To the Student

As you begin this course in Finite Mathematics, you may feel overwhelmed by the number of topics covered. Don’t worry! Your concerns are normal, and, with the help of your instructor and this text, you will be fine.

After teaching Finite Mathematics for over 30 years and dealing with the math-related questions of my four children, I know what you are going through. This text was written with you, the student, in mind, to help you master the basic concepts of Finite Mathematics.

The many learning aids featured in the text and displayed on the facing page, are there to make your study easier and more rewarding, helping you get the most from the time and effort you invest. If you attend class, work hard, and read and study this text, you will build the knowledge and skills you need to be successful. Here’s how you can use this book to your benefit:

1. **Please, read the material assigned to you.** First, and most important, this book is meant to be read! You will find that the text has additional explanations and examples that will help you.

2. **Start your study of a section with Preparing for this Section.** Quickly, this list of concepts will show you what to expect from the section. Reviewing now will make the section easier to understand and will actually save you time and effort.

3. **Read the list of Objectives.** Provided at the beginning of each section, these will help you recognize the important ideas and skills developed in the section.

4. **Complete the Now Work Problems.** After a concept has been introduced and an example given, you will see Now Work Problem #. Go to the exercises at the end of the section, work the problem cited, and check your answer in the back of the book. If you get it right, you can be confident in continuing on in the section. If you don’t get it right, go back over the explanations and examples to see what you might have missed. Then rework the problem. Ask for help if you miss it again.

If you follow these practices throughout the section, you will find that you have probably done many of your homework problems. In the exercises, every Now Work problem number is **yellow** with a pencil icon ☹️. All the odd-numbered problems have answers in the back of the book and worked-out solutions in the Student Solutions Manual. Be sure you have made an honest effort before looking at a worked-out solution.

5. **Use the Chapter Review feature.** This ensures familiarity with the definitions, formulas, and equations listed under Things To Know. If you are unsure of an item here, use the page reference to go back and review it. Go through the Objectives and be sure you can answer “Yes” to the question “I should be able to...” Review Exercises that relate to each objective are listed to help you.

6. **Lastly, do the problems in the Review Exercises.** Blue problem numbers indicate my suggestions for use in a practice test. Do some of the other problems in the review for more practice to prepare for your exam.

Please do not hesitate to contact me, through the publisher of this book, John Wiley and Sons, with any suggestions or comments that would improve this text. I look forward to hearing from you.

Best wishes,

Michael Sullivan
## Use the Learning Tools!

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<th>Feature</th>
<th>Description</th>
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<tr>
<td><strong>Chapter Opening and Project</strong></td>
<td>Each chapter begins with a situational problem and ends with the related project.</td>
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<tr>
<td><strong>OBJECTIVES</strong></td>
<td>Each section begins with a list of objectives. The objective heading also appears in the text where each objective is covered. Use these as an outline of the important elements of the section.</td>
</tr>
<tr>
<td><strong>PREPARING FOR THIS SECTION</strong></td>
<td>Most sections begin with a list of key concepts to review, with page numbers. This gets you ready for the section.</td>
</tr>
<tr>
<td><strong>NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS</strong></td>
<td>Special problems that support the PREPARING FOR THIS SECTION feature. Solve these problems to be sure you are ready for this section.</td>
</tr>
<tr>
<td><strong>NOW WORK PROBLEMS</strong></td>
<td>These follow most examples, and direct you to a related exercise. If you do this problem, you will have confidence in going on.</td>
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</table>

### Step-by-Step, Annotated Examples

Examples contain detailed, intermediate steps. Many include additional annotations. Read and study with pencil and paper handy.

### Graphing Calculator/Spreadsheet Examples and Exercises

These optional examples and problems require the use of a graphing utility or a spreadsheet program such as Microsoft Excel, and are marked by a special icon and purple numbers.

