G. Sterkenburg.

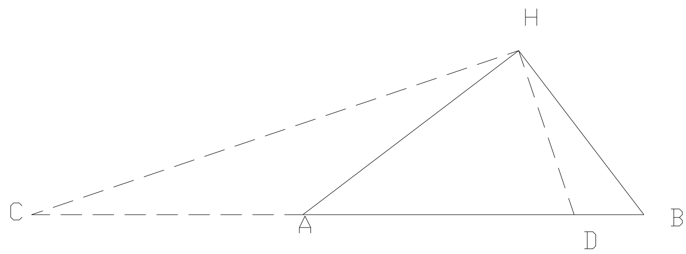

Fig. 14

FIFTEEN

In fig. 15 the const's is obvious giving four similar right trinangles ABH, AHE, HBE and HCD, from which the continued proportion (BH = a) : (HE = x) : (BE = y) : (CD = y/2) = (HA = b) : (EA = h – y) : (EH = x) : (DH = x/2) = (AB = h) : (AH = b) : (HB = a) : (HC = a/2) follows, giveng 18 simple proportions.

a. From the two simple proportions .

(1) a : y = h : a and

(2) b : h – y = h : b we get easily $h^2 = a^2 + b^2$.

c. This solution is original with the author, but, like cases 11 and 12, it is subordinate to case 1.

d. As the number of ways in which three or more similar right triangles may be constructed so as to contain related linear relations with but few unknowns involved is unlimited, so the number of

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possible proofs there from must be unlimited.

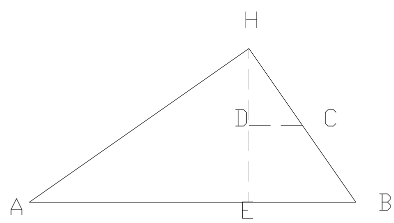


Fig. 15

SIXTEEN

The two following proofs, differing so much, in method, from those preceding, are certainly worthy of place among selected proofs.

1 St. – This proof rests on the axiom, "The whole is equal to the sum of its parts."

Let's AB = h, BH = a and AH =b, in the rt. tri. ABH, and let HC, C being the pt. where the perp. form H intersects the line AB, be perp. to AB. Suppose $h^2 = a^2 + b^2$. If $h^2 = a^2 + b^2$, then $a^2 = x^2 + y^2$ and $b^2 = x^2 + (h – y)^2$, or $h^2 = x^2 + y^2 + x^2 + (h – y)^2 = y^2 + 2x^2 + (h – y)^2 = y^2 + 2y(h – y) + (h – y)^2 = y + (h – y)^2$.

∴ h = y + (h – y), i.e. , AB = BC + CA, which is true.

The supposition is true, or $h^2 = a^2 + b^2$.

a. This proof is one of Joh. Hoffmann's 32 proofs. See Jure Wipper, 1880, p. 38, fig. 37.

2nd.—This proof is the "Reductio ad Absurdum" proof.

h^2<, = , or > ($a^2 + b^2$). Suppose it is less then , since $h^2 = [(h – y) + y]^2 +$

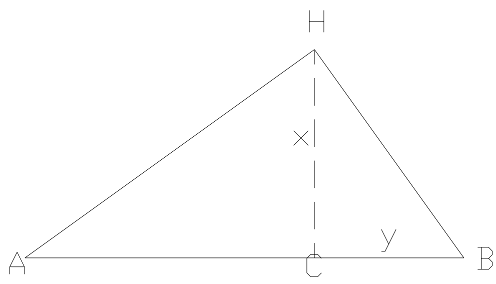


Fig. 16

$[(h – y) + x^2 + (h - y)]^2$ and $b^2 = [ax + (h – y)]^2$, then $[(h – y) + x^2 + (h – y) ^2 < [ax + (h – y)]^2 + a^2$.

$\therefore [x^2 + (h – y)^2]^2. < a^2[x^2 + (h – y)^2]$.

$\therefore a^2 > x^2 + (h – y) ^2$, which is absurd, For, if the supposition be true, we must have $a^2 < x^2 + (h – y)^2$, as is easily shown.

Similarly, the supposition that $h^2 > a^2+ b^2$, will be proven fasle.

Therefore it follows that $h^2 = a^2+ b^2$.

a. See Am. Math. Mo., V. III, p. 170.

SEVENTEEN

Take AE = 1, and draw EF perp. to AH perp. to AB. HC = (AC x FE)/FE, BC = (HC x FE)/AF, = (AC x FE) /AF x FE/AF = $ACxFE^2/AF^2$ and AB = AC x CB = AC + $CAxFE^2/ AF^2$ = AC (1 + $FE^2)/AF^2$ =$AC(AF^2 +FE^2) / AF^2$. (1).

But AB : AH = 1 : AF , whence AB = AH/ AF, and AH = AC /AF. Hence AB = AC $/AF^2$.(2)

$\therefore AC (AF^2 + EF^2)/AF^2 = AC /AF^2 \therefore AF^2 + FE^2 = 1$.

$\therefore$ AB : 1 AH : AF. $\therefore$ AH = AB x AF. (3).

And BH = AB x FE. (4).

$(3)^2 + (4)^2 = (5)^2$, or $AH^2 + BH^2 = AB^2 \times AF^2 + AB^2 \times FE^2 = AB^2 (AF^2 + FE^2) = AB^2$. $\therefore AB^2 = HB^2 + HA^2$, or $h^2 = a^2 + b^2$.

a. See Math. Mo., (1859) Vol. II, No, 2,Dec. 23, fig. 3.

b. An indirect proof follows. It is: If $AB^2 \neq (HB^2 + HA^2)$ let $x^2 = HB^2 + HA^2$ then $x = (HB^2 + HA^2)^{1/2} = HA\,(1 + HB^2/HA^2)^{1/2} = HA\,(1 + FE^2/FA^2)^{1/2} = HA[(FA^2 + FE^2)/FA^2]^{1/2} = HA/FA = AB$, since AB : AH = 1 : AF.

$\therefore$ If x = AB, $x^2 = AB^2 = HB^2 + HA^2$. Q.E.D.

b. See said Math. Mo. , (1859), Vol. II, No. 2, Dem. 24, fig. 3.

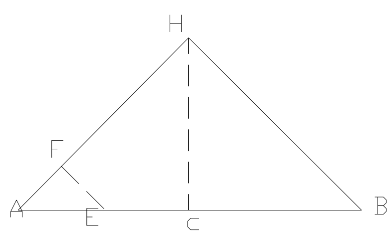


Fig. 17

EIGHTEEN

From sim. tri's ABC and BCH, HC = a^2/b. Angle ABC = angle CDA = rt. angle. From sim. tri's AHD and DHC, CD = ah/b; CB = CD. Area of tri. ABC on base AC = $(1/2)(b + a^2/b)a$. Area of ACD on base AD = (1/2) (ah/ b) h.

$\therefore (b + a^2/b)a = ah^2/b = (b^2 + a^2) / b\, xa = (ab^2 + a^3)/b$

$\therefore H^2 = a^2 + b^2$.

a. See Versluys, p. 72, fig. 79.

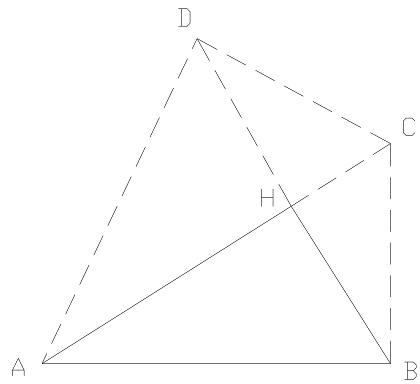


Fig. 18

NINTEEN

Tri's 1, 2 and 3 are similar. From tri's 1 and 2, AC = h^2 /a and CD = hb/a. From tri's 1 and 3, EF = ha/ b, and FB =h^2/b.

Tri's CFH = tri. 1 + tri. 2 + tri.3 + sq.AE.

So (1/2)(a + h^2/b) (b + h^2/a) = (1/2) ab + (1/2) h^2 (a/b) +(1/2) h^2 (a/b) + h^2, or ab + 2abh + h = ab + ha + hb + 2abh, or h = ha + hb. .: h = a + b. Q.E.D.

a. See Versluys, p. 23. fig. 80.

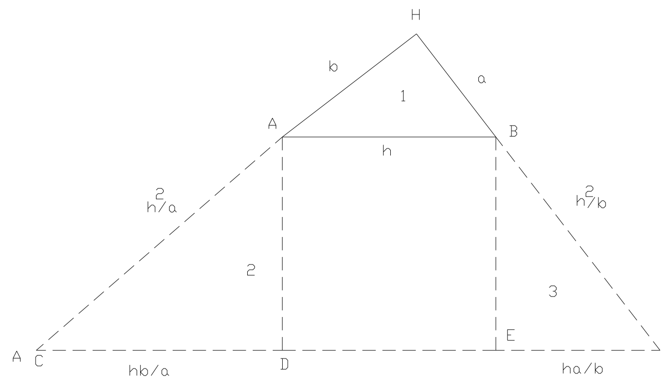


Fig. 19

TWENTY

Draw HC perp. to AB and = AB. Join CB and CA. Draw CD and CE perp. resp'y to HB and HA.

Area BHAC = area ABH + area ABC = (1/2) h^2. But area tri. CBH = (1/2) a^2 , and of tri. CHA = (1/2) b^2. :. (1/2) h^2 = (1/2) a^2 + (1/2) b^2 :. $h^2 = a^2 + b^2$.

a. See Versluys, p. 75, fig. 82, where credited to P.Armand Meyer, 1876.

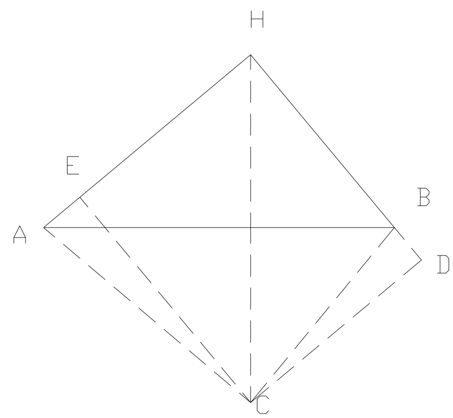


Fig. 20

TWENTY ONE

HC = HB = DE; HD = HA. Join EA and EC . Draw EF and HG perp. to AB and EK perp. to DC.

Area of trap. ABCD = area (ABH + HBC + CHD + AHD) = ab + $(1/2)a^2 + (1/2)\ b^2$. (1)

= area (EDA + EBC + ABE + CDE) = (1/2) ab + (1/2)ab + [(1/2)AB xEF = (1/2) AB xAG as tri's BEF and HAG are congruent) = ab + (1/2)(AB = CD)(AG + GB) = ab + $(1/2)h^2$. (2)

$\therefore$ ab + $(1/2)\ h^2$ = ab + $(1/2)\ a^2 + (1/2)\ b^2$. $\therefore$ $H^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 74, fig. 81.

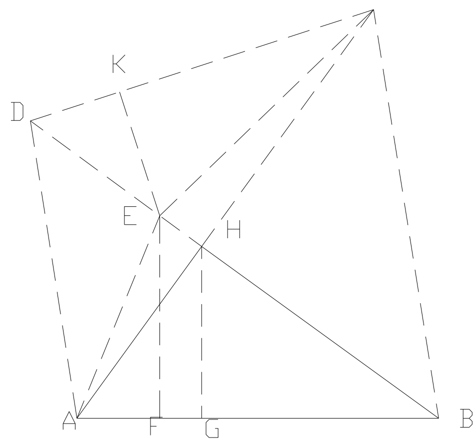


Fig. 21

TWENTY- TWO

In fig. 22 it is obvious that:

(1) Tri. ECD = (1/2) h. (2) Tri. DBE = (1/2) a . (3) Tri. HAC = (1/2) b

∴ (1) = (2) =(3) = (4) (1/2) h = (1/2) a + (1/2)b.

∴ h = a + b . Q.E.D.

a. See Fersluys, p.76 fig.83 credited to Meyer, (1876); also this work, p. 181, fig. 238 for a similar geometrin=c proof.

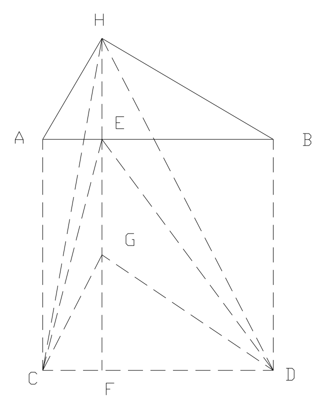


Fig. 22

TWENTY-THREE

For figure, use fig. 22 above, omitting lines EC and ED. Area of sq. AD= (2 area of tri. DBH = rect. BF) + (2area of tri. HAC = rect, AF) = 2x(1/2)a^2 + (1/2) b^2 = a^2+ b^2 = h^2∴ $h^2 = a^2 + b^2$ .Or use similar parts of fig. 315 in geometric proofs.

a. See Vers;uys, p. 76, proof 66, evedited to Meyer's, 1876, collection.

TWENTY-FOUR

In fig. 22, denote HE by x. Area of tri. ABH + area of sq. AD = (1/2) hx + h = area of (tri. ACH + tri. CDH + tri. DBH) = (1/2) b^2 + (1/2) h(h + x) + (1/2) a^2 = (1/2) b^2 + (1/2) h^2 + *1/2)hx + (1/2) a^2.∴ $h^2 = a^2 + b^2$.

a. See Versluys, p. 76 proof 67, and there credited to P. Armand Meyer's collection made in 1876.

b. Proofs Twenty-Two, Twenty- Three and Twenty-Four are only variations of the Mean Proportional Principle,-- see p. 51, this book.

TWENTY- FIVE

At A erect AC = to, and perp. to AB; and from C drop (CD=AH) prep. to AH. Join CH , CB and DB. Then AD = HB =a Tri. CDB = Tri.CDH = (1/2)CD xDH.

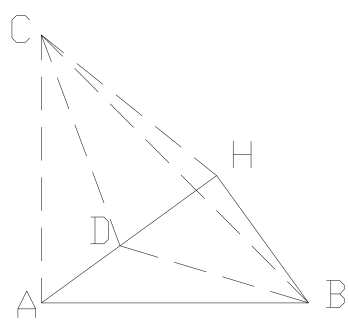


Fig. 23

Tri, CAB = Tri. CAD + tri DAB +(tri BDC = tri.CDH = tri.CAH + tri. DAB). ∴ (1/2) h = (1/2) a + (1/2)b.

a. See Versluys p. 77, fig. 84, one of Meyer's 1876, collection.

TWENTY- SIX

From A draw AC perp. to, and = to AB. Join CB and draw BF parallel and = to HA, and CD parallel to AH and = to HB. Join CF and BD.

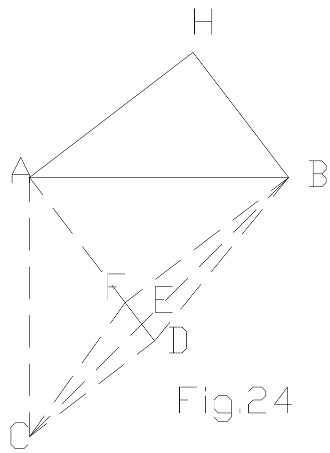


Fig.24

Tri, CBA = tri. BAF + tri. FAC + tri. CBF =tri. BAF + tri.FAC + tri. FDB (since tri. ECF = tri. EDB) = tri. FAC + tri. ADB. :. $(1/2)h^2 = (1/2) a^2 + (1/2)b^2$. :.$h^2 = a^2 + b^2$.

a. See Versluys, p. 77, fig. 85, being one of Meyer's collection.

TWENTY-SEVEN

From A draw AC perp. to, and = to AB. From C draw CF equal to HB and parallel to AH. Join CB, AF and HF and draw BE parallel to HA. CF = EB = BH = a. ACF and ABH are congruent; so are CFD and BED.

Quad, BHAC = tri. BAC+ tri. ABH = tri. EBH + tri. HFA + tri. ACF + tri. FCD + tri. DBE. ∴(1/2) h^2 = (1/2) a ^{2}+ (1/2)b^2. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p.78, fig.86 also see "Vriend de Wiskunde," 1898,by F.J.Vaes.

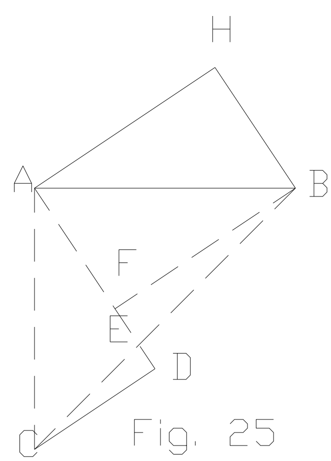


Fig. 25

TWENTY-EIGHT

Draw PKH perp. to AB and make PH = AB. Join PA, PB, AD and GB.

Tri's BDA and BHP are congruent; so are tri's GAB and AHP. Quad. AHBP = tri. BHP + tri. AHP. :.(1/2)h = (1/2)a + (1/2) b. :. h = a + b Q.E.D.

a. See Versluys, p. 79 fig. 88. Also the Scientifique Revue, Feb. 16 1889, H. Renan; also Fourrey, p. 77 and p.99 – Jal de Vuibert, 1879-80.

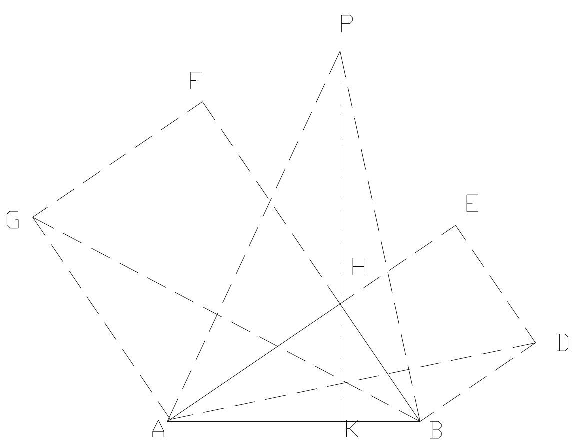


Fig. 26

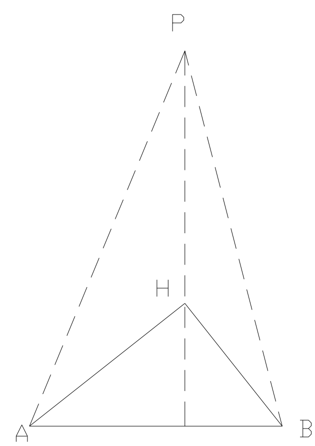


Fig. 27

TWENTY-NINE

Through H draw PK perp. to AB, making PH = AB and join PA and PB.

Since area ABHP = [area PHA + area PHB = (1/2)h xAK + (1/2)xBK = (1/2)(AK+ BK) = (1/2) h xh = (1/2) h^2] = (area AHP + area BPH = (1/2)b^2 + (1/2)a^2 :.$h^2 = a^2 + b^2$.

a. See Fersluys, p. 79, fig.89, being one of Meyer's, 1876, collection.

THIRTY

Draw PH perp. to AB, making PH = CD = AB. Join PA, PB, CA and CB.

Tri, ABC = (tri. ABH + quad. AHBC) = (quad,AHBC + quad. ACBP), since PC = HD. In tri. BPH, angle BPH = 180^0 - (angle BDH = 90 + angle HBD). So the alt, of tri. BPH from the vertex P = a, and its area = $(1/2)a^2$; likewise tri. AHP = $(1/2) b^2$. But as in fig. 27 above, area AHBP = $(1/2) h^2$:. h^2= a2 + b^2. Q.E.D.

a. See Versluys p. 80, fig. 90, as one of Meyer's 1876, collections.

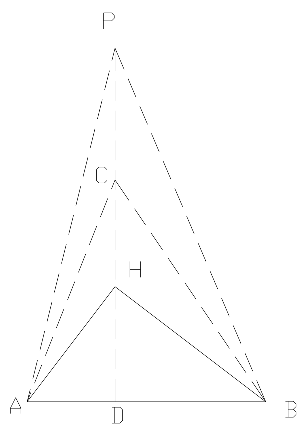


Fig. 28

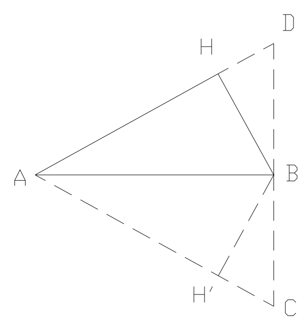


Fig. 29

THIRTY- ONE

Tri's ABH and BDH are similar, so DH = a^2/b and DB = ab/h. Tri, ACD = 2tri. ABH + 2 tri. DBH.

Area of tri. ACD = ah^2/b = area of 2 tri. ABH + 2 tri. DBH = ab + a^3/b. hence h = a + b Q.E.D.

a. See Versluys, P. 87, fig. 91.

THIRTY-TWO

Another Reductio ad Absurdum proof –See proof Sixteen above.

Suppoce $a^2 + b^2 > h^2$. Then $AC^2 + p^2 > b^2$, and $CB^2 + p^2 > a^2$. ∴ $AC^2 + CB^2 + 2p^2 > a^2 + b^2 > h^2$. As $2p^2 = 2(AC \times BC)$ then $AC^2 + CB^2 + 2ACxCB > a^2 + b^2$,or $(AC + CB)^2 > a^2 + b^2 >$ or $h^2 > a^2 + b^2 > h^2$ or $h^2 > h^2$ an absurdity. Similarly, if $a^2 + b^2 < h.^2$ ∴ $h^2 > a^2 + b$. Q.E.D.

See Versluys, p. 60, fig. 64.

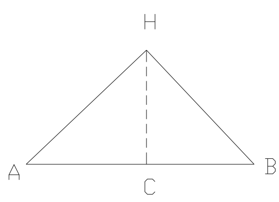


Fig. 30

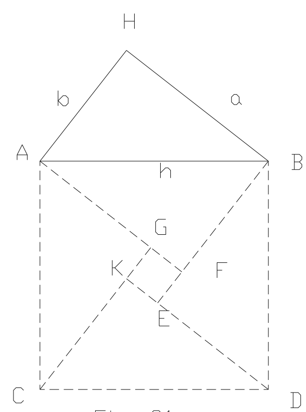


Fig. 31

THIRTY- THREE

Sq. AD = (area of 4 tri's = 4xtri, ABH + area of sq. KF) $= 4 \times (1/2)\,ab + (a-b)^2 = 2ab + b^2 - 2ab + a^2 = a^2 + b^2$.∴ $h^2 = a^2 + b^2$.

a. See Math.Mo., 1858-9; Vol. I p. 361, and it refers to this proof as given by Dr. Hutton, (Tracts, London, 1812, 3 Vol., 800 in history of Algebra.

THIRTY- FOUR

Let BH = x, and HF = y; then AH =x + y; sq. AC = 4 tri. ABH + sq. HE = 4 [x (x + y)/2] + y^2 = $2x^2 + 2xy + y^2 = x^2 + 2xy + y^2 + x^2 = (x+y)^2 + x$, ∴ Sq. of BH. ∴ $h^2 = a^2 + b^2$.Q.E.D.

a. This proof is dut to Rev. J. G. Excell, Lakewood,O., July,1928; also given by R.A. Bell, Cleveland, O., Dec. 28, 1931. And it appears in “Der Pythagoreisch Lehrsatz” (1930), by Dr. W. Leitzmann. In Germany.

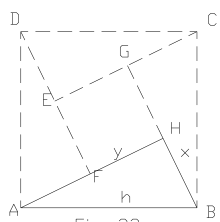


Fig. 32

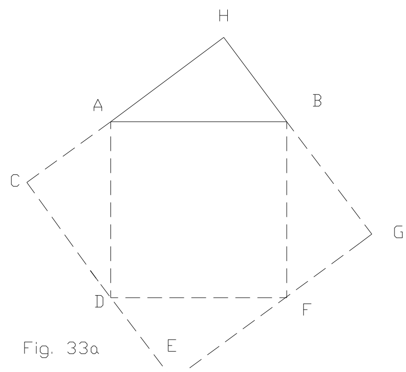


Fig. 33a

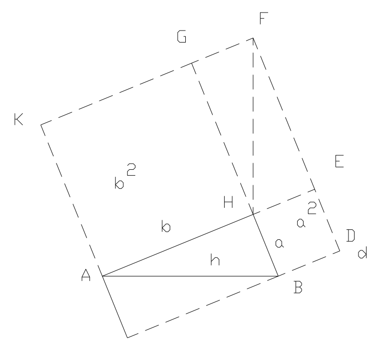


Fig.33b

THIRTY-FIVE

In fig.33a, sq. CG = sq. AF + 4tri.ABH = h + 2ab. ---------(1)

In fig. 33b, sq. KD = sq. + sq. HD + 4xtri.ABH = $a^2 + b^2$ + 2ab.-----(2)

But –sq. CG = sq. KD, by const’n. ∴ (1) =(2) or $h^2 + 2ab = a^2 + b^2 + 2ab$. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Math. Mo., 1809, Dem. 9, and there, p. 159, Vol . I credited to Rev. A.D. Wheeler, of Brunswick, Me.; alse see Fourrey, p.80, fig’s, a and b; also see “Der Pythagoreisch Lehrsatz” (1930), by Dr. W.leitzmann.

b. Using fig. 33a, a second proof is: Place 4 rt. trianbles BHA, ACD, DEF and FGB so that their legs form a square whose side is HC. Then it is plain that:

1. Area of sq. HE = $a^2 + 2ab + b^2$.

2. Area of tri. BHa = ab/2

3. Area of 4 tri's = 2ab

4. Area of sq. AF = area of sq. HE – area of the 4 tri's = $a^2 + 2ab + b^2 – 2ab = a^2 + b^2$.

5. But area of sq. AF = h .

6. $\therefore H^2 = a^2 + b^2$. Q.E.D.

This proof was devised by Maurice Laisnez, a high school boy in the Junior-Senior High Schoo of South Bend, Ind., and sent to me, May 16, 1939, by his class teacher Wilson Thornton.

THIRTY-SIX

Sq. AE = sq. KD – 4 ABH = $(a + b)^2 - 2ab$; and h^2 = sq. NH + 4ABH= $(b - a)^2 + 2ab$. Adding, $2h^2 = (a + b)^2 + (b – a)^2 = 2a^2 + 2b^2$. $\therefore H^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 72, fig. 78, also given by Saunderson (1682-1750); also see Fourrey, p. 92, and A. Marre. Also assigned to Baskara, the Hindu Mathematician, 12, century,A.D. Also said to have been known in China 1000. years before the time of Christ.

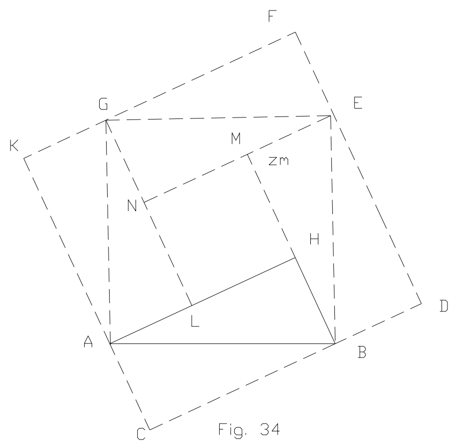


Fig. 34

THIRTY-SEVEN

Since tri's ABH and CDG are similar, and CH = b- a, then CD = h(b - a) /b, and DH = a(b-a)/b. draw GD. Now area of tri. CHD = (1/2) (b – a)xa (b-a)/b= (1/2) a $(b - a)^2$/b. ---(1)

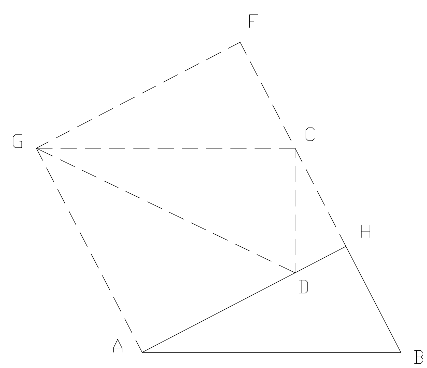


Fig. 35

Area of tri. DGA = $^1/_2$ GA X AD =$^1/_2$b x[b^2 – a(b- a)/b] = $^1/_2$ [b^2 – a (b – a)]----(2)

Area of tri. GDC= $^1/_2$ h[(b – a) /b]h= $^1/_2$ h^2(b – a) /b----(3)

∴ Area of sq. AF = (1) + (2) + (3) + tri.GCF= $^1/_2$ a(b – a)^2/b + $^1/_2$ [b^2 – a (b –a)] + $^1/_2$ h^2(b – a) /b + $^1/_2$ ab = b^2, which reduced and collected gives h^2(b – a) – (b – a) a^2 = (b – a) b^2. ∴ h^2 = a^2 + b^2 . Q.E.D.

a. See Versluys, p. 73-4, solution 62.

b. An Arabic work of Annairizo, 900 B.C. has a similar proof.

c. As last 5 proofs show, figures for geometric proof are figures for algebraic proofs also. Probably for each geometric proof there is an algebric proof.

B. The Mean Proportional Principal

The mean proportional principle leading to equivalency of areas of triangles and parallelograms, is very prolific on proofs.

By rejecting all similar right triangles other taan those obtained by dropping a perpendicular from the vertex of the right angle to the hypotenuse of a right angle and omitting all equations resulting from the three similar tight triangles thus formed, save only equations (3), (5) and (7), as given in proof *One*, we will have limited our field greately. But in this limited field the proofs possible are many, of which a few interesting ones will now be given.

In every figure under B we will let h = the hypotenuse, a = the shorter leg, and b = the longer leg of the given right triangle ABH.

THIRTY-EIGHT

Since AC: AH = AH:AB, AH^2 = AC x AB, and BH^2 = BC x BA. Then BH^2 + HA^2 = (AC + CB)HB = AB^2. ∴ h^2 = a^2 + b^2.

a. See Versluys, p. 82, fig. 92, as given by Leonardo Pisano, 1220, in Practica Geometieae; Wallis, Oxford, 1655; Math. Mo. 1859, Dem,4 and credited to Legender's Geom.; Wentworth's New Plane Geom., p. 158 (1895); also Chauvenet's Geom., 1891, p. 117, Prop. X. Also Dr. Leitzmann's work (1930), p 33, fig. 34. Also "Mathematics for the Million," (1937), p. 155, fig. 51(i), by Lancelot Hogben, F.R.S.

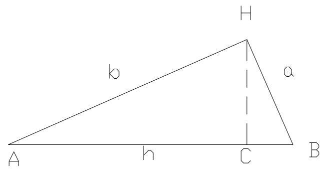


Fig. 36

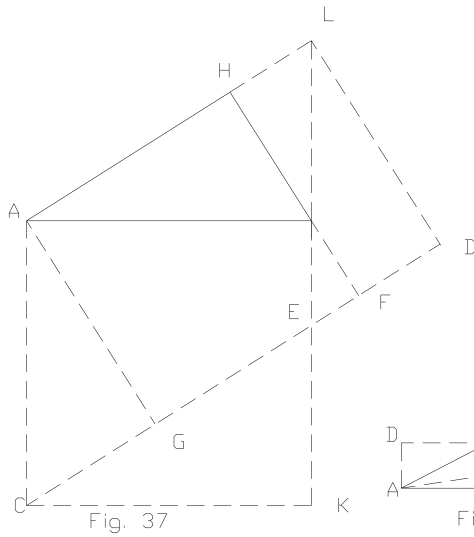


Fig. 37

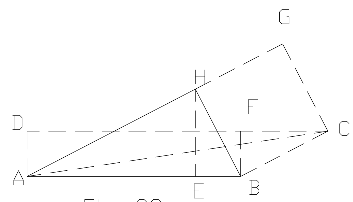


Fig. 38

THIRTY-NINE

Extend AH and KB to L, through C draw CD par. to AL, AG prep. to CD, and LD par. to HB, and extend HB to F.

BH^2 = AH x HL = FH x HL = FDLH = a^2. Sq. AK = paral. HCEL = paral. AGDL = $a^2 + b^2$. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 84. fig. 94, as given by Jules Camirs, 1889 in S. Revue.

FORTY

Draw AC. Through C draw CD par. to BA, and perp's AD, HE and BF.

Tri. ABC = ½ sq. BG = ½ rect, BD. ∴sq. BG = a^2 = rect. BD = sq. EF + rect.ED = sq. EF + (EA x ED =EH^2) = sq. EF + EH^2. But tri's ABH and BEH are similar. ∴ if in tri. BHE, $BH^2 = BE^2 + EH^2$, then in its similar, the tri. ABH, $AB^2 = BH^2 + AH^2$. ∴ $H^2 = a^2 + b^2$. Q.E.D.

See Sci. Am. Sup., Vol. 70 p. 382, Dec. 10 1910, fig. 7---one of the 108 proofs of Aurther E. Colburn, LL.M. of Dist. of Columbia Bar.

FORTY- ONE

Const'n obvious. Rect. LF = 2 tri. FBH + 2tri. ADB = sq. HD = = sq. LG + (rect. KF = KC x CF = AL x LB $=HL^2$) = sq. LG + HL^2.

But tri's ABH and BHL are similar. Then as in fig. 36, $\therefore h^2 = a^2 + b^2$.

See Sci. Am. Sup., V. 70, p. 359, one of Colburn's 108.

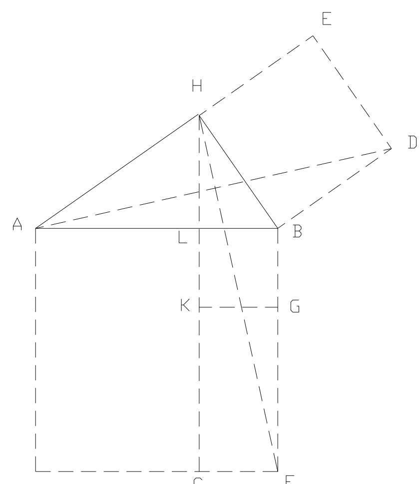


Fig. 39

FORTY-TWO

Construction as in fig. 38. Paral. BDKA =rect. AG = AB x BG = AB x BC = BH . And AB x AC = AH. Adding BH + AH = ABxBC + ABx AC = AB(AC + CB) = AB . : . $h^2 = a^2 + b^2$. Q.E.D.

See Wipper, 1880, p. 39, fig. 38 and there credited to Oscar Werner, as recorded in "Archiv. d. Math. und Phys., "Grunert, 1855; also see Versluys, p. 64. fig. 67 , and Fourrey, p.76.

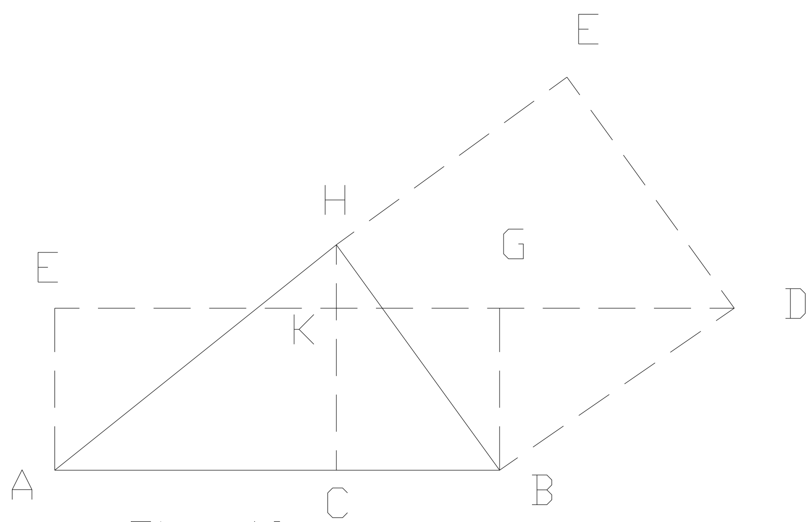


Fig. 40

FORTY-THREE

Two squares, one on AH const'd outwardly, the other on HB overlapping the given triangle.

Take HD and cons't rt. tri. CDG. Then tri's CDH and ABH are equal. Draw GE par. to AB meeting GKA produced. At E.

Rect. GK = rect. GA + sq.HK = (HA = HC)HG + sq.HK = HD^2 + sq. HK.

Now CG: DC = DC: (HC=GE) ∴DC^2 = GCx GE = rect.GK = sq.HK + sq. DB = AB^2 ∴ $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p.382, Dec. 10, 1910. Credited to A.E. Colburn.

FORTY-FOUR

AK = sq. on AB. Through G draw GDpar. to HL and meeting FL produced at D and draw EG.

Tri, AGE is common to sq. AK and rect. AD. ∴ tri. AGE= $^1/_2$ sq.AK = $^1/_2$ rect. AD. ∴sq. AK = rect. AD. Rect. AD =sq. HF + (rect. HD = sq. HC, see argument in proof 39). ∴sq. BE = sq.HC + HF, or $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., v. 70, p. 382,Dec. 10, 1910. Credited to A.E. Colburn.

b. I regard this proof, wanting ratio, as a geometic, rather than algefraic proof. E.S. Loomis.

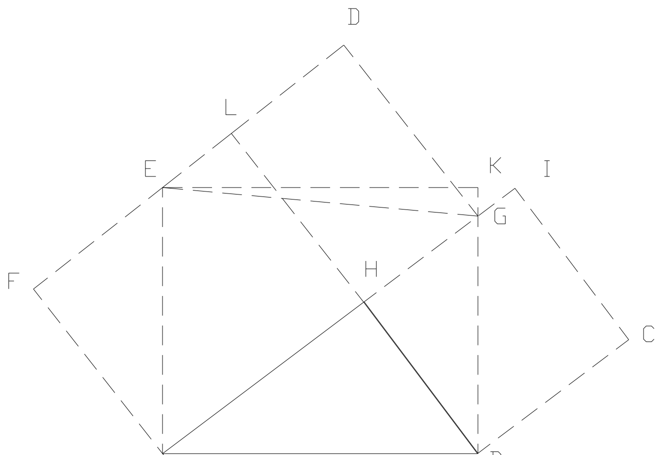


Fig. 42

FORTY-FIVE

HG =sq. on AH. Extend KB to M and through M draw ML par. to HB meeting GF extended at L and draw CM.

Tri. ACG = tri. ABH. Tri. MAC =$^1/_2$ rect. AL = $^1/_2$ sq. AK.:. sq. AK = rect. AL = sq. HG + (rect. HL = ML x MH). = HA x HM = HB = sq. HD + sq. HG :. $h^2 = a^2 + b^2$.

a. See Am. Sci. Sup., V. 70, p. 383, Dec. 10, 1910, Credited to A.E. Colburn.

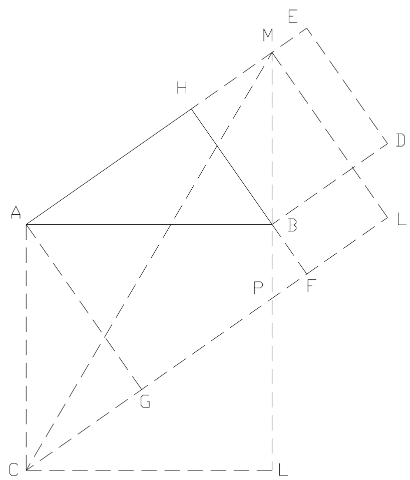


Fig. 43

FORTY-SIX

Extend KB to O in HE. Through O, and par. to HB draw NM, making OM and ON each = to HA. Extend GF to N, GA to L, making AL = to AG and draw CM.

Tri. ACL = tri. OPM = tri. ABH, and tri. CKP= tri, ABO.

∴ Rect. OL = sq. AK, having polygon ALPB in common. ∴sq. AK = rect. AM = sq. HG + rect. HN = sq. HG + sq. HD; see proof Forty- Four above. ∴ h = a + b. Q.E.D.

a. See Am. Sci. Sup., v. 70, p. 383. Credited to A.E. Colburn.

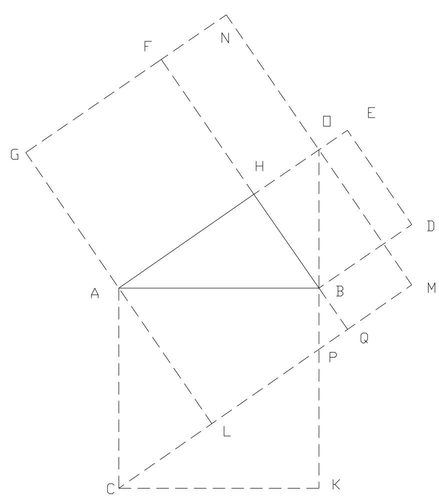


Fig. 44

FORTY- SEVEN

Trancposed sq. LE = sq. on AB

Draw through H, perp. to AB, GH, and produce it to meet MC produce at F. Take HK = GB, and through k draw LN par. and equal to AB. Complete the transposed sq. LE. Sq. LE = rect. DN + rect. DL = (DK x KN = LNxKN = AB x AG = HB) + (rect. LD = paral. AF = sq. AC) for tri.FCH = tri. RAM. and tri. CPR = tri. SLA.. :. sq. LE = HB + sq. AC, or $h^2 = a^2 + b^2$.

a. Original with the author of this work, Feb. 2 , 1926.

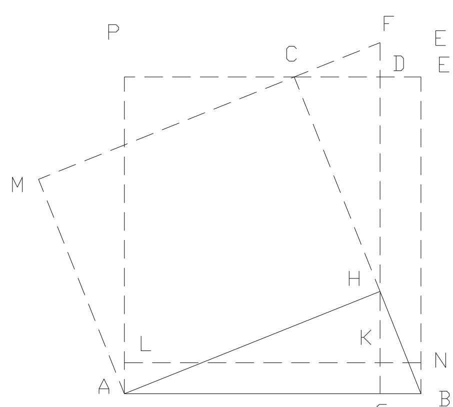


Fig. 45

FORTY-EIGHT

Construct tri. BHC and tri. AHF = tri. AHC, and through pts. F. H, and E draw the line GHL, making FG and EL each = AB, and complete the rect's FK and ED, and draw the lines HD and HK.

Tri, GKA = $^1/_2$ AK x AF = $^1/_2$ AB xAC - $^1/_2$ AH^2. Tri. HBD = $^1/_2$ BDxBE= $^1/_2$ ABxBC = $^1/_2$ HB^2. Whence AB xAC = AH^2 and ABx BC = HB^2. Adding, we get ABxAC +ABxBC= AB(AC + BC) = AB^2, or AB^2 = BH^2+ HA^2.: . $h^2 = a^2 + b^2$.

a. Original with the author, discovered Jan. 31, 1926.

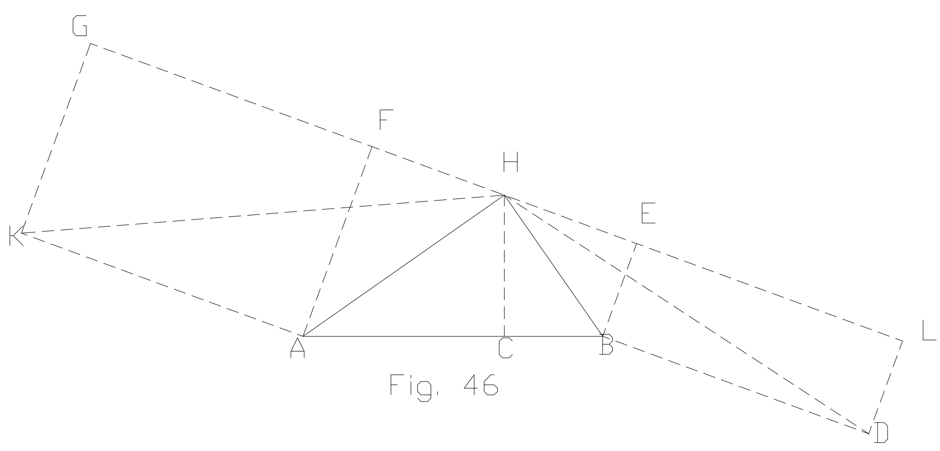


Fig. 46

FORTY-NINE

Construction, Craw HC, AE and BF each perp. to AB, making each equal to AB. Draw EC and FCD. Tri's. ABH and HCD are equal and similar.

Figure FCEBHA= paral. CB+ paral. CA = CH x GB + CH x GA= AB x GB + AB x AG = HB^2 + HA^2 = AB (GB + AG) = AB x AB = AB^2.

a. See Math, Teacher, V. XVI, 1915. Created to Goe. G. Evans, Charleston High School, Bosten, Mass.; also Versluys, p. 64 fig. 68, and p. 65, fig. 69; also Journal de Mathein, 1888, F.Fabre; and found in "de Mathein, 1889,"by A. E. B. Dulfer.

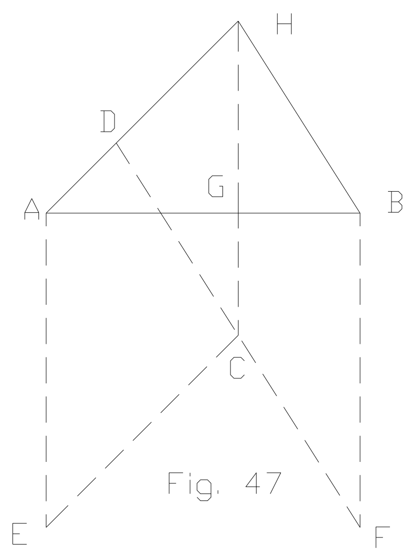


Fig. 47

FIFTY

I am giving this figure of Cecil Hawkins as it appears in Versluys' work, --- not reducing it to my scale of h = 1".

Let HB' = HB = a, and HA' = HA = b, and draw A'B' to D in AB.

Then angle BDA' is a rt. angle , since tri's BHA and B'HA' are congruent having base and altitude of the one res'ly perp. to base and altitude of the another.

Now tri. BHB' + tri. AHA' = tri. BA'B' + tri. AB'A' = tri. BAA

– tri.BB'A.: .$^1/_2$ a^2 + $^1/_2$ b^2 = $^1/_2$ (AB x A'D) - $^1/_2$ (AB x B'D) = $^1/_2$ [AB(A'B' + B'D)] - $^1/_2$ (AB x B'D) = $^1/_2$ AB x A'B' + $^1/_2$ AB x B'D- $^1/_2$ AB x B'D = $^1/_2$ AB x A'B' = $^1/_2$ h x h = $^1/_2$ h^2 .: . h^2 = $a^2 + b^2$. Q.E.D.

a. See Vers;uys. p. 71, fig. 76, as given by Cecil Hawkins, 1909, of England.

FIFTY-ONE

Tri. ACG = tri. ABH. ∴sq. HG = quad. ABFC = b^2 . Since angle BAC = rt. angle. ∴tri. CAB = $\frac{1}{2}h^2$. ∴b^2= quad. ABFC = $\frac{1}{2}h^2$ + tri. BFC = $\frac{1}{2}h^2 + \frac{1}{2}(b+a)(b-a)$.------(1)

Sq. HD = sq.HD'. Tri. OD'B= tri. RHB. ∴ Sq. HD'= quad. BRE'O = a^2 + tri. ABL – tri. AEL. ∴$a^2 = \frac{1}{2}h - \frac{1}{2}(b+a)(b-a)$.-------(2) (1) + (2) = (3). $a^2 + b^2 = \frac{1}{2}h^2 + \frac{1}{2}h^2 = h^2$. ∴ $h^2 = a^2 + b^2$. Q.E.D.

Or from (1) thus: $\frac{1}{2}h^2 + \frac{1}{2}(b+a)(b-a) = b^2 = \frac{1}{2}b^2 + \frac{1}{2}h - \frac{1}{2}a$. Whence $h^2 = a^2 + b^2$.

a. See Versluys, p. 67, fig. 71, as one of Meyer's collection, of 1896.

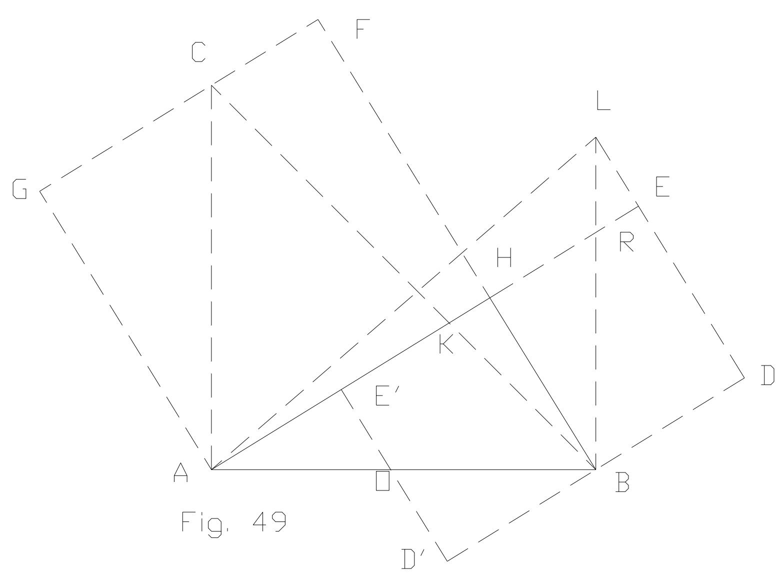


Fig. 49

FIFTY- TWO

Given the rt. tri, ABH. Through B draw BD = 2BH and par. to AH.

From D draw perp. DE to AB. Find mean prop'l between AB and AE which is Given the rt. tri. ABH. Through B draw BD = 2 BH and par. to AH. BF. From A, on AH, lay off AT = BF. Draw TE and TB, forming the two similar tri's AET and ATB, from which AT : TB = AE : AT, or $(b - a)^2 = h(h - EB)$, whence $EB = [h - (b - a)^2] / h$---- (1)

Also EB : AH = BD : AB. ∴ EB = 2ab/h. ----(2) Equating (1) and (2) gives $[h - (b - a)^2] / h = 2ab/h$, whence $h^2 = a^2 + b^2$.

a. Devised by the author, Feb. 28, 1926.

b. Here we introduce the circle in findingthe mean proportional.

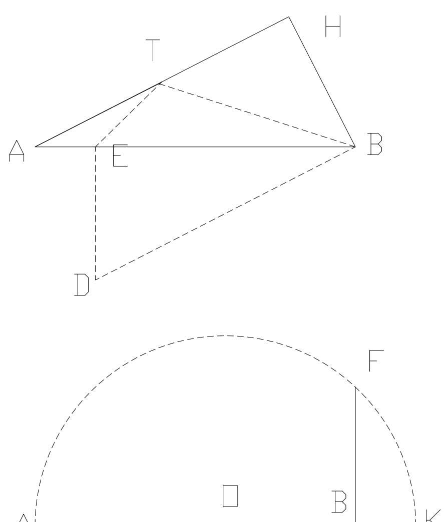


Fig. 50

FIFTY-THREE

An indirect algebraic proof, said to be due to the great Leibniz (1646-1716).

If (1) $HA^2 + HB^2 = AB^2$, then (2) $HA^2 = AB^2 - HB^2$, whence (3) $HA^2 = (AB + HB)(AB – HB)$.

Take BE and BC each equal to AB, and form B as center describe the semicircle CA'E. Join AE and AC , and draw BD perp. to AE. Now (4) HE = AB + HB, and (5) HC = AB – HB. (4) x (5) gives HE x HC = HA^2 , which is true only when triangles AHC and EHA are similar.

So (6) angle CHA = angle AEH, and so (7) HC :HA = HA : HE; since angle HAC = angle HEA = angle E, then angle CAH = angle EAH. ∴angle AEH + angle EAH = 90^0 and angle CAH + angle EAH 90^0 ∴ = Angle EAC = 90^0. ∴ Vertex A lies on the semicircle, or A coincides with A' .∴ .EAC is inscribed in a semicircle and is a rt. angle. Since equation (1) leads through the data drived from it to a rt. triangle, then starting with such a triangle and reversing the argument we arrive at $h^2 = a^2 + b^2$.

a. See Versluys, p. 61, fig. 65, as given by von Leibmiz.

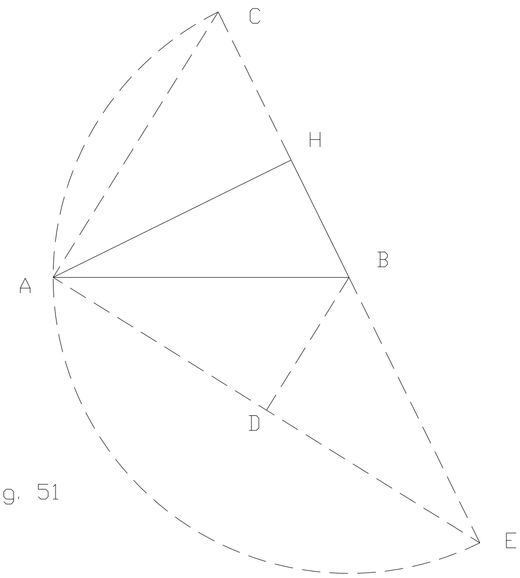


Fig. 51

FIFTY- FOUR

Let CB = x, CA = y and HC = p. $p^2 = xy$; $x^2 + p^2 = x^2 + xy = x(x + y) = a^2$. $y^2 + p^2 = y^2 + xy = y(x + y) = b^2$. $x^2 + 2p^2 + y^2 = a^2 + b^2$. $x^2 + 2xy + y^2 = (x + y)^2 = a^2 + b^2$.·. $h^2 = a^2 + b^2$. Q.E.D.

a. This proof was sent to me by J. Adams of The Hague, Holland. Received it March 2, 1934, but the author was not given.

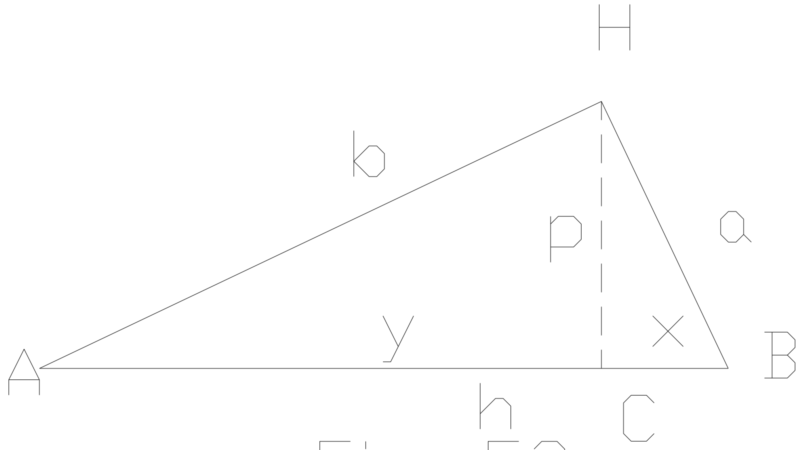


Fig. 52

FORTY-FIVE

Assume (1) $HA^2 + HB^2 = AB^2$, Draw HC perp. to AB. Then (2)$AC^2 + CH^2 = HA^2$. (3) $CB^2 + CH^2 = HB^2$, (4) Now AB = AC + CB, so (5) $AB^2 = AC^2 + 2AC \times CB + CB^2 = AC^2 + 2HC^2 + CB^2$. But (6) $HC^2 = AC \times CB$. (7)

$AB^2 = AC^2 + 2AC \times CB + CB^2$ and (8) AB = AC + CB. ∴(9) $AB^2 = AC^2 + 2AC \times CB + CB^2$. (2) + (3) = (10) $HB^2 + HA^2 = AC^2 + 2HC^2 + CB^2$, or (11) $AB^2 = HB^2 + HA^2$. ∴ (12) $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 62, fig. 66.

b. This proof is one of Hoffmann's, 1818, collection.

c. – *The Circle in Connection with the Right Triangle*

(I) – <u>Through the Use of One Circl</u>

From certain Linear Relations of the Chord, Secant and Tangent in conjunction with a right triangle, or with similar related right triangles, it may also be proven that : *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

And since the algebraic is the measure of transliteration of the geometric square the truth by any proof through the algebraic method involves the truth of the geometric method.

Furthermore these proofs through the use of circle elements are true, not because of straight line properties of the circle, but because of the law of similarity, as each proof may be reduced to the proportionality of the homologous sides of similar triangles, the circle being a factor only in this, that the homologous angles are measured by equal arcs.

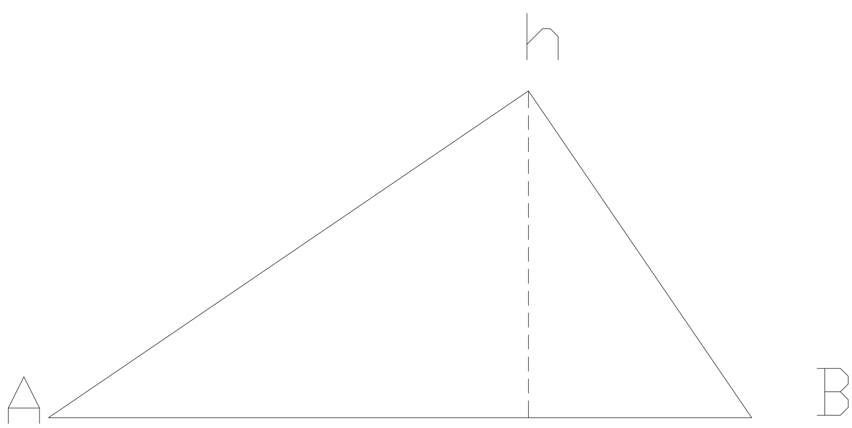


FIG.53c

(1) The Method by Chords.

FIFTY- SIX

In fig. 54 H is any pt. on the semicircle AHB.: . the tri. ABH is a rt. triangle. Complete the sq. AF and draw the perp. EHC.

$BH^2 = AB \times BC$ (mean proportional)

$AH^2 = AB \times AC$ (mean proportional)

Sq. AF = rect. BE + rect. AE = AB x BC + AB x AC = $BH^2 + AH^2$.

:. $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., v. 70, p. 383, Dec. 10, 1910. Credited to A.E. Colburn.

b. Also by Richard A. Bell, --- given to me Feb. 28, 1938. He says he produced it on Nov. 18, 1933.

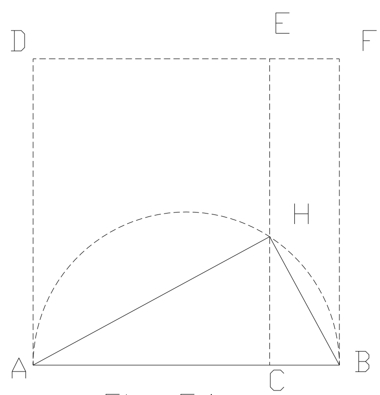


Fig. 54

FIFTY-SEVEN

In fig 55 take ER = ED and bisect HE. With Q as center describe semicircle AGR. Complete sq. EP. Rect. HD = HC x HE = AH x HE = HB^2 = sq. HF. EG is a mean proportional between EA and (ER = ED) . :.sq. EP = rect.AD = sq.AC + sq. HF. But AB is a mean prop'l between, EA and (ER x ED). :.EG = AB. sq. BL = sq. AC + sq. HF. :. $h^2 = a^2 + b^2$.

a. See Sic. Am. Sup., v. 70, p. 359, Dec. 3, 1910. Credited to A. E. Colburn.

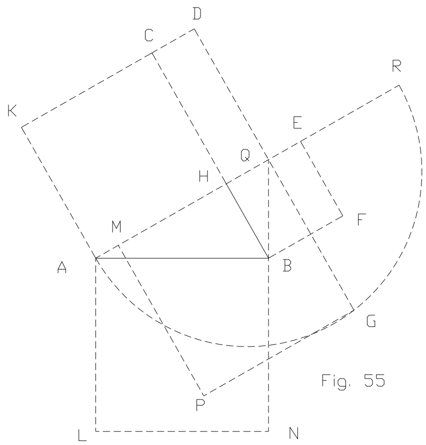


Fig. 55

FIFTY- EIGHT

In any circle upon any diameter, EC in 56; take any distance from the center less than the radius as BH. At H draw a chord AD perp. to the diameter, and join AB forming the rt. tri.ABH.

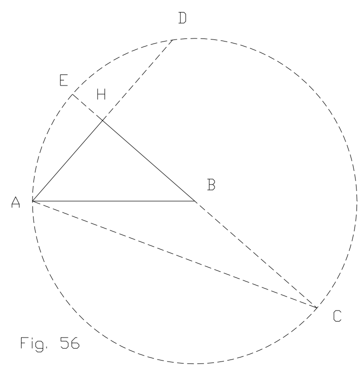


Fig. 56

a. Now HA x HD = HC x HE, or $b^2 = (h + a)(h - a)$ $\therefore$ $h^2 = a^2 + b^2$.

b. By joining A and C, and E and D, two similar rt. tri's are formed, giving HC : HA = HD : HE or, again $b^2 = (h + a)(h - a)$ $\therefore$ $h^2 = a^2 + b^2$.

But by joining C and D, the tri. DHC = tri. AHC, and since the tri. DEC is a particular case of one, fig. 1, as is obvious, the above proof is subordinate to, being but a particular case of the proof of one.

c.See Edwards' Geometry, p. 156, fig. 9, and Journal of Education, 1887, v. xxv, p. 404, fig. VII.

FIFTY- NINE

With B as center, and radius = AB, describe circle AEC.

Since CD is a mean proportional between AD and DE, andas CD = AH,

$b^2 = (h + a)(h – a) = h^2 – a^2 .·. h^2 = a^2 + b^2$.

a. See Journal of Education, 1888, Vol. XXVII, p. 327, 21st. proof; also Heath's Math. Monograph, No. 2, p. 30, 17th of the 26 proofs there given.

b. By analysis and comparison it is obvious, by substituting for ABH its equal, tri. CBD, that above solution is subordinate to that of Fifty-Six.

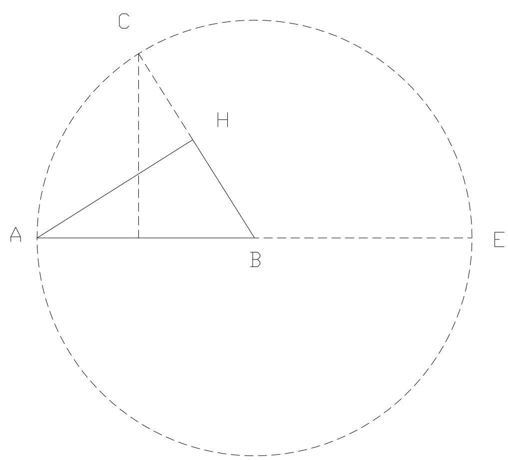


Fig. 57

SIXTY

In fig. 58 , in any circle draw any chord as AC perp. to any diameter as BD, and join A and B, B and C and D and C, forming the three similar rt. tri's ABH, CBH and DBC.

Whence AB : DB = BH : BC, giving AB x BC = DB x BH = (DH + HB) BH = DH x BH + BH^2 = AH x HC + BH^2; or $h^2 = a^2 + b^2$.

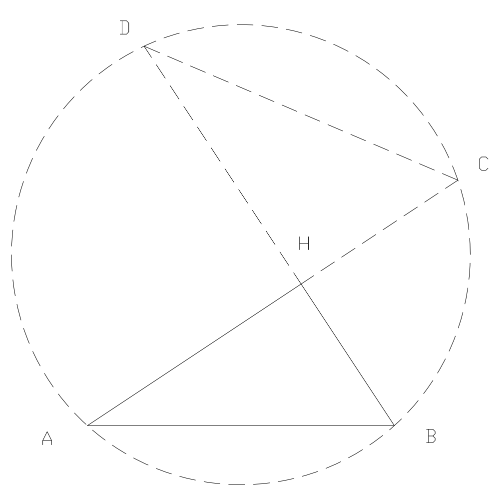


Fig.58

a. Fig. 58 is closely related to fig. 56.

b. For solutions see Edwards' Geom., p. 156, fig, 10, Journal of Education, 1887, V. XXVI, p. 21, fig. 14, Heath's Math. Monographs, No. 1. p. 26 and Am. Math. Mo. V. III, p. 300, solution XXI.

SIXTY-ONE

Let H be the center of a circle, and AC and BD two diameters perp. to each other. Since HA = HB, we have the case particular, same as in fig. under Geometric Solutions.

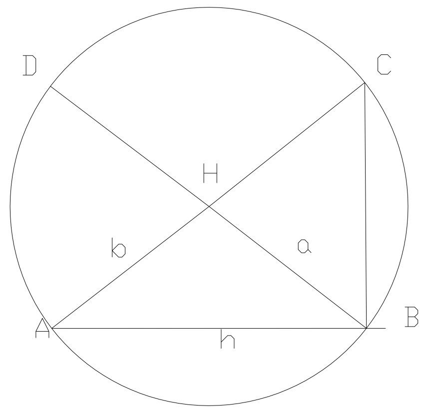


Fig. 59

Proof 1. AB x BC = BH^2 + AH x CH ⁚. $AB^2 = HB^2 + AH^2$.·. . $h^2 = a^2 + b^2$.

Proof 2. AB x BC = BD x BH = (BH + HD) x BH = BH^2 + (HD x HB = HA x HC) = $BH^2 + AH^2$. $h^2 = a^2 + b^2$.

a. These two proofs are from Math. Mo., 1859, Vol. 2, No.2, Dem,20 and Dem. 21, and are applications of Prop. XXXI, Book IV, Davies Legendre, (1858), p. 119; or Book III, p. 173, Exercise 7, Schuyler's Geom., (1876) of Book III, p. 165, Prop. XXIII, Wenworth's New Plane Geom., (1895).

b. But it does not follow that being true when HA = HB, it will be true when HA>or <HB. The author.

SIXTY- TWO

At B erect a perp. to AB and prolong AH to C, and BH to D. BH = HD Now AB^2 = AH x AC = AH (AH + HC) = AH^2 + (AH x HC = HB^2) = $AH^2 + HB^2$. $h^2 = a^2 + b^2$. Q.E.D.

a. See Verslus, p. 92, fig. 105.

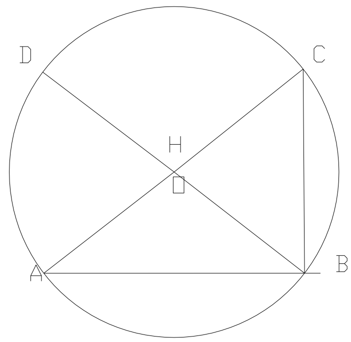


Fig. 60

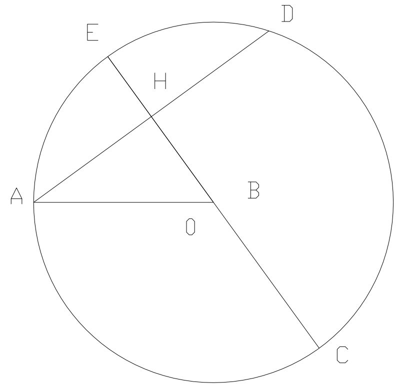


Fig.61

SIXTY- THREE

From the figure 61 it is evident that AH x HD = HC x HE, or $b^2 = (h + a)(h - a) = h^2 - a^2$. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 92, fig. 106, and credited to Wm. W. Rupert, 1900.

SIXTY- FOUR

With CB as radius describe semicircle BHA cutting HL at K and AL at M. Arc BH = arc KM . ∴ BN = NQ = AO = MR and KB = KA; also arc BHK = arc AMR = MKH = 90^0. So tri's BRK and KLA are congruent. HK = HL – KL = HA – OA. Now HL : KL = HA : OA. So HL – KL : HL = HA : OA :HA, or (HL – KL) /HL = (HA – OA)/ HA = (b- a) /b . ∴ KQ = (HK – KL) LP = [(b – a) /b]x $^1/_2$ b = $^1/_2$ (b – a).

Now tri. KLA = tri. HLA – tri.AHK $= ^1/_4 \; b^2 - ^1/_2 \; b \times ^1/_2 \; (b - a) = ^1/_4 \; ba$ $= ^1/_2$ tri. ABH, or tri. ABH = tri. BKR + tri. KLA, whence trap. LABR – tri.ABH = trap. LABR-(tri.BKR + tri.KLA) = trap. LABR-(tri.HBR + tri.HAL) = trap. LABR – tri. ABK ∴ tri. ABK = tri. HBR + tri.HAL; or 4 tri.HBR + 4tri. HAL. $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 93, fig. 107; and found in Journal de Mathein, 1897, credited to Brand. (10/23, '33, 9P. m. E. S.L.).

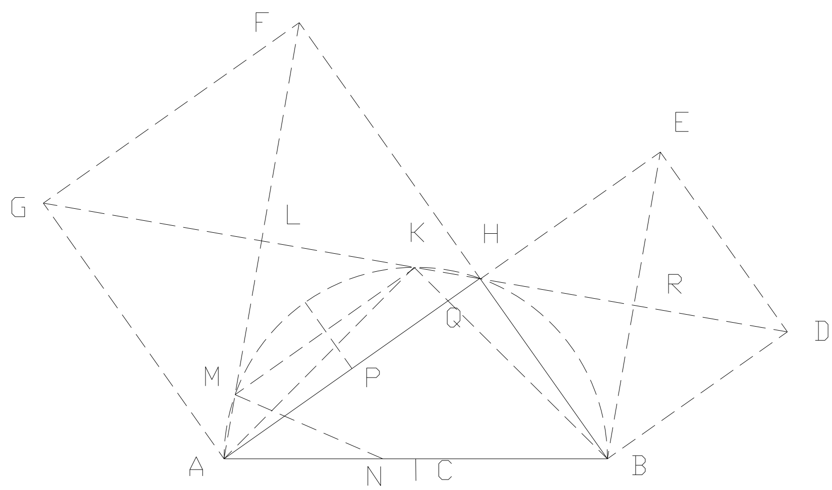


Fig. 62

SIXTY-FIVE

Fig. 63

The construction is obvious. From the similar triangles HDA and HBC, we have HD : HB = AD : CB, or HD x CB = HB x AD.----(1)

In like manner, from the similar triangles DHB and AHC, HD x AC = AH x DB. ----(2) Adding (1) and (2), HD x AB = HB x AD + AH x DB.---(3) ∴ $h^2 = a^2 + b^2$.

a. See Halsted's Elementry Geom., 6^{th} Ed'n, 1895 for Eq. (3) , p. 202; Edwards' Geom., p. 158, fig. 17; Am. Math. Mo. ,V. IV, p. 11.

b. Its rifst appearance in print, it seems, was in Runkle's Math. Mo. , 1859, and by Runkle credited to C. M. Raub, of Allentown, Pa.

c. May not a different solution be obtained from other proportions from these same triangles?

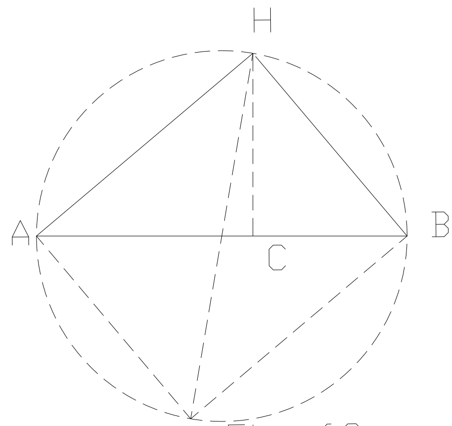


Fig. 63

SIXTY – SIX

FIG 64-65

Ptolemy's Theorem (A.D. 87- 168). If ABCD is any cyclic (inscribed) quadrilateral, then AD x BC + AB x CD = AC x BD.

As appears in Wentworth's Geometry, recised edition (1895), p. 176, Theorem 238. Draw DE making angleCDE = angleADB. Then the tri's ABD and CDE are similar; also the tri's BCD and ADE are similar. From these pairs of similar triangles it follows that AC x BD = AD x BC + DC x AB. (For full demonstration, see Teacher's Edition of Plane and Solid Geometry (1912) , by Geo. Wentworth and David E. Smith, p. 190, Proof 11.)

In case the quad. ABCD become a rectangle then AC = BD , BC= AD and AB = CD. So $AC^2 = BC^2 + AD^2$, or $c^2= a^2 + b^2$. ∴ a special case of Potlemy's Theorem gives a proof of the Pyth. Theorem.

a. As formulated by the author. Also see " A Companion to Elementry School Mathematics (1924), by F.C. Boon, B.A. , p. 107 proof 10.

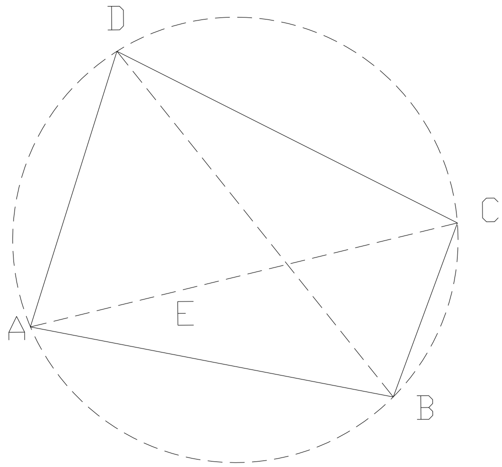


Fig.64

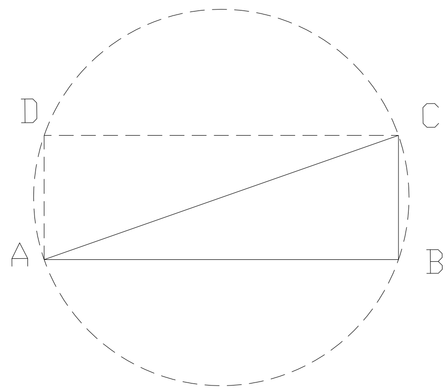


Fig.65

SIXTY- SEVEN

FIG. – 66

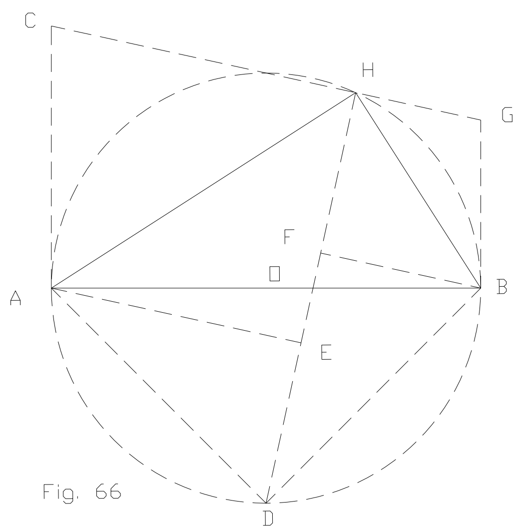


Fig. 66

Circumscribe about tri. ABH circle BHA. Draw AD = DB. Join HD. Draw CG perp. to HD at H, and AC and BG each perp. to CG; also AE and BF perp. to HD.

Quad's CE and FG are squares. Tri's HDE and DBF are congruent ∴ AE = DF = KH = AC. HD = HF + FD = BG + AC. Quad. ADBH = $^1/_2$ HD(BF + AE) = $^1/_2$ HD x CG. Quad. ABGC = $^1/_2$ (AC + BG) x CG = $^1/_2$ HD x CG. ∴tri.ADB = tri.ACH + 4tri.HBG. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See E. Fourray's C. Geom., 1907; credited to Piton-Bressant; see Versluys, p. 90, fig. 103.

b. See fig. 333 for Geom. Proof – so –called.

SIXTY-EIGHT

FIG. 67

Construction same as in fig. 66, for points C, D and G. Join DG. From H draw HE perp. to AB, and join EG and ED. From G draw Gk perp. to HE and GF perp. to AB, and extend AB to F. KF is a square, with diag. GE. ∴angle BEG = angle EBD = 45^0. ∴GE and BD are parallel. Tri. BDG = tri. BDE.---(1) Tri. BGH = tri. BGD. ----(2). ∴ (1) = (2) , or tri. BGD = tri. BDE. Also tri. HCA = tri. ADE. ∴ tri. BGH+ tri. HCA = tri. ADB. So 4 tri. ADB = 4 tri.BHG + 4 tri. HCA. $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 91. fig. 104, and credited also to Piton Bressant. as found in E. Fourrey's Geom., 1907, p. 79, IX.

b. See fig. 334 of Geom. Proofs.

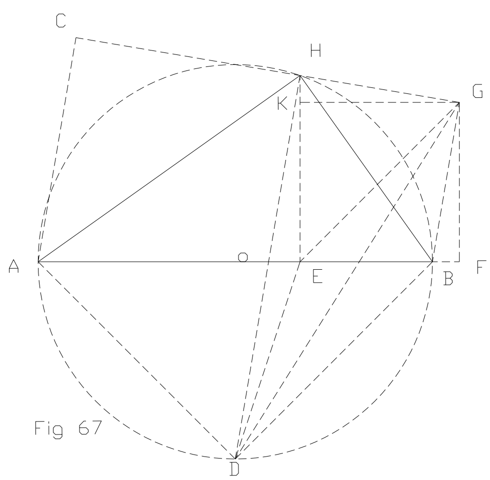


Fig 67

SIXTY- NINE

In fig. 63 above it is obvious that AB x BH = AH x BD + AD x BH.∴ $AB^2 = HA^2 + HB^2$ ∴ $h^2 = a^2 + b^2$.

a. See Math. Mo. , 1859, by Runkle, Vol. II No. 2, Dem. 22, fig. 11.

b. This is a particular case of Prop. XXXIII, Book IV, p. 121, Davies Legendre (1858) which is Exercise 10, in Schuylaet's Geom. (1876), Book III, p. 173, or Exercise 238, Wentworth's New Plane Geom. (1895), Book III, p. 176.

SEVENTY

On any diameter as AE = 2AH, const. rt. tri. ABH, and produce the sides to chords. Draw ED. From the sim. tri's ABH and AED, AB: AE = AH : AD, or h : b + HE = b : h + BD. ∴ h (h + BD) = b(b+HE) = b^2 + bxHE = b^2 + HF x HC = b^2 + HC^2.----(1)

Now conceive AD to revolve on A as a center until D coincides with C, when AB = AD = AC = h, BD = O, and BH = HC = a. Substituting in (1) we have $h^2 = a^2 + b^2$.

a. This is the solution of G. I. Hopkins of Manchester, N. H. See his Plane Geom. , p. 92, art.427; also see Jour. Of Ed., 1888, V.XXVII, p. 327, 16th prob. Also Heath's Math. Monographs, No. 2, p. 28, proof XV.

b. Special case . When H coincides with O we get (1) BC = (b + c)(b – a) /h and (2) BC = 2 b^2/ h – h.

c. See Am. Math. Mo., V. III, p. 300.

(2) The Method by Secants.

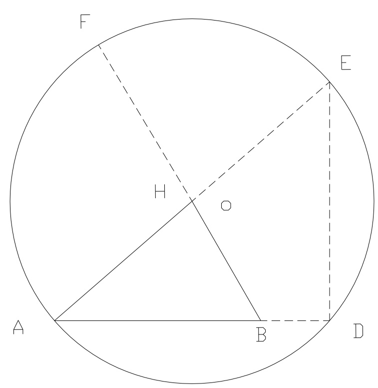


Fig. 68

SEVENTY- ONE

FIG. 69

With H as center and HB as radius describe the circle EBD.

The secants and their external segments bring reciprocally proportional, we have, AD : AB = AF: AE, or b + a : h = (h – 2CB = h – $2a^2$/h) : b – a , whence ∴ $h^2 = a^2 + b^2$.

a. In case b = a , the points A, E and F coincides and the proof still holds; for substituting b for a the above prop'n reduces to $h^2 – 2a^2$ as it should.

b. By joining E and B, and F and D, the similar triangles upon which the above rests are formed.

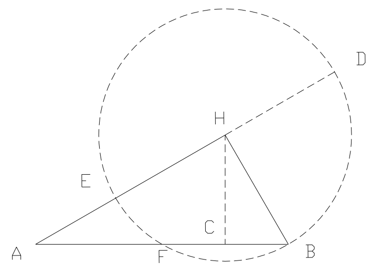


Fig. 69

SEVENTY – TW0

FIG. 70

With H as center and HB as radious describe circle FBD, and draw HE and HC to Middle of EB.

AE x AB = AF x AD, or (AD – 2BC) AB = (AH – HB) (AH + HB) . $\therefore AB^2 - 2BC \times AB = AH^2 - BH^2$. $\therefore AB^2 = HB^2 + HA^2$. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. Math. Mo., Vol. II, No. 2, Dem. 25,fig. 2. Derived from: Prop. XXIX, Book IV, p. 118, Davies Legendre (1858); Prop. XXXIII, Book III, p. 171, Schuyler's Geometry (1876); Prop. XXI, Book III, p. 163, Wentworth's New Plane Geom. (1895).

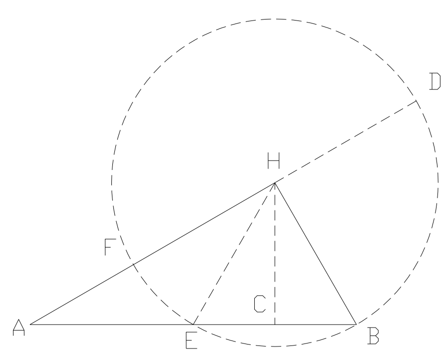


Fig. 70

SEVENTY- THREE

FIG. 71

AE : AH = AH : AD. ∴ AH^2 = AE x AD = AE (AB x BH) = AE x AB + AE x BH. So $AH^2 + BH^2$ = AE x AB + AE x HB + HB^2 = AE x AB + HB(AE + BH) = AB (AE + BH) = AB^2. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Math. Mo., (1859); Vol. II, No. 2, Dem. 26, p. 13; derived from Prop. XXX, Prop. XXXII, Cor. P. 172 (1876); Wentworth's Geom., Book III, Prop. XXII, p. 164. It is credited to C. J. Kemper, Harrinonburg, Va., and Prof.Charles A Young (1859), at Hudson, O. Also found in Fourrey's collection, p. 93, as given by J.J.I. Hoffmann, 1821.

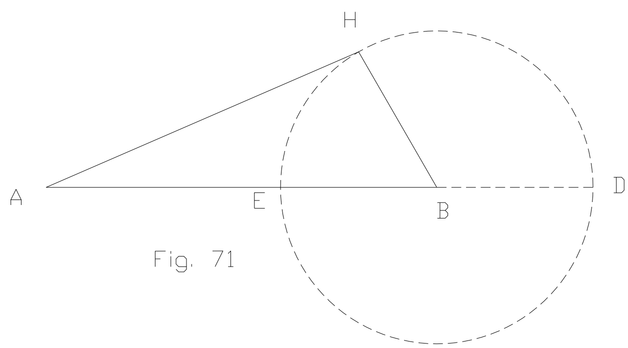


Fig. 71

SEVENTY-FOUR
FIG.72

In fig. 72, E will fall between A and F , or between f and B as HB is less than, equal to, or greater than HE. Hence there are three cases; but investigation of one case---- when it falls at middle point of AB ----- is sufficient.

Join L and B, and F and C, making the two similar triangles AFC and ALB; whence h: b + a = b – a : AF ; ∴ AF = $b^2 – a^2/h$. -----(1)

Join F and g. and B and D making the two similar tri's FGE and BDE, whence $^1/_2$ h = : a - $^1/_2$ h = a + $^1/_2$ h : FE, whence FE = $(a^2 - {}^1/_2\ h^2)/{}^1/_2\ h$.-----(2)

Adding (1) and (2) gives $^1/_2$ h = $(a^2 + b^2 - {}^1/_2\ h^2)/h$; whence $h^2 = a^2 + b^2$.

a. The above solutions given by Krueger, in "Aumerkungen uber Hrn. geh. R Wolf's Auszug ausder Geometrie," 1746. Also see Jury Wipper. p.41, fig. 42, and Am. Math. Mo. , V. IV, p. 11.

b. When G falls midway between F and B, then fig. 72 become fig. 69. Therefore cases 69. and 72 are closely related.

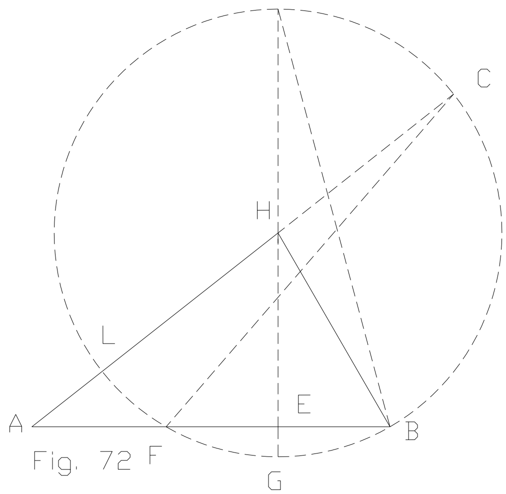


Fig. 72

SEVENTY- FIVE

In fig. 73a, take HF = HB. With B as center, and BF as radius describe semicircle DEG. G being the pt. where the circle intersects AB. Produce AB to D, and draw FG, FB, BE to AH produced, and DE, forming the similar tri's AGF and AED, from which (AG = a) : (AF = y) = (AE = y + 2FH) : (AD= x + 2BG) = y + 2z : x + 2r whence $x^2 + 2rx = y^2 + 2yz$ ----(1)

But if, see fig. 73b, HA = HB, (sq. GE = h^2) = (sq. HB = a^2) + (4tri. AHG =

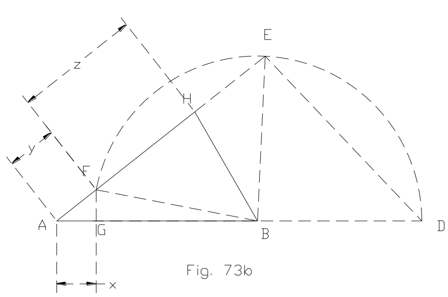


Fig. 73b

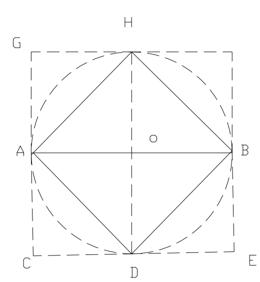


Fig. 73b

sq. AH = b^2), whence $h^2 = a^2 + b^2$; then, (see fig.73a.) when BF = BG, we will have $BG^2 = HB^2 + HF^2$, or $r^2 + z^2 + z^2$ (since z = FH). -----(2)

(1) + (2) = (3) $x^2 + 2rx + r^2 = y^2 + 2yz + z^2 + z^2$ or (4) $(x + r)^2 = (y + z)^2 + z^2$ ∴ (5) $h^2 = a^2 + b^2$, since x + r = AB = h, y + z = AH = b, and z = HB = a.

a. See Jury wiper, p. 36, where Wipper also credits it to Joh. Hoffmann. See also Wipper, p. 37, fig. 34, for another statement of same proof; and Fourrey, p. 94, for Hoffmann's proof.

SEVENTY- SIX

In fig. 74 in the circle whose center is O, and whose diameter is AB, erect the perp. DO. Join D to A and B, produce DA to F, making AF = AH, and produce HB to G making BG = BD, thus forming the two isosceles tri's FHA and DGB; also the two isosceles tri's ARD and BHS. As angle DAH = 2 angle at F, and angle HBD = 2 angle at G, and as angle DAH and angle HBD are measured by same arc HD, then angle at F = angle at G, ∴ arc AP = arc QB.

And as angles ADR and BHS have same measure, $^1/_2$ of arc APQ, and $^1/_2$ of arc BQP, respectively, then tri's ARD and BHS are similar, R is the intersection of AH and DG, and S the intersectionof BD and HF. Now since tri's FSD and GHR are similar, being equiangular, we have , DS : DF= HR : : HG ∴ DS : (DA + AF) = HR: (HB +BG)

∴ DS : (2BR + RH) = HR : (2BS + SD),.

∴ (1) $DS^2 + 2DS \times BS = HR^2 + 2HR \times BR$. And (2) $AH^2 = (HR + RA)^2 = HR^2 + 2HR \times RA + RA^2 = HR^2 + 2HR \times RA + AD^2$

(3) $HB^2 = BS^2 = (BD – DS)^2 = BD^2 + 2BDxDS + DS^2 = AD^2 – 2BDxDS – DS^2) = AD^2 – 2 (BS + SD) DS + DS^2 = AD^2 – 2BSxSD – 2DS^2 + DS^2 = AD^2 - 2BS \times DS – DS^2 = AD^2 – (2BS xDS - DS^2)$

(2) + (3) = (4) $HB^2 + HA^2 = 2 AD^2$ But as in proof, fig. 73b, we found, (eq.2). $r^2 = z^2 + z^2 = 2 z^2$. ∴ $2 AD^2$ (in fig.74) = $AB^2 = h^2 = a^2 + b^2$.

a. See Jury Wipper, p.44. fig. 43, and there credited to Hoh. Hoffmann, one of his 32 solutions.

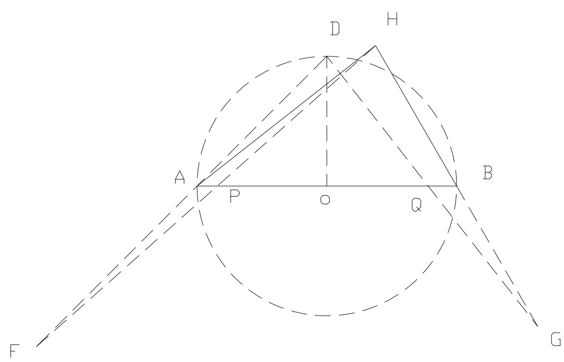


Fig.74

SEVENTY- SEVEN

In fig. 74, let BCA be any triangle, and let AD, BE and CF be the three perpendiculars from the three verticles, A, B, and C, to the three sides, BC, CA and AB, respectively. Upon AB, BC and CA as diameters describe circumference, and since the angles ADC, BEC and CFA are rt. angles, the since the circumferences pass though the points D and E, F and E, and F and D, respectively.

Since BC x BD = BA x BF, CB x CD = CA x CE, and AB x AF = AC x AE, therefore [BC x BD + CB x CD = BC(BD + CD) = BC^2] = [BA x BF + CA x CE = BA^2 + AB x AF + CA^2 + AC x AE = AB^2 + AC^2 + 2AB x AF (or 2AC x AE)].

When the angle A is acute (fig.75a) or obtuse (fig.75b) the sign is – or + respectively. And as angle A approaches 90^0 they become 0, and we have $BC^2 = AB^2 + AC^2$. ∴ when A = a rt. angle $h^2 = a^2 + b^2$.

a. See Olney's Elements of Geometry, University Edition, Part III, p. 252, art. 671, and Heath's Math. Monographs. No. 2 p. 35, proof XXIV.

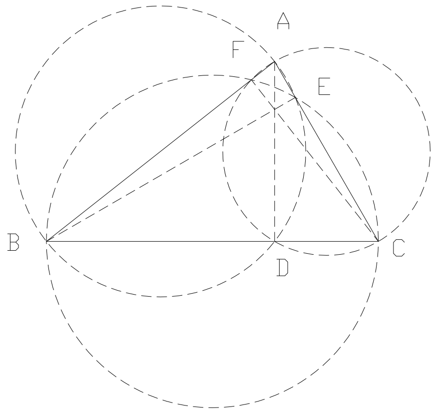


Fig.75a.

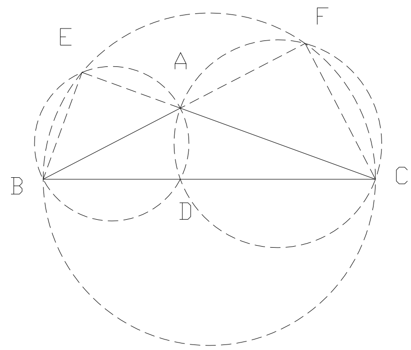


Fig. 75b

SEVENTY- EIGHT

FIG. 76

Produce KC and HA to M, complete the rect. MB, draw CN and AP perp.to HM.

Draw the semicircle ANC on the diameter AC. Let MN = x. Since the area of the paral. MFBA = the area of the sq. AK, and since, by the Theorem for the measurement of a parallelogram. (see fig. 308, this text), we have (1) sq. AK = (BF x AP = AM x AP) = a (a + x) . But in MCA, CN is a mean proportional between AN and NM. $\therefore$ (2) $b^2 = ax$. (1) – (2) = (3) $h^2 - b^2 = a^2 + ax - ax = a^2$. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. This proof is No. 99 of A. R. Colburn's 108 solutions, being devised Nov. 1, 1922.

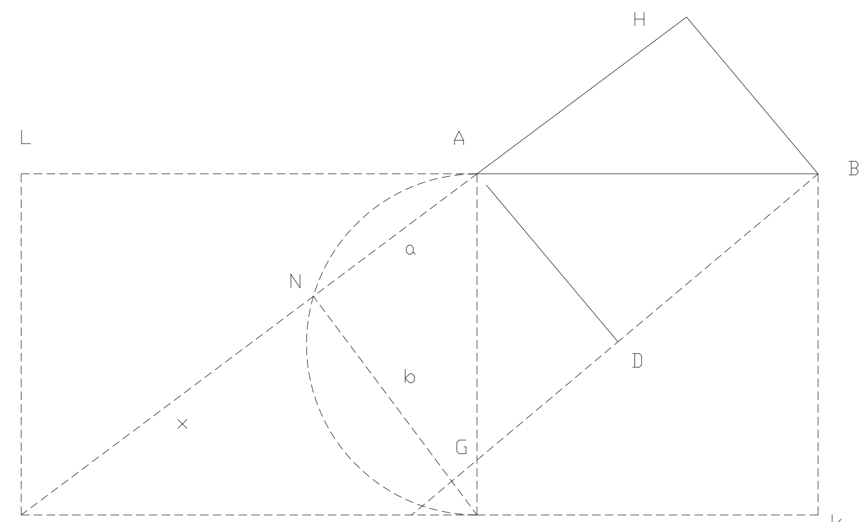


Fig. 76

(3)The Method by Tangents

1^{st}. ---- The Hypotenuse as a Tangent

SEVENTY-NINE

Draw in fig 77 HC perp. to AB, and with H as a center and HC as a radius describe circle GDEF.

From the similar tri's ACG and AEC, AC: or AC: b + r = **b – r : AC; ∴ (1) $AC^2 = b^2 - r^2$. From the similar tri's CBD and BFC, we get (2) $CB^2 = a^2 – r^2$. From the similar rt. tri's BCH and HCA, we get (3) BC x AC = r^2 ∴ (4) 2BC x AC = $2r^2$. (1) + (2) + (4) gives (5) $AC^2 + 2AC$ x BC + $BC^2 = a^2 + b^2 = (AC + BC)^2 = AB^2$.** $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. III, p. 300

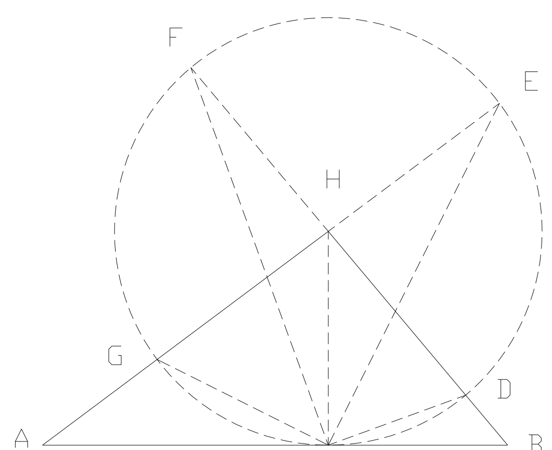


Fig.77

EIGHTY

O, the center of the circle, lies on the bisector of angle B, and on AH.

With the construction completed, from the similar tri's ACD and AHC, we get, calling OC = r, (AC = h – a) : (AH = b) = (AD = b – 2r) : (AC = h – a). $\therefore$ (1) $(h – a)^2 = b^2 - 2br$. But (2) $a^2 = a^2$. (1) + (2) = (3) $(h – a)^2 + a^2 = a^2 + b^2 - 2br$, or $(h – a)^2 + 2br + a^2 = a^2 + b^2$. Also (AC= h – a) : (AH = b) = (OC = OH = r) : (HB =a) . whence

(4) $(h – a) a = br$.

(5) $\therefore (h – a)^2 + 2(h – a) r + a^2 = a^2 + b^2$.

(6) $h^2 = a^2 + b^2$.

Or , in (3) above ,expand and factor gives

(7) $h^2 – 2a(h – a) = a^2 + b^2 - 2br$. Sub. For a(h – a) its equal, see (4) above, and collect, we have

(8) $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. IV, p. 81.

FIG.78

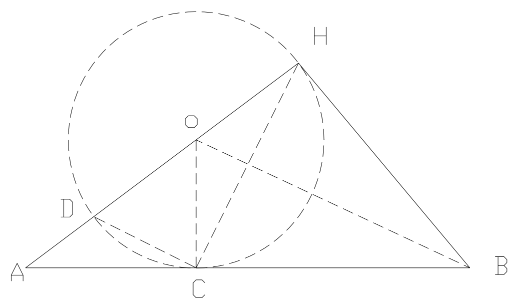


Fig. 78

EIGHTY – ONE

Having HB, the shorter leg, a tangent at C, any convenient pt. on HB, the construction is evident .

From the similar tri's BCE and BDC, we get BC : BD = BE= BE : BC, whence BC^2 = BD x BE = (BO +OD) BE = (BO + OC) BE. ----(1) From similar tri's. OBC and ABH, we get OB : AB = OC : AH. Whence OB/h = r/b; $\therefore$ BO = hr/b. ----(2) BC : BH = OC : AH. Whence BC = ar/b. ----(3) Substituting (2) and (3) in (1) , gives, $(a^2 r^2 /b^2)$ = [(hr/b)+ r] BE = [(hr + br)/b](BO – OC) = [(hr + br)/b] [(hr + br)/b]. ----(4) whence $h^2 = a^2 + b^2$. Q.E.D.

a. Special case is : when ,in fig. 79, O coincides with A, as in fig. 80

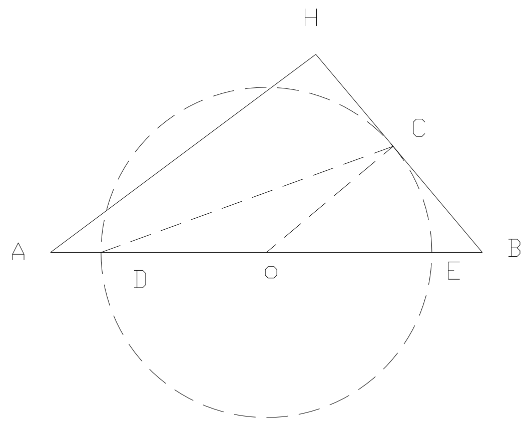


Fig. 79

EIGHTY – TWO

With A as center and AH as radius, describe the semicircle BHD.

From the similar triangle BHC and BDH we get, h – b : a = a : h + b, whence directly $h^2 = a^2 + b^2$.

a. This case is found in: Heth's Math. Monographs, No. 1, p. 22, proof VII; Hopkins' Plane Geom., p. 92, fig. IX; Journal of Education, 1887 V. XXVI, p. 21, fig. VIII; Am. Math. Mo. , V. III, p. 229; Jury Wipper, 1880, p. 39, fig. 39 where he says it is found in Hubert's Elements of Algebra, Wurceb, 1792, also in Wipper, p. 40, fig. 40, as one of Joh. Hoffmann's 32 proofs. Also by Richardsonin Runkle's Mathematical (Journal) Monthly, No. 11, 1859 ---one of Richardson's 28 proofs; Versluys. p. 89, fig. 99.

b. Many persons, independent of above sources, have found this proof.

c. When O, in fig. 80, is the middle pt. of AB, it becomes a special cse of fig. 79

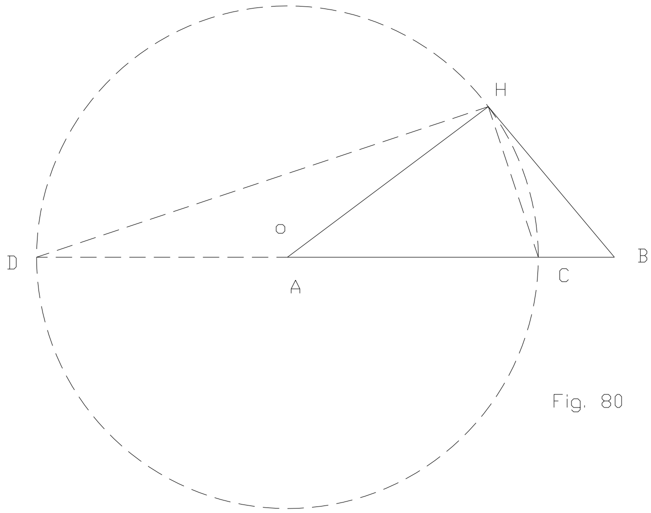


Fig. 80

EIGHTY- THREE

Assume HB< HA, and employ tang. HC and secant HE, whence HC^2 = HE x HD = AD x AE = AG x AF = BF x BG = BC^2 . Now employing like argument as in proof Eighty- One we get $h^2 = a^2 + b^2$.

a. When O is the middle point of AB, and HB = AH, then HB and HA are tangents, and AG = BF, secants, the argument is same as (c) , proof Eighty-Two by applying theory of limits.

b. When O is any pt. in AB, and the two legs are tangents. This is only another form of fig. 79 above, the general case. But as the general case gives, see proof, case above, $h^2 = a^2 + b^2$, therefore $h^2 = a^2 + b^2$. Q. Or if a proof by eplicit argument is desired, procedd as in fig.79

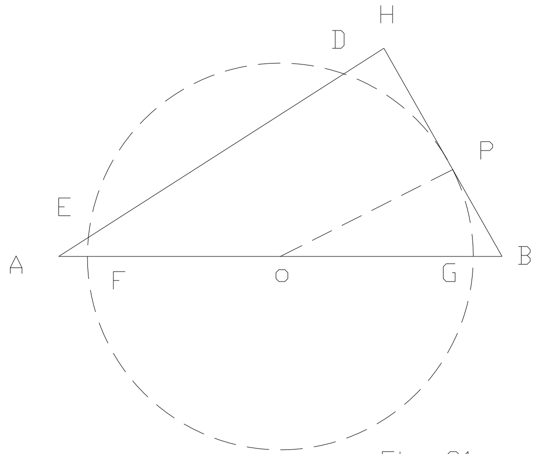


Fig. 81

EIGHTY –FOUR

By proving the general case, as in fig. 79, and then showing that same case is only a particular of the general, and therefore true immediately, is here contrasted with the following ling and complex solution of the assumed particular case.

The following solution is given in the Am. Math . Mo., V. IV, p. 80: "Draw OD perp. to AB. Then, $AT^2 = AE \times AF = AO^2 - EO^2 = AO^2 - TH^2$ ----(1)

$BP^2 = BF \times BE = BO^2 – FO^2 = BO^2 – HP^2$ -----(2)

Now, AO : OT = AD : OD;

∴ AO : OD = OT x AD.

And, since OD = OB , OT = TH = HP, and AD = AT + TD = AT + BP.

∴ AT x TH + HP x BP = AO x OB. ----(3)

Adding (1) , (2) , and (2) x (3) ,

$AT^2 + BP^2 + 2AT \times TH + 2HP \times BP = AO^2 - TH^2 + BO^2 - HP^2 + 2AO \times OB$;

∴ $AT^2 + 2\,AT \times TH + FH^2 + BP^2 + 2BP \times HP + HP^2 = AO^2 + 2AO \times OB + BO^2$.

∴ $(AT + TH)^2 + (BP + CP)^2 = (AO + OB)^2$.

∴ $AH^2 + BH^2 = AB^2$. Q.E.D.

$h^2 = a^2 + b^2$.

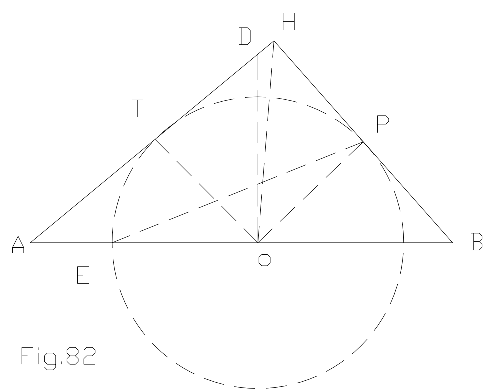


Fig.82

3rd. --- The Hypotenuse a Secant Not Passing Though the Center of the Circle, and Both Legs Tangents.

EIGHTY- FIVE

Through B draw BC parallel to HA, making BC = 2BH; with O the middle point of BC, as center, describe a circumference, tangent at Band E, and draw CD, forming the two similar rt. tri's ABH and BDC, whence BD : (AH = b) = (BC= 2a) : (AB =h) from which, DB = 2ab/h. (1)

Now, by the principal of tang. and sec. relations, $[AE^2 = (b - a)^2] = (AB = h)(AD = h - DB)$, whence

$DB = h - (b- a)^2 / h$ --(2)

Equating (1) and (2) gives $h^2 = a^2 + b^2$.

a. If the legs HB and HA are equal, by theory of linits same result obtains.

b. See Am. Math. Mo., V. IV, p. 81, No, XXXII.

c. See proof Fifty- Two above, and ovserve that this proof Eighty-Five is superior to it.

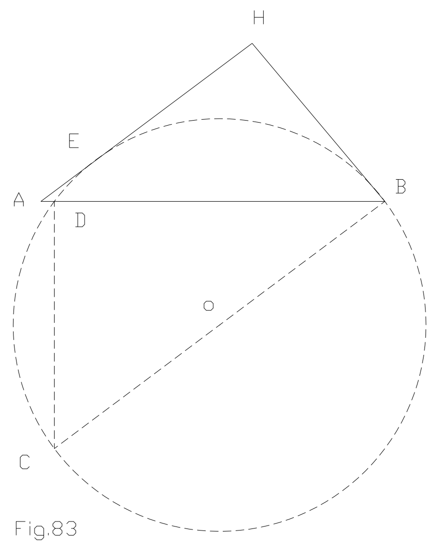


Fig.83

4TH.HYPOTENUSE AND BOTH LEGS TANGENTS

EIGHTY-SIX

The tangent points of the three sides are C, D and E.

Let OD = r = OE = OC, AB = h, BH= a and AH= b.

Now,

(1) $h + 2r = a + b$.

(2) $h^2 + 4hr + 4r^2 = a^2 + 2ab - b^2$.

(3) Now if $4hr + 4r^2 = 2ab$. then $h^2 = a^2 + b^2$.

(4) Suppose $4hr + 4r^2 = 2ab$.

(5) $4r(h + r) = 2ab$; $\therefore$ $2r(h+r) = ab$. (1) = (6) $2r = a + b - h$.

(6) In (5) gives

(7) $(a + b - h)(h + r) = ab.$

(8) $h(a + b - h - r) + ar + br = ab.$ (1) = (9) $r = (a + b - h - r).$

(9) In (8) gives

(10) $Hr + ar + br = ab.$

(11) But $hr + ar + br = 2$area tri. ABC.

(12) And $ab = 2$area tri. ABC. $\therefore$

(13) $hr + ar + br = ab = hr + r(a + b) = hr + r(h + 2r)$ $\therefore$

(14) $4hr + 4r^2 = 2ab.$

$\therefore$ the suppoitoin in (4) is true.

(15) $h^2 = a^2 + b^2$.Q.E.D.

a. This solution was devised by the author Dec. 13, 1901, *Before receiving* Vol. VIII, 1901, p. 258, Am. Math. Mo. where a like solution is given; also see Fourrey p. 94, where credited.

b. By drawing a line OC, in fig. 84, we have the geom.. fig. from which, May, 1891, Dr. L. A. Bauer, of Carnegie Institute, Wash. , D.C. deduced a proof through the equations (1) Area of tri ABH = $^1/_2$ $r(h + a + b)$, and (2) HD + HE = $a + b - h$. See pamphlet: On Rational Right- Angled Triangles Aug., 1912, by Artemus Martin for the Bauer proof. In same pamphlet is still another proof attributed to Lucius Brown of Hudson, Mass.

c. See Olney's Elements of Geometry, University Edition, p. 312, art. 971, or Scuyler's Elements of Geometry, p. 353, exercise 4; also Am. Math. Mo., V,VI, p. 12, proof XXVI; also Versluys. p. 90, fig.102; also Grunert's Archiv. der Mathein, and Physik; 1851, credited to Mollmann.

d. Remrk. – By ingenious devices, some if not all, of these in which the circle has been employed can be proved without the use of the Circle-not nearly so easily perhaps, but proved. The figure, without the circle, would suggest the device to be employed. By so doing new proofs may be discovered.

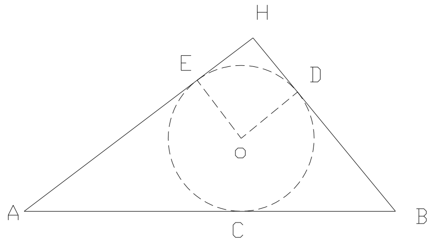


Fig. 84

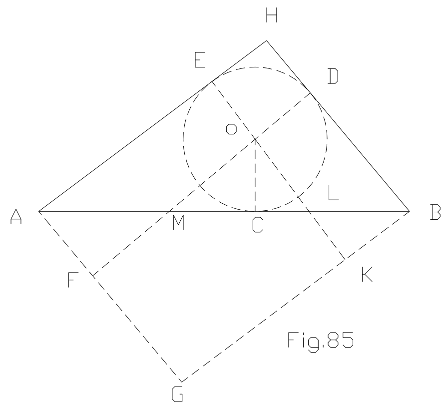


Fig.85

EIGHTY- SEVEN

Complete rect. HG. Produce DO to F and EO to K. Designate AC = AE by q and HE = HD by r.