## Do Your Homework!

<table>
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<th>Problem Type</th>
<th>Description</th>
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<tr>
<td>‘Are You Prepared?’</td>
<td>These assess your retention of the prerequisite material you’ll need for the section. Answers are given at the end of the section exercises.</td>
</tr>
<tr>
<td>Concepts and Vocabulary</td>
<td>These short-answer questions, mainly fill-in-the-blank and true/false items, assess your understanding of key definitions and concepts in the current section.</td>
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<tr>
<td>Skill Building</td>
<td>Correlated to section examples, these problems provide straightforward practice.</td>
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<tr>
<td>Applications and Extensions</td>
<td>These problems allow you to apply your skills to real-world problems, and to explore concepts learned in this section.</td>
</tr>
<tr>
<td>Discussion and Writing</td>
<td>“Discussion and Writing” problems are indicated by green problem numbers. These support class discussion, verbalization of mathematical ideas, and writing and research projects.</td>
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<tr>
<td><strong>NOW WORK PROBLEMS</strong></td>
<td>Many examples refer you to a related homework problem at the end of the section. These related problems are marked by a and yellow numbers.</td>
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And also to the Memory of Bob Walters,
My Production Manager for over 18 years,
A Dear Friend and an
Accomplished Professional
About the Author

Michael Sullivan

is Professor Emeritus in the Department of Mathematics and Computer Science at Chicago State University where he taught for 35 years before retiring. Dr. Sullivan is a member of the American Mathematical Society, the Mathematical Association of America, and the American Mathematical Association of Two Year Colleges. He is a past President of the Text and Academic Authors Association and represents that organization on the Authors Coalition of America. Mike has been writing textbooks since 1973, when this book was first published. Today he has 15 books in print, including three with John Wiley & Sons. He has four children: Kathleen, who teaches college mathematics; Michael, who teaches college mathematics; Dan, who is a director of sales for a college textbook publishing company; and Colleen, who teaches middle-school mathematics. Twelve grandchildren round out the family.
Other Books in this Series


This accessible introduction to calculus is designed to demonstrate how calculus applies to various fields of study. The text is packed with real data and real-life applications to business, economics, and the social and life sciences. Applications using real data enhance student motivation. Many of these applications include source lines to show how mathematics is used in the real world.


This book combines the content of Finite Mathematics and Brief Calculus in a single textbook.
Preface to the Instructor

The Eleventh Edition

The Eleventh Edition of *Finite Mathematics* builds upon the solid foundation of previous editions. While the elements of previous editions that proved successful remain, many improvements have been made. Virtually every change is the result of thoughtful comments and suggestions from colleagues and students who have used previous editions. I am sincerely grateful for this feedback and have tried to incorporate changes that improve the flow and usability of the text, and demonstrate the use of mathematics in contemporary settings.

Features in the Eleventh Edition

**Chapter Opening and Chapter Project**
Each chapter begins with a situation that gives meaning to the mathematics, and ends with a related project.

**A Look Back . . . A Look Forward**
Each chapter contains a discussion of the relationship between what has been learned earlier and what is to be learned in the chapter.

**Objectives**
Each section begins with a list of learning objectives. The objectives also appear in the text as headings and are repeated in the Chapter Review alongside pertinent Examples and Review Exercises related to the objective.
Preparing for This Section

Most sections begin with a list of key concepts to review in preparation for the section. Page references are provided for easy access. Related “Are You Prepared?” problems are given at the beginning of the Exercise set to help students assess their understanding of these concepts. Answers to these problems are given at the end of the Exercise set.

Now Work Problems

Following most Examples, the student is directed to a related problem to assess understanding before going further in the text. The problem is colored yellow and has an icon to identify it.

Step-by-Step Examples

Examples provide detailed, step-by-step solutions, most with explanatory annotations.

Using Technology

These worked examples show students how to use a graphing calculator and/or Microsoft Excel to solve complex, computation-heavy problems (often these examples are first solved in the text using paper-and-pencil methods). Where appropriate, Using Technology examples show TI-84 Plus or Excel screens. Exercises that require use of a calculator or spreadsheet are indicated by icons and purple problem numbers.

Exercise Sets

The Exercise sets are divided into categories:

“Are You Prepared?” Problems

These relate to the review topics listed in PREPARING FOR THIS SECTION. For convenience, page references to the review material are included.

Concepts and Vocabulary

Easy true/false and fill-in-the-blank problems test vocabulary and important ideas found in the section.