Then a = q + r , b = p + r, and h = p + q. Tri. FMA = tri. OMC and tri. COL = tri. KLB.

∴ tri. AGB = rect. FGKO= tri. ABH = $^1/_2$ rect. HG. Rect. FGKO = rect. AFOE + sq. ED + rect. OKBD.

So $pq = pr + r^2 + qr$. Whence $2pq = 2qr + 2r^2 + 2pr$.

But $p^2 + a^2 = p^2 + q^2$.

So $p^2 + 2pq + q^2 = (q^2 + 2qr + r^2) + (p^2 + 2pr + r^2)$ or $(p + q)^2 = (q + r)^2 + (p + r)^2$.

∴ $h^2 = a^2 + b^2$.

a. Sent to me by J. Adams, from The Hague, and credited to J.F. Vaes, XIII, 4 (1917).

(II) THROUGH THE USE OF TWO CIRCLES.

EIGHTY- EIGHT

Construction. Upon the legs of the re. tri. ABH, as diameters, construct circles and draw HC, forming three similar rt. tri's ABH, HBC and HAC.

Whence h : b = b : AC. ∴ $hAC = b^2$ ----(1)

Also h : a = a : BC. ∴ $hBC = a^2$ ----(2)

(1) + (2) = (3) $h^2 = a^2 + b^2$. Q.E.D.

a. Another form is :

(1) $HA^2 = HC \times AB$. (2) $BH^2 = BC \times AB = AB (AC + BC) + AB^2$ ∴ $h^2 = a^2 + b^2$. Q.E.D.

b. See Edwards' Elements of Geom., p. 161, fig. 34 and Am. Math. Mo., V. IV, p. 11; Math. Mo. (1859), Vol. II, No. 2, Dem. 27, fig. 13; Davies Legendre, (1876) Book III. Prop. XXXIII, cor., p. 172; Wentworth's New Plane Geom. (1895). Book III, Prop. XXII, p. 164, from each of said Propositions, the above proof Eighty- Eight may be drived.

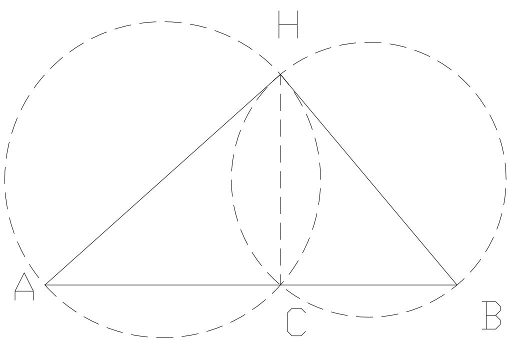


Fig. 86

EIGHTY- NINE

With the legs of the rt. tri. ABH as radii describe circumferences, and extend AB to C and F. Draw HC, HD, HE and HF. From the similar tri's AHF and HDH, AF : AH = AH : AD ∴ $b^2 = AF \times AD$. ----(1)

From the similar tri's CHB and HEB,

CB : HB = HB : BE . ∴ $a^2 = CB \times BE$. ----(2)

(1) + (2) = (3) $a^2 + b^2 = CB \times BE + AF \times AD$

$=(h + b) (h – b) + (h + a) (h – a)$

$h^2 - b^2 + h^2 - a^2$;

∴ (4) $2h^2 = 2a^2 + 2b^2$.

a. Am. Math. Mo., V. IV, p. 12; also on p. 12 is a proof by Richardson. But it is much more difficult than the above method.

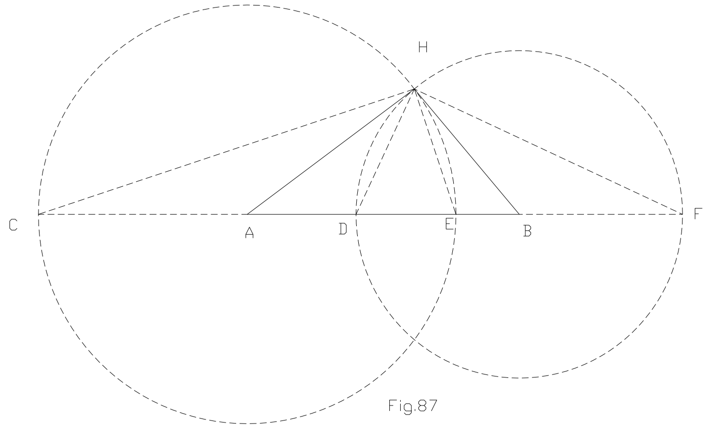


Fig.87

NINTY

For proof Ninty use fig.87

$AH^2 = AD (AB + BH)$ ----(1) $BH^2 = BE (BA + AH)$ ----(2) . (1) + (2) = (3) $BH^2 + AH^2 = BH (BA \_ AH) + AD (AB + BH) = BH \times BA + BE \times AH + AD \times HB + AD \times BH = HB (BE + AD) + AD \times BH + BE \times AH + BE \times AB - BE \times AB$

$= AB(BE + AD) + AD \times BH + BE (AH + AB) - (BE \times AB)$

$= AB (BE + AD) + AD \times BH + BE (AH + AE + BE) - BE \times AB$

$= AB (BE + AD) + AD \times BH + BE (BE + 2 AH) - BE \times AB$

$= AB (BE + AD) + AD \times BH + BE^2 + 2BE \times AH - BE \times AB$

$= AB (BE + AD) + AD \times BH + BE^2 + 2BE \times AE - BE (AD + BD)$

$= AB (BE + AD) + AD \times BH + BE^2 + 2BE \times AE - BE \times AD - BE \times BD$

$= AB (BE + AD) + AD \times BH + BE (BE + 2AE) - BE (AD + BD)$

$= AB (BE + AD) + AD \times BH + BE (AB + AH) - BE (AD + BD)$

$= AB (BE + AD) + AD \times BH + (BE \times BC = BH^2 = BD^2) - BE (AD + BD)$

$= AB (BE + AD) + (AD + BD) (BD - BE)$

$= AB (BE + AD) + AB \times DE = AB (BE + AD + DE)$

$= AB \times AB = AB^2 \quad \therefore h^2 = a^2 + b^2$. Q.E.D.

a. See Math. Mo. (1859), Vol. II, No. 2, Dem. 28, fig. 13 ---derived from Prop. XXX, Book IV, p. 119, Davies Legendre, 1858; also Am. Math. Mo. Vol. IV, p. 12, proof XXV.

NINETY – ONE

For proof Ninety-One use fig.87. This proof is known as the "Harmonic Proportion Proof."

From the similar tri's AHF and ADH.

AH : AD = AF : AH, of AC : AD = AF : AE

whence AC + AD : AF + AE = AD : AE

or CD : CF = AD : AE,

and AC - AD = AF – AE = AD : AE,

or DE : EF = AD : AE.

$\therefore$ OD : CF = DC : EF.

or (h + b – a) : (h + b + a) = (a – h + b) : (a + h + b)

$\therefore$ by ecpanding and collecting, we get $h^2 = a^2 + b^2$.

a. See Olney's Elements of Geom., University Ed'n, p. 312, art. 971, or Schuyler's Elements of Geom., p. 353, Exercise 4; also Am. Math. Mo. , V. IV, p. 12 proof XXVI.

D.---- RATIO OF AREAS

As in the three preceding divisions, so here in D we must rest our proofs on similar rt. triangles.

NINETY-TWO

Draw HC perp. to AB , forming the three similar triangles ABH, AHC, and HBC, and denote AB = h CB =y and HC = z.

Since similar aurfaces are proportional to the squares of their homologous dimensions, therefore,

$[\frac{1}{2} (x + y) z + \frac{1}{2} yz = h^2 + a^2] = [\frac{1}{2} yz + \frac{1}{2} xz = a^2 + b^2]$

$=[\frac{1}{2} (x + y) z + \frac{1}{2} yz = (a^2 + b^2) a^2]$

$\therefore h^2 + a^2 = (a^2 + b^2 + a^2 \therefore h^2 = a^2 + b^2$.

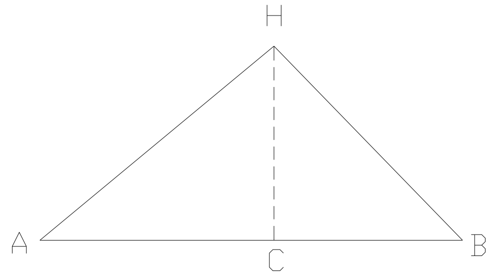


Fig. 88 &89

a. See Jury Wipper, 1880, p. 38, fig. 36, as found in Elements of Geometry of Bezout; Fourrey, p. 91, as in Wallis' Treatiese of Algebra, (Oxford), 1685; p.93 of Cours de Mathematiques, Paris, 1768.Also Heat's Math. Monographs, No. 2, p. 29, proof XVI; Journal of Education, 1888, V. XXVII, p. 327, 19th proof , where is is credited to L.J. Bullard, of Manchester, N.H.

NINETY-THREE

As the tri's ACH, HCB and ABH are similar, then tri. HAC: tri. BHC : tri. ABH = $AH^2 : BH^2 : AB^2$, and so tri. AHC + tri. BHC : tri. ABH = $AH^2 + BH^2 : AB^2$. Now tri. AHC + tri. BHC: tri. ABH = 1. $\therefore AB^2 = BH^2 + AH^2$. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 82, proof 77, where credited ot Bezout, 1768; also Math. Mo., 1859, Vol. II, Dem. 5, p. 45; also credited to Oliver; the School Visitor, Vol. 20, p. 167 says Pythagoras gave this proof ---but no documentary evidence.

Also Stanley Jashemski a school boy, age 19. of So. High School, Youngstown, O. , in 1934, sent me same proof, as an original discovery on his part.

b. Other proportions than the explicit one as given above may be deduced, and so other symbolized proofs, from same figure, are derivable---see Versluys, p. 83, proof 78.

NINETY-FOUR

Tri's ABH and ABH' are congruent; also tri's AHL and AHP: also tri's BKH and BPH . Tri. ABH = tri. BPH + tri. HAP = tri. BKH + tri. AHL. ∴ tri. ABH : tri. BKH : tri. AHL = $h^2 = a^2 + b^2$. and so tri. ABH : (tri.BKH + tri. AHL) = $h^2 : a^2 + b^2$, or $1 = h^2 + (a^2 + b^2)$. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See VErsluys p. 84, fig. 93, where it is attributed to Dr. H. A. Maber, 1908. Also see Dr. Leitzmann's work , 1930 ed'n, p. 35, fig. 35.

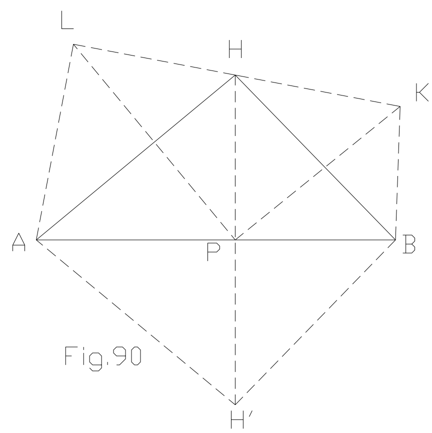


Fig.90

NINETY – FIVE

Complete the paral. HC , and the rect. AE, thus forming the similar tri's BHE, HAD and BAG. Denote the areas of these tri's by x, y and z respectively.

Then $z : y : x \; h^2 : a^2 : b^2$.

But it is obvious that $z = x + y$.

∴ $h^2 = a^2 + b^2$.

a. Original with the author, March 26, 1926,

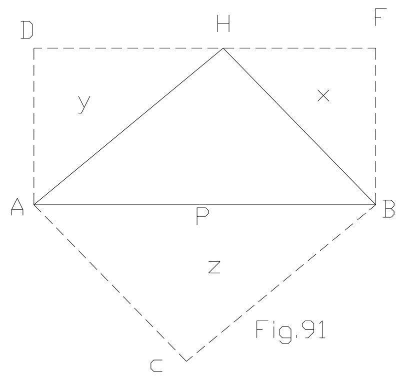


Fig.91

NINETY- SIX

Draw HL perp. to AB. Since the tri's ABH , AHL, and HBL are similar, so also the square AK, BE and HG, and since similar polygons are to each other as the squares of their homologous dimensions, we have

tri. ABH : tri. HBL : tri. HAL. ∵ sq. AK = sq. BE + sq.HG.

∴ = h^2 : a^2 : b^2.

a. Devised by the author, July 1, 1901, and sfterwards, Jan. 13 ,1934, found in Fourrey's Curio Geom., p. 91, where credited to R.P. Lamy,1685

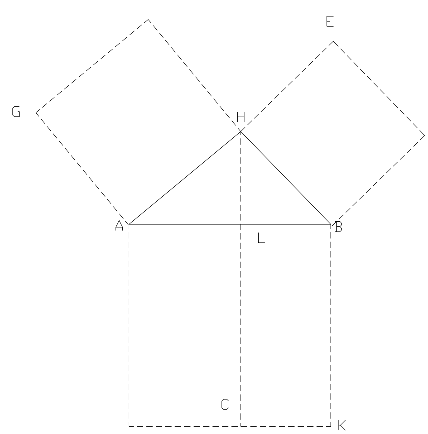


Fig. 92

NINETY-SEVEN

Use fig.92 and fig.1

Since, by ewuation (5) , see fig. 1 , proof one $BH^2 = BA \times BL$ = rect. LK, and in like manner, $AH^2 = AB \times AL$ = rect. AC, therefore sq. AK = rect. LK + rect. AC = sq. BE. + sq. HG.

$h^2 = a^2 + b^2$. Q.E.D.

a. Devised by the author July 2, 1901.

b. This principle of "mean proportional " can be use of in many of the here-in-after figures among the Geometric Proofs, thus giving variations as to the proof of said figures. Also many other figures may be constructed ased upon the use of the "mean proportional " relation; hence all such proofs. since they result from an algebraic relationship of corresponding lines of similar triangles, must be classed as algebraic proofs.

E. --- ALGEBRAIC PROOF, THROUGH THORY OF LIMIT.

NINETY-EIGHT

The so-called Pythagorean Theorem, in its simplest form is that in which the two legs are equal. The great Socrates (b, 500B.C.), by drawing replies from a slave, using his staff as a pointer and a figure on the pavement (see fig.93) as a model, made him (the slave) see that the equal tringles in the squares on HB and HA were just as many as like equal tri's in the sq. on AB, as is evident by inspection. (See Plato's Dailogues, Meno. Vol. I, pp. 256- 260, Edition of 1883, Jowett's translation,Chas. Scribner and Sons.)

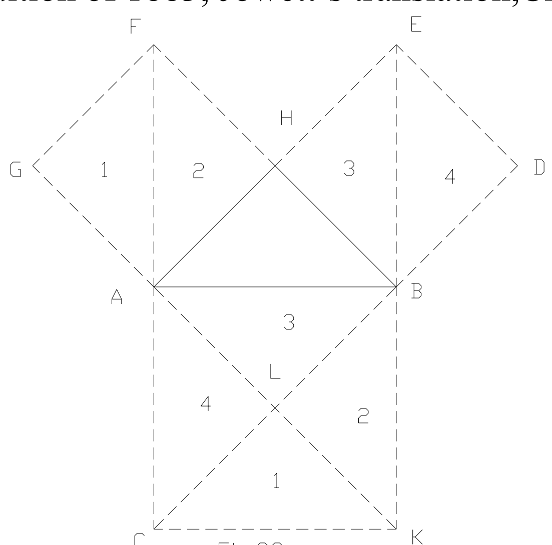


Fig.93

a. Omitting the lines AK, CB, BE and FA, which eliminates the numbered triangles, there remains the figure which, in Free Masonry, is called the Classic Form, the form usually found on the master's carpet.

b. The following rule is credited to Pythagoras. Let n be any odd number, the short side; square it, and from this square subtract 1; divide the remainder by 2, which gives the median side; add 1 to this quotient, and this sum is the hypotenuse; e.g., 5 = short side; $5^2 - 1 = 24$; 24 /2 = 12, the median side; 12 + 1 = 13 the hypotenuse. See said Rule of Pythagoras, above on p. 19.

NINETY- NINE

Starting with fig. 93 and decreasing the length of AH, which necessarily increases the length of AH which necessarily increases the length of HB, since AB remains constant, we decrease the sq. HC (see fig. 94a).

Now we are to prove that the sum of the two variable squares, sq. HD and sq. HC will equal the constant sq. HF.

We have, fig. 94a, $h^2 = a^2 + b^2$. ------(1)

But let side AH, fig. 93, be diminished as by x, thus giving AH, fig. 94a, or better, FD, fig. 93b. and let DK be increased by y, as determined by the hypotenuse h remaining constant.

Now, fig. 94b, when a = b, $a^2 + b^2 = 2$ area of sq. DP. And when $a < b$, we have $(a – x)^2$ = area of sq. DN and $(b + y)^2$ = area of sq. DR.

Also $c^2 – (b + y)^2 = (a – x)^2 + (b + y)^2 = c^2$-----(2)

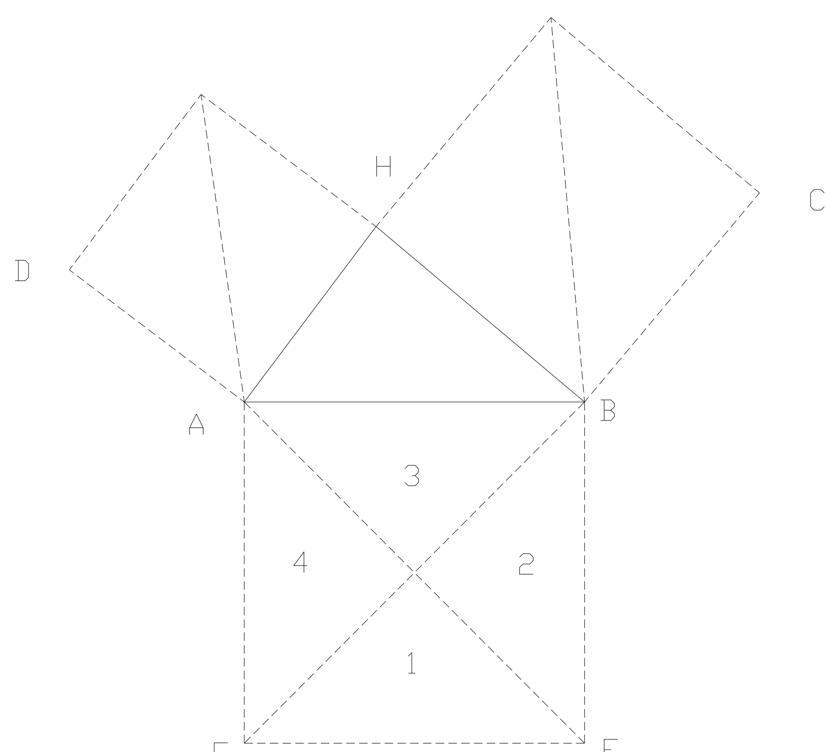


Fig. 94a

Is this true? Suppose it is; then, after reducing (2) – (1) = (3) – $2ax + x^2$

$+ 2by + y^2 = 0$, or (4) $2ax - x^2 = 2by + y^2$, which shows that the area by which (a^2 = sq. DP) is diminished = the area by which b^2 is increased. See graph 94b. ∴ the increase always equals the decrease.

But $a^2 - 2x(a - y) - x^2 = (a - x)^2$ approaches O when x approaches a in value.

∴ (5) $(a - x)^2 = 0$ when x = a, which is true and (6) $b^2 + 2by + y^2 = (b + y)^2 = c^2$ when x = a, for when x becomes a, (b + y) becomes c, and so, we have $c^2 = c^2$ which is true.

∴ equation (2) is true; it rests on the eq's (5) and (6), both of which are true.

∴ whether a < = or > b, $h^2 = a^2 + b^2$.

a. Devised by the author, in Dec. 1925. Also a like proof to the above is that of A. R. Colburn, devised Oct. 18, 1922, and is No. 96 in his collection of 108 proofs.

F. ----- ALGEBRAIC- GEOMETRIC PROOFS

In determining the equivalency of areas these proofs are algebraic; but in the final comparison of areas they are geometric.

ONE- HUNDRED

The construction, see fig. 95, being made, we have sq. FE = $(a + b)^2$.

But sq. FE = sq. AC + 4 tri. ABH.= $h^2 + 4\ ab/2 = h^2 + 2ab$.

Equating, we have

$= h^2 + 2ab = (a + b)^2 = a^2 + 2ab + b^2$. ∴ $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 382, Dec. 10, 1910, credited to A. R. Colburn, Washington, D.C.

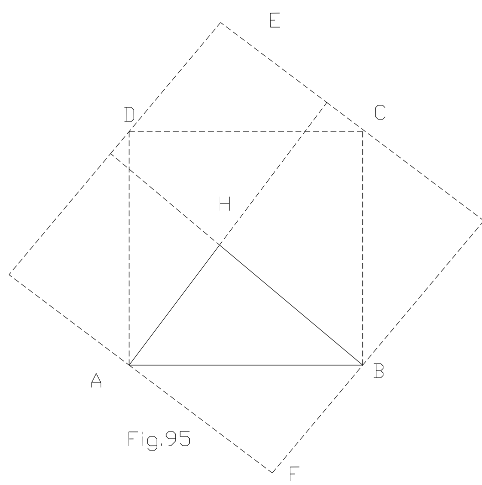


Fig.95

ONE- HUNDRED- ONE

Let AD = AG = x, HG = HC = y, and BC = BE = z. Then AH = x + y and BH = y + z.

With A as center and AH as radious describe arc HE; with B as center and BH as radious describe arc HD; with B as center, BE as radious describe arc EC; with A as center, radious AD, describe arc DG.

Draw the parallel lines as indicated. By inspecting the figure it becomes evident that if $y^2 = 2xz$, then the theorem holds. Now, since AH is a tangent and AR is a chord of same circle,

$AH^2 = AR \times AD$, or $(x + y)^2 = x(2y + 2z) = x^2 + 2xy + 2xz$.

Whence $y^2 = 2xz$.

$\therefore$ sq.AK = [$(x^2 + y^2 + 2xy)$ = sq. AL] + [$(z^2 + 2yz + (2xz = y^2))$] = sq.HP. $\therefore h^2 = a^2 + b^2$.

a. See Sci. Am. Supt. , V. 84, p. 362, Dce. 8, 1917, and credited to A. R. Colburn. It is No. 79 in his (then) 91 proofs.

b. This proof is a fine illustration of the flexibility of geometry. Its value lies, not in a repeated proof of the many times established fact, but in the effective marshaling and use of the elements of a proof, and even more also in the better insight which it gives us to the interdependence of the various theorems of geometry.

ONE- HUNDRED-TWO

Draw the bisectors of angles A, B and H, andfrom their common point C draw the perp's CR, CX and CT; take AN = AU = AP, and BZ = BP, and draw lines UV par. to AH, NM par. to AB and SY par. to BH. Let AJ = AP = x, BZ = BP = y, and HZ = HJ = z = CJ = CP = CZ.

Now 2 tri. ABH = HB x AH = $(x + z)(y + z) = xy + xz + yz + z^2$ = rect. HQ = sq. SX.

But 2 tri. ABH = 2AP x CP + 2BP x CP + (2 sq. HC = $2PC^2$) = $2xz + 2yz + 2z^2$.

= 2rect. HW + 2rect. HQ + 2sq. SX. $\therefore$ rect. PM = rect. HW + rect. HQ + sq. KX.

Now sq. AK = (sq. AO = sq. AW) + (sq. OK = sq. BQ) + (2 rect.PM = rect. HW + 2rect. HQ + 2sq. SX) = sq. HG + sq. HD $\therefore h^2 = a^2 + b^2$.

a. This proof was produced by Mr. F.S, Smedley, a photographer, of Berea, O., June 10, 1901.

Also see Jury Wipper, 1880, p. 34, fig. 31, credited to E. Mollmann, as given in "Archives d. Mathenatik, u. Ph. Grunert, " 1851. for fundamentally the same proof.

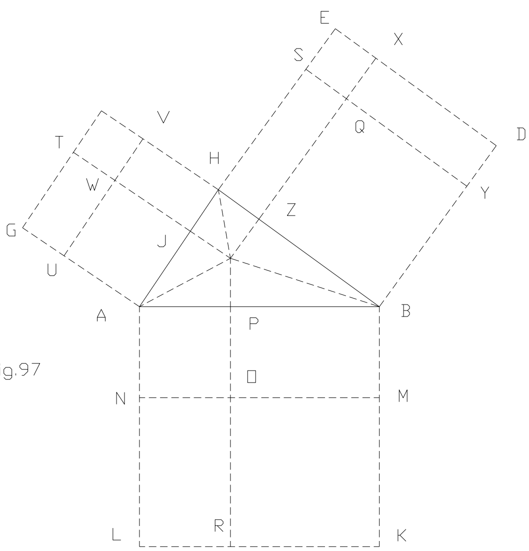


Fig.97

ONE HUNDRED THREE

Let HR = HE = a = SG. Then rect. GT = rect. EP, and rect. RA = rect. QB.

∴ **Tri's 2, 3, 4 and 5** are all equal. ∴ sq. AK = h^2 = (area of 4 tri. ABH + area sq. OM) = 2ba = $(b – a)^2$ = 2ab + b^2 - 2ab + a^2 = $a^2 + b^2$. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Math. Mo. , 1858- 59, Vol. I, p. 361, where above proof is given by Dr. Hutton tracts, London, 1812, 3 vol's, 820) in his History of Algebra.

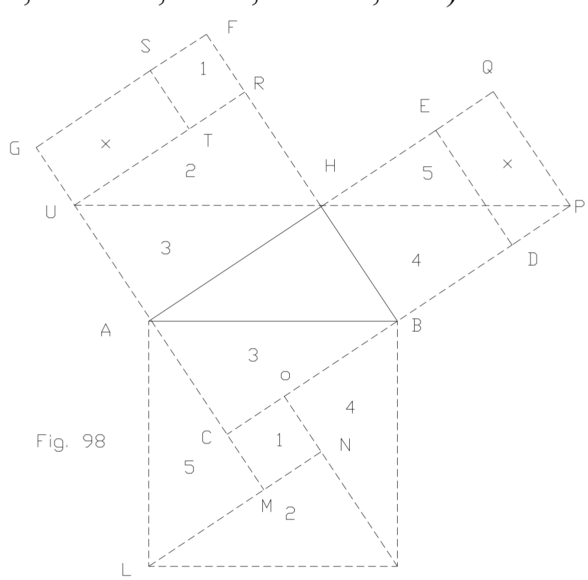


Fig. 98

ONE HUNDRED FOUR

Take AN and AQ = AH, KM and KR = BH, and through P and Q draw PM and QL parallel to AB; also draw OR and NS par. to AC. Then CR = h – a , SK = h – b and RS = a + b – h.

Now sq. AK = $CK^2 = CS^2 + RK^2 - RS^2 + 2$ CR x SK, or $h^2 = a^2 + b^2 - (a + b – h)^2 + 2(h – a) \times (h – b) = b^2 + a^2 - a^2 - b^2 - h^2 - 2ab + 2ah + 2bh + 2h^2 - ah – 2bh + 2ab$. $\therefore$ 2CR x SK = RS^2, or $2(h – a)(h – b) = (a + b – h)^2$, or

$2h^2 + 2ab – 2ah – 2bh = a^2 + b^2 + h^2 + 2ab + 2ah - 2bh$. $\therefore h^2 = a^2 + b^2$.

a. Original with the author, April 23, 1926.

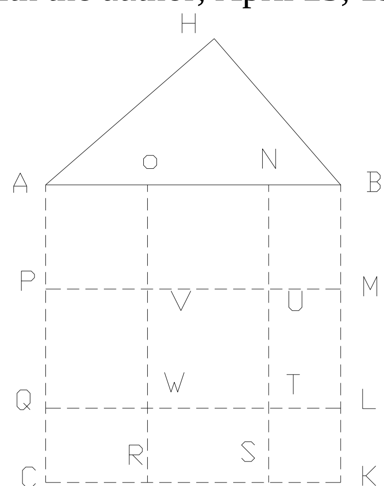


Fig.99

G.---Algebraic – Geometric Proofs Through Similar Polygons Other Than Squares.

1st. --- Similar Triangles.

ONE HUNDRED FIVE

Tri's ACB, BDH and HEA are three similar tri's constructed upon AB, BH and HA, and AK, BM and HO are three corresponding rect's, double in area to tri's ACB, BDH and HEA respectively.

Tri. ACB : tri, BDH : tri, HEA = $h^2 : a^2 : b^2$ = 2tri. ACB : 2tri.BDA = 2tri HEA = rect. AK : rect. BM : rect. HO. Produce LM and ON to their intersection P, and draw PHG. It is perp. to AB, and by the Theorem of Pappus, see fig.143, PH = QG. $\therefore$ by said theorem, rect. BM + rect. HO = rect. AK. $\therefore$ tri. BDH + tri. HEA = tri. ACB. $\therefore h^2 = a^2 + b^2$.

a. Devised by the author Dec. 7, 1933

ONE HUNDRED SIX

In fig. 100 extend KB to R, intersecting LM at S, and draw PR and HT par. to AB. Then rect. BLMH = paral. BSPH = 2tri. BPH = 2tri, (BPH = PH x QB) = rect. QK. In like manner, 2 tri. HEA = rect. AG.

Now tri. ABH : tri. BHQ : tri. HQA = h^2 : a^2 : b^2.= tri, ACB : tri. BDH : tri. HEA.

But tri. ABH = tri. BHQ + tri. HAQ, ∴ tri. ACB = tri. BDH + tri HEA. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Developed by author Dec. 7, 1933.

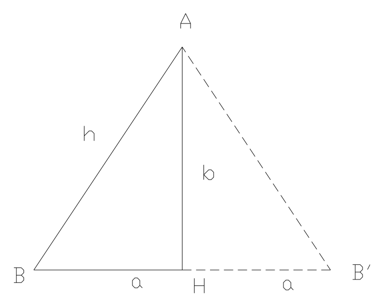


Fig. 101

ONE HUNDRED EIGHT

Any regular polygons can be resolved into as many equal isosceles tri's as the polygon has sides. As the tri's are similar tri's so whatever relations are established among these tri's AOB, BPH and HRA, the same relations will exist among the polygons O, P and R.

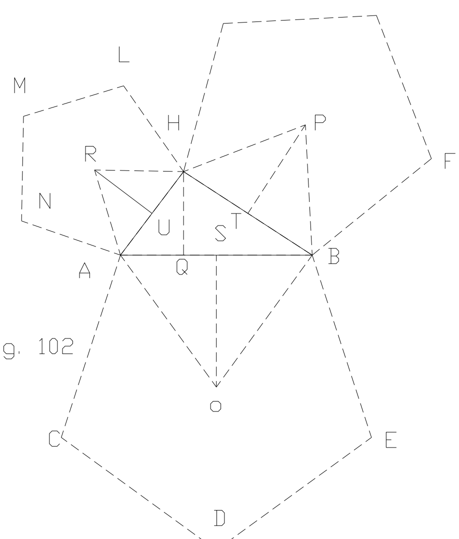


Fig. 102

As tri's AOB, BPH and HRA are similar isosceles tri's, it follows that these tei's are a particular case of proof One Hundred Six.

And as tri. ABH : tri. BHQ : tri. HAQ = h^2 : a^2 : b^2 . = tri. AOB : tri BPH : tri. HRA = pentagon O : pentagon P : pentagon R, since tri. ABH = tri. BHQ + tri. HAQ. ∴ polygon P + polygon R. ∴ = h^2 : a^2 : b^2 .

a. Devised by the author Dec. 7, 1933.

ONE HUNDRED NINE

Upon the three sides of the rt. tri. ABH are constructed the three similar polygons (having five or more sides—five in fig. 103), ACDEB, BFGKH and HLMNA. Prove algebraically that $h^2 = a^2 + b^2$, through proving that the sum of the areas of the two lesser polygon = the area of the greater polygon.

In general, an algebraic proof is impossible before transformation. But granting that $h^2 = a^2 + b^2$, it is sasy to prove that polygon (1) + polygon (2) = polygon (3), as we know that polygon (1) : polygon (2) : polygon (3) = $a^2 : b^2 : h^2$. But from this it does not follow that $a^2 + b^2 = h^2$.

See Beman and Smith's New Plane and solid Geometry (1899), p. 211, exercise 438.

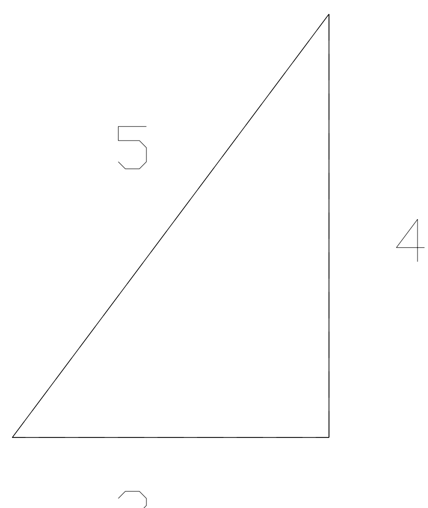
But an algebraic proof is always possible by transforming the three similar polygons into equivalent similar paral's and then proceed as in proof <u>One Hundred Six.</u>

Knowing that tri. ABH: tri. BHQ : HAQ = h^2 : a^2 : b^2 ----(1) and that P. (3) : P. (1) : P. (2). [P= polygon] = h^2 : a^2 : b^2-----(2); by equating tri. ABH; tri. BHQ : tri. AHQ = P. (3) : P. (1) : P. (2). But tri. ABH = tri. ABH = tri. BHQ + tri. HAQ. ∴ P. (3) = P. (1) + P. (2) . ∴ $h^2 = a^2 + b^2$ Q.E.D.

a. Devised by the suthor Dec.7, 1933.

b. Many more algebraic proofs are possible.

To evolve an original demonstration and put it in a form free from criticism is not the work of a tyro.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II. GEONETRIC PROOFS

All geometric demonstrations must result from the comparison of areas --- the foundation of which is superposition.

As the possible number of algebric proofs has been shown to be limitless, so it will be conclusively shown that the possible number of geometric proofs through dissection and comparison of congruent or equivalent areas is also "absolutely unlimited."

The geometric proofs are classified under ten type forms, as determined by the figure, and only a limited number, from the indefinite many, will be given; but among those given will be found all heretofore (to date, June 1940), recorded proofs which have come to me, together with all recently devised or new proofs.

The references to the authors in which the proof, or figure, is found or suggested, are arranged chronologically so far as possible.

The idea of throwing the suggested proof into the form of a single equation is my own; by means of it every essential element of the proof is set forth, as well as the comparison of the equivalent or equal areas.

The wording of the theorem for the geometric proof is : *The square described upon the hypotenuse of a right-angled triangle is equal to the sum of the, squares described upon the other two sides.*

TYPES

It is obvious that the three squares constructed upon the three sides of a right-angled-triangle can have eight different positions, as per selections. Let us designate the square upon the hypotenuse by *h*, the square upon the shorter side by a, and the square upon the other side by b, and set forth the eight arrangements; they are:

A. All squares h, a and b exterior.

B. a and b exterior and h interior.

C. h and a exterior and b interior.

D. h and b exterior and a inerior.
E. a exterior and h and b interior.
F. b exterior and h and a interior.
G. h exterion and a and b interior.
H. All squares h, a and b interior.

The arrangement designated above constitute the first eight of the following ten geometric types, the other two being:

I .A translation of one or more squares.
J. One or more squares omitted.

Also for some selected figures for proving Euclid I, Proposition 47, the reader is referred to H. d'Andre, N. H. Math. (1846), Vol. 5, p. 324.

Note. By "exterior" is meant constrivetd outwardly.

By " interior" is meant constructed overlappong the given right triangle.

A

This type includes all proofs derived from the figure determined by constructing squares upon each side of a right-angled triangle, each square being constructed outwardly from the given triangle.

The proofs under this type are classified as following.

(a) Those proofs in which pairs of the dissected parts are congruent.
Congruency implies superposition, the most fundamental and self-evident truth found in plane geometry.
As the ways of dissection are so various, it follows that the number of "dissection proofs" is unlimited.

(b) Those proofs in which pairs of the dissected parts are shown to be equivalent.
As geometricians at large are not in agreement as to the symbols denoting "congruency" and "equivalency" (personally the author prefers ≡ for congruency, and = for equivalency), the symbol used herein shall be = , the context deciding its import.

(a) PROOFS IN WHICH PA RS OF THE DISSECTED PARTS ARE CONGRUENT.
Paper Folding "Proofs, " Only Illustrative

ONE

Cut out a square piece of paper EF, and on its edge, using the edge of a second small square of paper, EH, as a measure, mark off EB, ED, LK, LG, FC and AQ.

Fold on DA, BG, KN, KC, CA, AB and BK. Open the sq. KG.

With scissors cut off tri. CFA from sq. HF, and lay it on sq. BC in position BHA, observing that it covers tri. BHA of sq. BC; next cut off KLC form sq's NL and HF and lay it on sq. BC in position of KNB so that MG falls on PO. Now observe that tri. KMN is part of sq. KG and sq. BC and that the part HMCA is part of sq. HF and sq. BC and that all of sq. BC is now covered by the two parts of sq. KG and the two parts of sq. HF.

Therefore the (sq. EH = sq. KG) + sq. HF = the sq. BC. Therefore the sq. upon the side BA which is sq. BC. = the sq. upon the side BH which is sq. BD + the sq. upon the side HA which is sq. HF. $\therefore h^2 = a^2 + b^2$, as shown with paper and scissors, and observation.

a. See "Geometric Exercises in Paper Folding " (T. Sundra Row's), 1905, p. 14, fig. 13, by Beman and Smith; also School Visitor, 1882, Vol. III, p. 209; also F.C. Boon, B. H. , in " A Companion to Elementry School Mathematics," (1924), p. 102, proof 1.

ONE

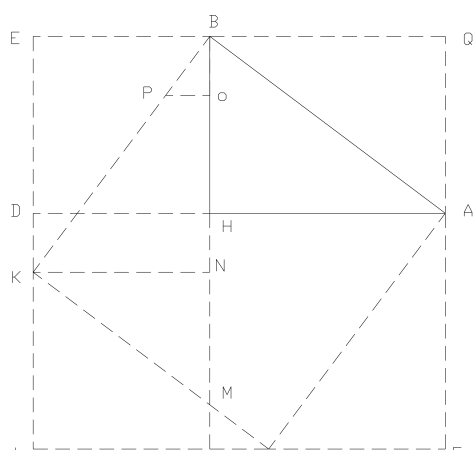


Fig. 104

Cut out a square piece of paper EF, and on its edge, using the edge of a second small square of paper, EH, as a measure, mark off EB, ED, LK, LG, FC and QA.

Fold on DA, BG, KN, KC, CA, AB and BK . Open the sq. EF and observe three sq's , EH, HF and BC, and that sq. EH = sq. KG.

With scissors cut off tri. CFA from sq. HF, and lay it on sq. BC in position BHA, observing that it covers tri. BHA of sq. BC; next cut off KLC from sq's NL and HF and lay it on sq. BC in position of KNB so that MG falls on PO. Now, observe that tri. KMN is part of dw. KG and sq. BC and that the part HMCA is part of sq. HF and sq. BC, and that all of sq. BC is now covered by the two parts of sq. KG, and the two parts of sq.HF.

Therefore the (sq. EH = sq. KG) + sq.HF= the sq. BC. Therefore the sq. upon the side BA which is sq. BC = the sq. upon the side BH which is sq. BD + the sq. upon the side HA which is sq. HF. $\therefore h^2 = a^2 + b^2$, as shown with paper and scissors, and observation.

a. See "Geometric Exercises in Paper Folding," (T. Sundra Row's), 1905, p. 14, fig. 13 by Beman and Smith; also School Visitor, 1882, Vol. III, p. 209; also F.C. Boon, B.H., in " A Companion to Elementary School Mathematics," (1924), p. 102, proof 1.

TWO

Cut out trehh sq's EL whose edge is HB, FA whose edge HA, and BC shoes edge is AB, making AH = 2 HB.

Then fold sq. FA along MN and OP, and separate into 4 sq's MP, QA, ON and FQ each equal to sq. EL.

Next fold the 4 paper sq's (U, R, S and T being middle pt's), along HU, PR , QS and MT, and cut,forming parts, 1,2,3, 4, 5,6,7 and 8.

Now place the 8 parts on sq. BC in positions as indicted, reserving sq. 9 for last place. Observe that sq. FA and EL exactly cover sq. BC. $\therefore$ sq. upon (HB = EL) + sq. upon AH. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. Beman and Smith's Row's (1905) , work, p. 15, fig. 14; also School Visitor, 1882, Vol. III, p. 208; also F. C. Boon, p. 102, proof 1.

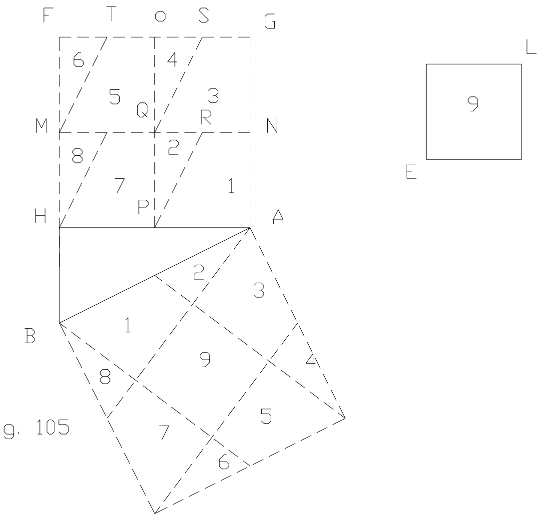


Fig. 105

<u>THREE</u>

Cut out three sq's as in fig. 105. Fold small sq. ((fig. 105) along middle and cut, forming 2 rect's; cut each rect. Along diagonal, forming 4 rt. tri's, 1, 2, 3 and 4. But from each corner of sq. FA (fig. 105), a rt. tri, each having a base HL = $^1/_2$ HP (fig. 105; FT = $^1/_2$ FM), giving 4 rt. tri's 5,6, 7 and 8(fig. 106), and a center part 9 (fig. 106), and arrange the pieces as in fig. 106, 105. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. See "School Visitor," 1882, Vol.III, p. 208.
b. Proofs <u>Two</u> and <u>Three</u> are particular and illustrative ---not general ---but useful as a paper and scissors exercise.
c. With paper and scissors, many other proofs, true under all conditions, may be produced, using figs. 110, 111,etc. as models of procedure.

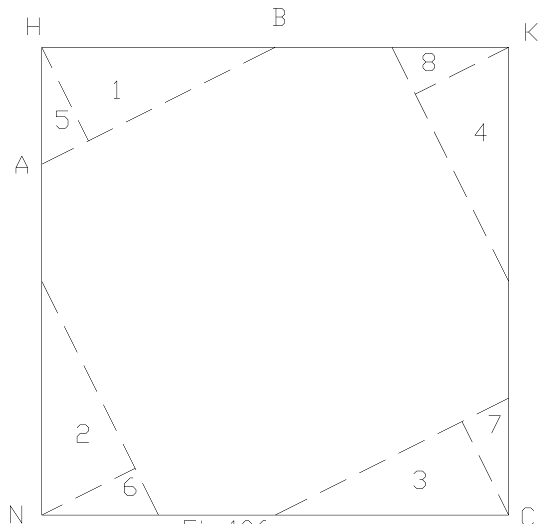


Fig.106

FOUR

Particualr case --- illustrarive rather than demonstrative.

The sides are to each other as 3, 4, 5 units. Then sq. AK contains 25 sq. units, HD 9 sq. units and HG 16 sq. units. Now it is evident that the no. of unit squares in the sq. AK = the sum of the units squares in the squares HD and HG.

∴ square AK = sq. HD + sq. HG.

a. That by the use of the lengths 3,4 and 5, or length having the ratio of 3 : 4 : 5, a right-angled triangle is formed was known to the Egyptians as early as 2000, B.C., for at that time there existed professional "rope – fasteners" ; they were employed to construct right angles which they did by placing three gegs so that a rope measuring off 3, 4 and 5 units would just reach around them. This method is in use today by carpenters and masons; sticks 6 and 8 feet long form the two sides and a "ten foot" stick forms the hypotenuse, thus completing a right-angled triangle, hence establishing the right angle.

But granting that the early Egyptians formed right angles in the "rule of thumb" manner described above, it does not follow, in fact it is not believed, that they knew the area of the square upon the hypotenuse to be equal to the sum of the areas of the squares upon the other two sides.

The discovery of this fact is credited to Pythagoras, a renowned philosopher and teacher, born at Samos about 570 B.C., after whom the theorem is called "The Pythagorean Theorem." (See p. 3)

b. See Hill's Geometry for Beginners, p. 153; Ball's History of Mathematics, pp. 7-10; Heath's Math. Monographs No. 1, pp. 15-17; The School Visitor, Vol. 20, p. 167

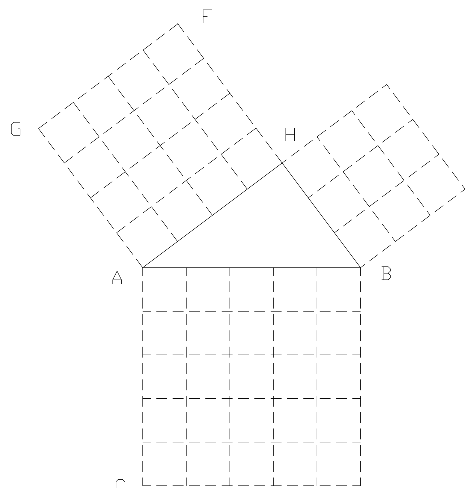


Fig. 107

FIVE

Another particular case is illustrated by fig. 108, in which BH = AH, showing 16 equal triangles.

Since the sq. AK contains 8 of these triangles,

∴ sq. AK= sq. HD = sq. HG.

∴ $h^2 = a^2 + b^2$.

a. For this and many other demonstrations by dissection, see H. Perigal, in Messenger of Mathematics, 1873, V. 2, p. 103; also see Fourrey, p. 68,

b. See Beman and Smith's New Plane and Solid Geometry, p. 103, fig. 1.

c. Also R.A. Bell, Cleveland, O., using sq. AK and lines AK and BC only.

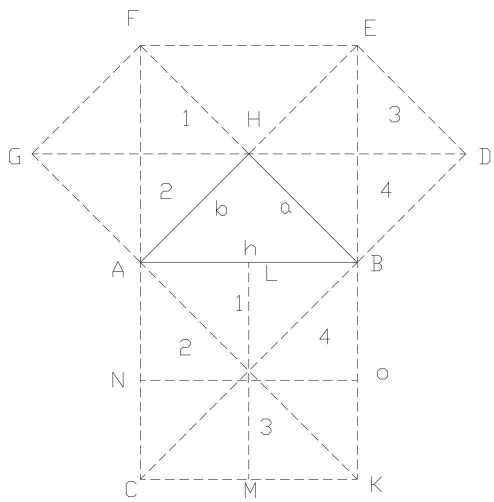


Fig.108

SIX

In fig. 108, omit lines AF, BE, LM and NO, and draw line FE; this gives the fig. used in "Grand Lodge Bulletin," Grand Lodge of Inowa, A.F. and A. M. , Vol. 30, Feb. 1929, No. 2, p. 42. The proof is obvious, for the 4 equal isosceles rt. tri's which make up sq. FB = sq. AK . $\therefore h^2 = a^2 + b^2$.

a. This gives another form for a folding paper proof.

SEVEN

In fig. 108, omit lines as in proof Six, and it is obvious that tri's 1,2, 3 and 4 in sq's HG and HD will cover tri's 1,2,3 and 4 in sq. AK, or sq. AK = sq. HD + sq. HG. $\therefore h^2 = a^2 + b^2$.

a. See Versluys (1914) , fig. 1,p. 9 of his 96 proofs.

EIGHT

In fig. 109, let HAGF denote the larger sq. HG, Cut the smaller sq. EL into two equal rectangles AN and ME, fig. 109, and form with these and the larger sq. the rect. HDEF. Produce DH so that HR = HF. On RD as diameter describe a semicircle DCR. Produce HF to C in the arc. Join CD, cutting FG in P, and AG is S. Complete the sq. HK.

Now tri's CPF and LBD are congruent as are tri's CKL and PED. Hence sq. KH = (sq. EL, fig. 105 = rect. AN + rect. ME, fig. 109) + (sq. HG, fig. 105 = quad. HASPF + tri. SGP, fig. 109). $\therefore h^2 = a^2 + b^2$.

a. See School Visitor, 1882, Vol. III p. 208.

b. This method, embidied in proof_ Eight , will transform any rect. Into a square.

c. Proofs Two to Eight inclusive are illustrative rather than demonstrative.

Demonstrative proofs

NINE

In fig. 110, through P, Q, R and S the centers of the sides of the sq. AK draw PT and RV par. to AH, and QU and SW par. to BH and through O, the center of the sq.HG, draw XH par. to AB and IY par. to AC, forming 8 congruent quadrilaterals; viz., 1, 2, 3 and 4 in sq AK, and 1,2,3 and 4 in sq. HG, and sq. 5 in sq. AK = sq. (5 = HD). The proof of their congruency is evident, since, in the paral. OB, (SB = SA) = (OH = OG = AP since AP = AS). (Sq.AK= 4 quad. APTS + sq. TV) = (sq. HG = 4 quad. OYHZ) + sq. HD. $\therefore$ sq. on AB = sq. on BH + sq. on AH. $\therefore h^2 = a^2 + b^2$.

a. See Mess, Math. , Vol. 2, 1873, p. 104, by Henry Perigal, F.R.A.S. , etc. , Maxmillan and Co., London and Cambridge. Here H. Perigal shows the great value of proof by dissection, and suggests its application to other theorems also. Also see Jury Wipper, 1880, p. 50, fig. 46; Ebene Geometric, Von G. Mahler Lepizig, 1897, p. 58, fig. 71, and school Visitor, V. III, 1882, p. 208, fig. 1, for a particular application of the above demonstration; Versluys, 1914, p. 37 taken from "Plane Geometry" of J. S. Mackay, as given by H. Perigal, 1830; Fourrey, p. 86, F. C. Boon, proof , p. 105; Dr. Leitzmann. p. 14, fig. 16.

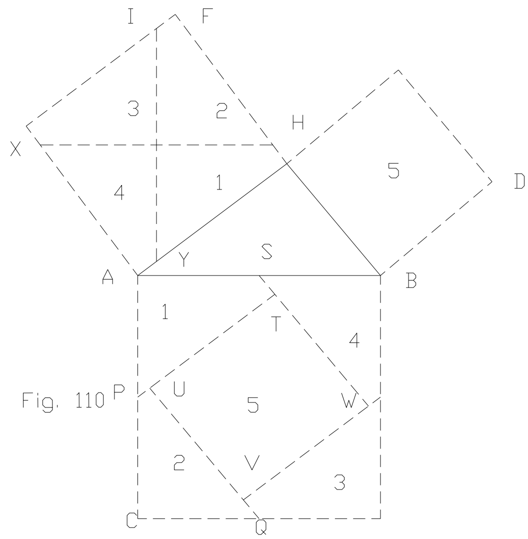


Fig. 110

b.See Todhunter's Euclid for simple proof extracted from a paper by De Morgan, in Vol. I of the Quarterly Journal of Math., and reference is also made there to the work "Der Pythagoraische Jehrsatz," Mainz, 1821, by J.J.I. Hoffmann.

c. By the above dissecition any of two squares may be transferred into one square, a fine puzzle for puplis in plane geometry.

d. Hence any case in which the three squares are exibited, as set forth under the first 9 types of II, Geometric Proofs, A to J inclusive (see Table of Contents for said, types) may be proved by this method.

e. Proof Nine is unique in that the smaller sq. HD is not dissected.

TEN

In fig. 111, on CK construct tri. CKL = tri. ABH; proudce CL to P making LP = BH and take LN = BH; draw NM, OA and BP each perp. to CP; at any angle of the sq. GH, as F, construct a tri. GSF = tri. ABH, and from any angle of the sq. HD, as H, with a radius = KM, determine the pt. R and draw HR, thus dissecting the sq's. as per figure.

It is readily shown that sq. AK = (tri, CMN = tri. BTP) + (trap. NMKL=

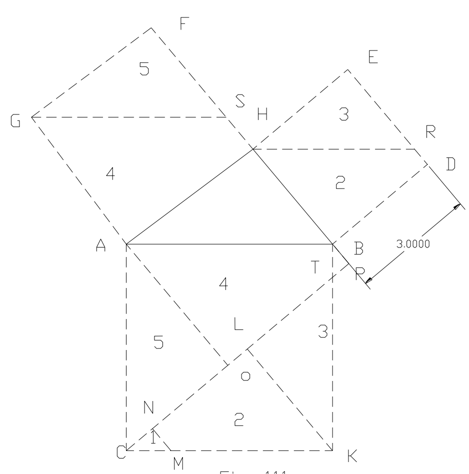


Fig. 111

trap. DRHB) + (tri. KTL = tri. HRE) + (quad. AOTB + tir. BTP = trap. GAHS) + (tri.ACO = tri. GSF) = (trap. DRHB + tri. HER = sq.BE) + (trap. GAHS + tri.GSF = sq.AF) = sq. BE + sq. AF ∴ sq. upon AB = sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. This dissectionand proof were devised by the author. On March 18,1926, to establish a Law of Dissection, by which, no matter how the three squares are arranged, or placed, their resolution into the respective parts as numbered in fig. 111, can ge readily obtained.

b. In many of the geometric proofs therein the reader will observe that the above dissection, wholly or patially, has been employed. Hence these proofs are but variation of this general proof.

ELEVEN

In fig. 112, conceive rect. TS cut off from sq. AF and placed in position of rect. QE, AS coinciding with HE; then DEP is a st. line since these rect. Were equal by construction. The rest of the construction and dissection is evident.

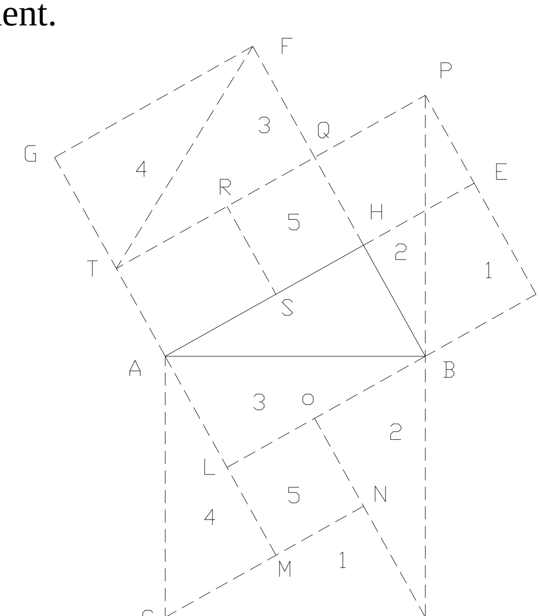


Fig. 112

Sq.AK = (tri. CKN = tri. PBD) + (tri KBO = tri. BPQ) + (tri.BAL = tri.TFQ) + (tri. ACM = tri. FTG) + (sq. LN = sq. RH) = sq.BE + rect. QE + rect.GQ + sq. RH = sq. BE + sq.GH. ∴ sq. upon AB = sq. upon BH + sq. upon AH ∵ $h^2 = a^2 + b^2$.

a. Original with the author afetr having carefully anallyzed the esoteric implications of Bhaskara's " Behold!" proof – see proof Two Hundred Twenty- Four, fig.325.

b. The reader will notice that this dissection contains some of the elements of the preceding dissection, that it is applicable to all three-square figures like the preceding, but that it is not so simple or fundamental, as it requires a transposition of one part of the sq. GH, ---- the rect. TS---, to the sq. HD, --- the rect. In position QE---, so as to form the two congruent rect's GQ and QD.

c. The student will note that all geometric proofs hereafter, which make use of dissection and congruency, are fundamentally only variations of the proofs established by proofs Nine, Ten and Eleven and that all other geometric proofs are based,either partially or wholly on the equivalensy of the corresponding pairs of parts of the figures under consideration.

TWELVE

This proof is a simple variation of the proof Ten above. In fig. 113, extend GA to M, draw CN and BO perp. to AM, take NP = BD and draw PS par. to AB. Then since it is easily shown that parts 1 and 4 of sq. AK = parts of 1and 4 of sq. HD, and parts 2 and 3 of sq. AK = 2 and 3 of sq. HG, ∴ sq. upon AH.

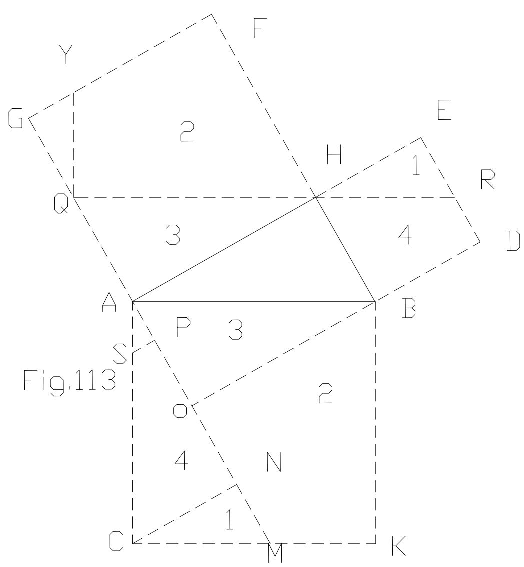


Fig.113

a.Original with the author March 28, 1926 to obtain a figure more readily constructed than fig.111.

b. See School Visitor, 1882, Vol. III, p. 208-9; Dr. Leitzmznn, p. 15, fig. 17, 4th Ed'n.

THRTEEN

In fig. 114, produce CA to O , KB to M, GA to V, making AV = AG, BD to U, and draw KX and CW par. resp. to BH and AH, GN and HL par. to AB, and OT par, to FB.

Sq. AK = [tri. CKW = tri. (HAL = trap. BDEM + tri. NST)] + [tri. KBX = tri.GNF= (trap.OQNF + tri.BMH)] + (tri. BAU = tri.OAT) + (tri. ACV = tri.AOG) + (sq. VX = paral. SN) = sq. BE + sq. HG. ∵ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Original with author March 28, 1926, 9.30 p.m.

b. A Variation of the proof Eleven above.

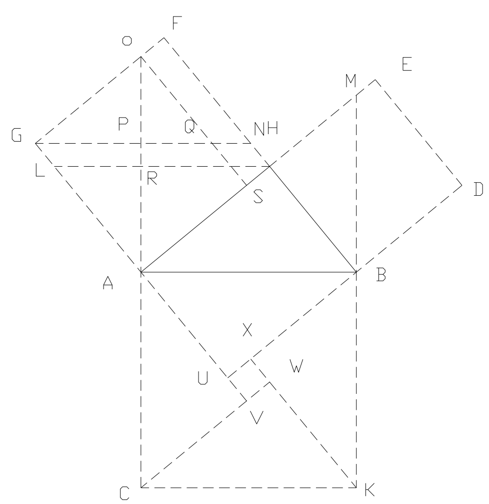


FIG. 114

FOURTEEN

Prodce CA to S, draw SP par. to FB , take HT, produce GA to M, making AM = AG, produce DB to L, draw KO and CN par. resp. to BH and AH, and draw QD. Rect. RH = rect. QB. Sq. AK = (tri.CKN = tri.ASG) + (tri. KBO = tri SAQ) + (tri. BAL = tri.DQP) + (tri.ACM = tri.QDE) + (sq.LN = sq, ST) = rect. PE + rect. GQ + sq. ST = sq. BE + rect. QB + rect.GQ + sq. ST = sq. BE + sq.GH. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

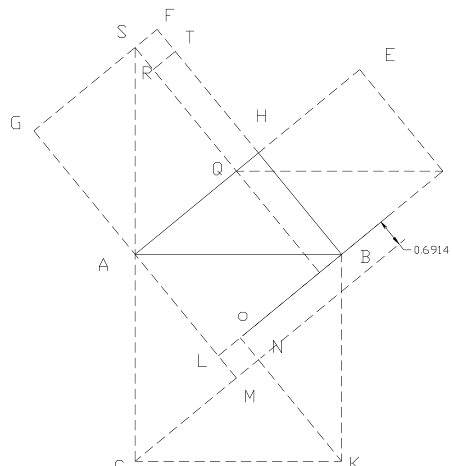


Fig. 115

a. Original with author March 28, 1926, 10 a.m.

b.This is another variation of fig. 112.

FIFTEEN

Take HR = HE and FS =FR = EQ = DP.

Draw RU par. to AH, ST par. to FH, QP par. to BH, and UP par. to AB. Extend GA to M, making AM = AG, and DB to L and draw CN par. to AH and KO par. to BH.

Place rect. GT in postioin of EP. Obvious that: sq. AK = parts (1+ 2 + 3) + (4+ 5 of rect. HP), ∴ Sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Math Mo., 1858-9, Vol. I, p. 231, where this dissectionis cerdited to David W. Hoyt, Prof. Math. and Mechanics, Polytechnic Collage, Phila. Pa.

b. The Math. Mo. was edited by J.D. Junkle, A,M. Cambridge Eng. He says this demonstration is essentially the same as the Indian demonstration is found in "Beja Gauita" and referred to as the figure of "The History of Algebra).

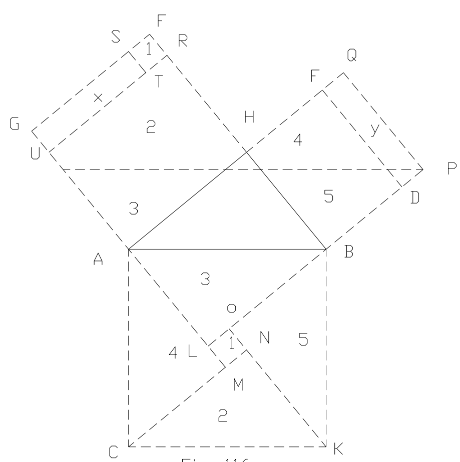


Fig. 116

SIXTEEN

In fig. 117, the dissection is evident and shows that parts 1,2, and 3 in sq. AK are confruent to parts 1, 2, and 3 in HG; also that parts 4 and 5 in sq. AK are congruent to parts 4 and 5 in sq. HD. ∴ (sq. AK = parts 1 + 2+ 3+ 4+ 5) = (sq. HG = parts 4+ 5). ∴ sq. on AB = sq. on BH + sq. on AH. ∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 27, fig.24, as given by Dr. Rudolf Wolf in "Handbook der Mathematik, etc.. " 1869; Journal of Eduction, V. XXVIII, 1888, p. 17, 27th proof, by C.W. Tyron, Louisville, Ky.; Beman and Smith's Plane and Solid Geom., 1895, p. 88, fig.5; Am. Math, Mo. V. IV, 1897, p. 169 proof XXXIX; and Heath's Math. Monographs, No, 2, p.33, proof XXII. Also The School Visitor, V. III, 1882, p.209, for an appliction of it to a particular case; Fourrey, p. 87, by Ozanam, 1778, R. Wolf, 1869.

b. See also "Recreations in Math. and Physics," by Ozanam; "Curiosities of Geometry," 1778, by Zie E. Fourrey; M, Krőger, 1896; Versluys, p.39, fig. 39 and p. 41, fig. 41, and a variation is that of Versluys (19140), p. 40 fig. 41.

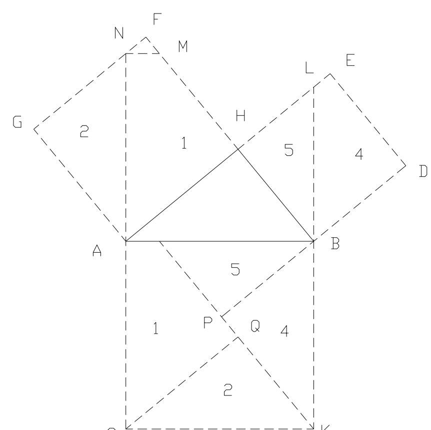


Fig.117

SEVENTEEN

Extend CA to M and KB to Q, draw MN par. to AB. Extend GA to T and DB to O. Draw CP par. to AB. Take OR = HB and draw RS par. to HB.

Obvious that sq. AK = sum of pats (4+5) + (1 + 2 + 3) = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon HA. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Conceived by the author, at Nashville, O., March 26, 1933, for a high school girl there, while present for the funeral of his cousin; also see School Visitor, Vol. 20, p. 167.

b. Proof and fig. 118, is practically the same as proof Sixteen, fig. 117.

On Dec. 17, 1939, there came to me this: Der Pythagoreische Leharats von Dr. W. Leitzmann, 4th Edition, of 1930 (1st Ed'n, 1911, 2nd Ed'n, 1917, 3rd Ed'n), in which appears on less than 23 proofs of the Pythagorean Proposition, of which 21 were among my proof herein.

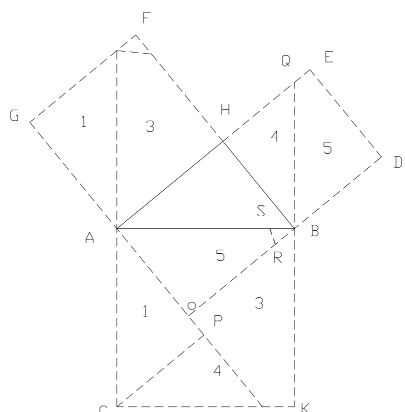


Fig. 118

This little book of 72 pages is an excellent treatise, and the bibliography, pages 70, 71, 72, is valuable for investigators, listing 21 works re this theorem.

My manuscript, for 2nd edition, credits this work for all 23 proof therein, and gives, as new proof, the two not included in the said 21.

EIGHTEEN

In fig. 119, the dissection is evident and shows that parts 1, 2 and 3 is rect. QC; also that parts 4, 5, 6 and 7 in sq. HD are congruent to parts 4, 5, 6 and 7 in rect. QR.

Therefore, sq. upon AB = sq. upon HB + sq. upon HA, ∴ $h^2 = a^2 + b^2$. Q. E. D.

a. See dissection, Tafel II, in Dr. W. Leitamann's work, 1930 ed'n ---on last leaf of said work. Not credited to any one, but is based on H, Dobriner's proofs.

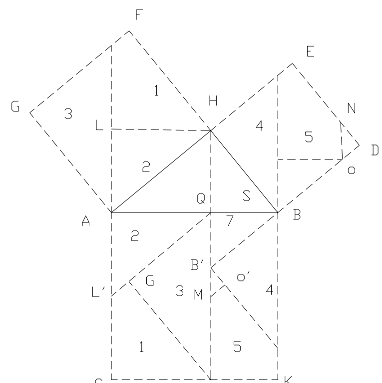


Fig.119

NINETEEN

In fig. 120 draw GD, and form F and E draw lines to GD par. to AC; then extend DB and GA, forming the rect. AB; through C and K draw lines par. respectively to AH and BH, forming tri's equal to tri. ABH. Through points L and M draw line par. to GD, Take KP = BD, and draw MP, and through L draw a line par. to MP

Number the parts as in the figure, It is obvious that the dissected sq's HG and HD, giving 8 triangles, can be arranged in sq. AK can be superimposed by their 8 equivelent tri's in sq's HG and HD. ∴ sq. AK = sq.HD + sq.HG. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See dissection, Tafel I, In Dr, W. Leitzmann work, 1930 ed'n, on 2nd last leaf. Not credited to any one, but is based on J.E. Bőttcher's work.

TWENTY

In fig. 121 the construction is readily seen, as also the vongruendy of the corresponding dissected parts, from which sq. AK = (quad. CPNA = quad. LAHT) + (tir. CPK = tri. ALG) + (tri. BOK = quad, DEHR + tri. TFL) + (tri. NOB = tri. RBD).

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Math, Mo. V, IV, 1897 , p. 169, proof XXXVIII.

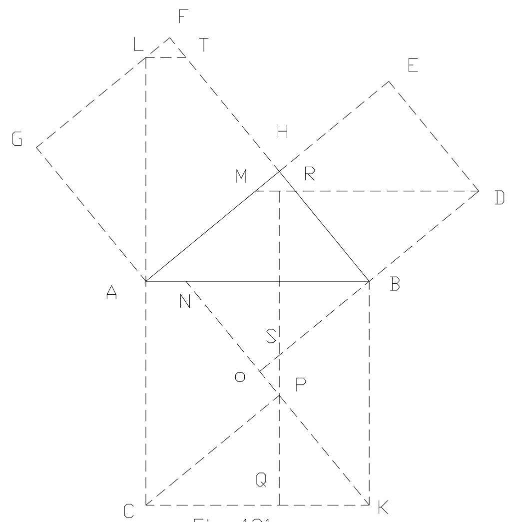


Fig. 121

TWENTY TWO

The construction and dissection of fig. 122 is obvious and the congruency of the corresponding parts being established, and we find that sq. AK = (quad ANMR = quad. AHWX) + (tri. CAN = tri. WFG) + (tri.CQM = tri.AXG) + (tri. MQK = tri. EDU) + (tri.POK = tri.THS) + (pentagon BLMOP= pentagon ETSBV) + (tri.BRL = tri. DUV). ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Origional with the author of this work, August 9, 1900, Afterwards, on July 4, 1901, I found same pfoof in Jury Wipper, 1880, p. 28, fig.25, as fiven by E. von Littrw in “Popularen Geometrie,” 1839; also see Vresluys, p. 42, fig. 43.

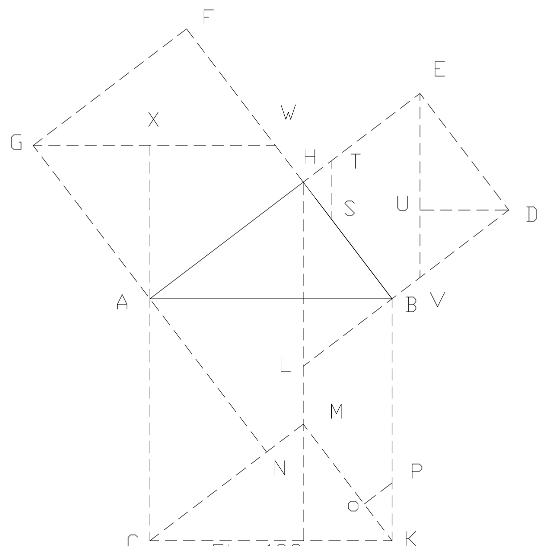


Fig. 122

TWENTY- TWO

Extend CA to Q, KB to P draw RJ through H, par. to AB, HS perp. to CK, SU and ZM par. to BH, SL and ZT par. to AH and take SV = BP, DN = PE, and draw VW par, to AH and NO par. to BP.

Sq. AK = parts (1+2+3 + 4= sq. HD) + parts (5+6+7 = sq. HG); so dissected parts of sq. HD + dissected parts of sq. HD + dissected parts of sq. HG (by superposition), equals the dissected parts of sq. AK.

∴ Sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 43, fig. 44.

b. Fig. and proof, of Twenty-Two is very much like that of Twenty – One.

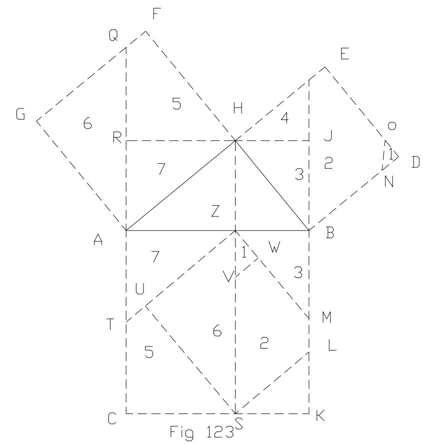


Fig 123

TWENTY-THREE

After showing that each numbered part found in the sq's HD and HG is correspondign numbered part in sq. AK, which is not difficult, it follows that the sum of the parts in sq. AK = the sum of the parts of the sq. HG.

∴ the sq. upon AK = the sq. upon HD + the sq. upon HA. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Geom. of Dr. H. Dobriner, 1898; also Versluys, p. 45, fig. 46, from Chr, Nielson; also Leitzmann, p. 13, fig. 15, 4th Ed'n.

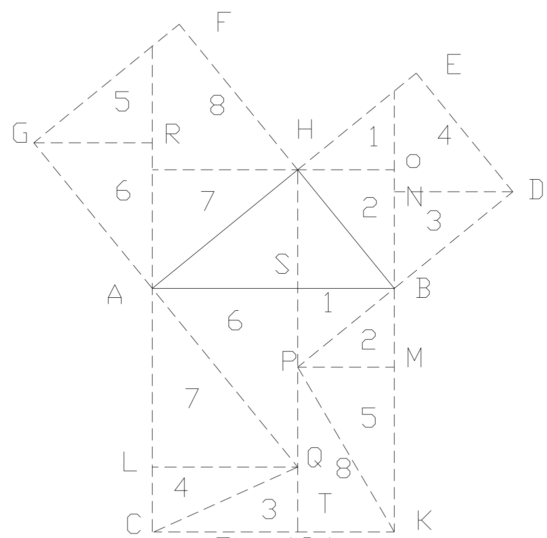


Fig. 124

TWENTY-FOUR

Proceed as in fig. 124 and after conuensy is establlished, it is evident that, since the eight dissected parts of sq. AK are congruent to the corresponding numbered parts found found is sq's HD and HG, parts (1+2+3 +3 +4 + 5+ 6 + 7 +8 in sq. AK) = parts (5+6+ 7 +8) +(1+2+3+4) in sq's HB and HG.

∴ sq. upon AB = sq. upon HD + sq. upon HA. ∴ $h^2 = a^2 + b^2$.

a. See Paul Epstein's (of Straatsbers), collection of proofs; also Versluys, p. 44, fig. 45; also Dr. Leitzmann's 4th ed'n, p. 13, fig. 14.

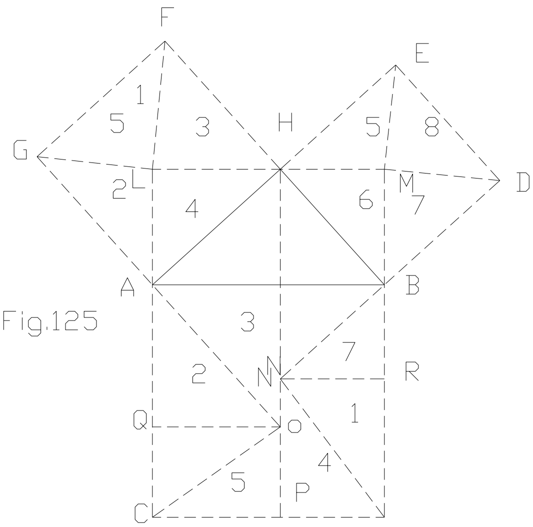


Fig.125

TWENTY-FIVE

Establish confruency of corresponding parts; then it follows that : sq. AK (= parts 1 and 3 of sq. HD + parts 3, 4 and 5 of sq. HG) = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon HA. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 38, fig. 38. This fig. is similar to fig. 111.

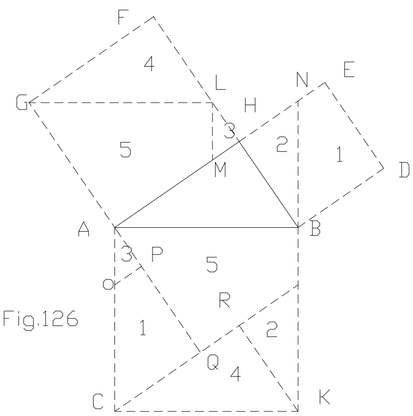


Fig.126

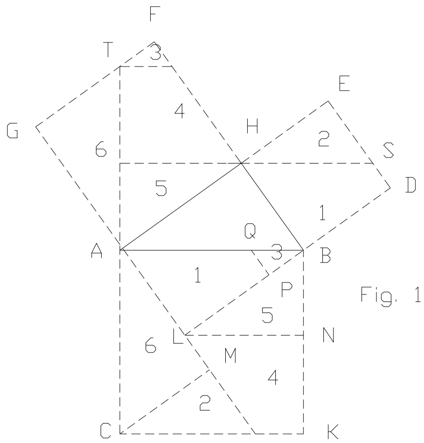


Fig. 127

TWENTY-SIX

Since parts 1 and 2 of sq. HD are congruent to like parts 1 and 2 in sq.AK, and parts 3, 4, 5 and 6 of sq. HG to like parts 3,4, 5 and in sq. Ak. $\therefore$ sq. upon AB = sq. upon HB + sq. upon HA. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a, This dissection by the author, March 26, 1933.

TWENTY-SEVEN

Take AU and CV = BH and draw UW par. to AB and VT par. to BK; from T draw TL par. to AH and TS par. to BH, locating pts. L and S; complete the sq's LN and SQ, making sides SR and LM par. to AB Draw SW par. to HB and CJ par. to AH, The 10 parts found in sq's HD and HG are congruent to corresponding parts in sq. AK. ∴ the sq. upon HA. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. This proof, and dissection, was sent to me by J.Adams, Chassestreet 31, The Hafue, Holland, April 1933.

b. All lines are either perp. or par. to the sides of the tri. ABH---a unique dissection,

c. It is a fine paper and scissors exercise.

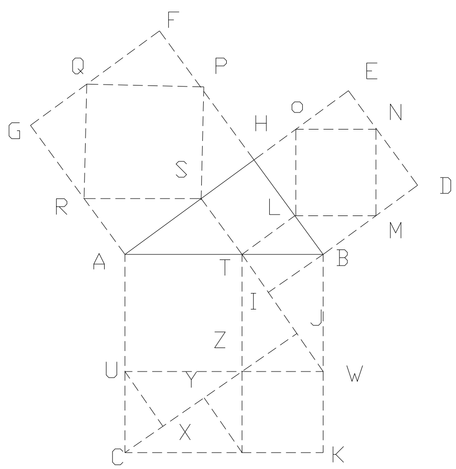


Fig. 128

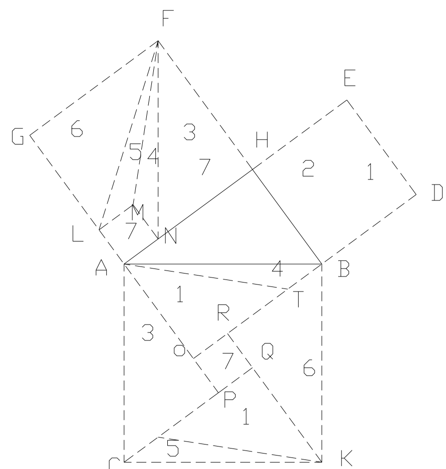


Fig. 129

TWENTY-EIGHT

Draw AF and BF; produce GA to P making AP = AG; produce DB to O; draw CQ par. to AH and KR par. to BH; construct sq. LN = sq. OQ; draw FL and FN; take AT and KS = to FM. Confruensy of corresponding numbered parts having been established, as is easily done, it follows that: sq. upon AB = sq. upon HB + sq. upon AH, ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Benjr von Gutheil, oberlehrer at Nuruberg, Germany, produced the above proof. He died in the trenches in France 1914. So wrote J. Adams (see a fig. 128), August 1933.

b. Let us call it the B. von Gutheil World War Proof.

c. Also see Dr. Leitzmann, p. 15, fig. 18

TWENTY-NINE

In fig. 130, extend CA to O, and draw ON and KP par. to AB and BH respectively, and extend DB to R. Take BM = AB and draw DM. Then we have sq. AK = (trap. ACKP = trap. OABN = pentagon OGAHN) + (tri. BRK = trap. BDLH + tri. MHL = tri. OFN) + (tri. PRB = tri.LED). ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Math. Mo. , V. VI, 1897, p. 170 , proof XLIV.

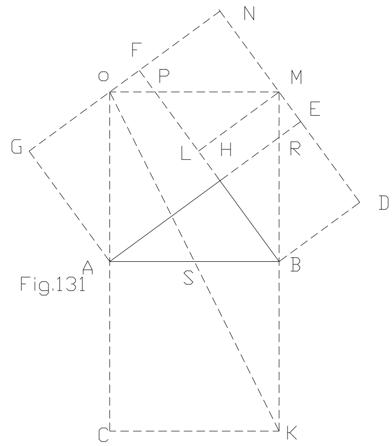


Fig.131

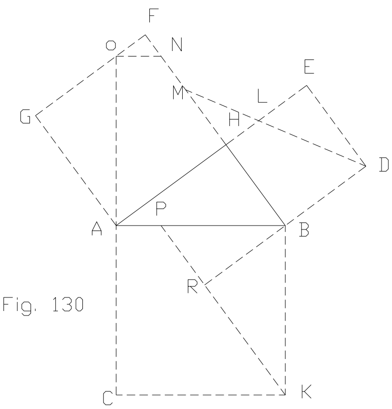


Fig. 130

THIRY

Fig. 131 objectifies the line to be drawn and how they are drawn is readily seen.

Since tri. OMN = tri. ABH, tri. MPL = tri. BRH, tri. BML = tri. AOG, and tri. OSA = tri. KBS (K is the pt. of intersection of the lines MB and OS) then sq. AK = trap. ACKS + tri. KSB = tri. KOM = trap. BMOS + tri. OSA = quad. AHPO = tri. ABH + tri. BML + tri. MPL = quad. AHPO + tri. OMN + tri. AOG + tri. BRH = (pentagon AHPOG + tri. OPF) + (trap. PMNF = trap. RBDE) + tri. BRH = sq. HG + sq. HD. ∴ sq. upon AB = sq. upon HD + sq. upon AH ∴ $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., V. 70, p. 383, Dec. 10,1910. It is No. 14 of A.R. Colburn's 108 proofs.

THIRY-ONE

Extended GA making AP = AG; extend DB making BN =BD = CP. Tri. CPK = tri. ANB = $^1/_2$ sq. HD = $^1/_2$ rect. LK. Tri. APB =$^1/_2$sq. HG = $^1/_2$rect. AM. Sq. AK = rect. AM + rect. LK.

∴ sq. upon AB = sq. upon HB + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. This is Huygens' proof 91657); see also Versluys, p. 25, fig.22.

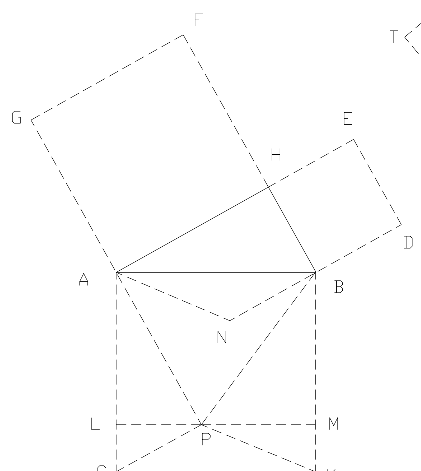


Fig. 132

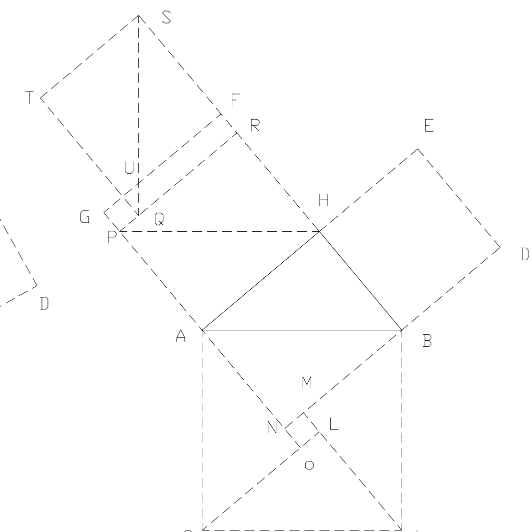


Fig. 133

THIRTY-TWO

Extend GA making AD = AO. Extend BD to N, draw CL and KM. Extend BF to S making FS = HB, complete sq. SU, draw HP par. to AB, PR par. tho AH, and draw SQ.

Then obvious, sq. AK = 4 tri. BAN + sq. NL = rect. AR + rect. TR + sq. GQ = rect. AR + rect. GQ + (sq. TF = sq.ND) = sq. HG + sq. HD. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. This proof is credited to Miss E.A. Coolidge, a blind girl. See Journer of Education, V. XXVIII, 1888, p. 17, 26th proof.

b. The reader will note that this proof employs exactly the same dissection and arrangement as found in the solution by the Hindu mathmatician, Bhaskara. See fig. 324, proof Two Hundred Twenty Five.

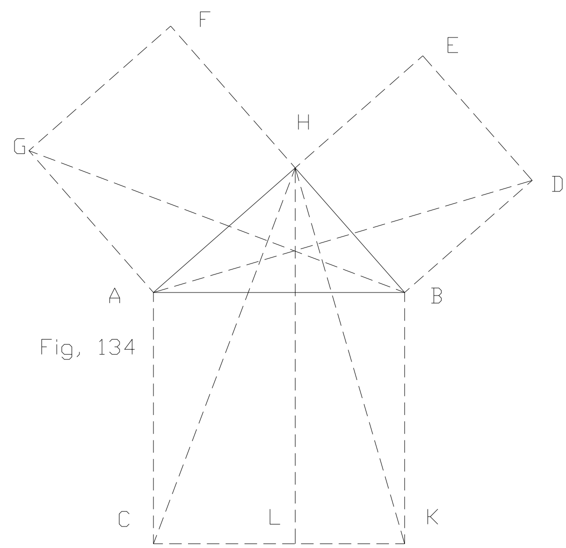
(b) THOSE PROOFS IN WHICH PAIRS OF THE DISSECTED PARTS ARE SHOWN TO BE EQUIVALENT.

THIRTY-THREE

Draw HL perp. to CK, and draw, HC, HK AD and BG. Sq. AK = rect. AL + rect. BL = 2 tri. HAC + 2 tri. HBK = 2 tri. GAB + 2tri DAB = sq.GH + sq. HD. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Euclid, about 300 B.C. discovered the above proof, and is has found a place in every standard text on geometry. Logically no better proof can be devised than Euclid's.

For the old descriptive form of this proof see Elements of Euclid by Todhunter, 1887, Prop. 47, Book I. For a modern model proof, second to none, see Beman and Smith's New Plane and Solid Geometry, 1899, p 102, Prop VIII, Book II. Also see Heath's Math. Monographs, No. 1, 1900, p. 18, proof I; Versluys, p. 10, fig. 3, and p. 76 proof 66 (algebraic); Fourrey, p. 70, fig. a;



Fig, 134

also The South Wales Freemason, Vol. XXXVIII, No. 4, April 1, 1938, p. 178, for a fine proof of Wor. Bro. W. England, F.S.P. , of Auckand, New Zealand. Also Dr. Jeitzmann's work (1930), p. 29, fig's 29 and 30.

b. I have noticed lately two or three American texts on grometry in which the above proof does not appear. I suppose the author wishes to show his originality or independence –possibly up-to-dateness. He shows something else. The leaving out of Euclid's proof is like the play of Hamlet with Hamlet left out.

c. About 870 there worked for a time, in Bagdad, Arabia, the celebrated physician, philosopher and mathematician Tabit ibn Qurra ibn Mervan (826-901), Abu Hasan, al- Harrani, a mative of Harran in Mesopotamia. He revised Ishaq ibn Honeiu's translation of Euclid's Elements, as stated at foot of the photostat.

See David Eugene Smith's "History of Mathematics," (1923), Vol. I pp.171-3.

d. The figure of Eucli's proof, Fig. 134 above, is known by the French as pop asinorum, by the Arabs as the "Figure of the Bride."

e. "The mathematical science of modern Europe dates from the thirteenth century, and received its first stimulus from the Moorish School in Spain and Africa. Where the Arab works of Euclid, Archimedes, Appollonius and Ptolemy were not uncommon………"

"First, for the geometry. As early as 1120 an English monk, named Adelhard (of Barth), had obtained a copy of Moorish edition of the Elements of Euclid; and another specimen was secured by Gerard of Cremona in 1186. The first of these was translated by Adelhard, and a copy of this fell into the hands of Giovanni Campanus, who in 1260 reproduced it as his own. The first pronted edition was taken from it and was issued by Ratdolt at Venice in 1482." A History of Mathematics at Cambridge , by W.W. R. Ball, edition 1889, pp. 3 and 4.

THIRTY- FIVE

Draw HN par. to AC, KL par. to BF, CN par. to AH, and extend DB to M. It is evident that sq. AK = hexagon ACNKBH = par. ACNH + par. HNKB = AH x LN + BH x HL = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards' Geom. 1895, p. 161, fig. (32) ; Versluys, p. 23, fig. 21, created to Van, Vieth (1805); also , as an original proof, by Joseph Zelson a sophomore in West Phila., Pa. High School, 1937,

b. In each of the 39 figures given by Edwards the author hereof devised the proofs as found herein.

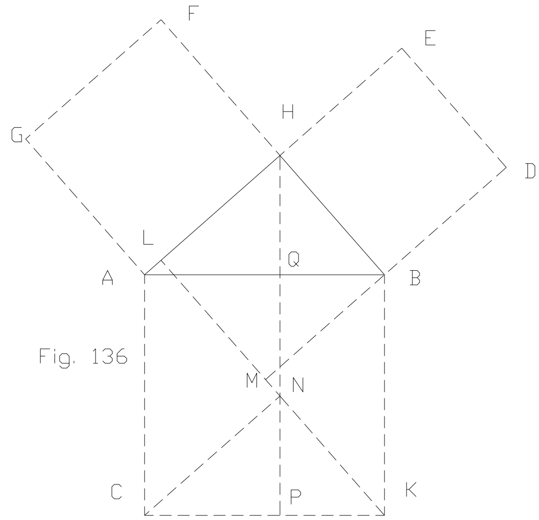


Fig. 136

THIRTY-SIX

In fig.136, produce HN to P. Then sq. AK = (rect. BP = paral. BHNK = sq. HD) + (rect. AP = paral. HACN= sq. HG)

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Math, Mo. (1859). Vol. 2, Dem. 17, fig.1.

THIRTY-SEVEN

In fig. 137, the constriction is evident. Sq. AK = rect. BL + rect. BM + paral. AM = paral. BN + paral. AO = sq. BE + sq. AF. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards' Geom., 1895, p. 160, fig. (28); Ebene Geometrie von. G. Mahlar, Leipzig, 1897, p 80, fig. 60; and Math. Mo., V. IV, 1897, p. 168, proof XXXIV; Versluys, p. 57, fig. 60, where it is credited to Hauff's work, 1803.

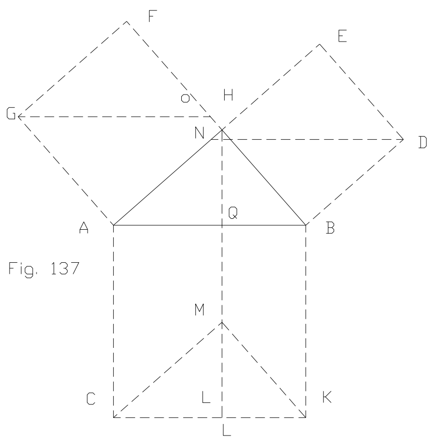


Fig. 137

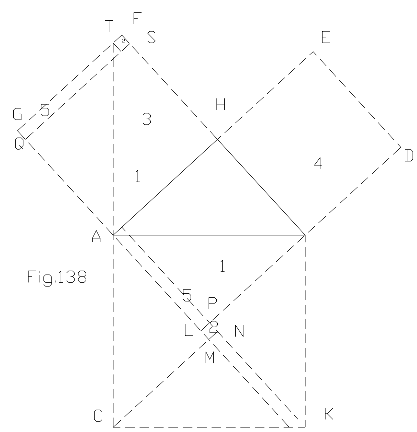


Fig.138

THIRTY-EIGHT

In fig. 138, the construction is evident, as well as the parts containing like numbers.

Sq. AK = tri. BAL + tri.CNK + sq. LN + (tri. ACM + tri. KBP) + tri, HQA + tri. QHS + sq. RF + (rect. HL = sq. HP + rect. AP + sq. HD + rect. GR) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Heath's Math. Monographs, No. 2, p. 33, proof XXI.

THIRTY-NINE

Produce CA to P, draw PHN, join NE, draw HO perp. to CR, CM par. to AH, join MK and MA and produce DB to L. From this dissection there results: Sq. AK = rect. AO +rect. BO = (2tri. MAC = 2tri. ACM = 2 tri. HAM = 2tri.AHP = sq. HG) + (rect. BHMK = 2tri. NHL = 2tri. HLN = 2tri. NEH = sq. HD).

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Deviced by the author Nov. 16, 1933.

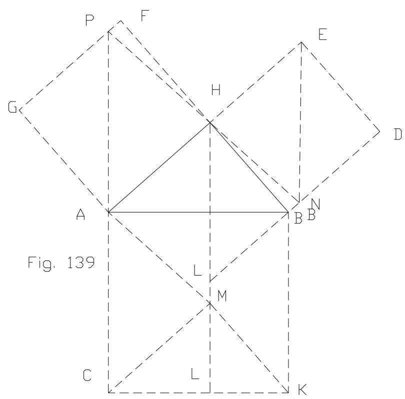


Fig. 139

FORTY

Fig. 140 suggests its construction, as all lines drawn are tither perp. or par. to a side of the given tri. ABH. Then we have sq. AK = rect. BL + rect. AL = paral. BHMK + paral. AHMC= paral. BHNP + paral.AHNO = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

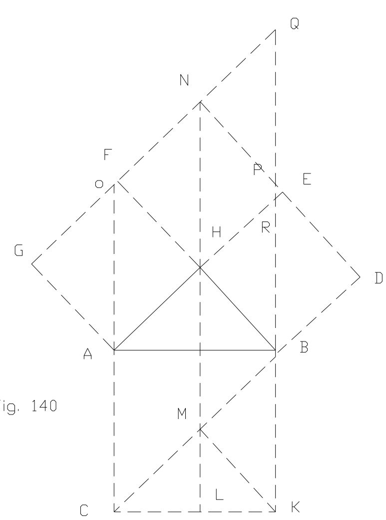


Fig. 140

a. This is known as Haynes' proof; see Math. Magazine, Vol. I, 1882, p. 25, and school Visitor, V. IX, 1888, p. 5, proof IV; also see Fourrey, p. 72, fig. a, in Edition arabe des Elements d' Euclides.

FORTY-ONE

Draw BQ perp. to AB meeting GF extended, HN par. to BQ, NP par. to HF, thus forming OARQ; draw OL par. to AB, CM par. to AH, AS and KT perp. to CM, and SU par. to AB, thus dissecting sq. AK into parts 1, 2. 3. 4 and 5.

Sq. AK = paral. AEQO, for sq. Ak = [(quad. ASMB = quad. AHLO) + (tri.CSA = tri.NFH = tri. OGH) + (tri.SUT = tri. OLF) = sq. HG] + [trap. CKUS= trap. NHRP = tri. NVW + trap. EWVA, since tri. EPR = tri WNV = trap, BDER) + (tri NPQ = tri. HBR) = sq. HD] = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. This proof and fig. was foumulated by the author Dec. 12, 1933, to show that, having given a paral. = those of the sq., the paral. can be dissected into parts, each in the square.

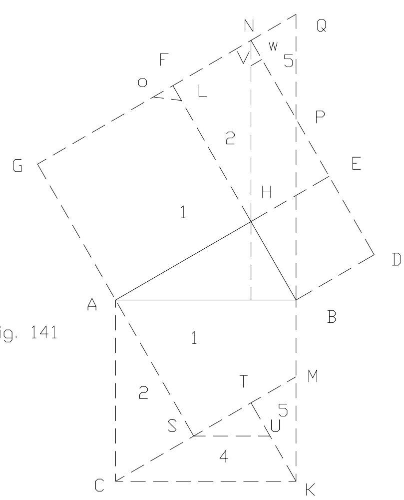


Fig. 141

FORTY-TWO

The construction of fig. 142 is easily seen. Sq. AK = rect. BL + rect. AL = paral. HBMN + paral. AHNO = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. This is Lecchio's proof, 1753. Also see Math. Mag., 1859. Vol. 2, No. 2, Dem. 3, and credited to Charles Young, Hudson, O., (afterwards Prof. Astronomy, Princeton Collage, N.J.); Jury Wipper, 1880, p. 26 fig. 22 (Historical Note); Olney's Geom., 1872, Part III, p. 251, 5th method; Jour. of Ecuaction, V. XXV, 1887, p. 404, fig, III; Hopkins' Plane Geom., 1891, p. 91, fig. II; Edwares' Geom., 1895, p. 159 fig. (25); Am. Math. Mo. V. IV, 1897, p. 169, XL; Heath's Math, Monographs, No. 1, 1900, p. 22, proof VI; Versluys, 1914, p. 18,fig.14

b. One reference says: "This proof is but a particular case of Pappus' Theorem."

c. Pappus was a Greek Mathematician of Alxandria, Egypt, supposed to have lived between 300 and 400 B.C.

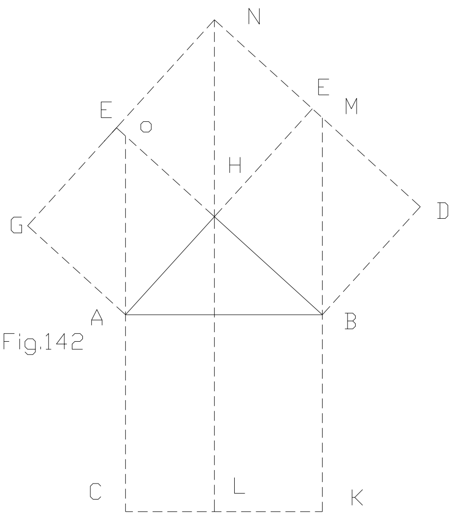


Fig.142

d. Theorem of Pappus: " If upon any two sides of any triangle, parallelograms are constructed, (see fig.143), their sum equals the possible resulting parallelogram determined upon the third side of the triangle.

e. See Chauvenet's Elem'y Geom. (1890), p. 147, Theorem 17. Also see F.C. Boon's proof, 8a, p. 106

f. Therefore the so-called Pythagorean Proposition is only a particular case of the theorem of Pappus; see fig. 144 herein.

THEOREM OF PAPPUS

Let ABH be any triangle; upon BH and AH construct any two dissimilar parallelograms BE and HG; produce GF and DE to C, their point of intersection; join C and H and produce CH to L making KL = CH; through A and B draw MA to N making AN = CH, and OB to P making BP = CH.

Since tri. GAM = tri. FHC, being equangualar and side GA = FH. ∴ MA = CH = AN; also BO = CH = BP = KL. Paral. EHBD + paral HFGA = paral. CHBO + paral. HCMA = paral. KLBP + paral. ANLK = paral. AP.

Also paral. HD + paral. HG = paral. MB, as paral.MB = paral. AP.

a. As paral. HD and paral. HG are not sililar, it follows that $BH^2 + AH^2 \neq AB^2$.

b. See Math. Mo. (1858), Vol. I, p. 358, Dem. 8, and Vol. II, pp. 45-52, in which this theorem is given by Prof.Charles A. Young, Hudson, O., now Astronomer, Princeton, N.J. Also David E. Smith's Hist. of Math. , Vol. I, pp. 136-7.

c. Also see Masonic Grand Lodge Bulletin, of Iowa, Vol. 30 (1929), No. 2, p. 44, fig.; also Fourrey, p. 101, Pappus, Collection, IV, 4th century, A.D. also see p. 105, proof 8, in "A Comanion to Elementry School Mathematics," (1924), by F.C. Boon, A.B.;also Dr. Leitzmann, p. 31, fig. 32, 4th Edition; also Heath, History II, 355.

d. See "Companion to Elementry School Mathematics, " by F.C. Boon, A.B. (1924), p. 14; Pappus lived at Alexandria about A.D. 300,though date is uncertain.

e. This Theorem of Pappus is a Generalization of the Pythagorean Theorem, Therefore the Pythagorean Theorem is only a corollary of the Throrem of Pappus.

FORTY- THREE

By theorem of Pappus, MN = LH. Since, HD and HG are rectangular, and assumed squares (Eucleid, Book I, Prop. 47) But by Theorem of Pappus, paral. HD+ paral. HG = paral. AK. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. By the author, Oct. 26, 1933.

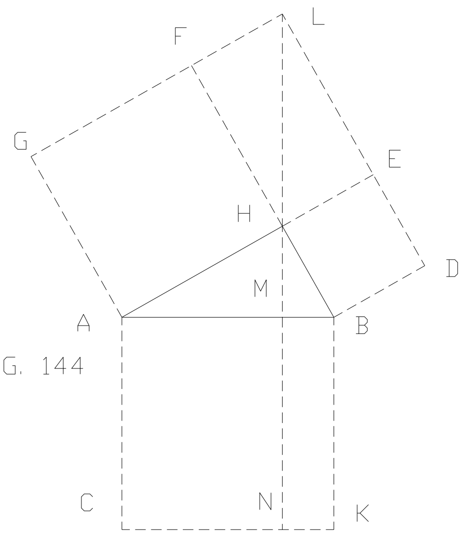


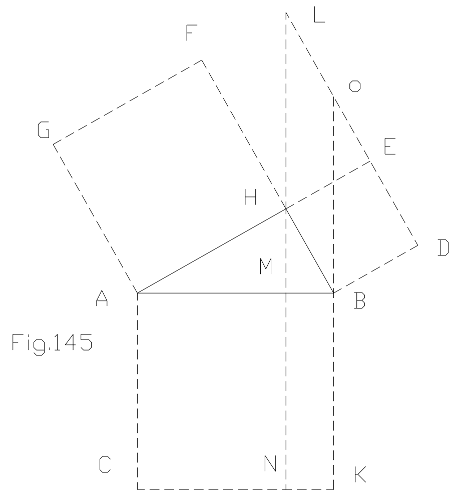
FIG. 144

FORTY – FOUR

Produce DE to L making EL = HF, produce KB to O, and draw LN perp. to CK. Sq. AK = rect. MK + rect. MC = [rect. BL (as LH =MN) = sq. HD] + (similarly, sq. HG).

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. See Versluts, p. 19 fig. 15, where credited to Nasir – Ed- Din (1201- 1274); also Fourrey, p. 72, fig.9.

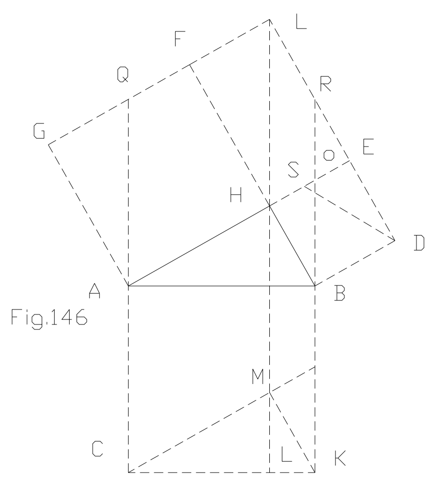


FORTY –FIVE

In the fig.146 extend DE and GF to P, CA and KB to Q and R respectively draw PL and KM perp. to AB and CN respectively. Take ES = HO and draw DS.

Sq. AK = tri. KNM + hexagon ACKMNB = tri. BOH + prntagon ACNBH = tir. DSE + pentagon QAORP = tri. DES + paral.AHPQ + quad. PHOR = sq. HG + tri. DES + paral. BP – tri. BOH = sq. HG + tri. DES + trap. HBDS = sq. HG + sq. HD. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Am. Math. Mo. , V, IV, 1897, p. 170. Proof XLV.



FORTY-SIX

The construction needs no explanation; from it we get sq. AK + 2 tri. ABH = hexagon ACLKBH = 2 quad, ACLH = 2 quad. FEDG = hexagon ABDEFG = sq. HA + 2 tri ABH.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Sccording to F.C. Boon, A.B. (1924), p. 107 of his "Miscellaneous Mathimatics," this proof is that of Leonafdo da Vinci (1452 -1519).

b. See Jury Wipper, 1880, p. 32, fig. 29, as found in "Aufangagfunden der Geomtrie" von Tempelhoff , 1769; Versluys, p. 56, fig. 59, where Tempelhoff, 1769, is mentioned; Fourrey, p. 74. Also proof 9, p. 107, in "A Companion to Elementary School Mathematics," by F.C. Boon, A.B.; also Dr. Leitzmann, p. 18, fig. 22, 4th Edition.

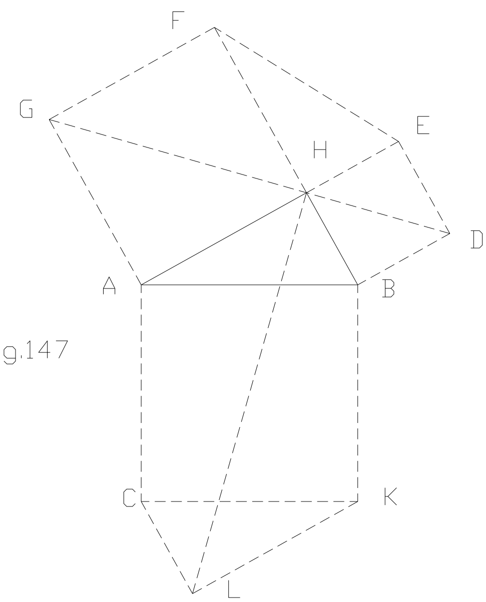


Fig.147

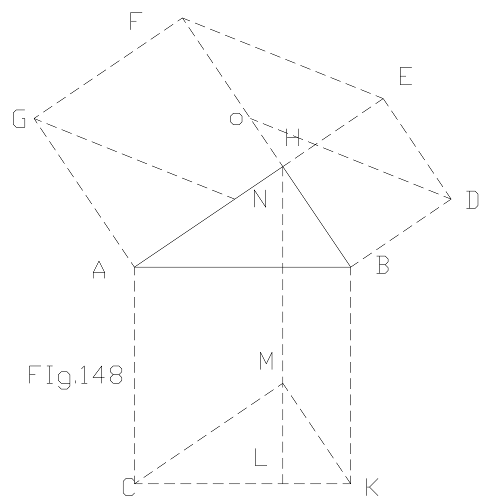


FIg.148

FORTY- SEVEN

In fig. 148 take BO = AH and AN = BH, and complete the figure; Sq. AK = rect. BL + rect. AL = paral. HMKB = paral. ACMH = paral. FODE + paral. DNEF = sq. DH + sq. GH. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 158, fig. (21), and Am. Math. Mo. V. VI, 1897, p. 169 proof XLI.

FORTY – NINE

In fig. 149 extend CA to Q and complete sq. QB. Draw GM and DP each par. to AB, and draw NO perp. to BF. This construction gives sq. AB = sq. AN = rect. AL + rect. PN = paral. BDRA + (rect. AM = paral. GABO) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 158, fig. (29), and Am. Math. Mo., V. VI, 1897, p. 168, proof XXXV.

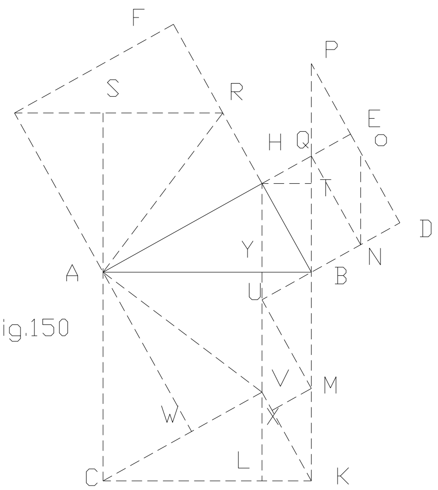


Fig.150

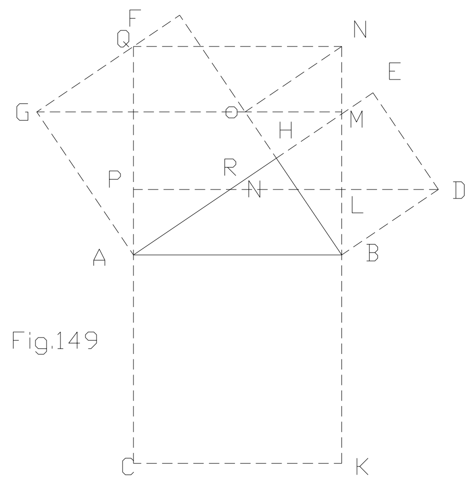


Fig.149

FORTY- NINE

In fig.150 extend KB to meet DE produced at P, draw QN par. to DE, NO par. to BP, GR and HT par. to AB, extend CA, to S, draw HL par. to AC, CV par. to AH, KV and MU par. to BH, MX par. to AH, extend GA to W, DB to U, and draw AR and AV. Then we will have sq. AK = tri. ACW + tri. CVL + quad. AWVY + tri. VKL + tri. KMX + trap. UVXM + tri. MBU + tri. BUY = tri. GRF + tri. AGS + quad. AHRS) + tri. BHT + tri. OND + trap. NOEQ + tri. QDN + tri. HQT) = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. This is E, von Litterow's proof, 1839; see also Am. Math Mo. V. IV, 1897, p. 169, proof XXXVII.

FIFTY

Extend GF and DE to P, draw PL perp. to CK, CN par. to AH meeting HB extended, and KO perp, to AH. Then there results: sq. AK [(trap. ACNH – tri.MNH = paral. ACMH = rect. AL) = (trap. AHPG – tri. HPF = sq. AG)] + [trap. HOKB – tri. OMH = paral. HMKB = rect. BL) = (trap. HBDP – tri. HEP = sq. HD)]

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. VI. 1897, p. 169, proof XLII.