Skill Building

These problems provide straightforward practice and are correlated to worked-out examples in the text.

Applications and Extensions

Applied problems that relate to real-world situations are given using a verbal description or a data set. Many of these are sourced using the most current data available. In some exercise sets, problems are given to challenge students to extend concepts into new areas.

Discussion and Writing

Indicated by exercise numbers colored green, these problems support class discussion, collaborative learning, verbalization and writing of mathematical ideas. Some involve research projects.
New to the Eleventh Edition

- Where appropriate, new applied problems and examples have been added and ones that appeared dated or not-so-relevant have been deleted.
- All problems that contained time-sensitive information were updated to reflect the most current data available. Every effort was made to present problems in the light of current facts. This proved particularly challenging in some cases due to the extraordinary economic times we have been in.
- Every section was read and revised with clarity and efficiency of exposition in mind.

Organizational Changes to the Eleventh Edition

- Chapter 2, Systems of Linear Equations; Matrices, of the 10th edition has been divided into two distinct chapters: Chapter 2, Systems of Linear equations, and Chapter 3, Matrices. With this change, all the chapters in the book are roughly the same length, making test construction and classroom time easier to manage. In addition, these changes allow for more flexibility in choosing the content of a course.
- Chapters 6, 7, and 8 of the 10th edition have been consolidated into two chapters: Chapter 7, Probability, and Chapter 8, Additional Probability Topics. If you consult the Contents of the 11th edition, you will see the effect of this change. Now probability can be taught without the burden of handling the more difficult topic of permutations and combinations first. This change also allows for more flexibility in course construction, with the added benefit that the difficulty of the material proceeds more evenly than before.
- The section on the Binomial Theorem now appears in Appendix A.

Using the Eleventh Edition Effectively and Efficiently with Your Syllabus

To meet the varied needs of diverse syllabi, this book contains more than is likely to be covered in a Finite Mathematics course. As the chart below illustrates, Finite Mathematics, Eleventh Edition, has been organized with flexibility of use in mind. Even within a given chapter, certain sections can be treated as optional. See the detailed notes following the flow chart.

Chapter 1 Linear Equations
This chapter serves as the basis for Chapters 2, 3, and 4. It can be used as the starting point of a course. Sections 1.3 and 1.4 are optional.

Chapter 2 Systems of Linear Equations
This chapter discusses various methods for solving systems of linear equations.

Chapter 3 Matrices
This chapter provides an introduction to the algebra of matrices. Section 3.6 is optional.
Preface to the Instructor

Chapter 4 Linear Programming with Two Variables
This chapter discusses linear programming problems that can be solved geometrically.

Chapter 5 Linear Programming: Simplex Method
This chapter requires Sections 2.1 and 2.2 of Chapter 2 and Chapter 4. Appendix B, Using LINDO to Solve Linear Programming Problems, may be used to avoid the heavy computation that the simplex method requires.

Chapter 6 Finance
This chapter can be used as the starting point of a course. Section 6.5 is optional and provides an alternative approach to annuities and amortization.

Chapter 7 Probability
This chapter can be used as the starting point of a course.

Chapter 8 Additional Probability Topics
This chapter continues the study of probability, including the use of permutations and combinations.

Chapter 9 Statistics
While not absolutely necessary, coverage of Chapter 7 in advance may prove to be helpful.

Chapter 10 Markov Chains; Games
This chapter requires Sections 3.1 and 3.2 and Chapter 7. The topics covered here, Markov Chains (10.1–10.3) and Games (10.4–10.6) are independent of each other.

Chapter 11 Logic
This chapter can be used as the starting point of a course.

Appendix A: Review
This material serves as a just-in-time review of intermediate algebra topics. When applicable, its content is referenced in the text in the PREPARING FOR THIS SECTION feature.

Appendix B: Using LINDO to Solve Linear Programming Problems
This material presents an introduction to the software package LINDO, for solving linear programming problems.

Appendix C: Graphing Utilities
This appendix provides an overview of some common uses of a graphing calculator in finite mathematics.