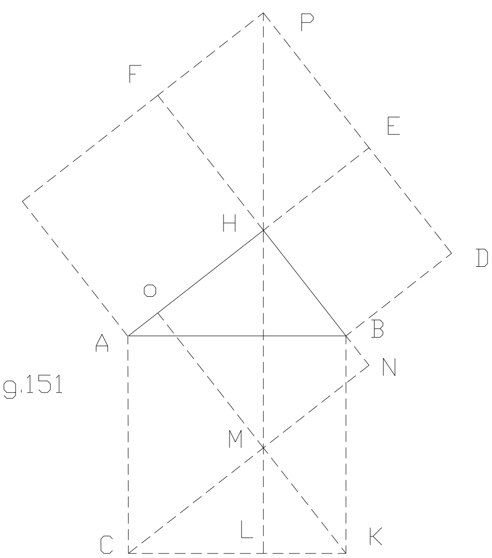


Fig.151

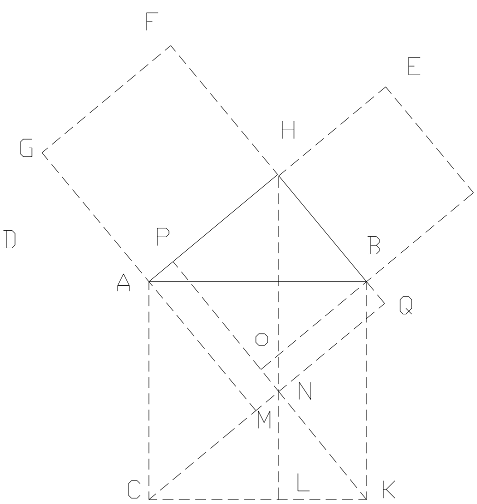


Fig.152

FIFTY- ONE

Extend GA to M making AM = AH, complete sq. HM, draw HL pero. to CK, draw CM par. to AH, and KN par. to BH; this construction gives: sq. AK = rect. BL + rect. AL = paral. HK + paral. HANC = sq. BP + sq. HM = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Vieth's proof --- see Jury Wipper, 1880, p. 24, fig. 19, as given by Vieth, in "Aufangsgrunden der Mathematik," 1805; also Am. Math. Mo., V. VI, 1897, p. 169, proof XXXVI

FIFTY-TWO

In fig, 153 construct the sq. HT draw GL, HM , and PN par. to AB; also KU par. to BH, OS par. to AB , and join EP. By analysis we find that sq. AK = (trap.CTSO + tri. KRU) + [tri. CKU + quad. STRQ + (tri. SON = tri. PRQ) + rect. AQ] = (trap. EHBV + tri. EVD) + [tri. GLF + HMA + (paral. SB = paral. ML)] = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. After three days of analyzing and classifying solutions based on the A type of figure, the above dissection occurred to me, July 16, 1890, from which I devised above proof.

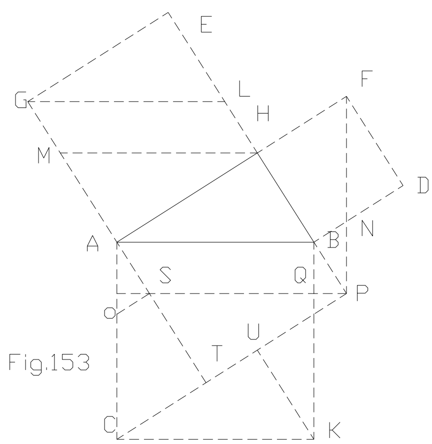


Fig.153

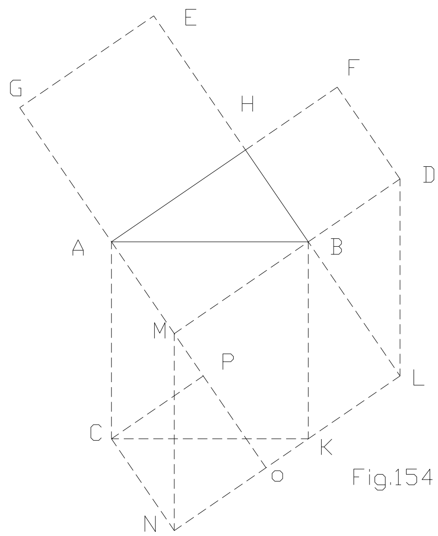


Fig.154

FIFTY-THREE

In fig. 154 through K draw NL, GA to O, DB to M, draw DL and MN par. to BK, and CN par. to AO.

Sq. AK = hexagon ACNKBM = paral. CM + paral. KM = sq. CO + sq. ML = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Edwards' Geom., 1895, p. 157, fig. (16).

FIFTY-FOUR

In fig. 155 extend HB to M making BM = AH, HA to P making AP = BH, draw CN and KM each par. to AH, CP and KO each perp. to AH, and draw HL perp. to AB. sq. AK = rect. BL + rect. AL = paral. RKBH + paral. CRHA = sq. RM + sq. CO = sq. HD + sq. HG.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. See AM. Math. Mo., V. IV, 1897, p. 169, proof XLIII.

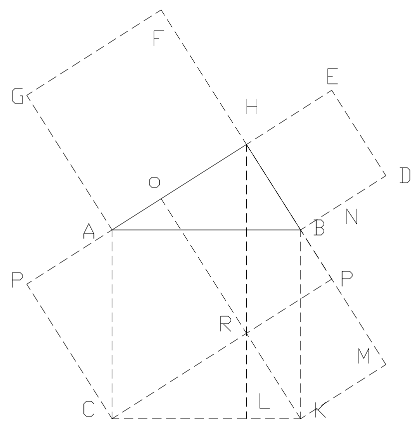


Fig. 155

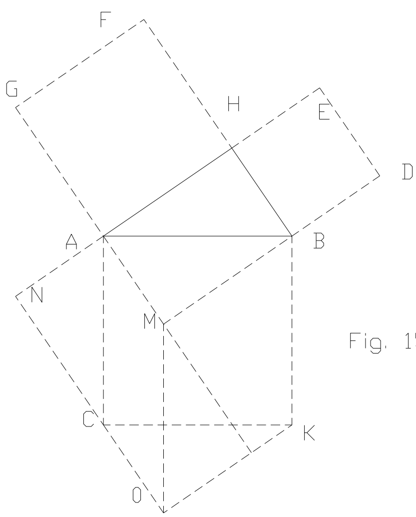


Fig. 156

FIFTY-FIVE

Extend HA to N making AN = HB, DB and GA to M, draw, through C, NO making CO = BH, and join MO and KO.

Sq. AK = hexagon ACOKBM = para. COMA + paral. OKBM = sq. HD + sq. HG.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. This proof is credited to C. French, Winchester, N. H. See Journal of Education, V. XXVIII, 1888, p. 159, fig. (26); Heath's Math, Monographas, No. 2, p. 31, proof XVIII.

FIFTY –SIX

Complete the sq's OP and HM, which are equal.

Sq. AK = LN – 4 tri. ABH = sq. OP – 4 tri. ABH sq. HD + sq. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluts, P. 54 fig. 56, taken from Delboeuf's work, 1860; Math. Mo., 1859, Vol. II, No. 2 , Dem. 18, fig.8; Fourrey, Curios. Gemo., 82, fig. e, 1683.

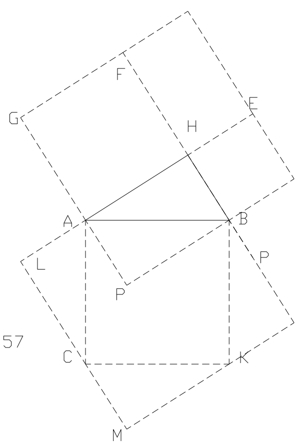


Fig. 157

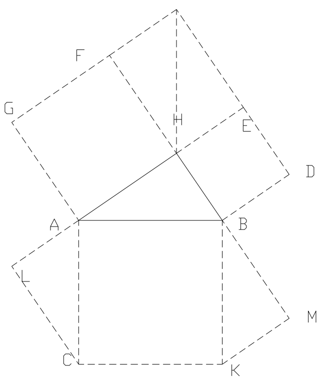


Fig. 158

FIFTY- SEVEN

Complete rect. FE and construct the tri's ALC and KMB, each = tri ABH.

It is obvious that sq. AK = pentagon CKMHL – 3 tri. ABH = pentagon ABDNG – 3 tri. ABH = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Versluys, p. 55, fig. 57.

FIFTY- SEVEN

In fig. 159 complete the squares AK, HD and HG, also the paral's FE, GC, AO, PK and BL. Fromthese we find that sq. AK = hexagon ACOKBP = paral. OPGN – paral. CAGN + paral. POLD - paral. BKLD = paral. LDMH – (tri. MAE + tri. LDB) + paral. GNHM – (tri. GNA + tri. HMF) = sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Olney's Geom., University Edition, 1872, p. 251. 8th method; Edwards' Geom. 1895, p. 160, fig. (30); Math. Mo. Vol. II 1859, No. 2, Dem, 16, fig. 8, and w. Rupert, 1900.

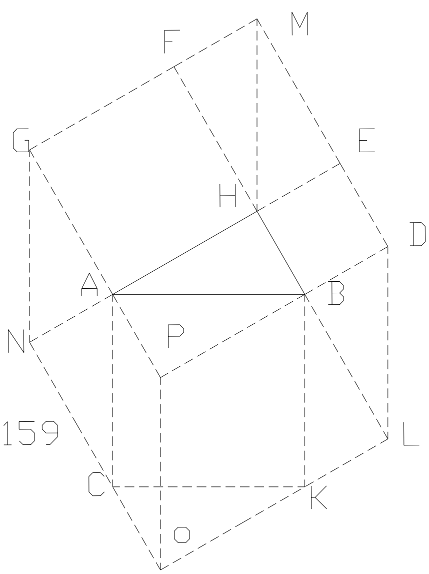


Fig.159

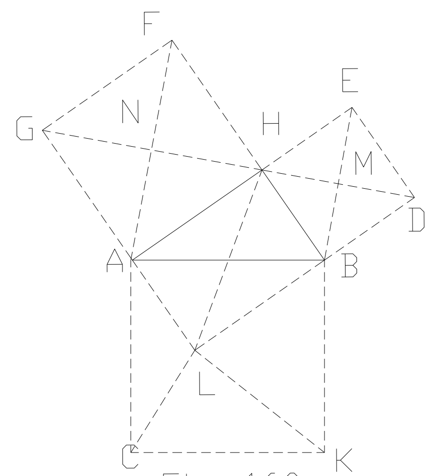


Fig. 160

SIXTY

In the figure draw the diag's of the sq's and draw HL. By the arguments established by the dissection, we have quad. ALBH= quad. ABMN (see proof, fig. 334).

Sq. AK = 2 (quad. AKBH – tri. ABH) = 2(quad. ABDG – tri. ABH = $^1/_2$ sq. EB + $^1/_2$sq. FA = sq. HD. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See E. Fourrey's Curios. Geom. p. 96, fig. a.

SIXTY- ONE

GL and DW are each perp. to AB, LN par. to HB QP and VK par. to BD, GR, DS, MP, NO and KW par. to AB and ST and RU perp. to AB Tri. DKV = tri. BPQ. AN = MC.

Sq. AK = rect. AO = (paral. ABDS = sq. HD)+ (rect. GU = paral. GABR = sq. GH). $\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys p. 28, fig. 24 ---- one of Werner's coll'n, credited to Dobriner.

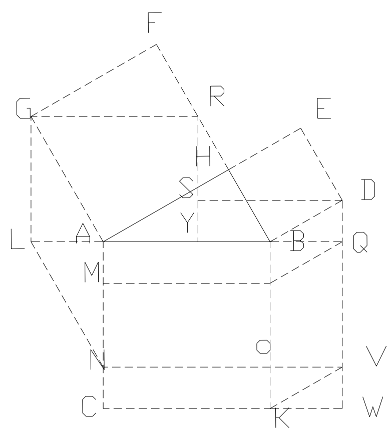


Fig.161

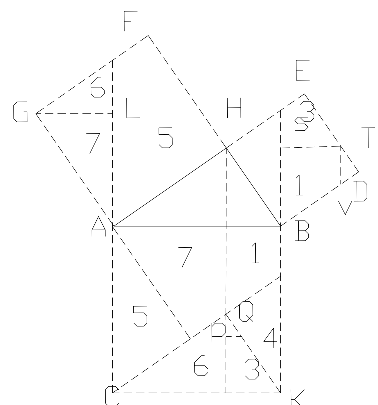


Fig.162

SIXTY- TWO

Constructed and numbered as here depicted, it follows that sq. AK = [(trap.XORB = trap. SBDT) + (tri. OPQ = tri. TVD) + (quad. PWKQ = quad. USTE) = sq. HD] + [(tri. CAN = tri. FMH) + (tri.CWO = tri. GLF) + (quad. ANOX =quad. GAML) = sq. HG.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 33, fig. 32, as given by Jacob de Gelder, 1806.

SIXTY-THREE

Extend GF and DE to N, complete the square NQ, and extend HA to P, GA to R and HB to L

From these dissected parts of the sq. NQ we see that sq. AK + (4 tei. ABH + rect. HM + rect. GE + rect.OA) = sq. NQ = (rect. PR = sq. HD + 2tri. ABH) + rect.HM + rect. AO = sq. AK + (4tri.ABH + rect. HM+ rect. GE + rect. AO – 2 tri. ABH – 2 tri ABH – ract. HM – rect. GE – rect. OA = sq. HD + sq. HG.

∴ sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Credited by Hoffmann, in "Der Phthagorasche Jehresatz," 1821, to Henry Boad, of London, Eng. See Jury Wipper, 1880, p.18, fig. 12; Versluys, p. 53 fig. 55; also see Dr. Leitzmznn, p. 20, fig. 23.

b. Fig. 163 employs 4 congruent triangles, 4 congruent rectangles, 2 congruent small squares, 2 congruent HG squares and sq. AK, if the line TB be inderted. Several variations of proof Sixty-Three may be produced from it, if difference is sought, especially if certain auxiliary lines are drawn.

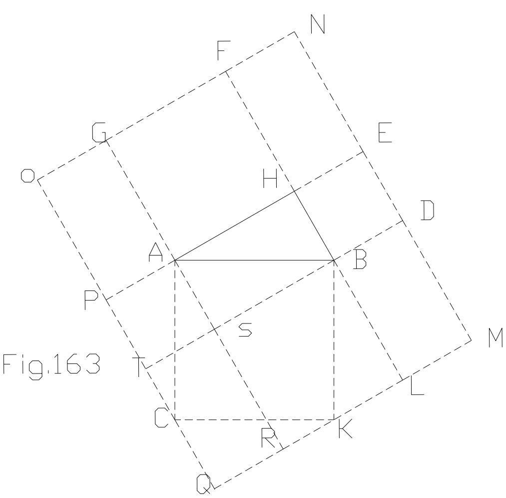


Fig.163

SIXTY-FOUR

In fig. 164, produce HB to L, HA to R meeting CK prolonged, DE and GF to O, CA to P, ED and FG to AB prolonged. Draw HN par. to, and OH perp. to AB. Obviously sq. AK = tri. RLH – (tri. RCA + tri. BKL + tri. ABH) = tri. QMO – (tri.QAP + tri. OHD + tri. ABH) = (paral.PANO = sq. HG) + (paral. HBMN = sq. HD).

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 30, fig. 28a; Versluys, p. 57, fig.61; Fourrey, p. 82, Fig. c and , by H. Bond, in Geometry, Londers, 1683 and 1733, also p. 89.

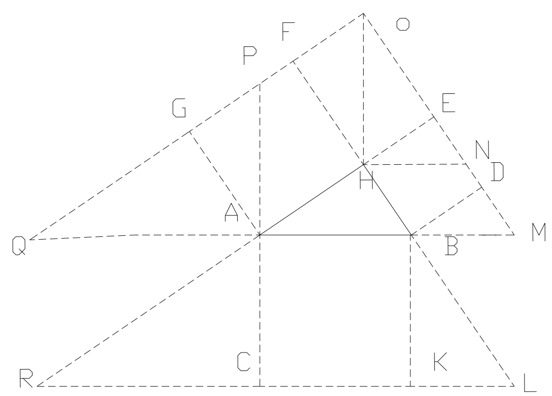


Fig.164

SIXTY-FIVE

In fig. 165 extend HB and CK to L, AB and ED to M, DE and GF to O, CA and KB to P and N respectively and draw PN. Now observe that sq. AK = (trap. observe that sq. AK = (trap. ACLB – tri. BLK) = [quad. AMNP = hexagon ABHNOP – (tri. NMB = tri. BLK) = (paral. BO = sq. HD) + (paral. AO = sq. AF)].

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. Devised by the author, July 7, 1901, but suggested by fig. 28b, in Jury Wipper, 1880,p. 31.

b. By omitting, from the fig., the sq. AK, and the tri's BLK and BMD; an algebraic proof through the mean proportional is easily obtaine.

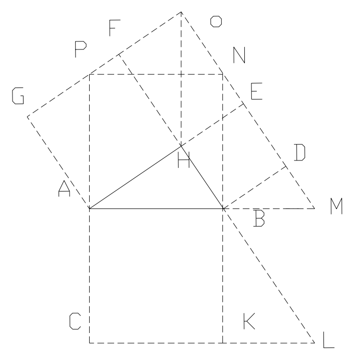


Fig. 165

SIXTY –SIX

In the construction make CM = HA = PL, LC = FP, MK = DE =NQ. ∴ OL = LM and MN = NO. Then sq. AK = tri. NLM - (tri. LCA + tri. CMK + tri. KNB) = tri.LNO – (tri. OPH + tri. HAB + tri. QOH) = paral. PLAH + paral. HBNQ = sq. HG + sq. HD. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 22, fig. 19, by J.D. Kruitbosch.

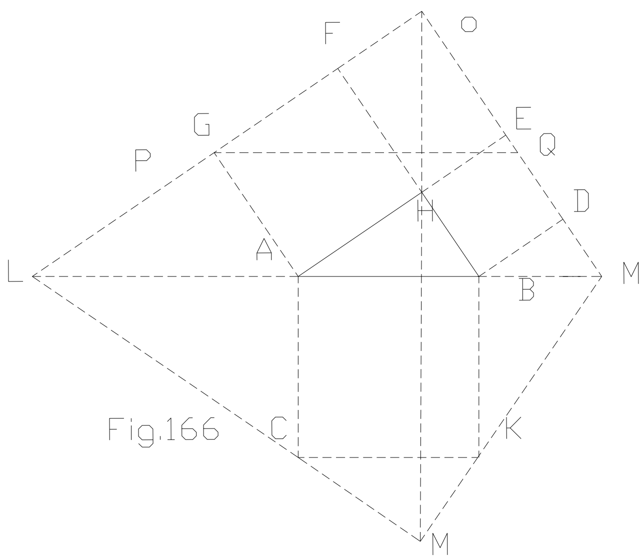


Fig.166

SIXTY-SEVEN

Make AM = AH, BP = BH complete paral. MC and PK. Extend FG and NM to L, DE and KB to S, CA to T, OP to R, and draw MP.

Sq. AK = paral. MC + paral. PK = PK = paral. LA + paral. RB = sq. GH + sq. HD.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. Math. Mo. (1859), Vol II, No. 2, Dem. 19, fig.9.

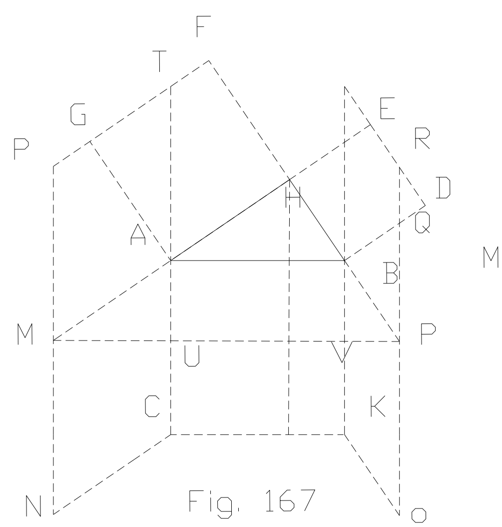


Fig. 167

SIXTY- EITHT

From P, the middle point of AB, draw PL, PM and PN perp. respectively to CK, DE and FG, dividing the sq's AK DH and FA into equal rect's.

Draw EF, PE, OH to R, PF and PC.

Since tri's BHA and EHF are congruent, EF = AB =AC. Since PH = PA, the tri's PAC, HPE and PHF have equal bases.

Since tri's having equal bases are to each other as their altitudes: tri. (HPE = EHP = sq. HD + 4): tri. (PHF = sq.HG + 4) = ER : FR $\therefore$ tri. HPE + tri. PHF : tri.PHF = (ER + FR = AC) : FR. $\therefore$ $^1/_4$ sq.HD + $^1/_4$ sq. HG : tri PHF = AC : FR. But (tri. PAC ¼ sq. AK) : tri. PHF = AC : FR. $\therefore$ $^1/_4$ sq. HD + $^1/_4$sq.HG : $^1/_4$sq. AK.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$. Q.E.D.

a. Fig. 168 is unique in that it is the first ever devised in which all auxiliary lines and all triangles used originate at the middle point of the hypotenuse of the given triangle.

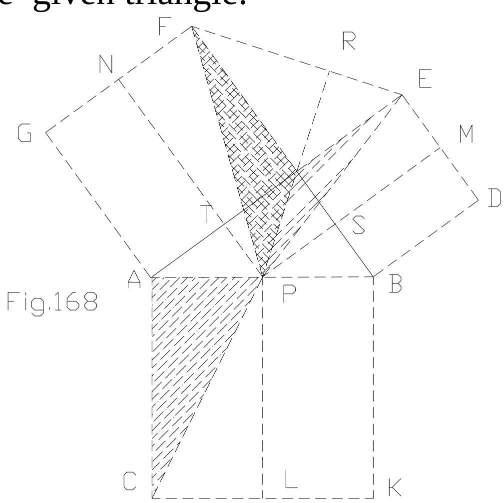


Fig.168

b. It was devised and proved by Miss Ann Condit, a girl, aged 16 years, of Central Junior Senior High School, South Bend, Ind., Oct. 1938. This 16-year – old girl has done what no great mathematician, Indian, Greek, or modern, is ever reported to have done. It should be known as the Ann Condit Proof.

SIXTY- NINE

Prolong HB to O making BO = HA; complete the rect. OL; on AC const. tri. ACM = tri. ABM = tri.ABH; on CK const. tri. CKN = tri. ABH. Join AN, AK , OA GB, GD, GE and FE.

It is obvious that tri.ACN = tri. ABO = tri. ABG = tri. EFG ; and since tri. DEG = [$^1/_2$ (DE) x (AE = AH + HE)] = tri. DBG = [$^1/_2$DB = DB x (BF= AE)] = tri. AKN = [$^1/_2$ (KN = DE) x (AN = AE)], then hexagon ACNKOB – (tri. CNK + tri. BOK) = (tri. CAN = tri. ABO = tri. ABG = tri. EFG) + (tri. AKN = tri. AKO = tri. GBD = tri. GEB) – (tri CNK + tri. BOK) = 2 tri. CNK = 2tri. GAB + 2 tri. ABD – 2tri. ABH = sq. AK = sq. HG + sq. HD. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. This fig. , and proof , is original; it was devised by Joseph Zelson, a junior in West Phila.,Pa., High School, and sent to me by his uncle, Louis G, Zelson, a teacher in a collage near St. Louis, Mo. on May 5, 1939. It shows a high intellect and a fine mentality.

b. The proof Sixty- Eight, by a girl of 16, and the proof Sixty- Nine by a boy of 18, are evidences that deductive reasoning is not beyond our youth.

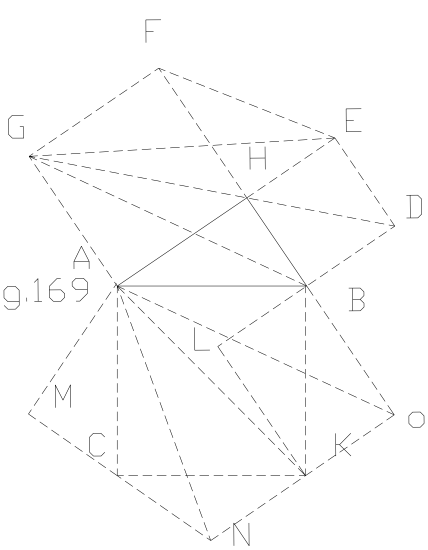


Fig.169

SEVENTY

Theorem.- If upon any convenient length, as AB, three triangles are constructed, one having the angle opposite AB obtuse, the second having that angle right, and the third having that opposite angle acute, and upon, right and acute angle squares are constructed, then the sum of the three squares are less than, equal to, or greater than, the square constructed upon AB, according as the angle is obtuse, right or acute.

In fig. 170, upon AB as diameter describe the semicir cumference BHA. Since all triangles whose vertex H' within the circumference BHA is obtuse as H', all triangles whose vertex H lies on that circumference is right at H, and all triangles whose vertex h_2 lies without said circumference is acute at H_2, let ABH' ABH and ABH_2 be such triangles, and on sides BH', AH' complete the squares H'D' and H'G'; on sides BH and AH complete squares HD and HG; on sides BH_2 and AH_2 complete square H_2D_2 and H_2G_2. Determine the points P', P and P_2 and draw P'H' toL' making N'L' = P'H' , PH to L making NL = PH, and P_2H_2 to L_2 making $N_2L_2 = P_2H_2$.

Therefore the paral. AK' = sq. H'D + sq. H'A'. (See d under proof <u>Forty two</u>, and pro of under fig. 143); the paral. (sq.) AK = sq. HD + sq. HG; and paral. AK_2 = sq. $H_2 D_2$ + sq. H_2G_2.

Now the area of AK' is less then the area of AK if (N'L' = P'H') is less than (NL = PH) and the srea of AK_2 is greater then the area of AK if ($N^2L_2 = p_2H_2$) is greater than (NL = PH).

In fig. 171 construct rect. FHEP in fig. 170, take HF' = H'F' in fig. 170 and complete F'H'E'P'; in like maner construct $F_2H_2E_2P_2$ equal to same in fig. 170. Since angle AH'B is always obtuse, angle E'H'F' is alwas acute and more acute E'H'F' becomes, the shorter P'H' becomes. Likewise, since angle AH_2B is always acute, angle $E_2H_2F_2$ is oburse, and the more obtuse it becomes the longer P_2H_2 becomes.

So first: As the variable acute angle F'H'E' approaches its superior limit, 90^0, the length H'P' increases and approaches the length HP; as said variable angle approaches, in degrees, its inferior limit, 0^0, the length of H'P' decreases and approaches, as its inferior limit, the length of the longer of the two lines H'A or H'B, P' then coinciding with either E' ro F', and the distance of P' (now E' or F') from a line drawn through H' parallel to AB', will be the second dimension of the parallelogram AK' on AB; as said angle F'H' E' continues to decrease, H' P' passes through its inferior limit and increases continually and approaches its superior limit ∞, and the distande of P' from the parallel line though the corresponding point of H' increases and again approaches the length HP.

∴ said distance is always less than HP and the parallelogram AK' is always less than the sq. AK.

And secondly: As the obruse variable angle $E_2H_2P_2$ approaches its inferior limit, 90^0, the length of H_2P_2 decereases and approaches the length of HP ; as said variable angle approaches its superior limit, 180^2, the length, of H_2P_2 increases and approaches ∞ in length, and the distance of P_2 from a line through the corresponding H_2 parallel to AB increases from the length HP to ∞, which distance is the second dimension of the parallelogram A_2H_2 on AB.

∴ the sq. upon AB = the sum of no other two squares except the two squares upon HB and HA.

∴ the sq. upon AB = the sq. upon BH + the sq. upon AH.

∴ $h^2 = a^2 + b^2$ and never $a'^2 + b'^2$.

a. This proof and figure was formulated by the author, Dec. 16, 1933.

B

This type includes all proofs derived from the figure in which the square constructed upon the hypotenuse overlaps the given triangle and the squares constructed upon the legs as in type A, and the proofs are based on the principle of equivalency.

SE ENTY- ONE

Fig, 172 gives a particular proof. In rt. tri. ABH, legs AH and BH are equal. Complete sq. AC on AB, overlapping the tri. ABH, and extend AH and BH to C and D, and there results 4 equal equivalent tri's 1, 2. 3 and 4.

The dq. AC = tri's[(1 + 2+ 3+ 4), of which tri. 1 + tri, (2= 2') = sq. BC and tri, 3 + tri. (4 = 4) = sq. AD].

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See fig.73b and fig. 91 herein.

b. This proof (better, illustration), by Richard, Bell, Feb. 22, 1938. He used only ABCD of fig. 172; also credited to Joseph Houston, a high school boy of South Bend Ind. , May 18, 1939. He used the full fig.

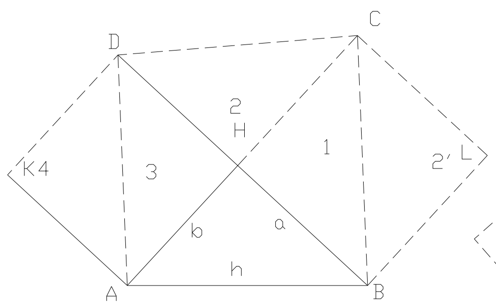


Fig. 172

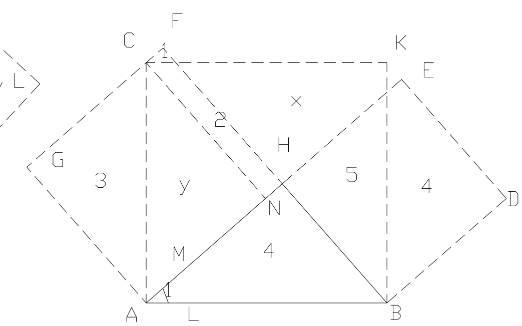


Fig.173

SEVENTY-TWO

Take AL = CP and draw LM and CN perp. to AH.

Since quad. CMNP = quad. KCOH, and quad. CNHP is common to both, then quad PHOK= tri. CMN, and we have: sq. AK = tri. ALM = tri. CPF of sq. HG) + (quad. LBHM = quad. OBDE of sq. HD) + (tri. OHB common to sq's AK and HD) + (quad. PHOK = tri. CGA of sq. HG) + (quad.CNHP common to sq's AK and HG) = sq.HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. This proof, with fig. , discovered by the author March 26, 1934, 1 p.m.

SEVENTY- THREE

Assuming the three squares constructed, as in fig. 174, draw GD ---it must pass through H.

Sq. AK = 2 trap. ABML = 2tri. AHL + 2 tri. ABH + 2 tri. HBM = 2 tri. AHL + 2 (tri. ACG = tri. ALG + tri. GLC) + 2 tri. HBM = 2 tri. AHL + 2 tri. ALG) + (2 tri. GLC = 2 tri. DMB) + 2tri HBM = sq. AF + sq. BE.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math, Mo., V. IV, 1897, p. 250, proof XLIX.

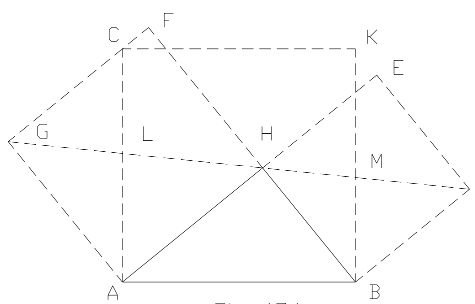


Fig. 174

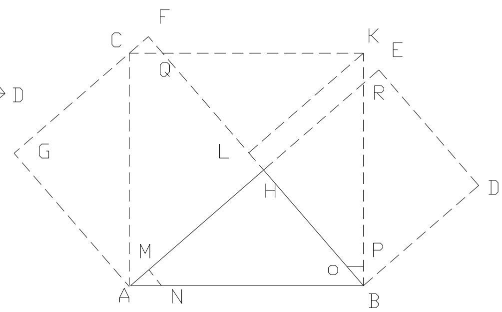


Fig. 175

SEVENTY –FOUR

Take HM = HB, and draw KL par. to AH and MN par. to BH.

Sq. AK = tri. ANM + trap. MNBH + tri. BKL + tri. KQL + quad. AHQC = (tri. CQF + tri. ACG + quad. AHQC) + (trap. RBDE + tri. BRH) = sq. AF + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. IV, 1897, p. 250, proof L.

b. If OP is drawn in place of MN, (LO = HB) the proof is prettier, but same in principale.

c. Also credited to R. A. Bell, Feb. 28, 1938.

SEVENTY-FIVE

In fig. 176, draw GN and OD par. to AB.

Sq. AK = rect. AQ + rect. OK = paral. AD + rect. AN = sq. BE + paral. AM = sq. HD + sq. HG.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. IV, 1897, p. 250, XLVI.

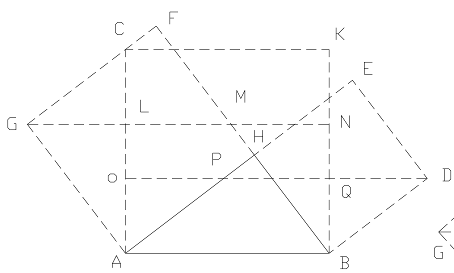


Fig. 176

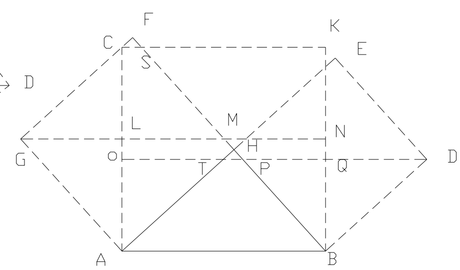


Fig. 177

SEVENTY- SIX

In fig. 177, draw GN and DR par. to AB and LM par. to AH. R is the pt. of intersection of AG and DO.

Sq. AK = rect. AQ + rect. ON + rect. LK = (paral. DA = sq. BE) + (paral. RM = pentagon RTHMG + tri. GSF) + (paral GMKC = trap. GMSC + tri. TRA) = sq. BE + sq. AF.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. IV, 1897, p. 250, proof XLVII; 1914, p. 12, fig. 7.

SEVENTY- SEVEN

In fig. 178, draw LM through H perp. to AB, and draw HK and HC.

Sq. AK = rect. LB + rect. LA = 2 tri. KHB + 2 tri. CAH = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Versluys, 1914, p. 12 , fig. 7; Wipper, 1880, p. 12, proof V; Edw. Geometry, 1895, p. 159, fig. 23; Am. Math. Mo., Vol. IV, 1897 p. 250. Proof LXVIII; E. Fourrey, Curiosities of Geometry, 2nd Ed'n, p. 76, fig. e, credited to Peter Warins, 1762'

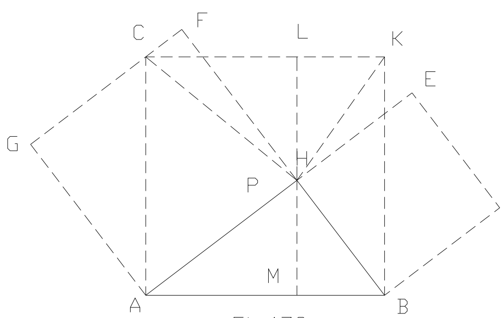


Fig.178

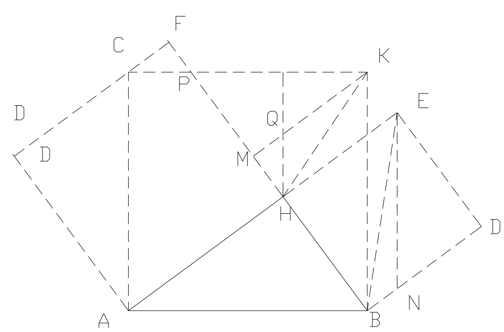


Fig. 179

SEVENTY- EIGHT

Draw HL par. to BK, KM par, to AH, KH and EB.

Sq. AK = (tri. ABH = tri, ACG) + quad. AHPC common to sq. AK and sq. AF + (tri. HQM = tri. CPF) + (tri. KMP = tri. END) + [paral. QHOK = 2(tri. HOK = tri. KHB – tri. OHB = tri. EHB – tri.OHB = tri.EOB) = paral. OBNE] + tri. OHB common to sq. AK and sq. HD.

∴ sq. AK = sq. HD + sq. AF

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. IV, 1897, p. 250, proof LI.

b. See Sci. Am. Sup., V. IV, 70, 1910, p. 382, for a geometric proof , unlike the above proof, but based upon a similar figure of the B type.

SEVENT – NINE

In fig. 180, extend DE to K, and draw KM perp. to FB,

Sq. AK = (tri. ABH = tri. ACG) + quad. AHLC common to sq. AK and sq. AF + [(tri. KLM = tri. BNH) = tri. BKM = tri. KBD = trap. BDEN + (tri. KNE = tri. CLF)]

$\therefore$ sq. AK = sq. BE + sq. AF.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 161, fig. (36); Am. Math. Mo., V. IV, 1897, p. 251, proof LII, Versluys, 1914, p. 36, fig.35, credited to Jenny de Buck.

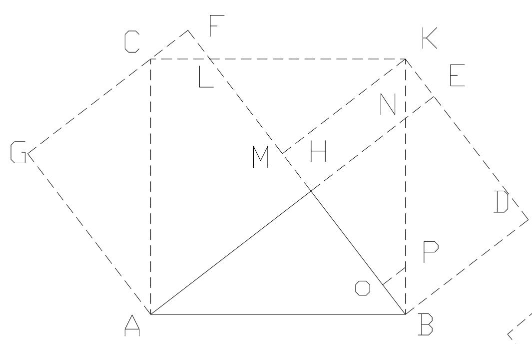


Fig. 180

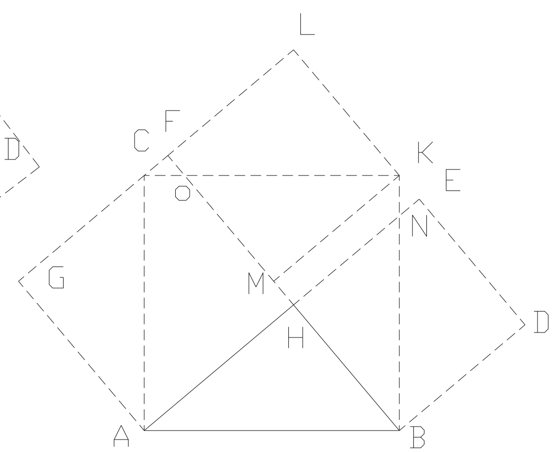


Fig. 181

EIGHTY

In fig. 181, ectend GF to L making FL = HB and draw KL and KM respectively par. to BH and AH.

Sq. AK = (tri. ABH =tri. CKL " trap. BDEN + tri. COF) + (tri. BKM = tri. ACG) + (tri. KOM = tri. BNH) + quad. AHOC common to sq. AK and sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. IV, 1897, p. 251, proof LVII.

EIGHTY – ONE

In fig. 182, ectend DE to L making KL = HN, and draw ML.

Sq. AK = (tri. ABH = tri. ACG) + (tri. BMK = $^1/_2$ rect. BL = [trap. BDEN + (tri. MKL = tri.BNH)] + quad. AHMC common to sq. AK and sq. AF = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 158, fig.(18)

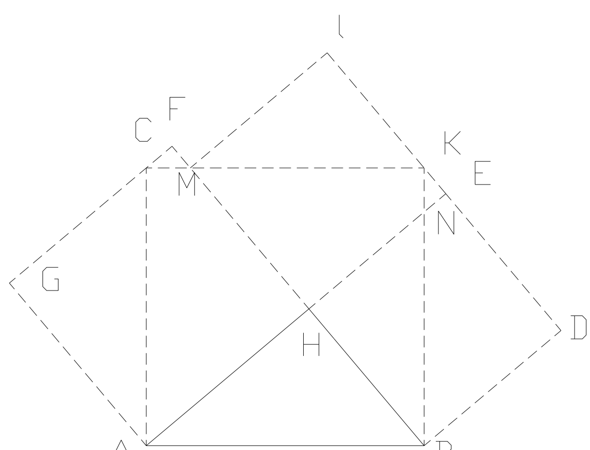


Fig.182

EIGHTY-TWO

In fig. 183, extend GF and DE to L and draw LH.

Sq. AK = hexagon AHBKLC + paral, HK + paral. HC = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

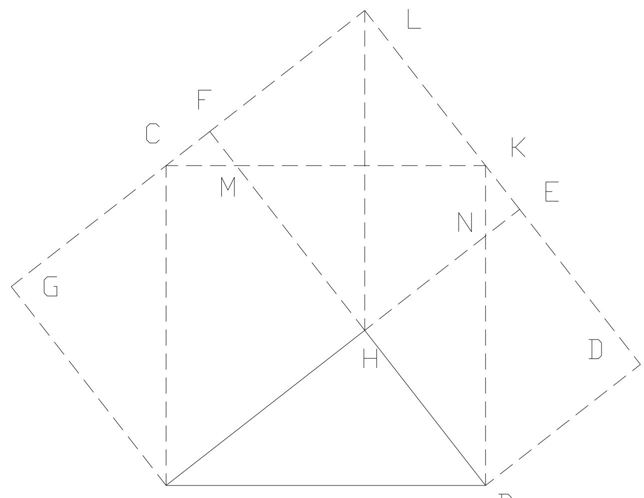


Fig. 183 &184

a. Original with the author, July 7, 1901; but old for it appears in Olney's Geom., university edition, 1872, p. 250, fig. 374; Jury Wipper, 1880, p. 25, fig. 20b, as given by M.V. Ash, in "Philosophical Transactions," 1683; Math. Mo. , V. IV, 1897, p. 251, proof LV; Heath's Math. Monographs, No. 1, 1900, p. 24, proof IX; Versluys, 1914, p. 55, fig. 58, credited to Henry Bond. Based on the Theorem of Pappus. Also see Dr. Leitzmann, p. 21, fig. 25, 4th Edition.

b. By extending LH to AB, an algebraic proof can be readily devised, thus increasing the no. of simple proofs.

EIGHTY-THREE

In fig. 184, extend GF and DE to L, and draw LH

Sq. AK = pentagon ABDLG – (3 tri. ABH = tri. ABH + rect. LH) + sq. HD + + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a.

b. See Journal of Education, 1887, V. XXVI, p. 21, fig. X; Math. Mo., 1855, Vol. II, No. 2, Dem, 12, fig. 2.

EIGHTY – FOUR

In fig. 185, extend H draw LM perp. to AB, and draw HK and HC.

Sq. AK = rect. LB + rect. LA = 2 tri. HBK + 2tri. AHC = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup. , V, 70, p. 383, Dec. 10, 1910, being No. 16 in A.R. Colburn's 108 proofs; Fourrey, p. 71, fig. e

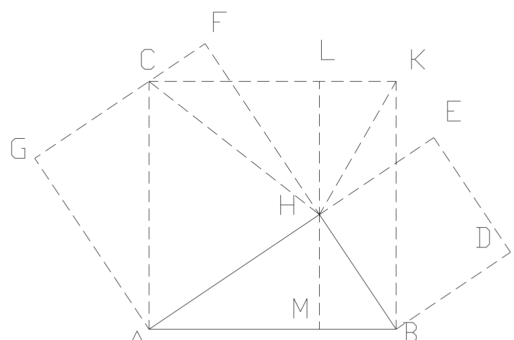


Fig. 185

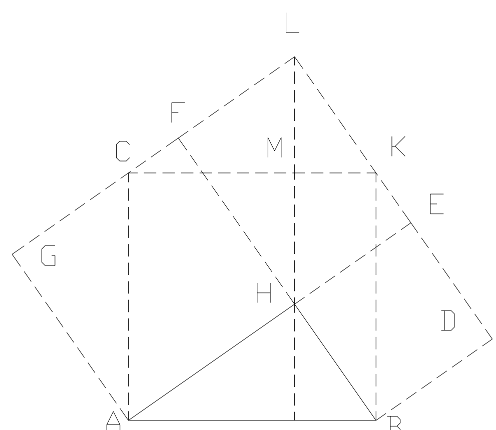


Fig. 186

EIGHTY-FIVE

In fig. 186, ectend GF and DE to L, and through H draw LN, N being the pt. of intersection of NH and AB.

Sq. AK = rect. MB + rect MA = paral. HK + paral. HC = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880. p. 13, fig. 5b, and p. 25, fig. 21, as given by Klagel in "Encyclopaedie," 1808; Edwards' Grom., 1895, p. 156, fig, (7); Ebene Geometrie, von G, Mahler, 1897, p. 87, art. 11; Am. Math. Mo. V. IV, 1897, p. 251, LIII; Math. Mo. , 1859, Vol.II, No. 2 fig. 2 Dem. 2, pp. 45-52, were credited to Charles A. Young, Hudson, O., now Dr.Young, astronomer, Princetion, N.J. This proof is and application of prop. XXXI, Book IV, Davies Legendre; Also Ash, M. v. of Dublin; also Joseph Zelson, Phila., Pa., a student in west Chester High School, 1939.

b. This figure will give an algebraic proof.

EIGHTY –SIX

In fig. 186 it is evident that sq. AK = hexagon ABDKCG – 2 tri. BDK = hexagon AHBKLC = (paral. KH = rect. KN) + paral. CH = rect. CN) = sq. HD + sq. HG. ∵ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.Q.E.D.

a. See Math. Mo. , 1858, Vol, I, p. 354, Dem. 8, where it is credited to David Trowgridge.

b. This proof is also based on theorem of Pappus. Also this geometric proof can easily be converted into an algebraic proof.

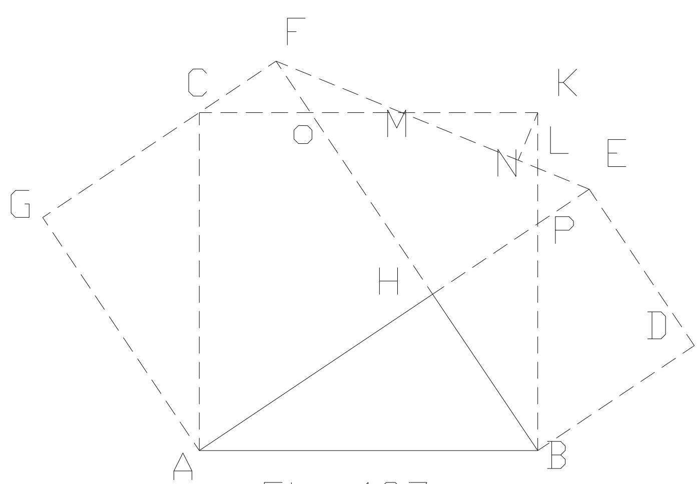


Fig. 187

EIGHTY-SEVEN

In fig. 187, extend DE to K, draw FE, and draw LM par. to AH

Sq.AK= (tri. ABH =tri. ACG) + quad. AHOC common to sq. AK and sq. AK + tri. BLH common to sq. AK and sq. HD +[quad. OHLK = pentagon OHLPN + (tri. MKN = tri. ONF)= tri.HEF = (tri. BDK = trap. DBEL + (tri. COF = tri. LEK)] =sq. HD + sq.HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. See Am. Maht. Mo. , V. IV, 1897, p. 251, proof LVI.

EIGHTY- EIGHT

In fig. 188, extend GF and BK to L, and through H draw MN par. to BK, and draw KM.

Sq. AK = paral. AOLC = paral. HL + paral. HC = (paral. HK = sq.AD) + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH.

∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 27, fig. 23, where it says that this proof was given to Joh. Hoffmann. 1800, by a friend; also Am. Math. Mo., 1897, V.IV, p. 251, proof LIV; Versluys, p. 20, fig. 16, and p. 21, fig. 18; Fourrey, p. 73, fig. b.

b. From this figure an algebraic proof is easily devised.

c. Omit line MN and we have R.A. Bell's fig. and a proof by congruency follows. He found it Jan. 31, 1922.

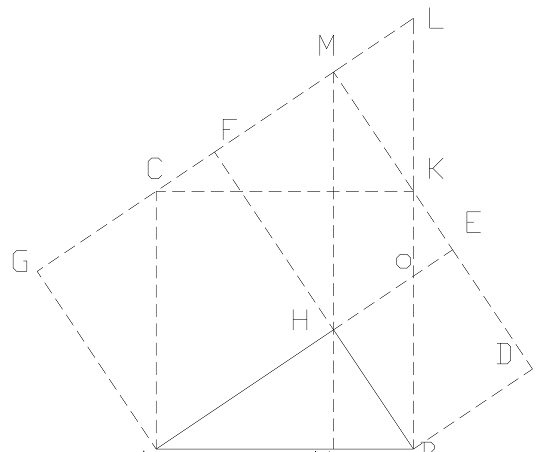


FIG. 188

EIGHTY-NINE

Extend GF to l making FL = BH, draw KL, and draw CO par. to FB and KM par. to AH

Sq.AK = (tri. ABH = tri. ACG) + tri. CAO common to sq's AK and HG + sq. MH common to sq's AK and HG + [pentagon MNBKC = rect. ML + (sq.HD)] = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Devised by the author, July 30, 1900, and afterwards found in Fourrey, p. 84, fig. c

NINTY

In fig. 190 produce GF and DE to L, and GA and DB to M. Sq. AK + 4 tri. ABH = sq.GD = sq. HD + sq. HG + (rect. HM = 2 tri. ABH) + (rect. LH =2 tri. ABH) whence sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

∴ $h^2 = a^2 + b^2$.

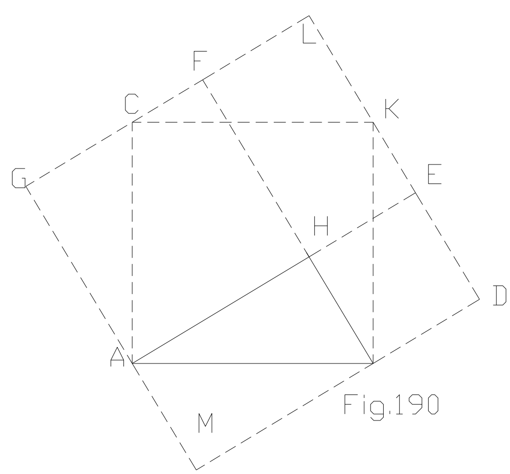


Fig.190

a. See Jury Wipper, 1880, p. 17, fig. 10, and is credited to Henry Boad, as given by Johann Hoffmann, in "Der Phthagoraische Lehrsatz," 1821; also see Edwards' Geom., 1895, p . 157, fig. (12). Heath's Math. Monographs, No.1. 1900, p.18, fig. 11; also attributed to Pythagoras, by W.W. Rouse Ball. Also see Pythagoras and his Philosophy in Sect. II, Vol. 10, p. 239, 1904, in proceedings of Royal Society of Canada, wherein the figure appears as follows lows:

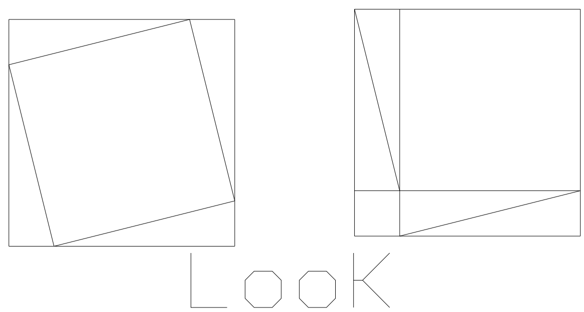


Fig. 191

NINTY-ONE

Tri's BAG, MKB, EMC, AEF, LDH and DLC are each = to tri. ABH.

∴ sq. AM = (sq. KF – 4 tri. ABH) = [(sq. KH + sq. HF + rect. HG) – 4 tri. ABH] = sq. KH + sq. HF. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See P.C. Cullen's pamphlet, 11 pages, with title, "The Pythagorean Theorem; or a New Method of Demonstrating it." Proof as above. Also Furrey, p. 80, as the demonstration of Pythagoras according to Bertschenschneider; see Simpson, and Elements of Geometry, Paris, 1766.

b. In No. 2, of Vol. I, of Scientia Baccalaureus, p. 61, Dr. Wm. B. Smith, of the Missoury State University, gave this method of proof as new. But see "School Visitor," Vol. II, No. 4, 1881, for same demonstration by Wm. Hoover, of Athens, O. , as"adapted from the French of Dalsme." Also see "Math, Mo. ," 1859, Vol. I, No. 5, p. 159; also the same journal, 1859, Vol. II, No. 2, pp. 45-52, where Prof. John M. Richardson, Collegiate Instute, Boudon, Ga., gives a collection of 28 proofs, among which, p. 47,is the one above, ascribed to young.

   See also Orlando Blanchard's Arthematic, 1852, pubkished at Cazenovia, N.Y., pp. 239-240; also Thomas Simpson's "Elemrnts of Geometry," 1760,p. 33, and p. 31, of his 1821 edition.

Prof. Saradaranjan Ray of India gives it on pp. 93-94 of Vol. I, of his Geometry, and says it "is due to the Persian Astronomer Nasir-uddin who flourished in the 13th century under Jengis Khan."

Ball, in his "Short Histoyr of Mathematics," fives same method of proof, p. 24, and thinks it is probably the one orifinally offered by Pythagoras.

Also see " Math, Magazine," by Atremas Martin, LL.D., 1892, Vol. II, No. 6, p. 97.Dr. Martin says: "Probably no other theorem has received so much attention from Mathematicians or been demonstrated in so many different ways as this celebrated proposition, which bears the namej of its supposed discoverer."

c. See T. Sundra Row, 1905, p. 14,by paper folding, "Reader, take two equal squares of paper and a pair of scissors, and quickly may you know that $AB^2 = BH^2 + AH^2$."

Also see Versluys, 1914, his 96 proofs. p. 41, fig. 42. The title page of Versluys is:

ZES EN NEENTIG BEWIJZEN

Voor Het

THEOREMA VAN PAYTHAGORAS

Verzameld en Geramgschikt

Door

J. VERSLUYS

Amesterdam----1914

NINTY-TWO

In fig. 193, draw KL par. and equal to BH, through H draw LM par. to BK, and draw AD, LB and CH.

Sq. AK = rect, MK + rect. MC = (paral. HK = 2 tri. BKL = 2 tri. ABD = sq. BE) + (2tri. AHC = sq. AF).

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. This figure and proof is taken from the following work, now in my library, the title page of which is shown on the following page.

The figures of this book are all grouped together at the end of the volume. The above figure is numbered 62, and is constructed for "Propositio XLVII," in "Librum Primum," which propostioin reads, "In rectangulis triangulis, quadratum quod a latere rectum angulum subtedente describitur; aequuale est eis, quae a lateribus rectum angulum continentibus desributur quadratis."

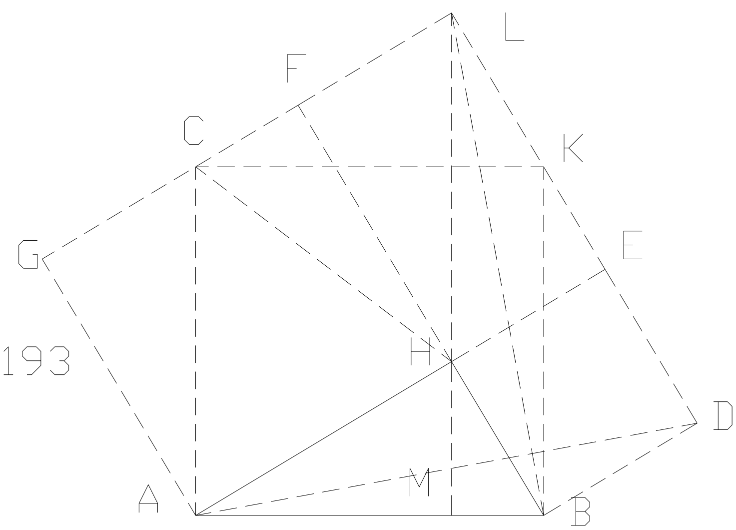


FIG. 193

“Euckides Elementorum Geometricorum

Libros Tredecim

Isidorum et Corporibus Regularibus, &

Procli

Propositiones Geometricas

------- - - - - - - - - - - -- --- - - -- - - - - - - - -- -

- - - -- - - - - -- - - - - - -- - - -- - - - - - - - - - - - - -

Claudius Richards

e Societate Jesu, Sacerdos, patria Ornacensis in Libero Comitatu

Burgundae, & Refius Mathematicarum

Professor: dican tique

Philippo IIII, Hispaniarum dt Indicarum Regi Cathilico.

Antwerpiae,

Ex Officina Hiesonymi Vredussii. M.DC. XLV.

Cum Gratia & Privilegio”

_ _ _ _

Then comes the following sentence:

“Proclus in hunc librum, celegrat Pythagoram Authorem huius propositionis, pro cuius demonstration dicitur Diis Sacrificasse hecatombam Taurorum.” Following this comes the “Supposito,” then the “Constructio,” and then the “Demonstratio,” which condensed and translated is: (as per fig. 193) triangle BKL equals triangle ABD; square BE equals twice triangle ABM and rectangle MK equals twice triangle BKL; therefore rectangle MK equals square BE. Also square AG equals twice triangle ACH; rectangle HM equals twice triangle CHA; therefore square AG equal rectangle HM, But square BK equals rectangle KM plus rectangle CM. Therefore square BK equals square AG plus square BD.

The work from which the above is taken is a book of 620 pages, 8 inches by 12 inches, bound in vellum, and , though printed in 1645 A.D., is well preserved. It once had a place in the Sunderland Library, Blenheim Palace, England, as the book plate shows---- on the book plate is printed --- “from the Sunderland Library, Blenheim Palace, Purchased, April, 1882.”

The work has 408 diagrams, or geometric figures, is entirely in Latin, and highly embellished.

I found the book in a second – hand bookstore inToronto, Canada, and on July 15, 1891, I purchased it. (E. S. Loomis.)

C

This type includes all proofs derived from the figure in which the square in which the square constructed upon the longer lag overlaps the given triangle and the square upon the hypotenuse.

Proofs by dissection and superposition are possible but none were found.

NINETY- THREE

In fig. 194, extended KB to L, take GN = BH and draw MN par. to AH. Sq. AK = quad. AGOB common to sq’s AK and AF + (tri. COK = tri. ABH + tri.BLH) + (teap. CGNM = trap. BDEL) + (tri. AMN = tri.BOF) =sq. HD + sq. HG.

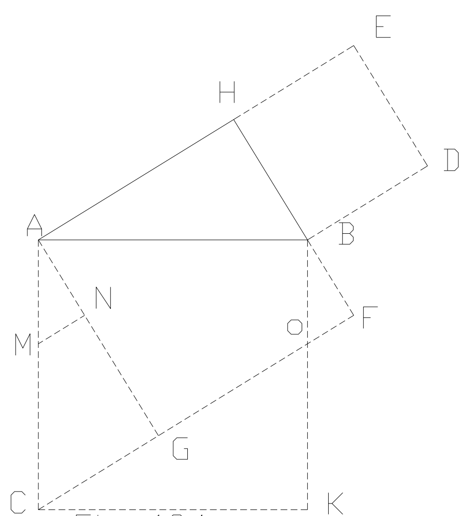


Fig. 194

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. IV, 1897, p. 268, proof LXI.

b. In fig. 194, omit MN and draw KR perp. to OC; then take KS = BL and draw ST perp. to OC. Then the fig. is that of Richard A.Bell, of Cleveland, O., devised July 1, 1918, and given to me Feb. 28, 1938, along with 40 other proofs through dissection, and all derivation of proofs by Mr. Bell (who knows practically nothing as to Eculidian Geometry) are found therein and credited to him, on March 2, 1938. He made no use of equivalency.

NINETY-FOUR

In fig. 195, draw DL par. to AB, through G draw PQ par to CK, take GN = BH, draw ON par. AH and LM perp. to AB.

Sq. AK = quad, AGRM common to sq's AK and AF + (tri. ANO = tri. BRF) + (quad. OPGN = quad. LMBS) + (rect. PK = paral. ABDL = sq. BE) + (tri. GRQ = tri. AML) = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Devised by the author, July20, 1900.

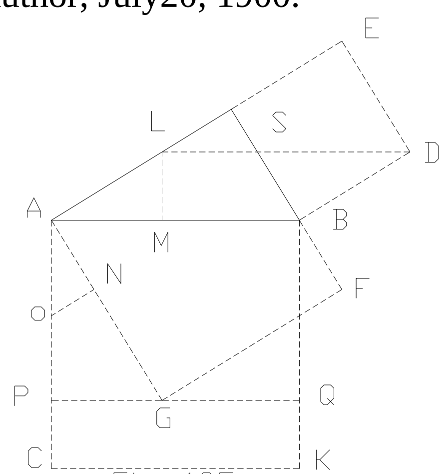


Fig. 195

NINETY- FIVE

In fig. 196, through G and D draw MN and DL each par. to AB, and draw GB.

Sq. AK = rect. MK + rect, MB = paral. AD + 2 tri. BAG = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. IV, 1897, p. 268, proof LXII.

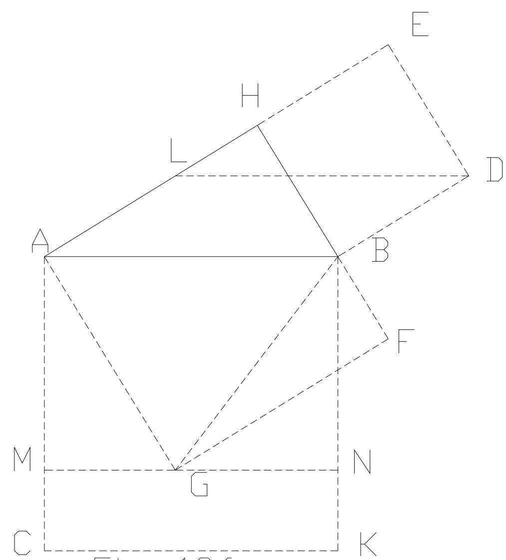


Fig. 196

NINETY-SIX

In fig. 197, extend FG to G, draw EB, and through C draw HN, and draw DL par. to AB.

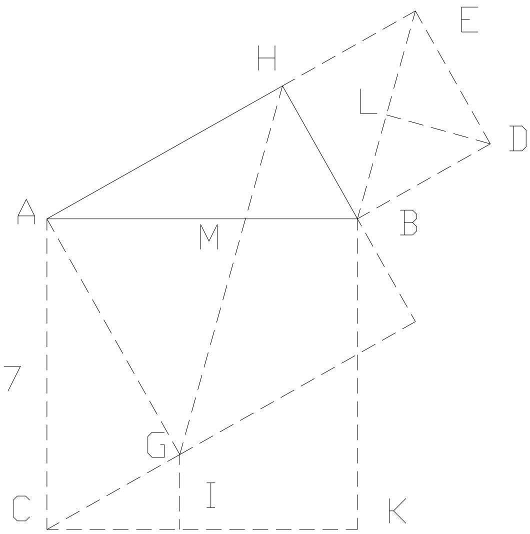


Fig. 197

Sq. AK = 2 [quad. ACMN = (tri. CNG = tri. DBL) + tri. AGM common to sq. AK and AF + (tri. ACG = tri. AMH + tri.ELD)] = 2 tri. AGH + 2 tri. BDE = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. VI, 1897, p. 268 proof LXIII.

NINETY-SEVEN

In fig. 198, extend FG to C, draw HL par. to AC, and draw AD and HK. Sq. AK = rect. BL + rect. AL = (2tri. KBH = 2 tri. ABD + paral. ACMH) = sq. BE+ sq. AF.

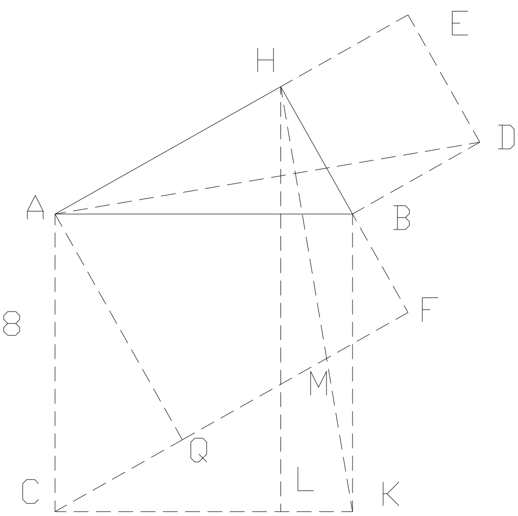


Fig.198

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 11, II; Am. Math. Mo., V. IV, 1897, p. 267, proof LVIII; Fourrey, p. 70, fig. b; Dr. Leitzmaan's work (1920), p. 30, fig. 31.

NINETY-EIGHT

In fig. 199, through G draw MN par. to AB, draw HL perp. to CK, and draw AD, HK and BG.

Sq. AK = rect. MK + rect. AN = (rect. BL = 2 tri. KBH = 2 tri. ABD) + 2 tri. AGB = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. IV, 1897, p. 268, proof LXI.

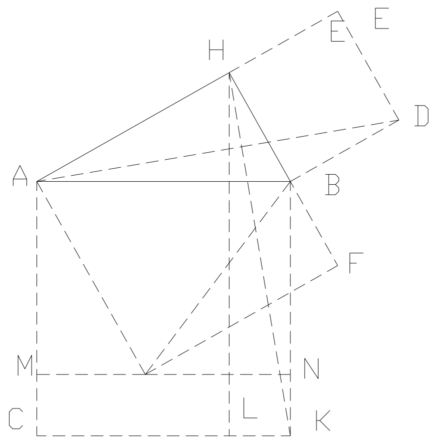


Fig. 199

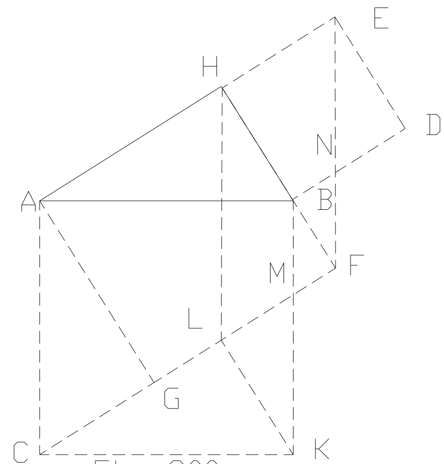


Fig. 200

NINETY-NINE

In fig. 200, extend FG to C, draw HL par. to BK and draw EF and LK. Sq. AK = quad. AGMB common to sq's AK and AF + (tri. ACG= tri. ABH) + (tri. CLK = trap. EHBN + tri. BMF) + (tri. KML = tri. END) =sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. IV. 1897, p. 268, proof LXIV.

ONE-HUNDRED

In fig. 201, draw FL par. to AB, extend FG to C, and draw EB and FK. Sq. AK = (rect. LK =2tri. CKF = 2 tri. ABE = 2tri. ABH + tri.HBE = tri. ABH+ tri. FMG + sq. HD) + (rect. AN = paral. MB).

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. V. IV, 1897,p. 269, proof LXVII.

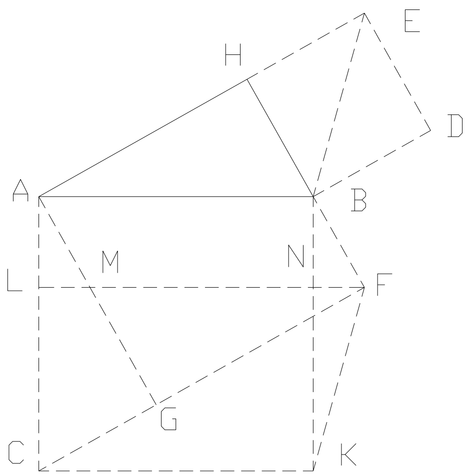


Fig. 201

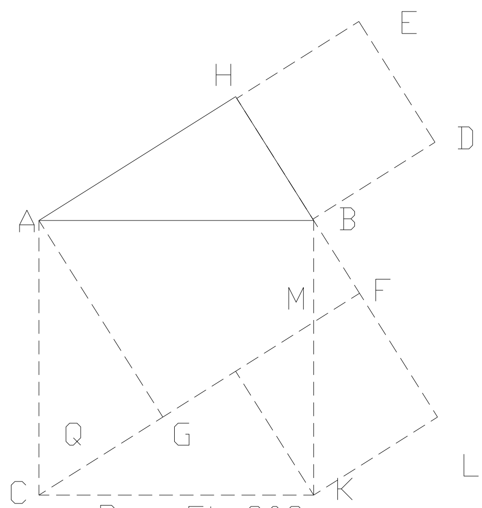


Fig.202

ONE- HUNDRED-ONE

In fig.202,extend FG to C, HB to L, draw KL par. to AH, and take NO = BH and draw OP and NK par. to BH.

Sq. AK = quad, AGMB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. ACG = tri. ABH) + (tri. CPO = tri. BMF) + (trap. PKNO + tri.KMN = sq. NL = sq. HD) = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 157, fig. (14).

ONE – HUNDRED- TWO

In fig. 203, extend HB to L making FL = BH, draw HM perp. to CK and draw HC and HK.

Sq. AK = rect. BM + rect. AM = 2 tri. KHB + 2tri. HAC = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

See Edward's Geom., 1895, p. 161, fig. (37).

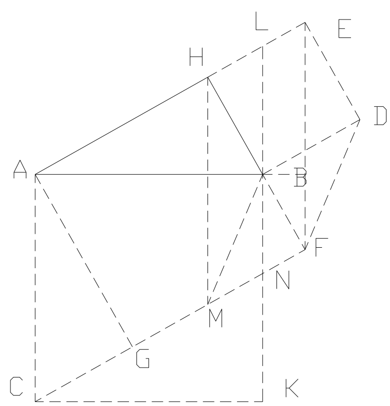


Fig. 204

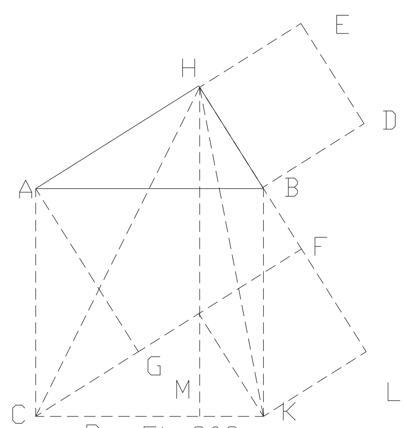


Fig.203

ONE-HUNDRED-THREE

Draw HM, LB and EF par. to BK, Join CG, MB and FD.

Sq. AK = paral. ACNL = paral. HN + paral. HC = (2tri. BHM = 2 tri. DEF = sq. HD) + sq. HG = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. V. IV, 1897,p. 269, proof IXIX.

ONE-HUNDRED-FOUR

In fig. 205, extend FG to C, draw KN par. to BH, take NM= BH, draw ML par. to HB, and draw MK, KF and BE.

Sq. AK = quad, AGBO common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CLM = tri. BOF) + [(tri. LKM = tri. OKF) + tri. KON = tri. BEH] + (tri.MKN= tri. EBD) =(tri. BEH + tri. EBD) + (quad. AGOB + tri. BOF +tri. ABC) = sq. HD +sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. V. IV, 1897,p. 269, proof LXVIII.

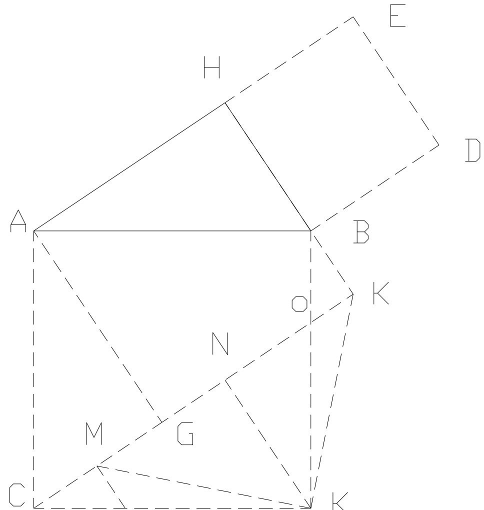


Fig. 205

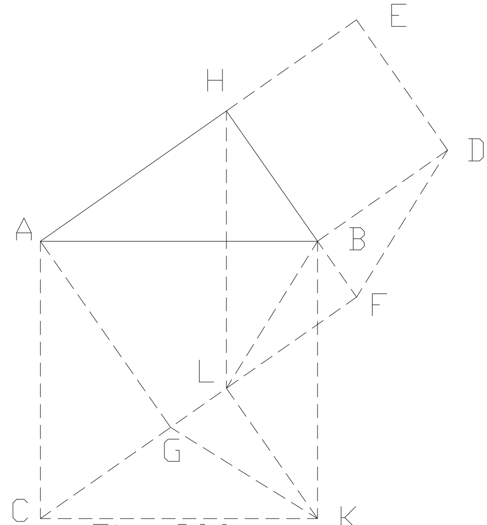


Fig. 206

ONE-HUNDRED-FIVE

In fig. 206, extend FG to H, draw HL par. to AC, KL par. to HB, and draw KG, LB, FD and EF.

Sq. AK = quad. AGLB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CKG = tri. EFD = ½ sq. HD) + (tri. GKL = BLF) + (tri. BLK = ½ paral. HK = ½ sq. HD) = ($^{1}/_{2}$ sq. HD + $^{1}/_{2}$sq. HD) + (quad. AGLB + tri. ABH + tri. BLF) = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. V. IV, 1897,p. 268, proof LXV.

ONE-HUNDRED-SIX

In fig. 207, extend FG to C and N, making FN = BD, KB to O, (K being the vertex opp. A in the sq. CB) draw FD, FE and FB, and draw HL par. to AC.

Sq. AK = paral. ACMO = paral. HM + paral. HC = [(paral. DHLF = rect. EF) – paral. EOMF = 2tri. EBF = 2tri. DBF = rect. DF) + sq. HD] = sq. HD + sq. AF.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. See Am. Math. Mo. V. IV, 1897,p. 268, proof LXVI.

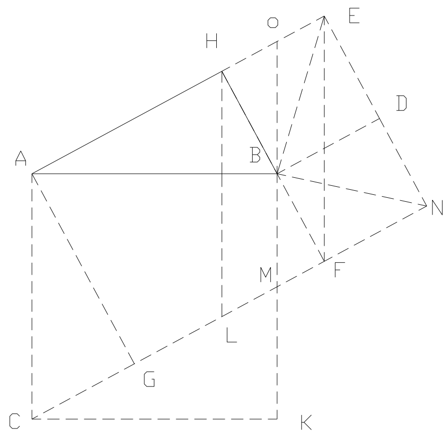


Fig. 207

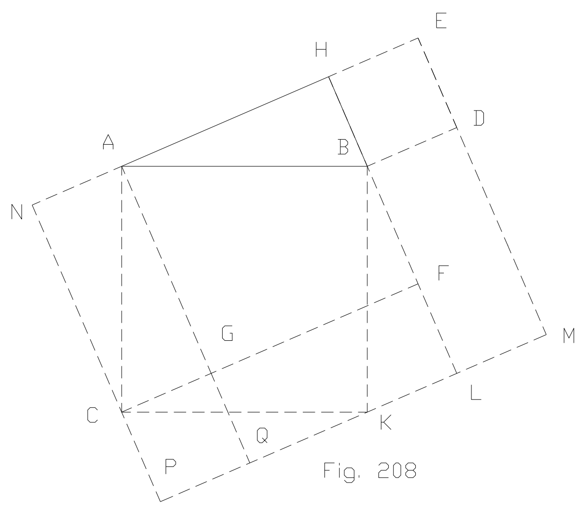


Fig. 208

ONE –HUNDRED-SEVEN

In fig. 208, through C and K draw NP and PM par. respectively to BH and AH, and extend ED to M, HF to L, AG to Q, AH to N and FG to C.

Sq. AK, + rect. HM + 4 tri. ABH = rect. NM = sq. HD + sq. HG + (rect.   = rect. HM) + (rect. ML = 2tri. ABH) + (rect.   = 2tri.ABH).

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. Credited by Jon. Hoffmann, in " Der Pythagoraische Lehrsatz," 1821, to Henry Boad of London; see Juty Wipper, 1880, p. 19, fig. 15.

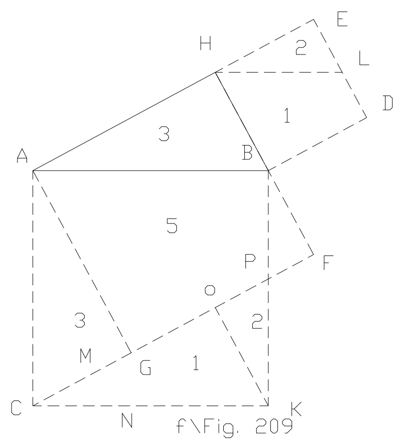
ONE-HUNDRED-EIGHT

By dissection. Draw HL par. to AB, CF par. to AH and KO par. to BH. Number parts as in figure.

Whence: sq. AK = parts [(1+ 2) = (1+2) in sq. HD)] + parts [(3 + 4 + 5) = (3 + 4 + 5 in sq. HG)] =sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Devised by the author to show a proof of type – C figure, by dissection, Dec,1933.



f\Fig. 209

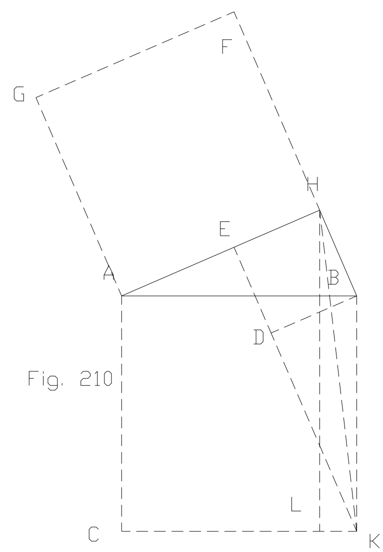


Fig. 210

ONE – HUNDRED-NINE

In fig. 210, extend ED to K, draw HL perp. to CK and draw HK.

Sq. AK = rect. BL + rect. AL = (2tri. BHK = sq. HD) + (sq.HE by Euclid’s proof).

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 11, fig. 3; Versluys, p. 12, fig. 4, given by Hoffmann.

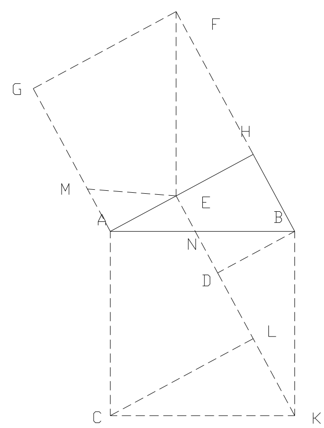
ONE –HUNDERD –TEN

In fig. 211, extend ED to K , draw CL par. to AH, EM par. to AB and draw FE.

Sq. AK = (quad. ACLN = quad. EFGM) + (tri. CLK + tri. ABH = trap. AHEN + tri. EMA) + (tri. KBD = tri. EFH) + tri. BND common to sq’s AK and HD = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Edwards’ Geom., 1895, p. 155, fig. (2)



Fig, 211

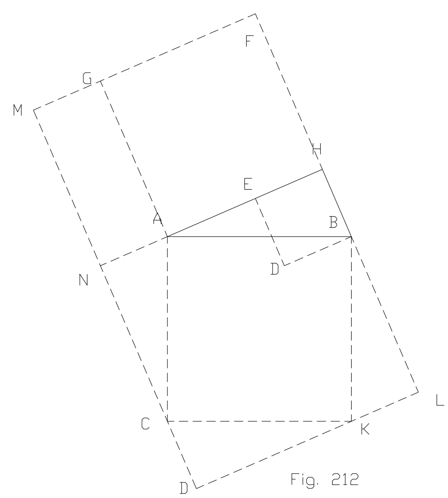


Fig. 212

ONE – HUNDRED- ELEVEN

In fig. 212, extend FB and FG to L and M making BL = AH and GM = BH, complete the rectangle FO and extend AH to N, and ED to K.

Sq. AK + rect. MH + 4 tri. ABH = rect. FO = sq. HD + sq. HG +(rect. NK = rect. HM) + (rect. MA = 2tri. ABH) + (rect. DL= 2tri. ABH); collecting we have sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Credited to Henry Boad by Joh. Hoffmann, 1821, see Jury Wipper, 1880, p. 20, fig. 14.

ONE-HUNDRED- TWELVE

In fig. 213, extend ED to K, draw HL par. to AC, and draw CM.

Sq. AK = rect. BL + rect. AL = paral. HK + paral. HC = sq. HD + sq. HG.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. Devised by the author, Aug. 1, 1900.

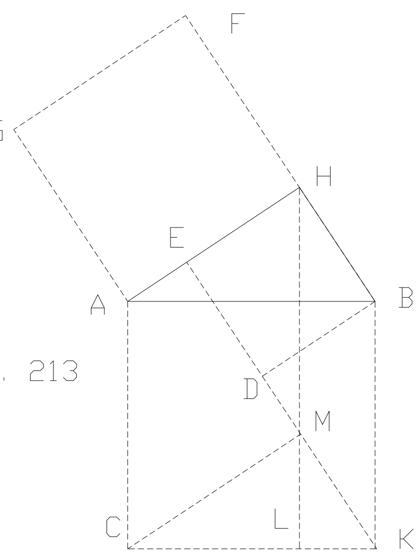
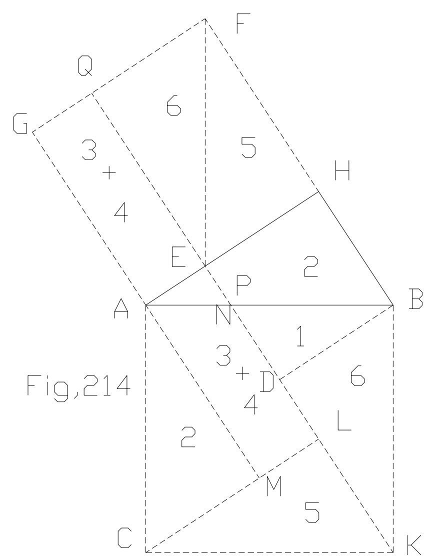


Fig. 213



Fig,214

ONE – HUNDRED- THIRTEEN

In fig. 214, extend ED to K and Q , draw CL perp. to EK, extend GA to M, take MN = BH, draw NO par. to AH, and draw FE.

Sq. AK = (tri. CKL = tri.FEH) + (tri. KBD = tri.EFQ) + (trap. AMLP + tri. ANO = rect. GE) + tri. BPD common to sq's AK and BE + (trap. CMNO = trap. BHEP) = sq. HD + sq. HG.

[c] sq. upon AB = sq. upon BH + sq. upon AH. $\therefore h^2 = a^2 + b^2$.

a. Origional with the author, Aug. 1, 1900.

ONE- HUNDRED- FOURTEEN

Employ fig. 214, numbering the parts as there numbered, then at once:sq. AK =sum of 6 parts[(1+2=sq.HD) + (3+4+5+6 =sq.HG) =sq. HD +sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Formulated by the author, Dec. 19, 1933,

ONE-HUNDRED- FIFTEEN

In fig. 215, extend HA to O making OA + HB, ED to K, and join OC, extend BD to P and join EP. Number parts 1to 11 as in figure. Now: sq. AK = parts 1 + 2+ 3+ 4+ 5; trapezoid EPCK = [(EK + PC)/2] x PD = KD xPD = AH x AG = sq. HG = parts 7+4+ 10 +11 +1. Sq. HD parts 3+ 6.

∴ sq.AK = 1+ 2+3+ 4+5= 1+(2 = 6 + 7+8) + 3 + 4 +5= 1 + (6+ 3) + 7 + 8 +4+5 = 1+ (6+3) + (7+8 = 11) + 4 + 5 = 1 +(6 + 3) + 11 + 4 + 5= 1 + (6+3) + 11 + 4 + (5= 2 -4, since 5+4 +3 = 2 +3) = 1 + (6+ 3)+ 11 +4+ 2- 4 = 1 + (6 + 3) + 11 + 4 + (2= 7 +4 +10) – 4 = 1 +(6 +3) + 11 + 4 + 7 + 10 + 11 +1) + (6+ 3) =sq. HG+ sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

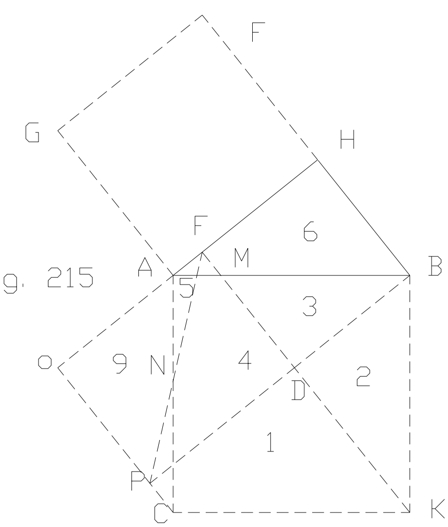


Fig. 215

a. This figure and proof formulated by Joseph Zelson, see proof Sixty Nine, a fig. 169, It come to me on May 5, 1939.

b. In this proof, as in all proofs received I omitted the column of “reasons” for steps of the demonstration, and reduced the argumentation form many (in Zelson’s proof over thirty) steps to a compact sequence of essentials, thus leaving, in all cases, the reader to recast the essentials in the form as given in our accepted modern texts.

By so doing a saving of as much as 60% of page space results—also hours of time for thinker and printer.

ONE- HYNDRED-SIXTEEN

In fig. 216, through D draw LN par to AB extend ED to K, and draw HL and CD.

Sq.AH = (rect. AN = paral. AD = sq. DH) + (rect. MK = 2tri. DCK = sq. GH).

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Contrived by the author, Aug. 1, 1900.

b. As in types A, B and C, many other proofs may be derived from the D type of figure.

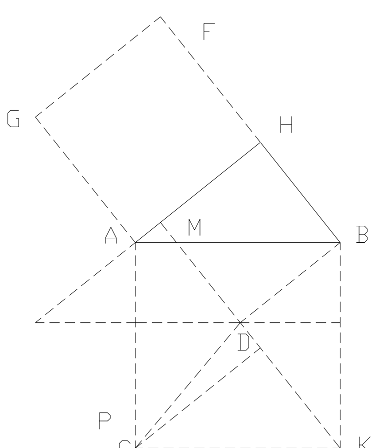


Fig.216

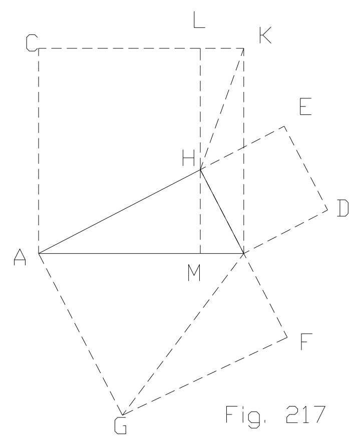


Fig. 217

E

This type includes all proofs derived from the figure in which the squares constructed upon the hypotenuse and the longer leg overlap the given triangle.

ONE- HUNDRED-SEVENTEEN

In fig. 217, through H draw LM par. to KB, and draw GB, HK and HC.

Sq. AK =rect. LB + rect. LA = 2(tri. HBK = sq. HD) + (2tri. CAH = 2 tri. BAG = sq. AF). ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 14, VI; Edwards' Geom.,1895, p. 162, fig. (38); Am. Math. Mo., Versluys, p. 14, fig. 9, one of Hofmann's collection, 1818, Fourrey, p. 71, fig. g; Math. Mo., 1859, Vol. II, No. 3, Dem. 13, fig. 5.

<u>ONE-HUNDERD-EIGHTEEN</u>

In fig. 218, extend DE to K and draw DL and CM par. respectively to AB and BH.

Sq. AK = (rect. LB = paral. AD sq. BE) + (rect. LK = paral. CD = trap. CMEK = trap. AGFB) + (tri. KDN = tri. CLM) =sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., V. V, 1898, p. 74 LXXIX.

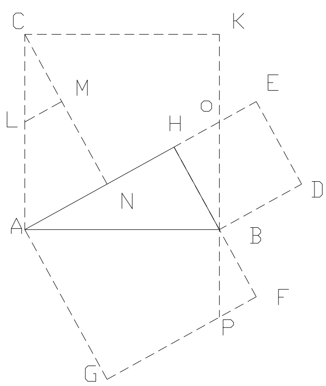


Fig.219

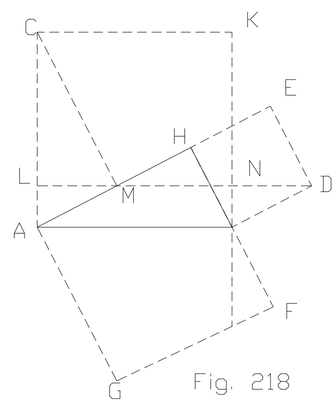


Fig. 218

<u>ONE-HUNDRED-NINTEEN</u>

In fig. 219 extend KB to P draw CN par. to HB, take NM= HB, and draw ML par. to AH.

Sq. AK = (quad. NOKC = quad. GPBA) + (tri. CLM = tri. BPF) + (trap. ANML = trap. BDEO) + tri. ABH common to sq's AK and AF + tri. BHO common to sq's AK and HD = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Am. Math. Mo., Vol. V, 1898, p. 74, proof LXXVII; School Visitor, Vol, III, p. 208, No. 410.

ONE-HUNDRED- TWENTY

In fig. 220, extend DE to K, GA to L, Draw CL par. to AH and draw LD and HG.

Sq. AK = 2[trap. ABNM = tri. AOH common to sq's AK and AF + (tri. AHM = tri. AGO) + tri. HBN common to sq's AK and HD + (tri. BHO = tri. BDN)] = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Am, Math. Mo., Vol. V, 1898, p. 74, proof LXXVI.

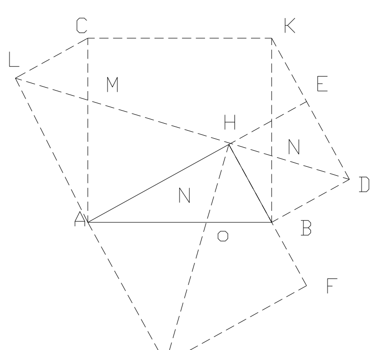


Fig.220

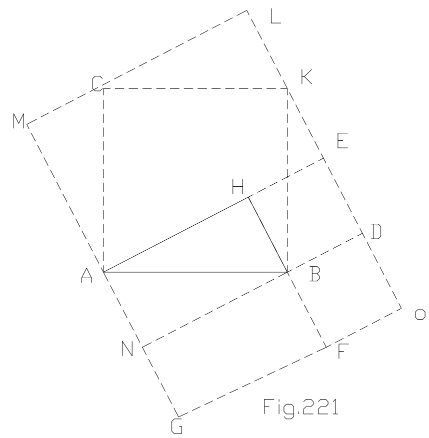


Fig.221

ONE – HUNDRED –TWENTY – ONE

Extend GF and ED to O, and complete the rect. MO, and extend DB to N.

Sq. AK rect. MO – (4tri. ABH + rect. NO) = {(rect. AL + rect. AO) – (4 tri. ABH + rect. NO)} = 2 (rect. AO = rect. AD + rect. NO) = (2 rect. AD +2 rect. NO – rect.NO – 4 ABH) - (2 rect. AD + rect. NO – 4 tri. ABH) = (2rect.AB + 2rect. HD + rect. NF + rect. BO – 4 tri. ABH) = [rect. AB + (rect. AB + rect. NF) + rect. HD + (rect. HD + rect. BO) – 4 tri. ABH] = 2 tri. ABH + sq. HG + sq. HD +2tri. ABH – 4 tri. ABH) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. This formula and conversion is that of the author, Dec. 22, 1933, but the figure is an given in Am. Math. Mo., Vol. V, 1898, p. 74, where see another somewhat different proof, No. LXXVII. But same figure furnishes.

ONE- HUNDRED-TWENTY – TWO

In fig. 221, extend GF and ED to O and complete the rect. MO, Extend DB to N.

Sq. AK = rect. NO + 4 tri. ABH = rect. MO = sq. HD + sq. AF + rect. BO + [rect. AL = (rect. HN = 2 tri. ABH) + sq. HG = 2tri. ABH + rect. NF)], which coll'd gives sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Credited to Henry Boad by Joh. Hoffmann, in "Der Pythagoraishe Lehrsatz," 1821; see Jury Wipper, 1880, p.21, fig. 15.

ONE- HUNDRED-TWENTY –THREE

In fig. 222, draw CL and KL par. respectively to AH and BH and draw through H, LP.

Sq. AK = hexagon AHBKLC = paral. LB = paral. LA = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Devised by the author March 12, 1926.

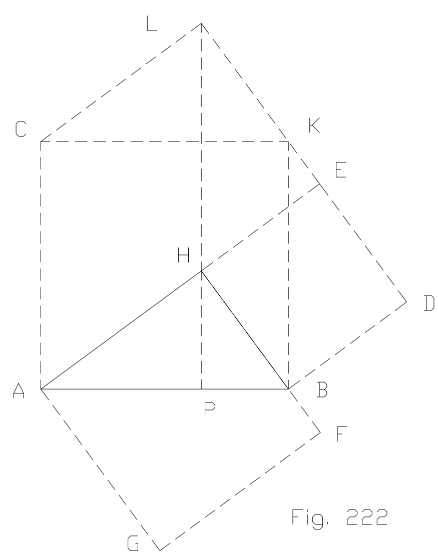


Fig. 222

ONE-HUNDRED- TWENTY-FOUR

Rect. LM = [sq. AK = parts 2 common to sq. AK and sq. HD + 3+ 4+5 common to sq. AK and sq. HG. + parts 6+ (7+8 = sq. HG) + 9 + 1 + 10 + 11 = (sq. AK = sq. HG + parts {(6 = 2) +1 = sq. HD} + parts (9 + 10 + 11= 2 tri. ABN + tri. KPE] = [(sq. AK = sq. HD + sq. HG) +(2 tri. ABH + tri. KPE)], or rect. LM – (2 tri. ABH + tri KPE) = [sq. AK= sq. HD + sq. AH].

∴ AK = sq. HD + sq. HA. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Original with the author, June 17, 1939.

b. See Am. Math. Mo. , Vol. V, 1898, p. 74, proof LXXVIII for another proof, which is: (as per essentials):

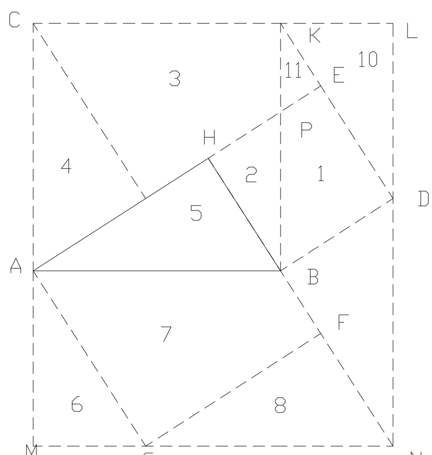


Fig. 223

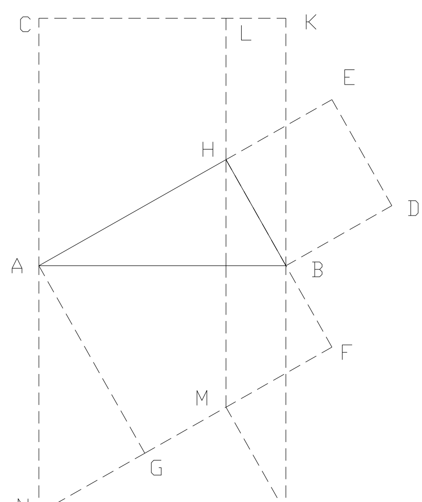


FIG.224

ONE – HUNDRED- TWENTY – FIVE

In fig. 223, extend CA, HB, DE and CK to M, N, K and L reapectively, and draw MN, LN and CO respectively par. to AB, KB and HB.

Sq. AK +2 tri. AGM + 3 tri. GNP + trap. AGFB = rect. CN = sq. HD + sq. HG + 2tri. AGM + 3 tri. GNF + trap. COEK, which coll'd gives sq. AK = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

ONE-HUNDRED- TWENTY-FIVE

In fig. 224 extend KB and CA respectifely to O and N, through H draw LM par. to KB, and draw GN and MO respectively par. to AH and HB.

Sq. AK = rect. LB + rect. LA paral. BHMO = paral. HANM = sq. HD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Original with the author, August 1, 1900.

b. Many other proofs are derivable from this type of figure.

c. An algebraic proof is easily obtained form fig. 224.

F

This type includes all proofs derived from the figure in which the squares constructed upon the hypotenuse and the shorter leg overlap the given triangle.

ONE- HUNDRED TWENTY- SEVEN

In the fig. 225, draw KM par. to AH.

Sq. AK= (tri. BKM = tri. ACG) + (tri. KLM = tri. BND) + quad. AHLC common to sq's AK, and AK + (tri. ANE = tri. CLF) + trap. NBHE common to sq's AK and EB = sq. HD + sq. HG.

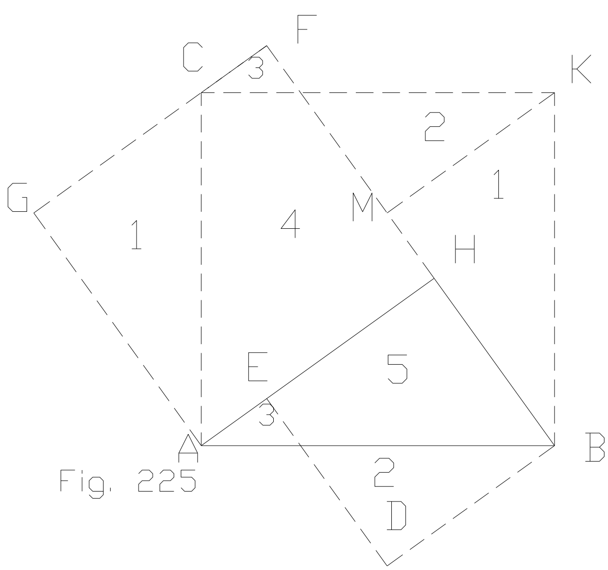


Fig. 225

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. The Journal of Education, V. XXVIII, 1888, p. 17, 24th proof, credits this proof to J.M. Me-Cready, of Black Hawk, Wis.; see Edwards' Geom., 1895, p. 89, art. 73; Heath's Math. Monographs, No. 2, 1900, p. 32, proof XIX; Scientific Review, Feb. 16, 1889, p. 31, fig. 30, R.A. Bell, July 1, 1938, one of his 40 proofs.

b. By numbering the dissected parts, an obvious proof is seen.

ONE – HUNDRED TWENTY- EIGHT

In fig. 226, extend AH to N making HN = HE through H draw LM par. to BK and draw BN, HK and HC.

Sq. AK =rect. LB = rect. LA = (2 tri. HBK = 2 tri. HBN =sq. HD) + (2 tri. CAH = 2 tri. AHC = sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Original with the author, August 1, 1900.

b. An algebraic proof may be resolved from this figure.

c. Other geometric proofs are easily derived from this form of figure.

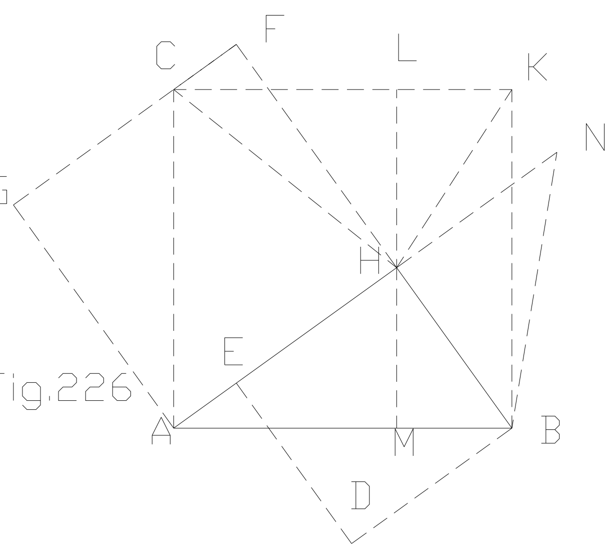


Fig.226

ONE- HUNDRED – TWENTY- NINE

In fig. 227, draw LH perp. to AB and extend it to meet ED produced and draw MB, HK and HC.

Sq. AK = rect. LB = rect. LA = (paral. HMBK = 2 tri. MBH = sq. BE) + (2tri. CAH = 2tri. AHC = sq. AF) = sq. BE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a.See Jutry Wipper, 1880, p. 14, fig. 7, Versluys, p. 14, fig. 10;Fourrey, p. 71 fig. f.V. V, 1898 p. 73, proof LXX; A. R. Bell

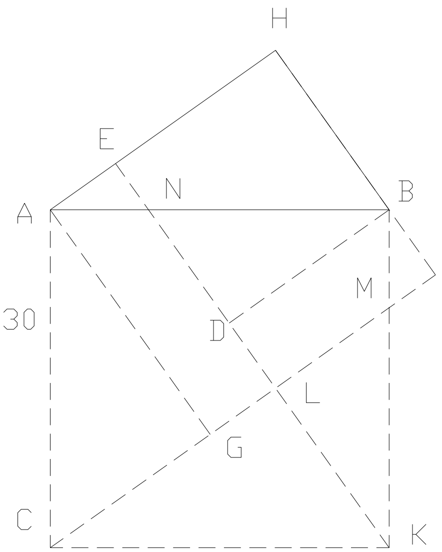


Fig.230

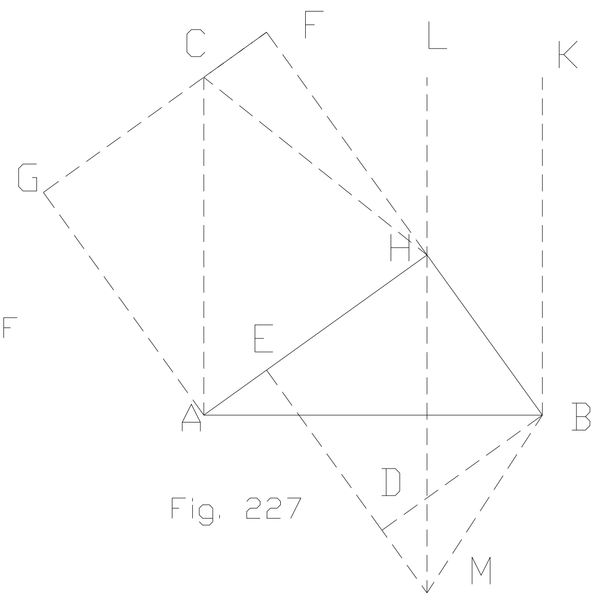


Fig. 227

Feb. 24, 1938.

b.In Sci. Am. Sup., V.70, p. 359, Dec. 3, 1910 is a proof by A.R. Colburn, by use of above figure, but the argument is not that given above.

ONE-HUNDRED- THIRTY – TWO

In fig. 230, extend FG to C and ED to K.

Sq. AK = (tri. ACG = tri. ABH of sq. HG) + (tri. CKL = trap. NBHE + tri BMF) + (tri. KBD = tri. BDN of sq. HD + trap. LMBD common to sq's AK and HG) = sq. HD = sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 159, fig. (24); Sci. Am. Sup., V. 70, p. 382, Dec. 10, 1910, for a proof by A. R. Colburn on same form of fugure.

ONE- HUNDRED – THIRTY THREE

The construction is obvious. Also that m+ n = o + p; also that tri. ABH and tri. ACG are congruent. Then sq. AK = 4o + 4p +q = 2 (o+p) + 2 (o+ p) + q = 2(m + n) + 2 (o +p) + q = 2 (m + o) + (m + 2n + o + 2p + q) = sq. HD + sq. AH.

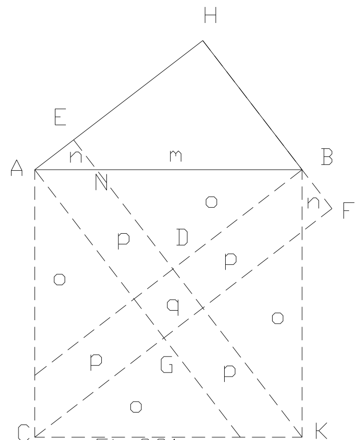


Fig.231

. ∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 48, fig. 49, where credited to R. Joan, Neponucen Reichenberger, Philosophia et Mathesis Universa, Regensburg, 1774.

b. By using congruent tri's and trap's the algevraic appearance will vanish.

ONE – HUNDERED – THIRTY- FORE

Having the construction, and the parts symbolized, it is evident that: sq. AK = 30 + p + r + s = (30 + p) + (o + p = s) + r = 2 (o + p) +2o + r = (m + o) + m +2n +o + r) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Versluys, p. 48, fig. 50; Fourrey, p. 86.

b. By expressing the dimensions of m,n,o, p, r and s in terms of a, b, and h an algebraic proof results.

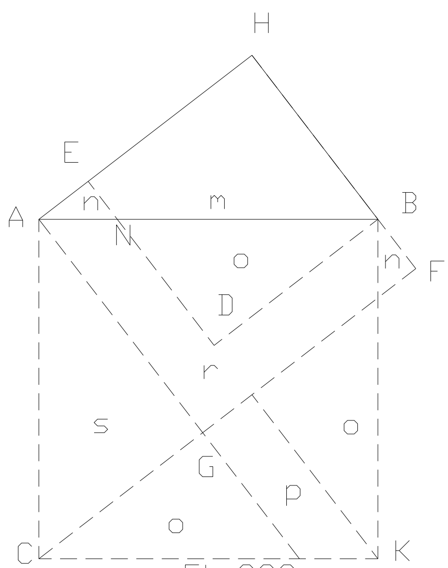


Fig.232

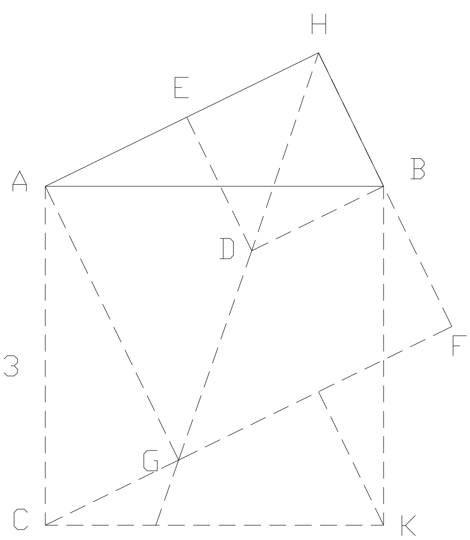


Fig.233

ONE- HUNDRED-THIRTY- FIVE

Complete the three sq's AK, HG and HD, draw CG, KN, and HL through G. Then

Sq. AK = 2[trap. ACLM = tri. GMA common to sq's AK and AF + (tri. ACG = tri. AMH of sq. AF + tri. HMB of sq. HD) + (tri. CLG = tri. BMD of sq. HD)] = sq. HD + sq. HG. ∴ $h^2 = a^2 + b^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH.

a. See Am. Math. Mo. , V. V, 1898, p. 73, proof LXXII.

ONE – HUNDRED- THIRTY –SIX

Draw CL and LK par. respectively to HB and HA, and draw HL.

Sq. AK = hexagon ACLKBH – 2 tri. ABH = 2 quad. ACLH - 2 tri. ABH = 2tri.ACG + (2 tri. CLG =sq. HD) + (2 tri. AGH = sq. HG) – 2 tri. ABH = sq. HD + sq. HG + (2 tri ACG = 2 tri. ABH – 2 tri ABH = sq. HD – sq. HG.

∴ AK = sq. HD + sq. HA. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$. Q.E.D.

a. Original by author Oct. 25, 1933.

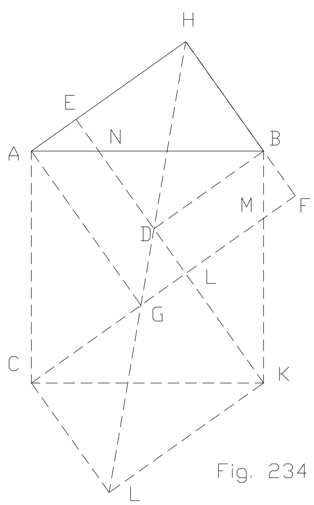


Fig. 234

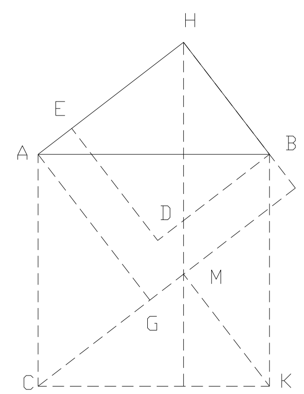


Fig.235

ONE- HUNDRED –THIRTY – SEVEN

In fig. 235, extend FG to C, ED to K and draw HL par. to BK.

Sq. AK = rect. BL + rect. AL =(paral. MKBH = sq. HD) + (paral. CMHA = sq. HG) = sq. HD + sq. HG.

∴ AK = sq. HD + sq. HA. ∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. Journal of Education, V. XXVII, 1888, p. 327, fifteenth proof Edwards' Geom., 1895, p. 158, fig. (22) ; Am. Math. Mo., V.V, 1898, p. 73, proof LXXI; Heath's Maht. Monographs, No. 2, p. 28, proof XIV; Versluys, p. 13, fig. 8--- also p. 20, fig. 17, for same figure, but a somewhat different proof, a proof credited to Jacob Gelder, 1810; Math. Mo., 1859, Vol. II, No. 2, Dem. 11; Fourrey, p. 70, fig. d.

b. An algebraic proof is easily devided from this figure.