The chapters from the Eighth Edition on Relations, Functions, and Induction (former Chapter 12) and Graphs and Trees (former Chapter 13) are available on the web at www.wiley.com/college/sullivan.
Acknowledgments

There are many colleagues I would like to thank for their input, encouragement, patience, and support in the development of the 11th edition. They have my deepest thanks and appreciation. I apologize for any omissions.

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As this book went to press, Bob Walters passed away after a long and valiant struggle with lung disease. To say he will be missed is an understatement.

And to the entire Wiley sales force for their continued support and confidence. Finally, I welcome comments and suggestions for improving this text. Please do not hesitate to contact me through the publisher, John Wiley & Sons.

Sincerely,
Michael Sullivan
Supplements

Instructor Supplements

Instructor’s Solutions Manual
Solutions to all the exercises in the text, matching the step-by-step methods used in worked examples in the textbook.

Instructor’s PowerPoint Slides
Provided online, PowerPoint presentations include key figures, definitions, theorems, and procedures.

Companion Website
The companion website has many instructor and student resources in digital formats. It can be accessed at www.wiley.com/college/sullivan.

Computerized Test Bank
The Computerized Test Bank, created in Diploma software, is based on questions from the text and allows for varied question types. Using the Computerized Test Bank, professors can freely edit and/or create their own test questions. This software also has the ability to generate questions algorithmically and to create multiple versions of a test.

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- Related media, examples, and sample practice items reinforce the learning objectives.
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- Print Tests
- Concept Mastery

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- Instructor’s Solutions Manual
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Linear Equations

You have been using a prepaid cell phone for a long while and have started to realize it is getting too expensive. To check out various cell phone plans to see which one is cheapest, you search online and are immediately overwhelmed by all of the choices. Is an unlimited talk time plan really the best way to go? Do you really need texting or Web capabilities? You are on a tight budget as a student and want to make the best choice possible so money will be left over each month for other expenses. The mathematics of this chapter provides the background for making the best decision to save the most money. The Chapter Project at the end of the chapter will help you choose the most economical plan to suit your needs.

1.1 Lines
1.2 Pairs of Lines
1.3 Applications in Business and Economics
1.4 Scatter Diagrams; Linear Curve Fitting
   • Chapter Review
   • Chapter Project
   • Mathematical Questions from Professional Exams

A Look Forward
In this chapter we make the connection between algebra and geometry through the rectangular coordinate system. The idea of using a system of rectangular coordinates dates back to ancient times, when such a system was used for surveying and city planning. Apollonius of Perga, in 200 B.C., used a form of rectangular coordinates in his work on conics, although this use does not stand out as clearly as it does in modern treatments. Sporadic use of rectangular coordinates continued until the 1600s. By that time, algebra had developed sufficiently so that René Descartes (1596–1650) and Pierre de Fermat (1601–1665) could take the crucial step, which was the use of rectangular coordinates to translate geometry problems into algebra problems, and vice versa. This step was important for two reasons. First, it allowed both geometers and algebraists to gain critical new insights into their subjects, which previously had been regarded as separate but now were seen to be connected in many important ways. Second, the insights gained made possible the development of calculus, which greatly enlarged the number of areas in which mathematics could be applied and made possible a much deeper understanding of these areas.
1.1 Lines

PREPARING FOR THIS SECTION Before getting started, review the following:
• Algebra Essentials (Appendix A, Section A.2, pp. A–16 to A–20 and A–24)

NOW WORK THE “ARE YOU PREPARED” PROBLEMS ON PAGE 16

OBJECTIVES
1. Graph linear equations (p. 3)
2. Graph a vertical line (p. 6)
3. Find the slope of a line and interpret it (p. 7)
4. Graph a line given a point on the line and the slope (p. 10)
5. Use the point–slope form of a line (p. 10)
6. Find the equation of a horizontal line (p. 11)
7. Find the equation of a line given two points (p. 12)
8. Use the slope–intercept form of a line (p. 12)
9. Solve applied problems involving linear equations (p. 14)

Rectangular Coordinates

We locate a point on the real number line by assigning it a single real number, called the coordinate of the point. For work in a two-dimensional plane, we locate points by using two numbers.