ONE –HUNDRED-THIRTY- EIGHT

Draw HL perp. to CK and extend ED and FG to K and C resp'ly

Sq. AK = rect. BL + rect. AL = (tri. MLK = quad. RDSP + tri. PSB) + [tri. BDK – (tri. SDM = tri. ONR) = (tri. BHA – tri. REA) = quad.RBHE] + [tri. CKM = tri. ABH) = (tri. CGA = tri. MFA) + quad. GMPA] = tri. RBD + quad. RBHE + tri. APH + tri. MEH + quad. GMPA = sq.HD + sq. HG.

∴ AK = sq. HD + sq. HA. ∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 46, fig's 47 and 48, as given by M. Rogot, and made known by E. Fourrey in his "Curiosities of Geometry," on p. 90.

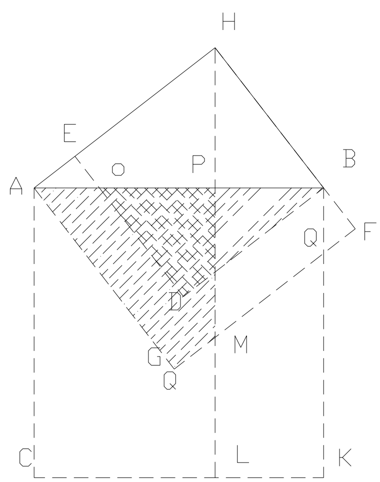


Fig. 236

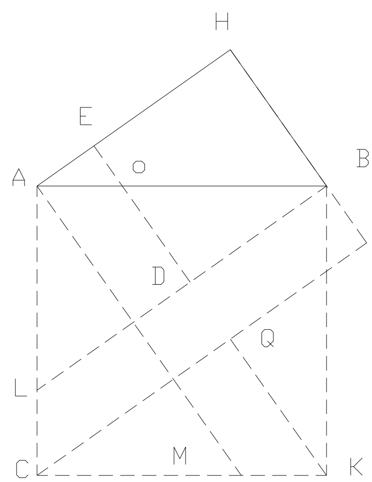


Fig. 237

ONE – HUNDRED- THIRTY-NINE

In fig. 237, extend AG, ED, BD and FG to M,K,L and C respectively.

Sq. AK = 4 tri. ALP + 4 quad. LCGP + sq. PQ + tri. AOE – (tri. BNE = tri. AOE) = (2 tri. ALP + 3 quad. LCGP + sq. PQ + tri. AOE = sq. HG) + (2tri. ALP + quad. LCPG – tri. AOE = sq. HD) = sq. HD + sq. HG. ∴ AK = sq. HD + sq. HA.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 29, fig. 26, as given by Reichenberger, in Philosophisa et Mathesis Universa, etc., " Ratisbonae, 1774; Versluys, p. 48, fig. 49; Fourrey, p. 36.

b. Mr. Richard A. Bell, of Cleaveland , O., submitted, Feb. 28, 1938, 6 fig's and proofs of the type G, all found between Nov. 1920 and Feb. 28, 1938. Some of his figures are very simple.

ONE-HUNDRED- FORTY

In fig. 238, extend ED and FG to K and C respectively, draw HL perp. to CK and draw HC and HK.

Sq. AK = rect. BL + rect. AL = (paral. MKBH = 2 tri. KBH = sq. HD) + (paral. CMHA = 2 tri. CHA = sq. HG) = sq. HD + sq. HG.

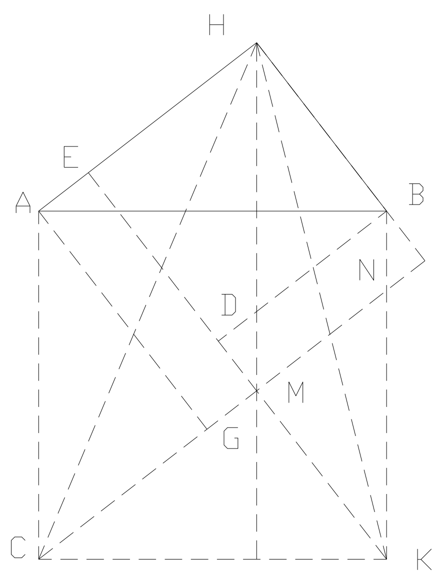
∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Jury wiper, 1880, p. 12, fig. 4.

b. This proof is only a variation of the one preceding.

c. From this figure an algebraic proof is obtainable.

Fig. 238



ONE-HUNDRED-FORTY-ONE

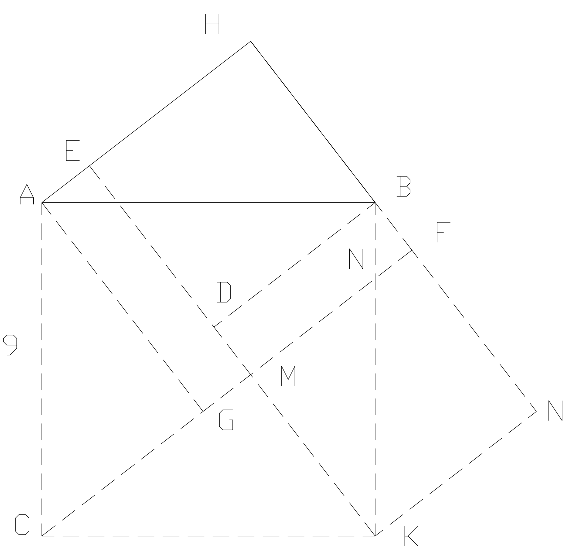
In fig. 239, extend FG to C, HF to L making FL = HB, and draw KL and KM respectively par. to AH and BH.

Sq. AK = [{(tri. CKM =tri. BKL) – tri. BNF = trap. OBHE} + (tri. KMN = tri BOD) = sq. HD] + {(tri. ACG = tri. ABH) +(tri. BOD + hexagon AGNBDO) = sq. HG} = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. As taken from "Philosophia et Mathesis Universa, etc.," Ratisbonae, 1774, by reichenberger; see Jury Wipper, 1880, p. 29, fig. 27.

Fig. 239



ONE – HUNDRED- FORTY – TWO

In fig. 240, extend HF and HA respectively to N and L, and complete the sq. HM, and extend ED to K and BG to C.

Sq. AK = sq. HM – 4 tri. ABH = (sq. FK = sq. HD) + sq. HG + (rect. LG =2 tri. ABH) + rect. OM = 2tri. ABH = sq. HD + sq. HG + 4 tri. ABH – 4 tri. ABH = sq. HD + sq. HC.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Similar to Henry Boad's proof, London, 1733; See Jury Wipper, 1880, p. 16, fig. 9; Am. Math. Mo., V. V, 1898, p. 74, proof LXXIV.

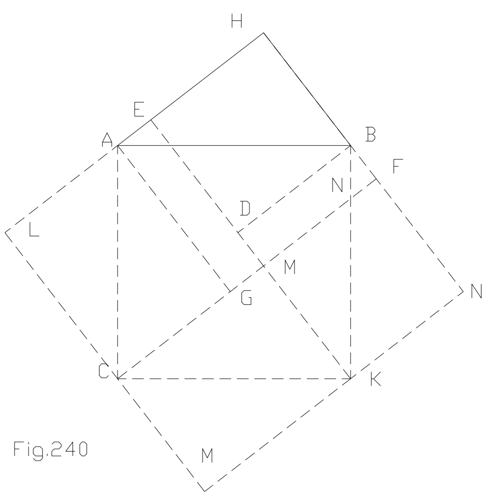


Fig.240

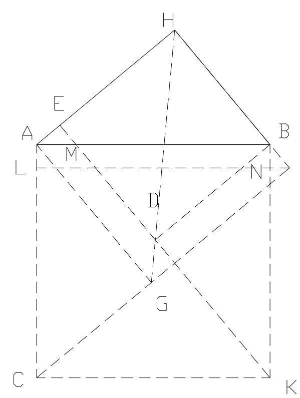


Fig.241

ONE –HUNDRED- FORTY- THREE

In fig. 241, extend FG and ED to CandK respectively, draw FL par. to AB, and draw HD and FK.

Sq. AK = (rect. AN = paral. MB) + (rect. LK = 2 tri. CKF = 2 tri. CKO + 2 tri. FOK = tri. FOK = tri. FMG + tri. ABH + 2tri. DBH) = sq. HD + sq. HG.
∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See Am. Math. Mo. Vol. V, 1898, p. 74, proof LXXIII.

ONE-HUNDRED- FORTY- FOUR

In fig. 242, produce FG to C through D and G draw LM and NO par. to AB, and draw AD and BG.

Sq. AK = rect. NK + rect. AO = (rect. AM = 2 tri. ADB = sq. HD) + (2 tri. GBA = sq. HG) = sq. HD + sq.HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. This is No. 15 of A. R. Colburn's 108 proofs; see his proof in Sci. Am. Sup., V. 70, p. 383, Dec. 10, 1910.

b. An algebraic proof from this figure is easily obtained.

$$2\text{tri. BAD} = hx = a^2 \text{ ----- } (1)$$

$$2\text{tri. BAG} = h(h - x) = b^2 \text{-----}(2)$$

(1) + (2) = (3) $h^2 = a^2 + b^2$.-----(E.S.L.)

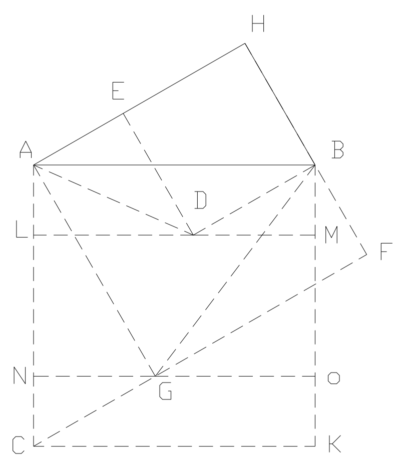


Fig.242

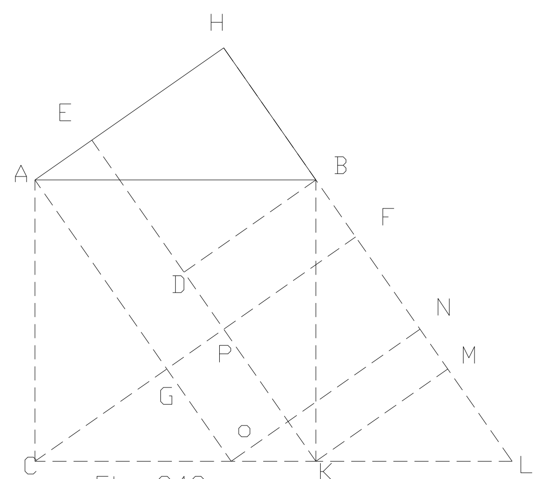


Fig. 243

ONE – HUNDRED – FORTY- FIVE

In fig. 243, produce HF and CK to L, ED to K, and AG to O, and draw KM and ON par. to AH.

Sq. AK = paral. AOLB = [trap. AGFB + tri. OLM = tri. ABH) = sq. HG] + {rect. GN = tri. CLF – (tri. COG= tri. KLM) – (tri. OLN = tri. CPK) = sq. FK = sq. HD} = sq.HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. This proof is due to Prin. Geo.M. Phillips, Ph.D., of the West Chester State Normal School, Pa. 1875; see Heath's Math. Monographs, No. 2, p. 36, proof XXV.

ONE –HUNDRED- FORTY –SIX

In fig. 244, ectend CK and HF to M, ED to K, and AG to O making GO = HB, draw NO par. to AH, and draw GN.

Sq. AK paral. ALMB = paral. GM + paral. AN = [(tri. NGO – tri. NPO = trap. RBHE) + (tri. KMN = tri. BRD)] = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devided by the author, March 14, 1926.

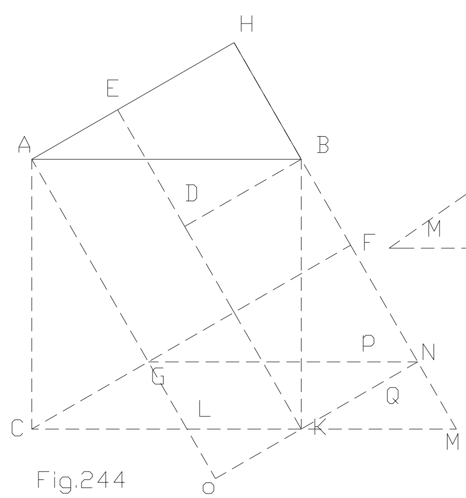


Fig.244

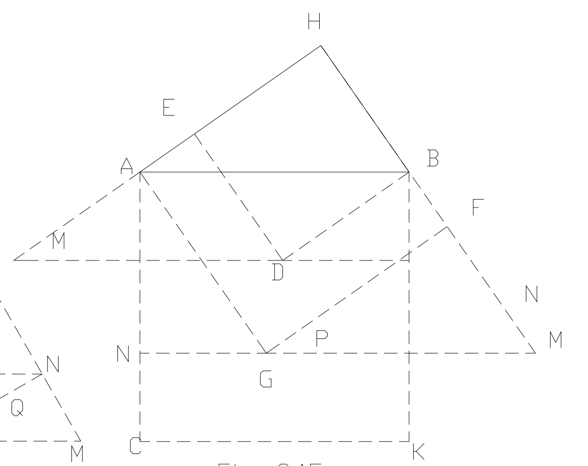


Fig. 245

ONE – HUNDRED- FORTY –SEVEN

Through D draw DR par. AB meeting HA at M, and through G draw NO par. to AB meeting HB at P, and draw HL perp. to AB.

Sq. AK = (rect. NK = rect. AR – paral. AMDB = sq. HB) + (rect. AO = paral. AGPB = sq. HG) =sq. HD = sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Versluvys, p. 28, fig. 25. By Werner.

ONE – HUNDRED – FORTY – EIGHT

Produce HA and HB to O and N resp'ly making AO = HB and BM = HA, and complete the sq. HL.

Sq. AK = sq. HL – (4 tri. ABH = 2rect. OG) = [(sq. GL = sq. HD) + sq. HG + 2 rect. OG) – 2 rect. OG= sq. HD + sq. HG. ∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Versluys, p. 52,fig. 54, as found in Hoffmann's list and in "Des Pythagoraische Lehrsatz." 1821.

ONE – HUNDRED – FORTY – NINE

Produce CK and HB to L, AG to M, and KO par. to AH.

Sq. AK = paral. AMLB = AGFB + rect. GN + (tri. MLN = tri. ABH) =sq. GH + (rect. GN = sq. PO = sq. HD) = sq. HG + sq. HD. ∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. By Dr. Geo. M. Philips, of West Chester, Pa., in 1875; Versluys, p. 58, fig. 62.

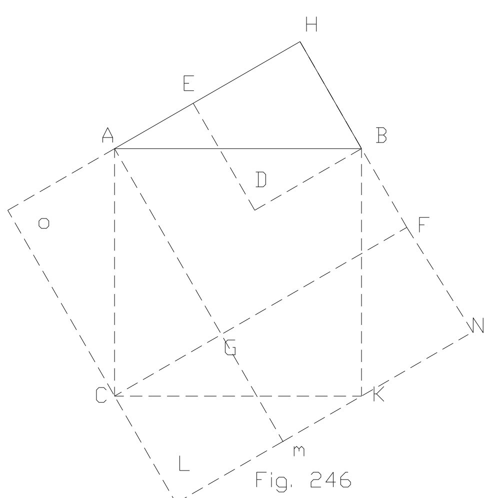


Fig. 246

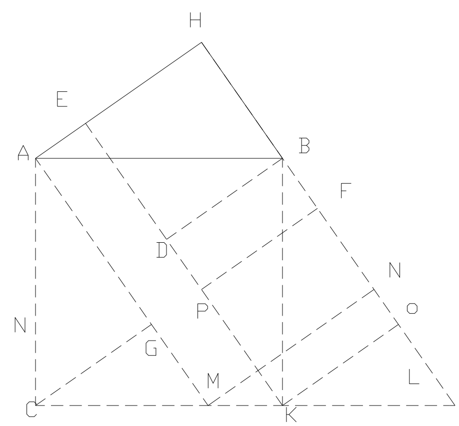


Fig. 247

H

This type includes all proofs devised from the figure in which the squares constructed upon the hypotenuse and the two legs overlap the given triangle.

ONE – HUNDRED-FIFTY

Draw through H, LN perp. to AB, and draw HK, HC, NB and NA.

Sq. AK = rect. LB + rect. LA = paral. KN + paral. CN = 2tri. KHB + 2 tri. NHA = sq. HD + sq. HG.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Math. Mo. , 1859, Vol. II, No. 2, Dem. 15, fig. 7.

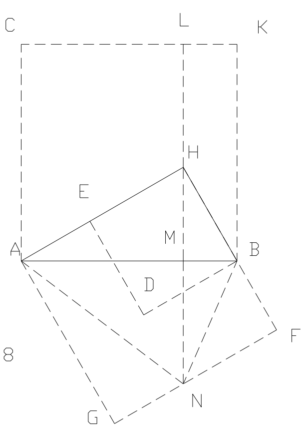


Fig. 248

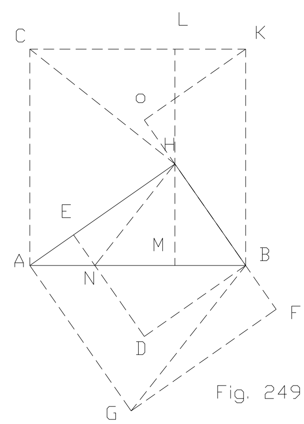


Fig. 249

ONE – HUNDRED- FIFTY – ONE

Through H draw LM perp. to AB. Extend FH to O making BO = HF, draw KO, CH, HN and BG.

Sq. AK = rect. LB + rect. LA = (2 tri. KHB = 2tri. BHA = sq. HD) + (2 tri. CAH = 2 tri. AGB = sq. AF) = sq. HD + sq. AF.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Original with the author Arterwards the first part of it was discovered to be the same as the solution in Am. Math. Mo., V. V, 1898, p. 78, proof LXXXI; also see Fourrey, p. 71, fig. h, in his “Curiosities.”

b. This figure gives readily an algebraic proof.

ONE –HUNDRED – FIFTY – TWO

In fig. 250, extend ED to O, draw AO, OB, HK and HC, and draw CM perp. to AH.

Sq. AK = rect. LB + rect. LA= (paral. HOBK = 2tri. OBH = sq. HD) + (paral. CAOH = 2 tri. OHA = sq. HD) =sq. HD = sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See Olney's Geom., 1872, Part III, p. 251, 6th method; Journal of Education, V. LXXX 1887, p. 21, fig. XIII; Hopkins' Geom., 1896, p. 91, fig. VI; Edw. Geom., 1895, p. 160, fig. (31); Am. Math. Mo., 1898, Vol, V, p. 74, proof LXXX; Heath's Math. Monographs, No. 1, 1900, p. 26, proof XI.

b. From this figure deduce an algebraic proof.

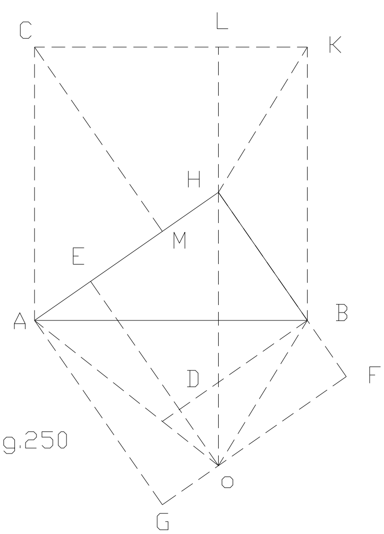


Fig.250

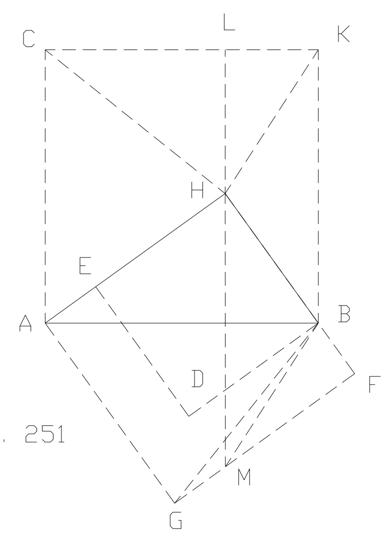


Fig. 251

ONE – HUNDRED- FIFTY- THREE

In fig. 251, draw LM perp. to AB through H, extend ED to M, and draw BG, BM, HK and HC.

Sq. AK = rect. LB + rect. LA + (paral. KHMB = 2tri. MBH = sq. HD) + (tri. AHC = 2tri. AGB = sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 15, fig. 8; Versluys, p. 15, fig. 11.

b. An algebraic proof follows the “mean prop’1” principle.

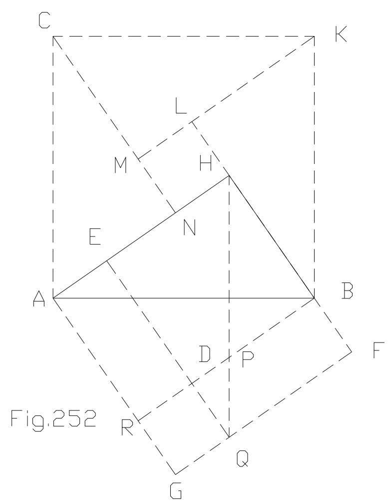
ONE –HUNDRED- FIFTY- FOUR

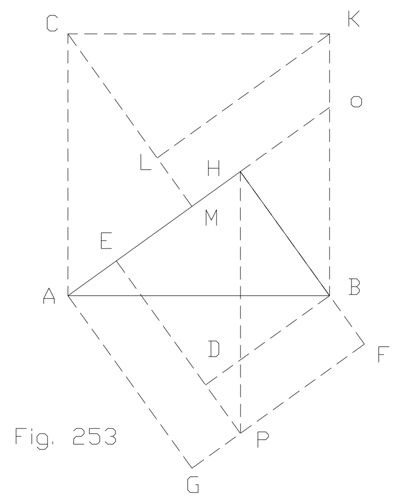
In fig. 252, extend ED to Q, BD to R, draw HQ perp. to AB, CN perp. to AH, KM perp. to CN and extend BH to L.

Sq. AK = tri. ABH common to sq’s AK and HG + (tri. BKL = trap. HEDP of sq. HD + + tri. QPD of sq. HG) + (tri. KCM = tri. BAR of sq. HG) + (tri. CAN= trap QFBP of sq. HG + tri. PBH of sq. HD) + (sq. MN = sq. RQ) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Edwards’ Geom., 1895, p. 157, fig. (13); Am. Math. Mo., V, V, 1898, p. 74, proof LXXXII.





ONE- HUNDRED- FIFTY – FIVE

In fig. 253, extend ED to P, draw HP, draw CM perp. to AH, perp. to CM.

Sq. AK = tri. ANE common to sq’s AK and NG + trap. ENBH common to sq’s AK and HD + (tri. BOH = tri. BND of sq. HD) + (trap. KLMO= trap. AGPN) + (tri. KCL = (tri. PHE of sq. HG) + (tri. CAM = tri. HPF of sq. HG) = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Original with the author, August 3, 1890.

b. Many other proofs may be devised from this type of figure.

ONE – HUNDERED- FIFTY- SIX

In fig. 254, extend GA to M making AM = AG, GF to N making FN = BH,complete the rect. MN, and extend AH and DB to P and O resp'ly and BH to R.

Sq. AK = rect. MN – (rect. BN + 3 tri. ABH + trap. AGFB) = (sq. HD = sq.DH) + sq. HG + rect. BN + {rect. AL = (rect. HL = 2 tri. ABH) + (sq. AP = tri. ABH + trap. AGFB)} = sq. HD + sq. HG + rect. BN + 2tri. ABH + tri. ABH + trap. AGFB – rect. BN – 3 tri. ABH – trap. AGFB = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See Jury Wippre, 1880, p. 22, fig. 16, credited by Joh. Hoffmann in "Der Pythagoraische Lehrsatz," 1821, to Henry Boad, of London, England.

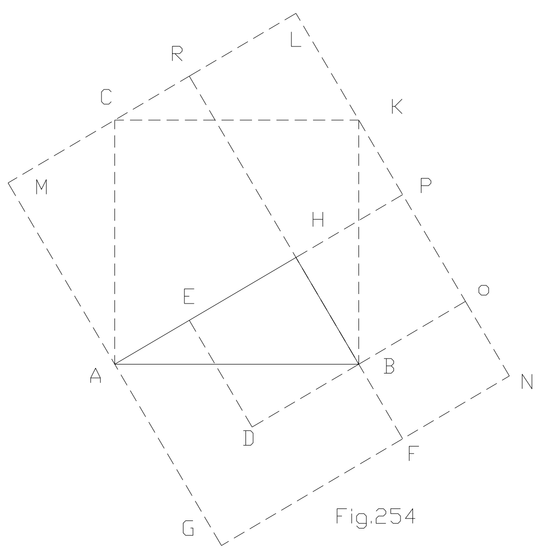


Fig.254

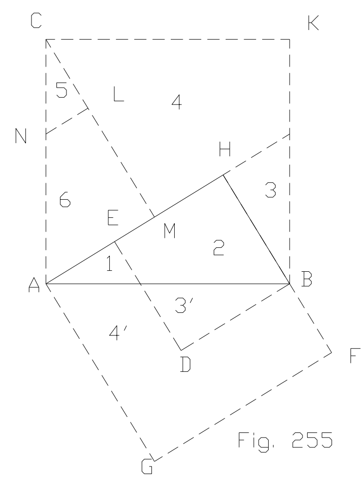


Fig. 255

ONE- HUNDRED- FIFTY-SEVEN

In fig. 255, we have sq. AK = parts 1+2+3+4+5+6; sq. HD= parts 2+3'; sq. HG = parts 1+4' +(7=5) + (6=2) ; so sq. AK(1 + 2 +3+4+5 +6) =sq. HD [2 + (3' =3)] + sq. HG[1 + (4' + 4) + (7 = 5) + (2 + 6).

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. Richard A. Bell, of Cleaveland, O., devided above proof, Nov.30 1920 and gave it to me Feb. 28, 1938. He has 2 other, among his 40 like unto it.

I

This type includes all proofs derived from a figure in which there has been a translation from its normal position of one or more of the constructed squares.

Symbolizign the hypotenuse-square by h, the shorter-leg-square by a, and the longer-leg –square by b, we find, by inspection, that there are seven distinct cases possible in this I-type figure, and that each of the first three cases have four possible arrangements, each of the second three cases have two possible arrangements, and the seventh case has but one arrangement, thus giving 19 sub-types, as follows:

(1) Translation of the H-square, with

(a) The a – and b – squares constructed outwardly.

(b) The a-sq. const'd out'ly and the b-sq. overlapping.

(c) The b-sq. const'd out'ly and the a-sq. overlapping.

(d) The a- and b – sq's const'd overlapping.

(2) Translation of the a – square, with

(a) The h – and b – sq's const'd out'ly.

(b) The h – sq. const'd out'ly and the b- sq. overlapping.

(c) The b-sq. const'd out'ly and the h-sq. overlapping

(d) The h – and b – sq's cons'd overlapping.

(3) Translation of the b –square,with

(a) The h- and a –sq's const'd out'ly

(b) The h – sq. cons'd out'ly and the a- sq. overlapping.

(c) The a- sq. const'd out'ly and the h- sq. overlapping .

(d) The h- and a-sq's const'd overlapping.

(4) Translation of the h- and a –sq's, with

(a) The b- sq. const'd out'ly.

(b) The b-sq. overlapping.

(5) Translation of the h – and b –sq's with

(a) The a – sq. const'd out'ly.

(b) The a – sq. const'd overlapping.

(6) Translation of the a – and b- sq's with

(a) The h- sq. const'd out'ly.

(b) The h-sq. const'd overlapping.

(7) Translation of all three, h-, a- and b-squares.

From the sources of proofs consulted, I discovered that only 8 out of the possible 19 cases had received consideration. To complete the gap of the 11 missing ones I have devised a proof for each missing case, as by the Law of Dissection (see fig. 111, proof Ten) a proof is readily produced for any position of the squares. Like Agassiz's student, after proper observation he found the law, and then the arrangement of parts (scales) produced desired results.

ONE – HUNDRED –FIFTY – ONE

Case (1), (a).

In fig. 256, the sq. upon the hypotenuse, hereafter called the h- sq. has been translated to the position HK. From P the middle pt. of ABdraw PM making HM = AH; draw LM, KM and CM; draw KN = LM,perp. to LM produced, and CO = AB, perp. to HM.

Sq. HK = (2tri. HMC = HM x CO = sq. AH) + (2 tri. MLK = ML x KN = sq. BH) = sq. BH + sq. AH.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Original with the author , August 4, 1900. Several other proofs from this figure is possible.

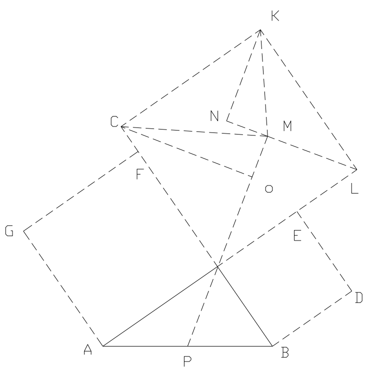


Fig. 256

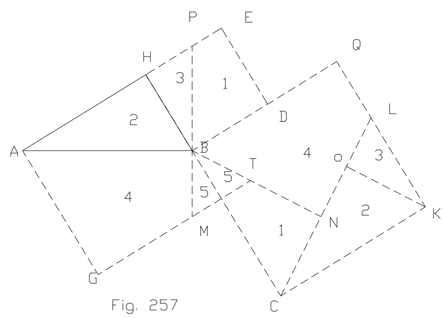


Fig. 257

ONE-HUNDRED- FIFTY- NINE

Case (1), (b).

In fig. 257, the position of the sq's are evident, as the b-sq. overlaps and the h-sq. is translated to right of normal position. Draw PM perp. to AB through B, take KL = PB, draw LC, and BN and KO perp. to BN and KO perp. to LC, and FT perp. to BN.

Sq. BK = (trap. FCNT = trap. PBDE) + (tri. COK = tri.ABH) + (tri.KLO = tri. BPH) + (quad. BOLQ + tri. BTF =trap. GFBA) = sq. BH + sq. AH.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. One of my dissection devices.

ONE –HUNDRED-SIXTY

Case (1), (c).

In fig. 258 draw RA and produce it to Q, and draw CO, LM and KN each perp. to RA.

Sq. CK= (tri. COA = tri. PDB) + (trap. CLMO + trap. PBHE) + (tri. NRK = tri. AQG) + (quad. NKPA + tri. RML = trap. AHFQ) =sq. HB + sq. CK.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised by author, to cover Case (1) , (c).

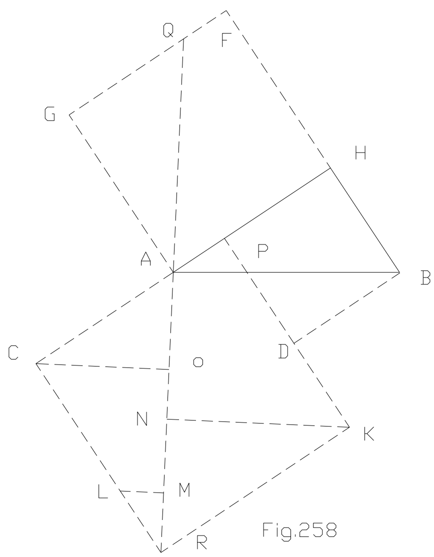


Fig.258

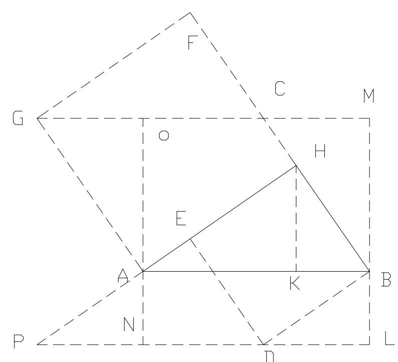


Fig. 259

ONE – HUNDRED- SIXTY – ONE

Produce HA to P making AP = HB, draw PN par. to AB, and through A draw ON perp. to and = to AB, complete sq. OL, produce MO to G and draw HK perp. to AB.

Sq. OL = (rect. AL = paral. PDBA = sq. HD) + (rect. AM = paral. ABCG = sq. HG = sq. HB + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See fersluys, p. 27, fig. 23, as found in "Feirnd of Wisdom," 1887, as given by J. de Gelder, 1810, in Geom. of Van Kunze, 1842.

ONE – HUNDED- SIXTY – TWO

Case (1) , (d).

Draw HO perp. to AB and equal to AH, and KP par. to AB and equal to HB; draw CN par. to AB, PL, EF and extend ED to R and BD to Q.

Sq. CK = (tri. LKP = trap. ESBH of sq. HD + tri. ASE of sq.HG) + (tri. HOB = tri. SDB of sq. HD + trap. AQDS of sq. HG) + (tri. CNH = tri. FHE of sq. HG) + (tri. CLT = tri. FER of sq. HG) + sq. TO = sq. DG of sq. HG = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. Conceived, by author to cover case (1), (d).

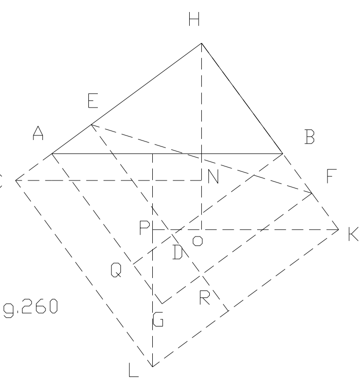


Fig.260

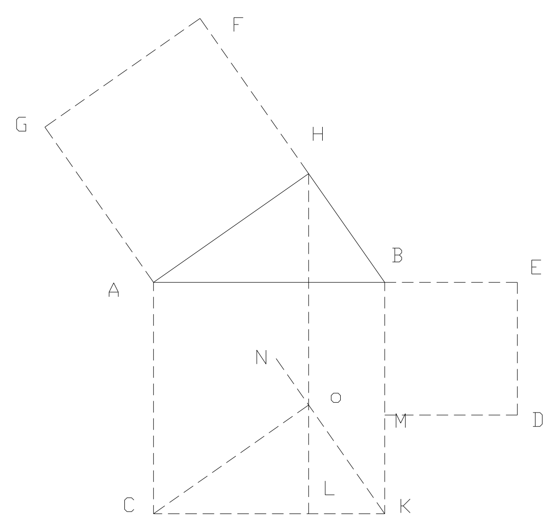


fIG. 261

ONE – HUNDRED- SIXTY THREE

Case (2), (a).

In fig. 261, with sq's placed as in the figure, draw HL perp. to CK, CO and BN par. to AH, making BN = BH, and draw KN.

Sq. AK = rect. BL + rect. AL =(paral. OKBH = sq. BD) + (paral. COHA = sq. AF) = sq. BD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised, by author, to cover Case (2) , (a).

ONE –HUNDRED – SIXTY FOUR

In fig. 262, the sq. AK = parts 1+2+ 3+ 4+ 5+ 6+ 16, sq. HD = parts. (12=5) + (13= 4) of sq. AK. Sq. HG = parts (9 = 1) + (10 + 2) + (11 = 6) + (14= 16) + (15 + 3) of sq. AK.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. This dissection and proof is that of Richard A. Bell, devised by him July 13, 1914, and given to me Feb. 28, 1938.

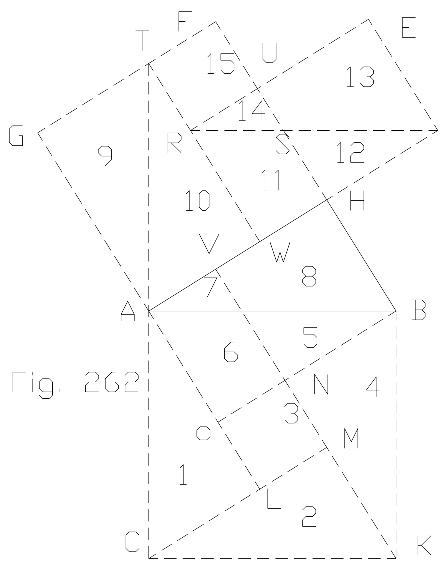


Fig. 262

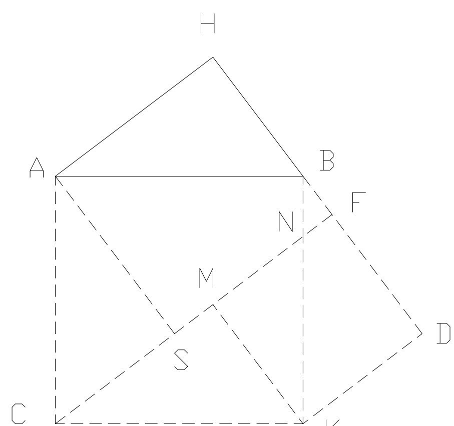


Fig. 263

ONE – HUNDRED – SIXTY –FIVE

Case (2) , (b). ----- For which are more proofs extant then for any other of these 19 cases --- Why? Because of the obvious dissection of the resulting figures.

In fig. 263, extend FG to Sq. AK = (pentagon AGMKB = quad. AGNB common to sq's AK and AF + tri. KNM common to sq's AK and FK) + (tri. ACG = tri. BNF + trap. NKDF) + (tri. CKM = tri. ABH) = sq. FK + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Hill's Geom. for Beginners, 1886, p. 154, proof I; Beman and Smith's New Plane and Solid Geom., 1899, p. 104, fig. 4; Versluys, p. 22 fig. 20, as given by Schlőmilch, 1849; also F.C. Boon, proof 7, p. 105; also Dr. Leitzmann, p. 18, fig. 20, also Jpseph Zelson, a 17 year-old boy in West Phila., Pa. High School, 1937.

b. This figure is of special interest as the sq. MD may occupy 15 other positions having common with side or sides produced of sq. HG. One such solution is that of fig. 256.

ONE –HUNDRED – SIXTY –SIX

In fig. 264, extend FG to C. Sq. AK = quad. AGPB common to sq's AK and AF + (tri. ACG = tri. ABH) + (tri. CME = tri. BPF) + (trap. EMKD common to sq's AK and EK) + (tri. KPD = tri. MLX) = sq. DL + sq. AF. ∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 161, fig. (35); Dr. Leitzmann, p. 18, fig. 21. 4th Edition.

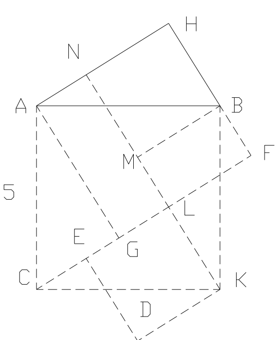
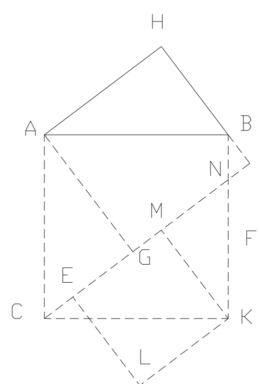


Fig. 265



Fig, 264

ONE – HUNDRED-SIXTY- SEVEN

In fig. 265, extend FG to C, and const. sq. HM = sq. LD, the sq. translated.

Sq. AK = (tri. ACG = tri. ABH) + (tri.COE = tri. BPF) + (trap. EOKL common to both sq's AK and LD, or = trap. NQBH) + (tri. KPL = tri. KOD = tri. BQM) + [(tri.BQM + polygon AGPBMQ) = quad, AGPB common to sq's AK and AF] = sq. LD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup. , V. 70, p. 359, Dec. 3, 1910, by A.R. Colburn.

b. I think it better to omit Colburn's sq. HM (not necessary), and thus reduce it to proof above.

ONE-HUNDRED – SIXTY- EIGHT

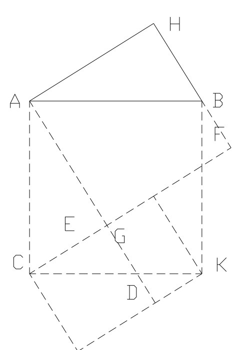
In fig. 266, extend ED to K and draw KM par. to BH.

Sq. AK = quad. AGNB common to sq's AK and AF + (tri. ACG = tri. ABH)= (tri. CKM + trap. CEDL + tri. BNF) + (tri. KNM = tri. CLG) =sq. GE + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 156, fig. (8).

Fig. 266



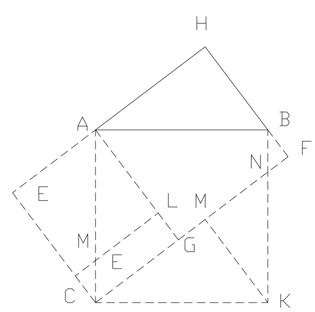


Fig. 267

ONE-HUNDRED- SIXTY- NINE

In fig. 267, extend ED to C and draw KP par. to HB.

Sq.AK quad. AGNB common to sq's AK and HG + (tri. ACG = tri. CAE = trap. EDMA + tri. BNF) + (tri. CPK =tri.ABH) + (tri. PKN = tri. ABH) + (tri. PNK = tri. LAM) = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. VI, 1899, p. 33, proof LXXXVI.

ONE – HUNDRED- SEVENTY

In fig. 268, extend ED to C,DN to B, and draw EO par. to AB, KL perp. to DB and HM perp. to EO.

Sq. AK = rect. AO + rect. CO = paral. AELB + paral. ECKL = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , Vol. VI, 1899, p. 33, LXXXVII.

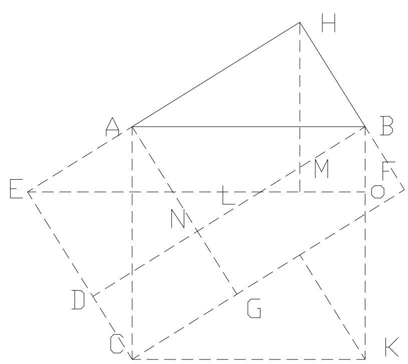


Fig. 268

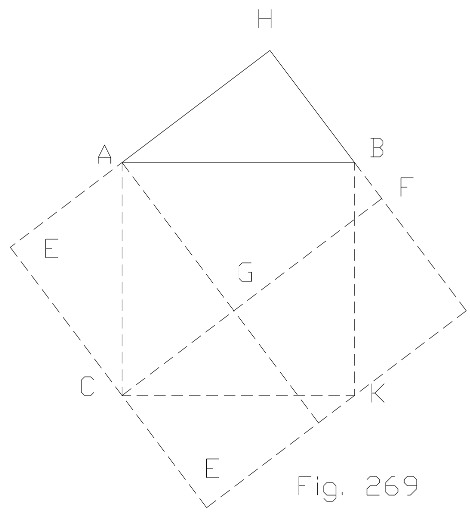


Fig. 269

ONE – HUNDRED- SEVENTY- ONE

In fig. 269, extend HF to L and complete the sq. HE.

Sq. AK = sq. HE – 4 tri. ABH = sq. CD + sq. HG + (2 rect. GL = 4 tri. ACG) – 4 tri. ABH = sq. CD + sq. HC.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. This is one of the conjectured proofs of Pythagoras; see Ball's Short Hist. of Math., 1888, p. 24; Hopkin's Plane Geom., 1891, p. 91, fig. IV; Edwards' Geom., 1895, p. 162, fig. (39); Beman and Smith's New Plane Geom., 1899, p. 103. fig. 2; Heath's Math. Monographs, No. 1, 1900, p. 18, proof II.

ONE – HUNDRED-SEVENTY- TWO

In fig. 270 extend FG to C, draw HN perp. to CK and KM par. to HB.

Sq. AK = rect. BN + rect. AN = paral. BHMK + paral. HACM = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. VI, 1899, p. 33, proof LXXXVII.

b. In this figure the given triangle may be either ACG, CKM, HMF or BAL; taking either of these four triangles several proofs for each is possible. Again, by inspection, we observe that the given trianble may have any one of seven other positions within the square AGFH, right angles coinciding. Furthermore the square upon the hypotenuse may be constructed as to the figure there will result several proofs unlike any, as to dissection, given heretofore.

c. The simplicity and applicability of figures under Case (2), (b) makes it worthy of note.

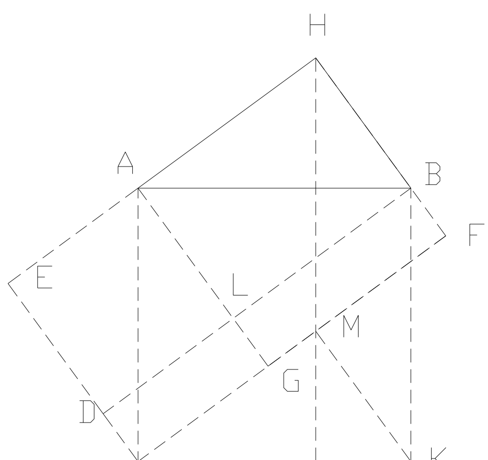


Fig.270

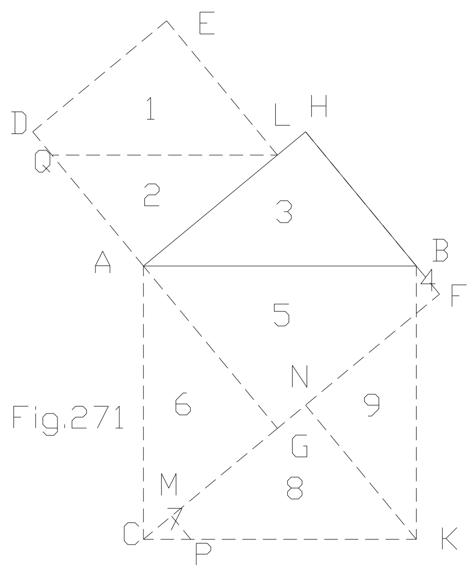


Fig.271

ONE-HUNDRED – SEVENTY – FOUR

In fig. 271, sq. AK = sections [5+ (6 +3) + (7 = 4)] + [(8 = 1) + (9 = 2)] = sq. AE.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. Devised by Richard Bell, Cleveland, O., on July 4, 1914, one of his 40 proofs.

ONE – HUNDRED- SEVENTY – FOUR

Case (2) , (c).

In fig. 272, ED being the sq. translated, the construction is evident.

Sq. AK = quad. AHLC common to sq's AK and AF + (tri. ABC = tri. ACG) + (tri. BKD = trap. LKEF + tri. CLF) + tri. KLD common to sq's AK and ED = sq. ED + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 22, fig. 17, as given by von Houff, in "Lehrbefriff der reinen Mathematik," 1803; Heath's Math. Monograph, 1900, No. 2, proof XX; Versluys, p. 29, fig. 27; Fourrey, p. 85--- A. Marre, from Sanscrit, "Yoncti Bacha"; Dr. Leitzmann, p. 17, fig, 19, 4th edition.

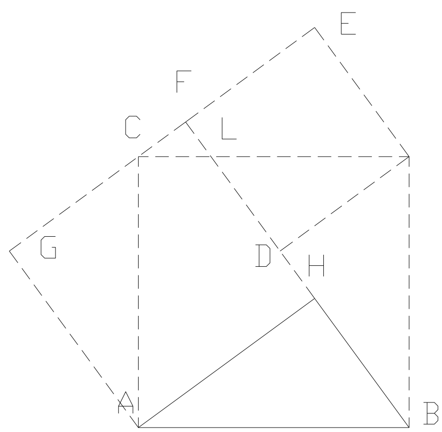


Fig.272

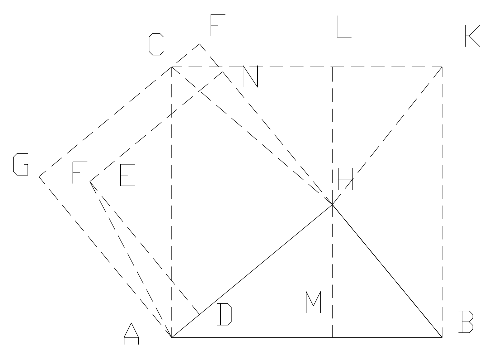


Fig. 273

ONE – HUNDRED- SEVENTY- FIVE

Having completed the three squares AK, HE and HG , draw, through H, LM perp. to AB and join HC, AN and AE.

Sq. AK = [rect. LB = 2(tri. KPH = tri. AEM) = sq. HD] + [rect. LA = 2 (tri. HAC = tri. ACH) = sq. HG] = sq. HD + sq. HG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Math. Mo. (1859), Vol. II, No. 2, Dem. fig. 6.

ONE –HUNDRED- SEVENTY – SIX

In fig. 274,since parts 2+3 = sq. on BH = sq. DE, it is readily seen that the sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Devised by Richard A. Bell, July 17, 1918, being one of his 40 proofs. He submitted a second dissection proof of same figure, also his 3 proofs of Dec. 1 and 2, 1920 are similar to the above, as to figure.

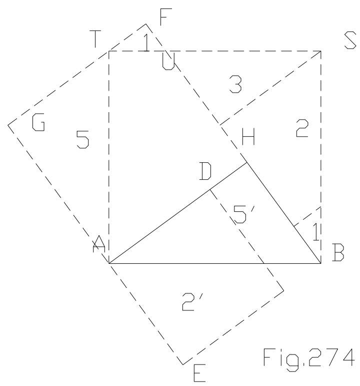


Fig.274

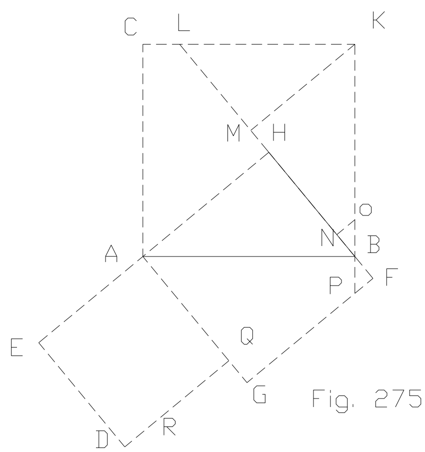


Fig. 275

∴

ONE-HUNDRED- SEVENTY –SEVEN

Case (2), (d).

In fig. 275, extend KB to P, CA to R, BH to L, draw KM perp.to BL, take MN = HB, and draw NO par. to AH.

Sq. AK = tri. ABH common to sq's AK and AF + (tri. BON = tri. BPF) + (trap. NOKM = trap. DRAE) + (tri. KLM = tri. ARQ) + quad. AHLC = quad. AGPB) = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. VI, 1899, p. 34, proof XC.

ONE – HUNDRED – SEVENTY – EIGHT

In fig. 276, upon CK const. tri. CKP = tri. ABH, draw CN par. to BH, KM par. to AH, draw ML and through H draw PO.

Sq. AK =rect. KO + rect. CO = (paral. PB = paral. CL =sq. AD) + (paral. PA = sq. AF) = sq. AD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Original with the author, July 28, 1900.

b. An algebraic proof comes readily from this figure.

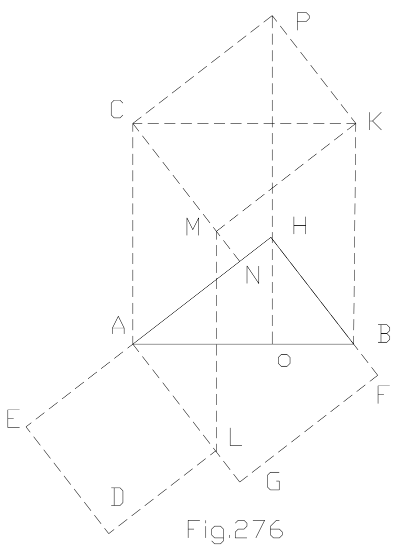


Fig.276

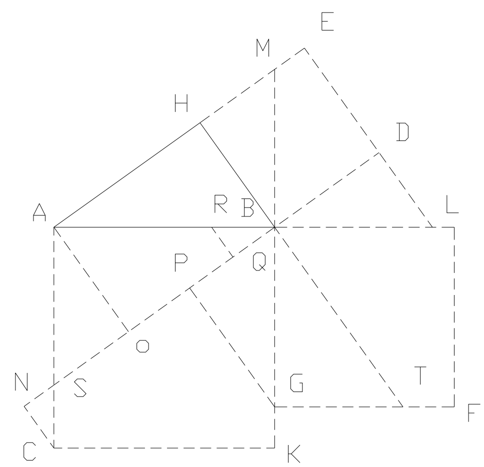


Fig.277

ONE- HUNDRED- SEVENTY- NINE

Case (3), (a).

In fig. 277, produce DB to N, HB to T, KB to M, and draw CN, AO, KP and RQ perp. to NB.

Sq. AK = (quad, CKPS + tri. BRQ = trap. BTFL) + (tri. KBP = tri. TBG) + (trap. OQRA = trap. MBDE) + (tri. ASO = tri. BMH) = sq. HD + sq. GL.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised for missing Case (3), (a), March 17,1926.

ONE- HUNDRED- EIGHTY

Case (3), (b).

In fig. 278, extend ED to K and through D draw GM par. to AB.

Sq. AK = rect. AM + rect. CM = (paral. GB = sq. HD) + (paral. CD = sq. GF) = sq. HD + sq. GF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., Vol. VI, 1899, p. 33, proof LXXXV.

b. This figure furnishes an algebraic proof.

c. If any of the triangles congruent to tri. ABH is taken as the given triangle, a figure expressing a different relation of the squares is obtained, hence covering some other case of the 19 possible cases.

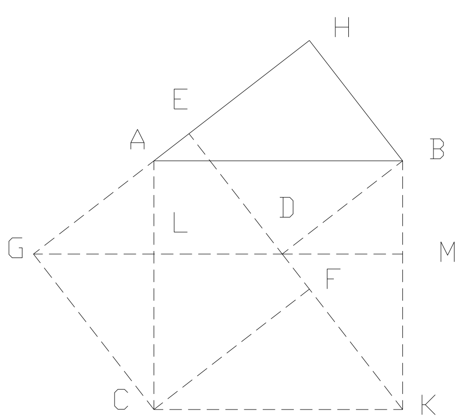


Fig278

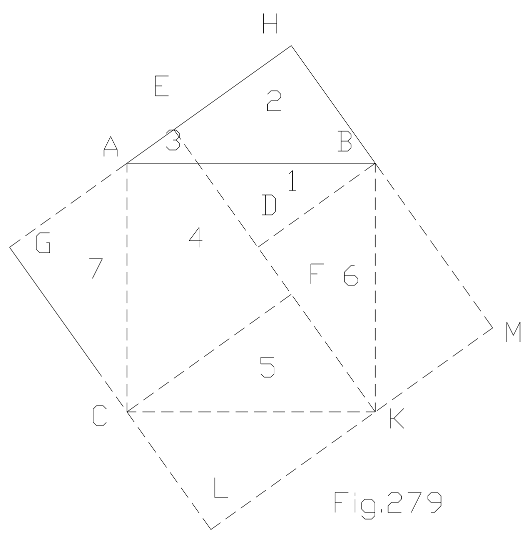


Fig.279

ONE – HUNDRED- EIGHTY – ONE

Extend HA to G making AG = HB, HB to M making BM = HA, complete the square's HD, EC, AK and HL. Number the dissected parts, omitting the tri's CLK and KMB.