We begin with two real number lines located in the same plane: one horizontal and the other vertical. Call the horizontal line the x-axis, the vertical line the y-axis, and the point of intersection the origin O. Now assign coordinates to every point on these number lines as shown in Figure 1, using a convenient scale. We usually use the same scale on each axis. However, if the graph represents an application problem, scales appropriate to the application are used, which often results in different scales being used on each axis.

The origin O has a value of 0 on both the x-axis and the y-axis. We follow the usual convention that points on the x-axis to the right of O are associated with positive real numbers, and those to the left of O are associated with negative real numbers. Points on the y-axis above O are associated with positive real numbers, and those below O are associated with negative real numbers. In Figure 1, the x-axis and y-axis are labeled as x and y, respectively, and we have used an arrow at the end of each axis to denote the positive direction.

The coordinate system described here is called a rectangular or Cartesian* coordinate system. The plane formed by the x-axis and y-axis is sometimes called the xy-plane, and the x-axis and y-axis are referred to as the coordinate axes.

Any point P in the xy-plane can then be located by using an ordered pair (x, y) of real numbers. Let x denote the signed distance of P from the y-axis (signed in the sense that, if P is to the right of the y-axis, then x > 0, and if P is to the left of the y-axis, then x < 0), and let y denote the signed distance of P from the x-axis. The ordered pair (x, y), also called the coordinates of P, then gives us enough information to locate the point P in the plane.

* Named after René Descartes (1596–1650), a French mathematician, philosopher, and theologian.
For example, to locate the point whose coordinates are \((-3, 1)\), go 3 units along the \(x\)-axis to the left of \(O\) and then go straight up 1 unit. We plot this point by placing a dot at this location. See Figure 2, in which the points with coordinates \((-3, 1), (-2, -3), (3, -2),\) and \((3, 2)\) are plotted.

The origin has coordinates \((0, 0)\). Any point on the \(x\)-axis has coordinates of the form \((x, 0)\), and any point on the \(y\)-axis has coordinates of the form \((0, y)\).

If \((x, y)\) are the coordinates of a point \(P\), then \(x\) is called the \(x\)-coordinate of \(P\), and \(y\) is the \(y\)-coordinate of \(P\). We identify the point \(P\) by its coordinates \((x, y)\) by writing referring to it as “the point \((x, y)\),” rather than “the point whose coordinates are \((x, y)\).”

The coordinate axes divide the \(xy\)-plane into four sections, called quadrants, as shown in Figure 3. In quadrant I, both the \(x\)-coordinate and the \(y\)-coordinate of all points are positive; in quadrant II, \(x\) is negative and \(y\) is positive; in quadrant III, both \(x\) and \(y\) are negative; and in quadrant IV, \(x\) is positive and \(y\) is negative. Points on the coordinate axes belong to no quadrant.

\[
\begin{array}{c|c}
\text{Quadrant II} & \text{Quadrant I} \\
\hline
x < 0, \ y > 0 & x > 0, \ y > 0 \\
\text{Quadrant III} & \text{Quadrant IV} \\
\hline
x < 0, \ y < 0 & x > 0, \ y < 0 \\
\end{array}
\]

\[\text{NOW WORK PROBLEM 9.}\]

\[\text{COMMENT} \quad \text{On a graphing utility, you can set the scale on each axis. Once this has been done, you obtain the viewing rectangle. See Figure 4 for a typical viewing rectangle. You should now read Section C.1, The Viewing Rectangle, in Appendix C.}\]

\section{Graph Linear Equations}

A linear equation in two variables \(x\) and \(y\) is an equation equivalent to one of the form

\[Ax + By = C \tag{1}\]

where \(A, B, C\) are real numbers and \(A\) and \(B\) are not both zero.

Examples of linear equations are

\[
\begin{align*}
3x - 5y - 6 &= 0 & \text{This equation can be written as} \\
3x - 5y &= 6 & A = 3, \ B = -5, \ C = 6 \\
-3x &= 2y - 1 & \text{This equation can be written as} \\
-3x - 2y &= -1 & A = -3, \ B = -2, \ C = -1 \\
\text{or as} & \\
3x + 2y &= 1 & A = 3, \ B = 2, \ C = 1
\end{align*}
\]
Chapter 1 Linear Equations

\[ y = \frac{3}{4}x - 5 \]

Here we can write
\[ -\frac{3}{4}x + y = -5 \quad A = -\frac{3}{4}, B = 1, C = -5 \]

or
\[ 3x - 4y = 20 \quad A = 3, B = -4, C = 20 \]

\[ y = -5 \]

Here we can write
\[ 0 \cdot x + y = -5 \quad A = 0, B = 1, C = -5 \]

\[ x = 4 \]

Here we can write
\[ x + 0 \cdot y = 4 \quad A = 1, B = 0, C = 4 \]

The graph of an equation is the set of all points \((x, y)\) whose coordinates satisfy the equation. For example, \((0, 4)\) is a point on the graph of the equation \(3x + 4y = 16\), because when we substitute \(0\) for \(x\) and \(4\) for \(y\) in the equation, we get
\[ 3 \cdot 0 + 4 \cdot 4 = 16 \]

which is a true statement.

It can be shown that if \(A, B,\) and \(C\) are real numbers, with \(A\) and \(B\) not both zero, then the graph of the equation
\[ Ax + By = C \]

is a line. This is the reason we call it a linear equation. Conversely, any line is the graph of an equation of the form \(Ax + By = C\).

Since any line can be written as an equation in the form \(Ax + By = C\), we call this form the general equation of a line.

Given a linear equation, we can obtain its graph by plotting two points that satisfy its equation and connecting them with a line. Usually the easiest two points to use are the intercepts.

**Definition**

**Intercepts**

The points at which the graph of a linear equation crosses the axes are called intercepts. The **x-intercept** is the point at which the graph crosses the \(x\)-axis; the **y-intercept** is the point at which the graph crosses the \(y\)-axis.

For example, the line shown in Figure 5 has the intercepts \((0, -4)\) and \((3, 0)\).

**Steps for Finding the Intercepts of a Linear Equation**

To find the intercepts of a linear equation \(Ax + By = C\), where \(A \neq 0\) or \(B \neq 0\), follow these steps:

**STEP 1** Let \(y = 0\) and solve for \(x\). This determines the \(x\)-intercept of the line.

**STEP 2** Let \(x = 0\) and solve for \(y\). This determines the \(y\)-intercept of the line.
EXAMPLE 1 Finding the Intercepts of a Linear Equation

Find the intercepts of the equation $2x + 3y = 6$. Graph the equation.

**SOLUTION**

**STEP 1** To find the $x$-intercept, we need to find the number $x$ for which $y = 0$. We let $y = 0$ in the equation and proceed to solve for $x$:

\[
\begin{align*}
2x + 3y &= 6 \\
2x + 3(0) &= 6 & y &= 0 \\
x &= \frac{6}{2} & x &= 3 & \text{Simplify.} \\
&= 3 & \text{Solve for } x.
\end{align*}
\]

The $x$-intercept is $(3, 0)$.

**STEP 2** To find the $y$-intercept, we let $x = 0$ in the equation and solve for $y$:

\[
\begin{align*}
2x + 3y &= 6 \\
2(0) + 3y &= 6 & x &= 0 \\
3y &= 6 & \text{Simplify.} \\
y &= 2 & \text{Solve for } y.
\end{align*}
\]

The $y$-intercept is $(0, 2)$.

Since the equation is a linear equation, its graph is a line. We use the two intercepts $(3, 0)$ and $(0, 2)$ to graph it. See Figure 6.

EXAMPLE 2 Graphing a Linear Equation

Graph the equation: $y = 2x + 5$

**SOLUTION** This equation can be written as

\[
-2x + y = 5
\]

This is a linear equation, so its graph is a line. The intercepts are $(0, 5)$ and $\left(-\frac{5}{2}, 0\right)$, which you should verify. To find a third point, arbitrarily let $x = 10$. Then $y = 2x + 5 = 2(10) + 5 = 25$, so $(10, 25)$ is a third point on the graph. See Figure 7.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
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<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$-\frac{5}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
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NOW WORK PROBLEM 13.
When a line passes through the origin, it has only one intercept. To graph such lines, we need to locate an additional point on the graph.