Sq. (AK = 1+ 4+ 5 + 6) = parts (1 common to sq's HD and AK) + (4 common to sq's EC and AK) + (5 = 2 of sq. HD + 3 of sq. EC) = (6 = 7 of sq. EC) = parts (1 + 2) + parts (3 + 4 + 7) =sq. HD + sq. EC.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See "Geometric Exercises in Paper Folding" by T. Sundra Row, edited by Beman and Smith (1905) p. 14.

ONE- HUNDRED EIGHTY- TWO

In fig. 280 extend EF to K, and HL perp. to CK.

Sq. AK = rect. BL + rect. AL = paral. BF + paral AF = sq. HD = sq.GF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Am. Math. Mo. , V. VI, 1899, p. 33, proof lXXXIV.

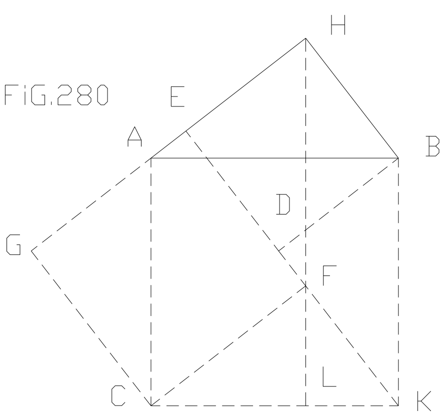


FiG.280

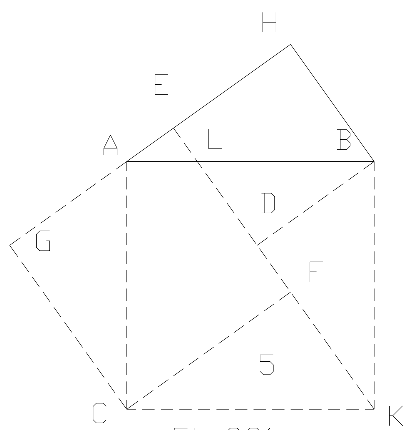


Fig.281

ONE-HUNDRED- EIGHTY – THREE

In fig. 281, extend EF to K.

Sq. AK = quad. ACFL common to sq's AK and GF + (tri. CKF = trap. LBHE + tri. ALE) + (tri. KBD = tri. CAG) + tri. BDL common to sq's AK and HD = sq. HD + sq. AK.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Olney's Geom., Part III, 1872, p. 250, 2nd method; Jury Wipper, 1880, p. 23, fig. 18; proof by E. Forbes, Winchester, N.H. as given in Jour. of Ed'n V. XXVIII, 1888, p. 17, 25th proof; Jour. of Ed'n V. XXV, 1887, p. 404, fig. II; Hopkin's Plane Geom., 1891, p. 91, fig. III; Edwards' Geom., 1895, p. 155, fig. (5) ; Math. Mo. , V. VI, 1899, p. 33, proof LXXXIII; Heath's Math. Monographs, No. 1, 1900, p. 21, proof V; Geometric Exercises in Paper Folding, by T. Sundra Row, fig. 13 p. 14 of 2nd Edition fo The Open Court Pub. Co., 1905. Every teacher of geometry should use this paper folding proof.

Also see Versluys, p. 29, fig. 26, 3rd paragraph, Clairaut, 1741, and found in "Yoncti Bacha"; also Math. Mo. 1858, Vol. I, p. 160, Dem. 10, and p. 46, Vol. II, where credited to Rev. A. D. Wheeler.

b. By dissection an easy proof results. Also by algebra, as (in fig. 281) CKBHG = $a^2 + b^2 + ab$; whence readily $h^2 = a^2 + b^2$.

c. Fig. 280 is fig. 281 with the extra line HL; fig. 281gives a proof by congruency, while fig.280 gives a proof by equivalency, and it also fives a proof, by algebra, by the use of mean proportional.

d. Versluys, p. 20, connects this proof with Macay; Van Schooter, 1657; J.C. Sturm, 1689; Dobriner; and Clairaut.

<u>ONE – HUNDRED- EIGHTY –FOUR</u>

In fig. 282, from the dissection it is obvious that the sq. upon AB = sq. upon AH.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devesed by R. A. Bell, Cleveland, O. , on Nov. 30, 1920, and given to the author Feb. 28, 1938.

<u>ONE- HUNDRED – EIGHTY-FIVE</u>

Case (3), (c).

In fig. 283, draw KL perp. to CG and extend BH to M.

Sq. AK = (tri. ABH = tri. CKF) + tri. BNH common to sq's AK and HD + (quad. CGNK = sq. LH + trap. MHNK + tri. KCL common to sq's AK and FG) + tri. CAG = trap. BDEN + tri. KNE) = sq. HD + sq. FG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Sci. Am. Sup., Vol. 70, p. 383, Dec, 10, 1910, in which proof A.R. Colburn makes T the given tri., and then substitutes part 2 for part 1 , part 3 for part 4 and part 5, thus showing sq. AK = sq. HD + sq. FG; also see Verslulys, p. 31, fig. 28, Geom., of M, sauvens, 1753 (1716).

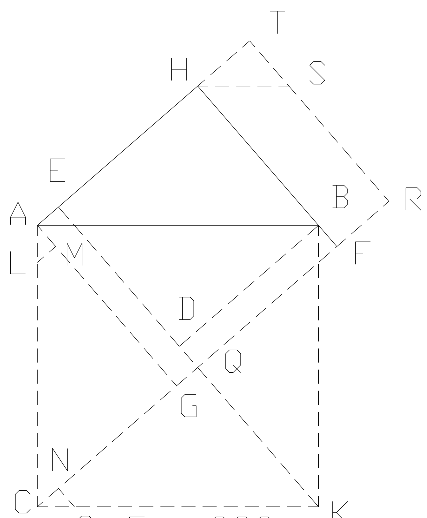


Fig. 282

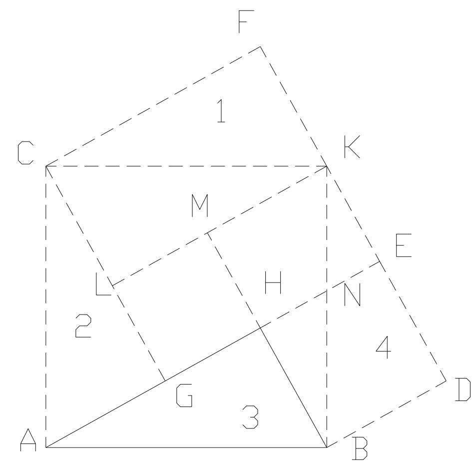


Fig. 283

ONE – HUNDRED – EIGHTY –SIX

In fig. 284, the construction is evident, FG being the translated b – square.

Sq. AK = quad. GLKC common to sq's AK and CE + (tri. CAG = trap. BDEL + tri. KLE) + (tri. ABH = tri. CKF) + tri. BLH common to sq's AK and HD = sq. HD = sq. CE.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Halsted's Elements of Geom., 1895, p. 78, theorem XXXVII; Edwards' Geom. 1895, p. 156, fig. (6); Heath's Math. Monographs, No. 1, 1900, p. 27, proof XIII.

ONE –HUNDRED – EIGHTY – SEVEN

In fig. 285, it is obvious that the pares in the sq. HD and HF are the same in number and congruent to the parts in the square AK.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. One of R.A. Bell's proofs, of Dec. 3, 1920 and received Feb. 28, 1938.

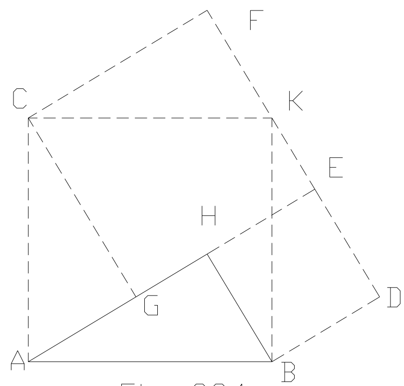


Fig. 284

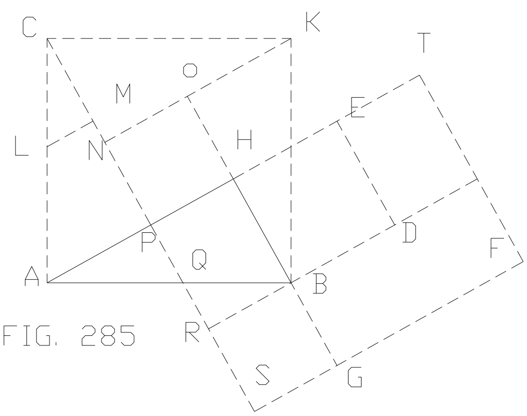


FIG. 285

ONE- HUNDRED – EIGHTY – EIGHT

Case (3) , (d).

In fig. 286, produce AG to O, draw CN par. to HB,and extend CA to G.

Sq. AK = trap. EMBH common to sq’s AK and HD + (tri. BOH = tri. BMD) + (quad. NOKC = quad. FMAG) + (tri. CAN = tri. GAL) + tri. AME common to sq’s AK and EG = sq. HD + sq. LF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Am. Math. Mo., Vol. VI, 1899, p. 34, proof LXXXIX.

b. As the relative position of the given triangle and the translated square may be indefinitely varied, so the number of proofs must be indefinitely great, of which the following two are examples.

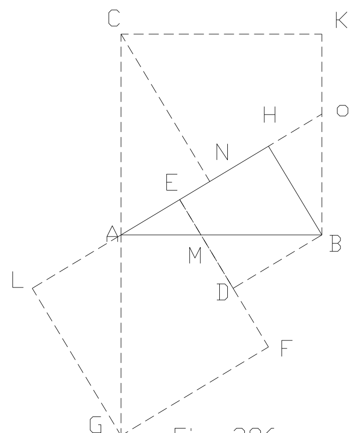


Fig. 286

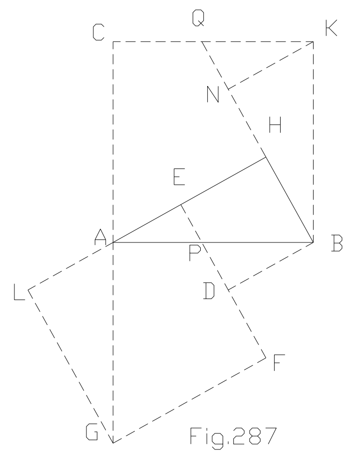


Fig.287

ONE –HUNDRED- EIGHTY- NINE

In fig. 287, produce BH to Q, HA to L and ED to F, and draw KN perp. to QB and connect A and G.

Sq. AK = tri. APE common to sq’s AK and EG + trap. PBHE common to sq’s HD and AK + (tri. BKN = tri. GAL) + (tri. NKQ = tri. DBP) + (quad. AHQC = quad. GFPA) = sq. HD + sq. HA.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. This fig. and proof due to R. A. Bell of Cleveland, O. He gave it to the author Feb. 27, 1938.

ONE- HUNDRED –NINTY

In fig. 288, draw LM through H.

Sq. AK = rect. KM + rect. CM = paral. CH = sq. HD + (sq. on AH = sq.NF)

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Original with the author, July 28, 1900.

b. An algebraic solution may be devised from this figure.

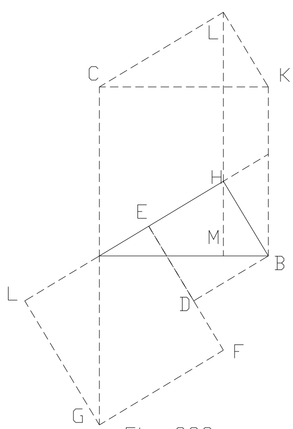


Fig. 288

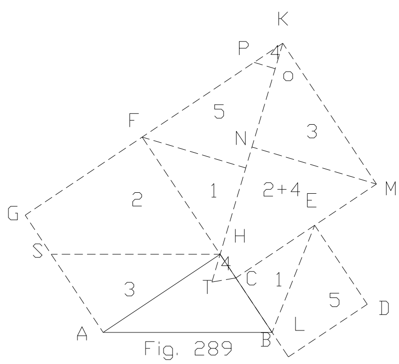


Fig. 289

ONE – HUNDRED – NINTY-ONE

Case (4), (a).

In fig. 289, extend KH to T making NT = AH, draw TC, draw FR, MN and PO perp. to KH and draw HS par. to AB.

Sq. CK = (quad. CMNH + tri. KPO = quad SHFG) + tri. MKN = tri. HAS) + (trap. FROP = trap EDLB) + (tri. FHR = tri ECB) = sq. CD + sq GH.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised by author for case (4) , (a) March 18, 1926.

ONE –HUNDRED – NINTY – TWO

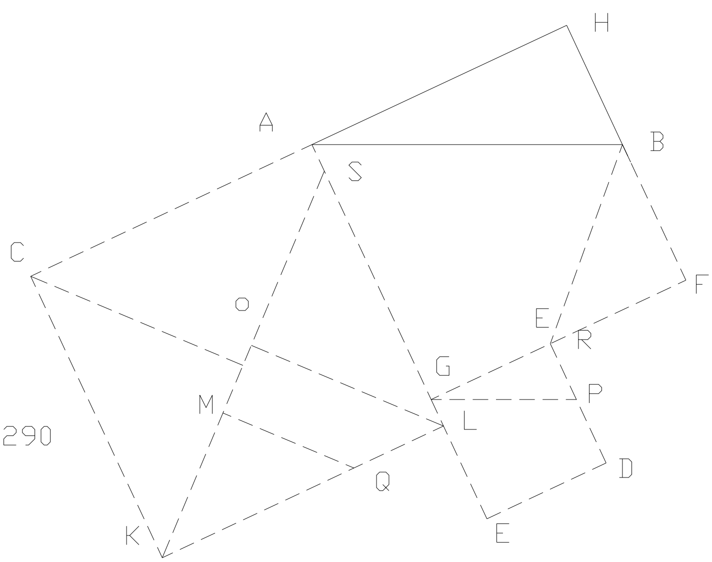
Case (4), (b).

In fig. 290 draw GP par. to AB, take LS =AH, draw KS, draw LO, CN and QM perp. to KS, and draw BR.

Sq. AK = (tri. CNK = tri. ABH) + (tri, KQM = tri.FBR) + (trap. QLOM = trap. PGED) + (tri. SOL = tri. GPR) + (quad. CNSA = quad. AGRB) = sq. GD + sq. AF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised by author for Case (4), (b).



ONE – HUNDRED –NINTY- THREE

Case (5), (a).

In fig. 291, CE and AF are the translated sq's; produce GF to O and complete the sq. MO; produce HE to S and complete the sq. US; produce OB to Q, draw MF, draw WH, draw ST and TX = HB and draw XY pero. to WH. Since sq. MO = sq. AF, and sq. US = sq. CE, and since sq. RW = (quad, URHV + tri. WYX = trap. MFOB + (tri. HST= tri. BHQ) + (trap. TSYX = trap BDEQ) + (tri. UVW= tri MFN) = sq. HD + (sq. NB = sq.AF).

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised March 18, 1926, for Case (5), (a), by author.

ONE – HUNDRED – NINTY- FOUR

Extend HA to G making AG = HB to D making BD = HA. Complete sq's PD and PG. Draw HQ perp. to CK and through P draw LM and TU par. to AB. PR = CO = BW.

The translated sq's are PD = BE' and PG = HG'.

Sq. AK = parts (1 + 2 + 3 +4+5 + 6 + 7 + 8) = parts (3+4+ 5 + 6 = sq. PD) + parts (1 + 2 + +7 + 8) = sq. PG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See Versluys, p. 35, fig. 34.

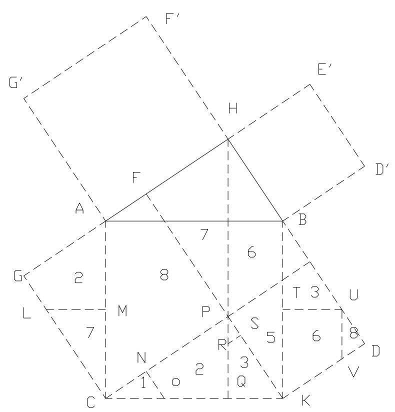


Fig. 292

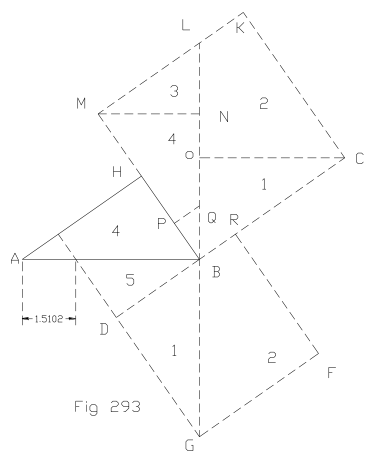


Fig 293

ONE – HUNDRED- NINTY – SIX

Case (6), (a).

In fig. 294, extend LE and FG to M thus completing the sq. HM, and draw DM.

Sq. AK + 4 tri. ABC =sq. HM, = sq. LD + sq. DF + (2 rect. HD = 4 tri. ABC), from which sq. AK = sq. LD + sq. DF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. This proof is credited to M. McIntosh of Whitwater, Wis. See Jour. of Ed'n 1888, Vol. XXVII, p. 327, seventeenth proof.

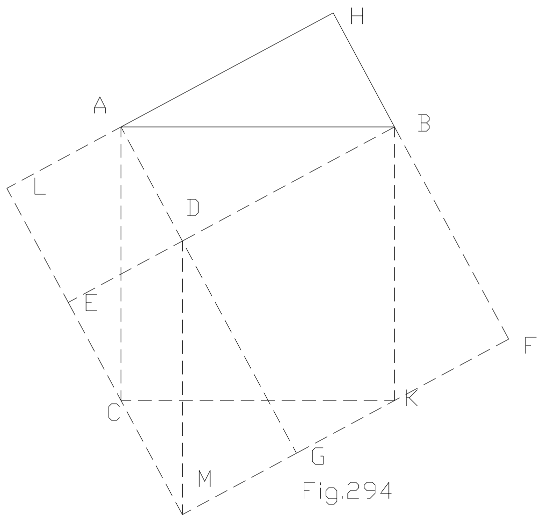


Fig.294

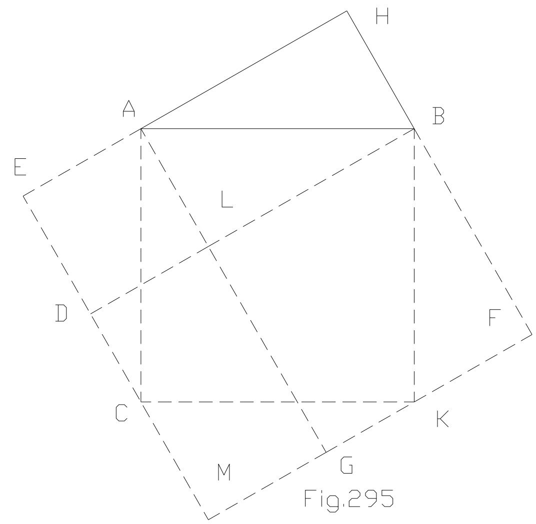


Fig.295

ONE – HUNDRED – NINTY- SEVEN

Sq. AK = sq. HM – (4 tri. ABH = 2 rect. HL =sq. EL + sq. LF + 2 rect. HL – 2 rect. HL = sq. EL + sq. LF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Journal of Education, 1887, Vol. XXVI, p. 21, fig. XII; Iowa Grand Lodge Buletin, F and A.M., Vol. 30, No. 2, p. 44, fig. 2, of Feb. 1929. Also Dr. Leitzmann, p. 20, fig. 24, 4th Ed'n.

b. An algebraic proof is $h^2 = (a + b)^2 - 2ab = a^2 + b^2$.

ONE – HUNDRED – NINTY – EIGHT

In fig. 296, the translation is evident.Take CM = KD. Draw AM; then draw

AM; then draw GR, CN and BO par. to AH and DU par. to BH. Take NP = BH and draw PQ par to AH.

Sq. AK = (tri. CMN = tri. DEU) + (trap. CNPQ = trap. CNPQ = trap. TKDU) + (quad.OMRB + tri. AQP = trap. FGRQ) + tri. AOB = tri. GCR) = sq. EK + sq. FC.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

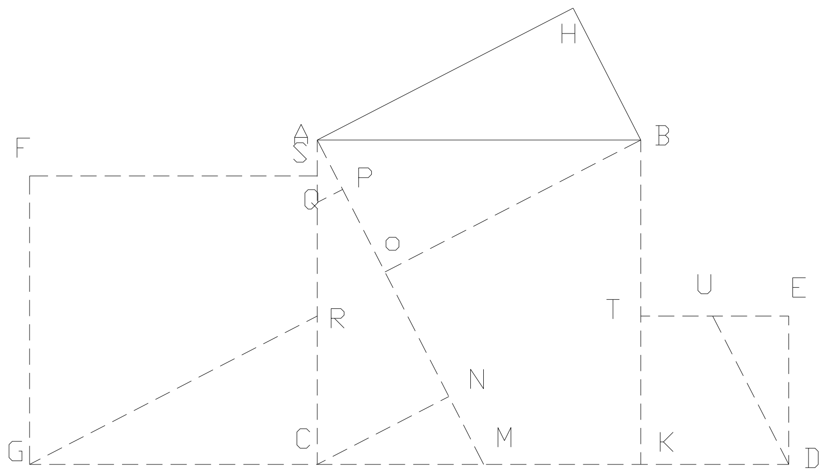


Fig. 296

a. Devised by the author, March 28, 1926.

ONE – HUNDRED – NINTY – NINE

In fig. 297, the translation and construction is evident.

Sq. AK = (tri. CRP = tri. BVE) +(trap. ANST = trap. BMDV) + quad. NRKB + tri. TSB = trap. AFGC) + tri. ACP common to sq. ME + sq. FP.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised by author, March 26, 1926, 10 p.m.

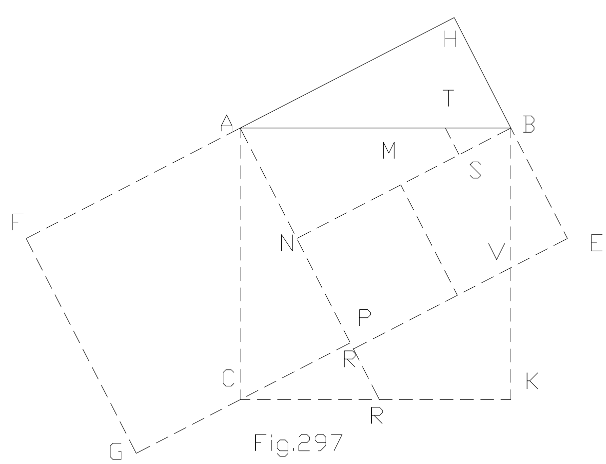


Fig.297

TWO – HUNDRED

In fig. 298, the sq. on AH is translated to position of GC, and the sq. on HB to position of GD. Complete the figure and conceive the sum of the two sq's EM + TC + sq. LN and the dissection as numbered.

Sq. AK = (tri, ACP = tri. DTM) + (tri. CKQ = tri. TDE) + (tri. KBR = tri. CTO) + (tri. BAS + tri. TNC) + (sq. SQ = sq. LN) = sq. EL + sq. GC.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised by author, March 22, 1926.

b. As sq. EL having a vertex and a side in common with a vertex and a side of sq. GC, either externally (as in fig. 298), or internally, may have 12 different positions, and as sq. GC may have a vertex and s a side in common with the fixed sq. AK, or in common with the given triangle ABH, giving 15 different positions, there is possible 180- 3 = 177 different figures, hence 176 proofs other than the one given above, using the dissection as used here, and 178 more proofs by using the dissection as given in proof <u>Ten</u>, fig. 111.

c. This proof is a variationof that given in proof <u>Eleven,</u> fig. 112.

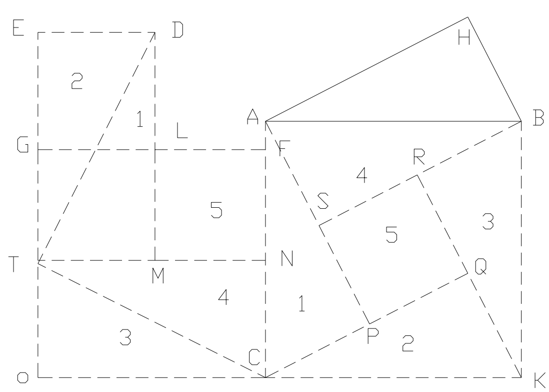


Fig. 298

TWO- HUNDRED – ONE

In fig. 299, the constructionis evident, as FO is the translation of the sq. on AH, and KE is the translation of the sq. on BH.

Since rect. CN = rect. QE, we have sq. AK = (tri. LKV = tri. CPL0 + (tri. KBW = tri. LFC) + (tri. BAT = tri. KQR) + (tri.ALU =tri. RSK) + (sq. TV = sq. MO) = rect. KR + rect. FP + sq. MO = sq. KE + sq. FO.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised by the author, March 27, 1926

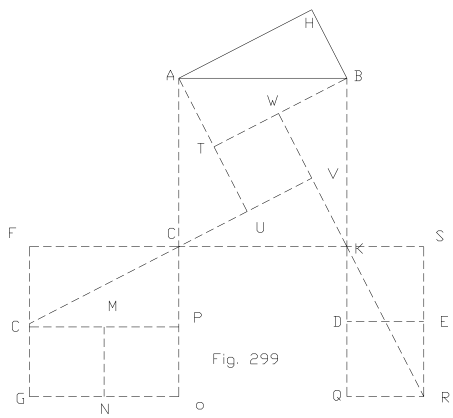


Fig. 299

TWO – HUNDRED – TWO

In fig. 300 the translation and construction are easily seen.

Sq. AK = (tri. CKN = tri. LFG) + (trap. OTUM = trap. RESA) + (tri. VOB = tri. RAD + (quad. ACNV + tri. TKU = quad. MKFL) = sq. DS + sq. MF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Devised by the author, March 27, 1926, 10.40 p.m.

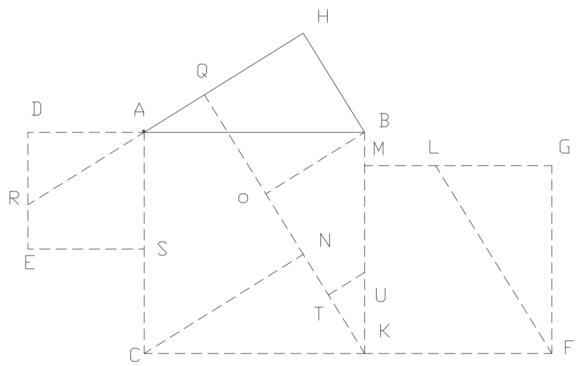


Fig. 300

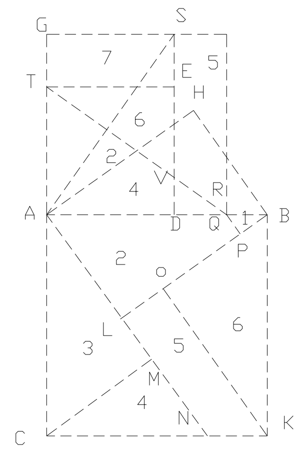


Fig. 301

TWO- HUNDRED- THREE

AR = AH and AD = BH. Complete sq's on AR and AD. Extend DE to S and draw SA and TR.

Sq. AK = (tri. QPB = tri. VDR of sq. AF) + (trap. AIPQ = trap. ETAU of sq. AE) + (tri. CMA = tri. SGA of sq. AE) + (tri. CNM = tri. UAD of sq. AE) + (trap. NKOL = trap. VRFS of sq. AF) + (tri. OKB = tri. DSA of sq. AF) = (parts 2+ 4 = sq. AE) + (parts. 1+3+ 5+ 6= sq. AF) .

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. Devised by author, Nov. 16, 1933.

TWO – HUNDRED- FOUR

In fig. 302, complete the sq. on EH, draw BD par. to AH, and draw AL and KF perp. to BD.

Sq. AK = sq. HG – (4 tri. ABH = 2 rect. HL) = sq. EL + sq. DK + 2 rect. FM - 2 rect. HL =sq. EL + sq. DK.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Edwards' Geom., 1895, p. 158, fig. (19).

b. By changing position of sq. FG, many other proofs might be obtained.

c. This is a variation of proof, fig. 240.

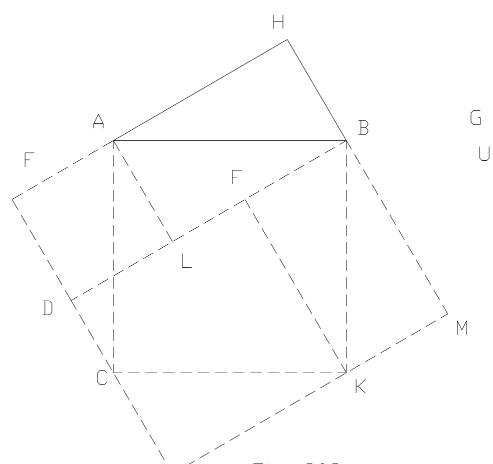


Fig. 302

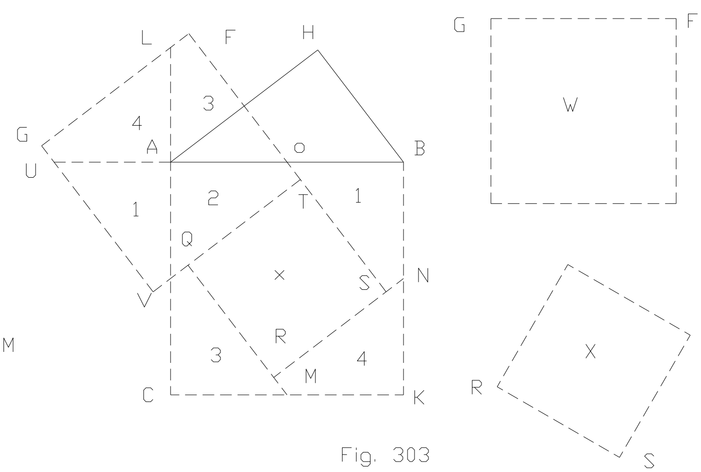


Fig. 303

TWO – HUNDRED- FIVE

In fig. 303, let W and X be sq's with sides equal resp'y to AH and BH. Place them as in figure, A being center of sq. W. and O, middle of AB as center of FS. ST = BH, TF = AH. Sides of sq's FV and QS are perp. to sides AH and BH.

It is obvious that:

Sq. AK = (parts 1+ 2+ 3 + 4= sq.FV) + sq. QS = sq. X + sq. W.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Messenger o fMath., Vol. 2, p. 103, 1873, and there credited to Henry Perigal, F.R.S.A.S.

TWO –HUNDRED – SIX

Case (6), (b).

In fig. 304, the construction is evident. Sq. AK = (tri. ABH + trap. KEMN + tri. KOF) + (tri. BOH = tri. KLN) + quad. GOKC common to sq's AK and CF + (tri. CAG = tri. CKE) = sq. MK + sq. CF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. See Hopkins' Plane Geom., 1891, p. 92, fig. VIII.

b. By drawing a line EH, a proof through parallelogram, may be obtained. Also an algebraic proof.

c. Also any one of the other three triangles, as CAG may be called the given triangle, from which other proofs would follow. Furthermore since the tri. ABH may have seven other positions leaving side of sq. AK as hypotenuse, and the sq. MK may have 12 positions having a side and a vertex in common with sq. CF, we would have 84 proofs, some of which have been or will be given; etc., etc., as to sq. CF, one of which is the next proof.

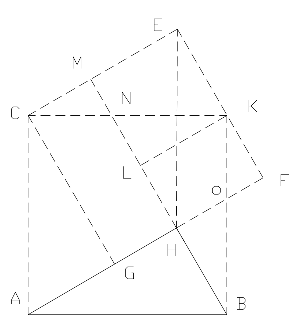


Fig. 304

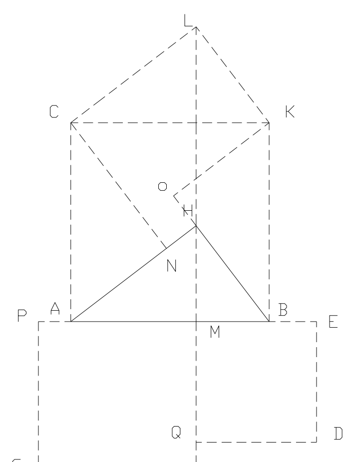


Fig. 305

TWO – HUNDRED – SEVEN

In fig. 305, through H draw LM and draw CN par. to BH and KO par. to AH.

Sq. AK = rect. KM + rect. CM = paral. KH + paral. CH = HB x KO + AH x CN = sq. on BH + sq. on AH =sq. MD + sq. MG.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Original with the author January 31, 1926, 3 p.m.

TWO – HUNDRED – EIGHT

In fig. 306, extend AB to X, draw WU and KS each = to AH and par. to AB, CV and HT pepr. to AB, GR and FP par. to AB and LW and AM pepr. to AB.

Sq. WK = (tri. CKS = tri. FPL = trap. BYDX of sq. BD + tri. FON of sq. GF) + (tri. BEX of sq. BD trap. WQRA of sq. GF) + (tri. WUH = tri. LWG of sq. GF) + (tri. WCV = tri. WLN of sq. GF) + (sq. VT = paral. RO of sq. GF) = sq. BD + sq. GF.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Origional with the author, Aug. 8, 1900.

b. As in fig. 305 many other arrangements are possible each of which will furnish a proof of or proofs.

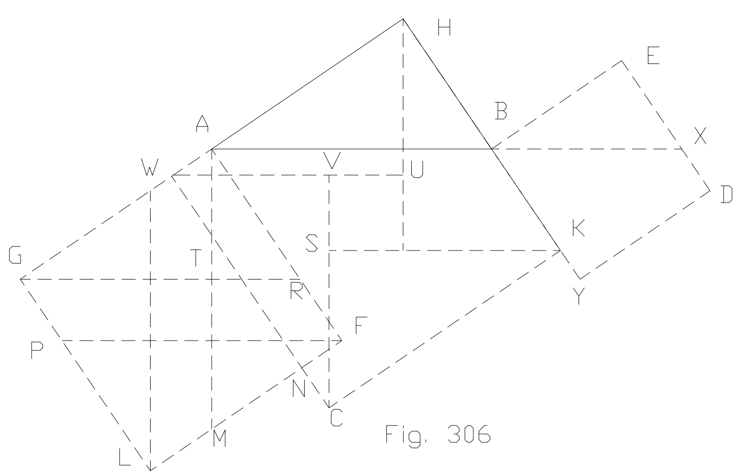


Fig. 306

J

(A) ----- Proofs determined by arguments based upon a square.

This type includes all proofs derived from figures in which one or more of the squares are not graphically represented. There are two leading classes or sub-types in this type ---first, the class in which the determination of the proof is based upon a square; second, the class in which the determination of the proof is based upon a triangle.

As in the I- type, so here, by inspection we find 6 sub- classes in our first sub- type which may be symbolized thus.

(1) The h- square omitted, with

(a) The a- and b- square const'd outwardly--- 3 cases.

(b) The a- sq. const'd out'ly and the b-sq. overlapping –3 cases.

(c) The b-sq. const'd out'ly and the a- sq. overlapping ---3 cases.

(d) The a- and b- sqares overlapping --- 3 cases.

(2) The a-sq. omitted, with

(a) The h- and b- sq's const'd out'ly overlapping ---3 cases.

(b) The h-sq. const'd out'ly and b-sq. overlapping ---3 case.

(c) The b- sq. const'd out'ly and the h- sq. overlapping ---3 cases.

(d) The h- and b- sq's const'd and overlapping ---3 cases.

(3) The b –sq. omitted with

(a) The h- and a – sq's const'd out'ly ---3 cases.

(b) The h-sq. const'd out'ly and the a- sq. overlapping – 3 cases.

(c) The a-sq. const'd out'ly and the h- sq. overlapping ---3 case.

(d) The h- and a-sq's const'd overlapping---3 cases.

(4) The h- and a – sq's omitted, with

(a) The b-sq. const'd out'ly

(b) The b-sq. const'd overlapping.

(c) The b-sq. translated—in all 3 cases.

(5) The h- and b-sq'd omitted, with

(a) The a-sq. const'd out'ly.

(b) The a-sq. const'd overlapping.

(c) The a-sq. translated – in all 3 cases.

(6) The a- and b- sq's omitted, with

(a) The h- sq. const'd out'ly.

(b) The h –sq. const'd overlapping.

(c) The h – sq. translated—in all 3 cases.

The total of these enumerated cases is 45. We shall give but a few of these 45, leaving the remainder to the ingenuity of the interested student.

(7) All three squares omitted.

TWO – HUNDRED – NINE

Case (1), (a).

In fig. 307, produce GF to N a pt., on the perp. to AB at B, and extend DE to L, draw HL and AM perp. to AB. The tri's AGM and ABH are equal.

Sq. HD + sq. GH = paral. HO = paral LP) + paral. MN = paral MP = AM x AB = ABx AB = AB^2.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Devised by author for case (1), (a). March 20, 1926.

b. See proof No. 88, fig. 188. By omitting lines CK and HN in said figure we have fig. 307. Therefore proof No. 209 is only a variation of No. 88, fig. 188.

   Analysis of proofs given will show that many supposedly new proofs are only modifications of some more fundamental proof.

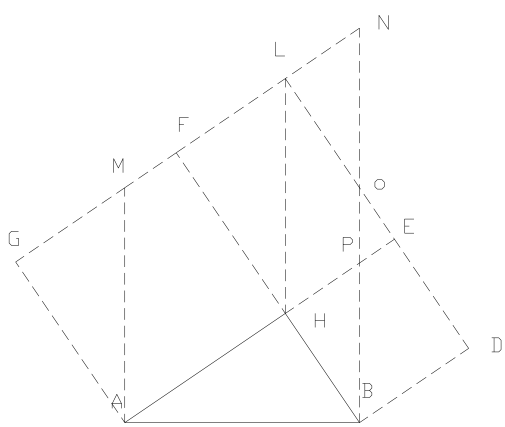


Fig. 307

TWO- HUNDRED – TEN

(Not a Pythagorean Proof)

While case (1), (b) may be proved in some other way, we have selected the following as being quit unique. It is due to the ingenuity of Mr. Arthur R. Colburn of Washington, D.C., and is No. 97 of his 108 proofs.

It rests upon the following Theorem on Parallelogram, which is: "If from one end of the side of a parallogram a straight line be drawn to any point in the opposite side,or the opposite side extended, and a line from the other end of said first side be drawn perpendicular to the first line, or its extention, the product of these two drawn lines will measure the area of the parallogram. " Mr. Colburn formulated this theorm and its use is discussed in Vol. 4, p. 45, of the " Mathematics Teacher." Dec., 1911. I have not seen his proof, but have demonstrated it as follows:

In the paral. ABCD, from the end A of the side AB, draw AF to side DC produced, and from B, the other end of side AB, draw B perp. to AF. Then AF x BG = area of paral. ABCD.

Proof: From D lay off DE = CF, and draw AE and BF forming the paral. ABFE = paral. ABCD. ABF is a triangle and is one-half of ABFE. The area of ABFE = 2 tri. FAB = ½ FA x BG; therefore the area of paral. ABFE = 2 times the area of the tri. FAB, or FA x BG. But the area of paral. ABFE = area of paral. ABCD.

∴ AF x BG measures the area of paral. ABCD. Q.E.D.

By means of this Paralleogram Theorem the Pythagorean Theroem can be proved inmany cases, of which is one.

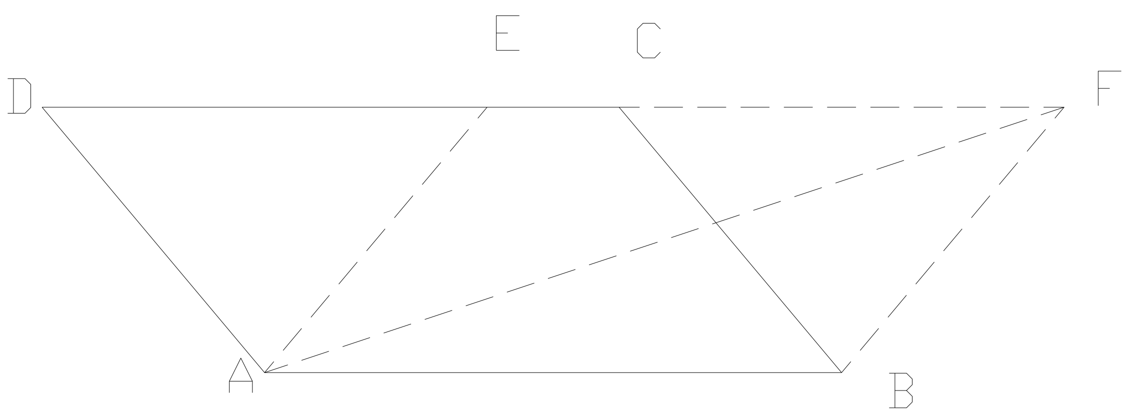


Fig. 308

TWO- HUNDRED – ELVEN

Case (1), (b).

In fig. 309, extend GF and ED to L completing the paral. AL, draw FE and extend AB to M. Then by the paral. theorem:

(1) EF x AM = AE x AG.

(2) EF x BM = FL x BF.

(1) – (2) = (3) EF (AM – BM) = AE x AG – FL x BF (3) = (4) (EF = AB) x AB = AGFH + BDEH, or sq. AB = sq. HG + sq. HD.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. This is No. 97 of A.R. Colburn's 108 proofs.

b. By inspecting this figure we discover in it the five dissected parts as set forth by my Law of Dissection. See proof Ten, fig. 111.

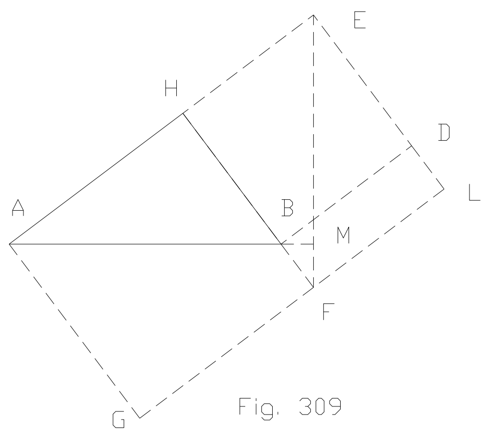


Fig. 309

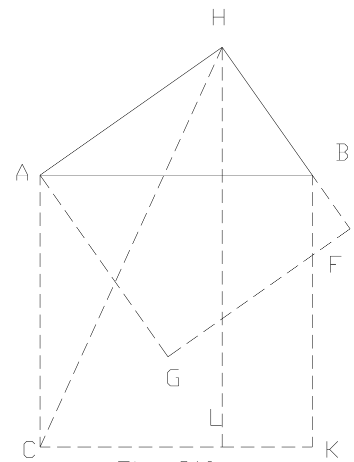


Fig. 310

TWO – HUNDRED – TWELVE

Case (2), (b).

Tri. HAC = tri. ACH.

Tri. HAC = ½ sq. HG

Tri. ACH = ½ rect. AL.

∴ rect. AL =sq. HG. Similarly rect. BL = sq. on HB. But rect. AL + rect. BL = sq. AK.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. Sent to me by J. Adams from The Hague. Holland. But the author not given. Received it March 2, 1934.

TWO – HUNDERED THIRTEEN

Case (2), (c).

In fig. 311, produce GA to M making AM = HB, draw BM, and draw KL par. to AH and CO par. to BH.

Sq. AK = 4 tri. ABH + sq. NH = 4x (AH x BH) / 2 + $(AH – BH)^2$ = 2AH x BH + AH^2 – 2AH x BH + BH^2 = $BH^2 + AH^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Original with author, March, 1926.

b. See Sci. Am. Sup., Vol. 70, p. 383, Dec. 10, 1910, fig. 17, in which Mr. Colburn makes use of the tri. BAM.

c. Another proof, by author, is obtained by comparison and substitution on dissected parts as numbered.

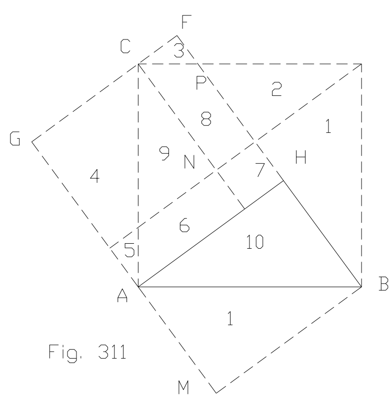


Fig. 311

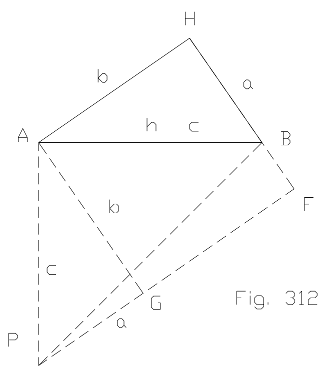


Fig. 312

TWO – HUNDERED- FORTEEN

Case (4), (b).

In fig. 312, produce FG to P making GP = BH, draw AP and BP.

Sq. GH = b^2 = tri. BHA + quad. ABFG = tri. APB + tri. PFB = $\frac{1}{2}c^2 + \frac{1}{2}b^2 – \frac{1}{2}a^2$. ∴ $c^2 = a^2 + b^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$.

a. Proof 4, on p. 104, in "A Comanion of Elementary School Mathematics," (1924) by F. C. Boon, B. A. Pub. By Longmans, Green and Co.

TWO – HUNDRED – FIFTEEN

In fig. 313, produce HB to F and complete the sq. AF. Draw GL perp. to AB, FM par. to AB and NH perp. to AB.

Sq. $AF = AH^2 = 4 (AO \times HO) /2 + [LO^2 = (AO – HO)^2] = 2AO \times HO + HO^2$ $(AO = AH^2/AB)^2 + (HO = AH \times HB / AB)^2 = AH^4 /AB^2 + AH^2 x HB^2 / AB^2 = AH^2 (AH^2 + HB^2) / AB^2$. $\therefore 1 = (AH^2 + BH^2) / AB^2$. $\therefore AB^2 = BH^2 + AH^2$.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH. $h^2 = a^2 + b^2$. Q.E.D.

a. See Am. Math. Mo. , Vol. VI, 1899, p. 69, proof CIII; Dr. Leitzmann, p. 22, fig. 26.

b. The reader will ovserve that this proof proves too mich, as it first proves that $AH^2 = AO^2 + HO^2$, which is the truth sought. Tringles ABH and AOH are similar, and what is true as to the relations of the sides of tri. AHO must be true, by the law of similarity, as to the relations of the sides of the tri. ABH.

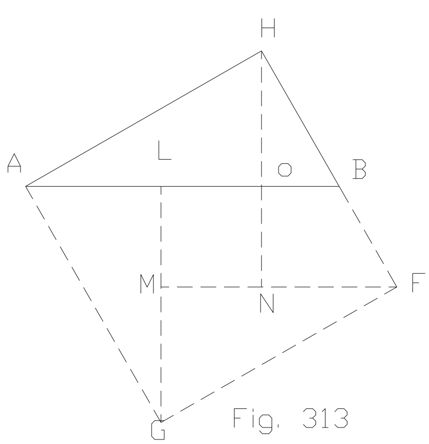


Fig. 313

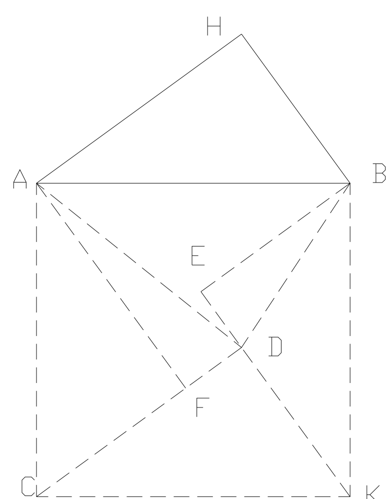


Fig. 314

TWO – HUNDRED – SIXTEEN

Case (6), (a). Thisis a popular figure with authors.

In fig. 314, draw CD and KD par. trespictively to AH and BH, draw AD and BD, and draw AF perp. to CD and BE perp, to KD extended.

Sq. AK = 2 tri. CDA + 2 tri. BDK = CD x AF + KD x EB = $CD^2 + KD^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Original with the author, August 4, 1900.

TWO – HUNDRED – SEVENTEEN

In fig. 315, extend AH andBH to E and F respectively making HE = HB and HF = HA, and through H draw LN perp. to AB, draw CM and KM par. respectively to AH and BH, complete the rect. FE and draw LA, LB HC and HK.

Sq. AK = rect. BN + rect. AN = paral. BM + paral AM = (2 tri. HMK = 2 tri. LHB = sq. BH) + (2 tri. HAL = 2 tri. LAH = sq. AH).

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. Original with author March 26, 1926, 9 p.m.

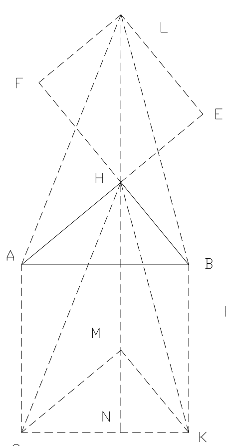


Fig.315

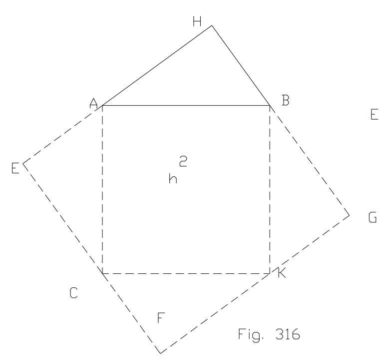


Fig. 316

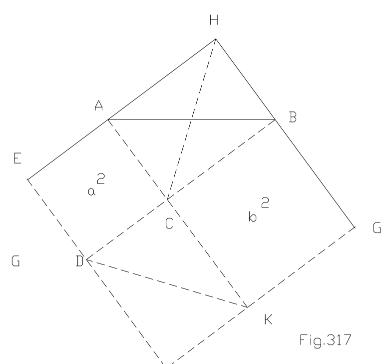


Fig.317

TWO – HUNDRED – EIGHTEEN

In fig. 316, complete the sq's HF and AK; in fig. 317 complete the sq's HF, AD and CG, and draw HC and DK. Sq. HF – 4 tri. ABH = sq. AK = h^2. Again sq. HF – 4 tri. ABH = $a^2 + b^2$. ∴ $h^2 = a^2 + b^2$.

∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Math. Mo., 1858, Dem. 9, Vol. I, p. 159, and credited to Rev. A. D. Wheerler of Brundwick, Me., in work of Henry Boad, London, 1733.

b. An algebraic proof: $a^2 + b^2 + 2ab = h^2 + 2ab$. $\therefore h^2 = a^2 + b^2$.

c. Also, two equal squares of paper and scissors.

TWO- HUNDRED- NINTEEN

In fig. 318, extend HB to N and complete the sq. HM.

Sq. AK = sq. HM – 4 (HB xHA)/2 = $(LA + AH)^2$ – 2HB x HA = LA^2 + 2LA x AH + AH^2 – 2 HB x HA = $BH^2 + AH^2$.

$\therefore$ sq. upon AB = sq. upon BH + sq. upon AH.

a. Credited to T. P. Sqowell, of Rochester, N. Y. See The Math. Magazine, Vol. I, 1882, p. 38; Olney's Geom. Part III, 1872, p. 251, 7th. method; Jour of Ed'n, Vol. XXVI, 1877, p. 21, fig. IX; also Vol. XXVII, 1888, p. 327, 18th proof, by R.E. Binford, Independence, Texas' The School Visitor, Vol. IX,1888, p. 5, proof II; Edwards' Geom. 1895, p. 159,fig. (27); Am. Math. Mo., Vol. VI, 1899, p. 70, proof XCIV; Heath's Math. Monographs, No. 1, 1900, p. 23, proof VIII; Sci. Am. Sup., Vol. 70, p. 359, fig. 4, 1910; Henry Boad's work, London, 1733.

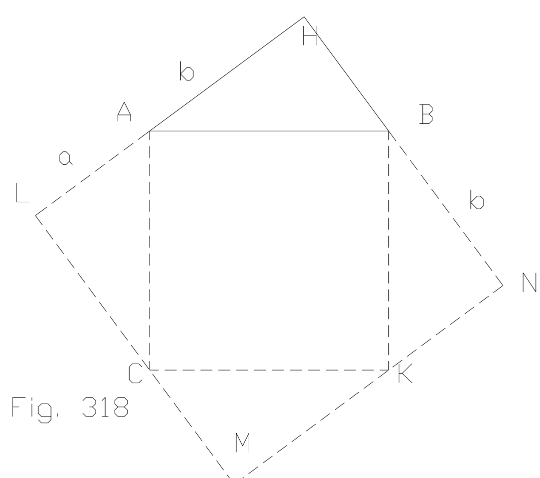


Fig. 318

b. For algebraic solutions, see p. 2, in a pamphlet by Artemus Martin of Washington, D.C., Aug. 1912, entitled " On Rational Right – Angled

Tringles"; and a solution by A.R. Colburn, in Sci. Am. Supplement, Vol. 70, p. 359, Dec. 3, 1910.

c. By drawing the line AK, and considering the part of the figure to the right of said line AK, we have the figure from which the proof known as Garfield's Solution follows ---see proof Two Hundred Thirty – one, fig. 330.

TWO – HUNDRED – TWENTY

In fig. 319, extend HA to L and complete the sq. LN.

Sq. AK = sq. LN – 4 x (HBxHA) /2 = $(HB + HA)^2 – 2HB \times HA = HB^2 + 2HB \times HA + HA^2 – 2HB \times HA$ = sq HB + sq.HA. ∴ sq. upon AB = sq. upon BH + sq. upon AH. ∴ $h^2 = a^2 + b^2$.

a. See Jury Wipper, 1880, p. 35, fig. 32, as given in " Hubert's Rudimenta Algebrae." Wurceb, 1762; Versluys, p. 70, fig. 75.

b. This fig. 319 is but a variation of fig. 240, as also is the proof.

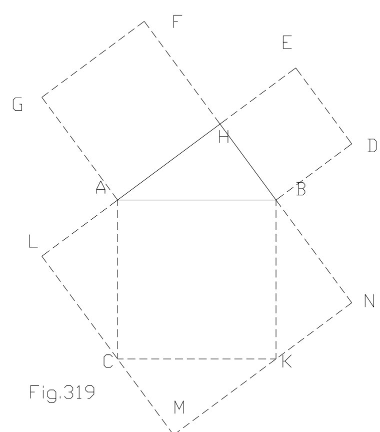


Fig.319