**Example 3** Graphing a Linear Equation

Graph the equation: \(-x + 2y = 0\)

**Solution** This is a linear equation, so its graph is a line. The only intercept is (0, 0). To locate another point on the graph, let \(x = 4\). (This choice is arbitrary; any choice of \(x\) other than 0 could also be used.) Then

\[
-4 + 2y = 0 \quad \Rightarrow \quad -x + 2y = 0, \quad x = 4
\]

\[
2y = 4 \quad \Rightarrow \quad y = 2
\]

So, \(y = 2\) when \(x = 4\) and (4, 2) is a point on the graph. See Figure 8.

**Example 4** Graphing a Vertical Line

Graph the equation: \(x = 3\)

**Solution** This is a linear equation \((1 \cdot x + 0 \cdot y = 3)\), so its graph is a line. Since \(x = 3\), no matter what \(y\)-coordinate is used, the corresponding \(x\)-coordinate always equals 3. Consequently, the graph of the equation \(x = 3\) is a vertical line with \(x\)-intercept \((3, 0)\) as shown in Figure 9.

As suggested by Example 4, we have the following result:

**Theorem**

**Equation of a Vertical Line**

A vertical line is given by an equation of the form

\[
x = a
\]

where \((a, 0)\) is the \(x\)-intercept.

**Example 5** Finding the Equation of a Vertical Line

Find an equation for the vertical line containing the point \((-1, 6)\).

**Solution** The \(x\)-coordinate of any point on a vertical line is always the same. Since \((-1, 6)\) is a point on the vertical line, its equation is \(x = -1\).

Now work Problem 17(a).
3 Find the Slope of a Line and Interpret It

Consider the staircase illustrated in Figure 10. Each step contains exactly the same horizontal run and the same vertical rise. The ratio of the rise to the run, called the slope, is a numerical measure of the steepness of the staircase. For example, if the run is increased and the rise remains the same, the staircase becomes less steep. If the run is kept the same, but the rise is increased, the staircase becomes more steep. This important characteristic of a line is best defined using rectangular coordinates.

Definition

Slope of a Line

Let \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \) be two distinct points. If \( x_1 \neq x_2 \), the slope \( m \) of the nonvertical line containing \( P \) and \( Q \) is defined by the formula

\[
m = \frac{y_2 - y_1}{x_2 - x_1} \quad x_1 \neq x_2
\]

If \( x_1 = x_2 \), the slope \( m \) is undefined (since \( x_1 = x_2 \) results in division by 0) and the line is vertical.

If \( y_1 = y_2 \), the slope \( m \) is 0 and the line is horizontal.

Figure 11(a) provides an illustration of the slope of a nonvertical line; Figure 11(b) illustrates a vertical line.

As Figure 11(a) illustrates, the slope \( m \) of a nonvertical line may be given as

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}} = \frac{\text{Change in } y}{\text{Change in } x}
\]

The change in \( y \) is usually denoted by \( \Delta y \), read “delta \( y \),” and the change in \( x \) is denoted by \( \Delta x \).

The slope \( m \) of a nonvertical line measures the amount \( y \) changes, \( \Delta y \), as \( x \) changes from \( x_1 \) to \( x_2 \), \( \Delta x \). This is called the average rate of change of \( y \) with respect to \( x \). Then the slope \( m \) is

\[
m = \frac{\Delta y}{\Delta x} = \text{Average rate of change of } y \text{ with respect to } x
\]
EXAMPLE 6 Finding the Slope of a Line and Interpreting It

The slope \( m \) of the line containing the points \((1, 5)\) and \((3, -2)\) is
\[
m = \frac{\Delta y}{\Delta x} = \frac{5 - (-2)}{1 - 3} = \frac{7}{-2} = -\frac{7}{2}
\]

We interpret the slope to mean that for every 2-unit change in \( x \), then \( y \) will change by \(-7\) units. That is, if \( x \) increases by 2 units, then \( y \) decreases by 7 units. The average rate of change of \( y \) with respect to \( x \) is \(-\frac{7}{2}\). See Figure 12.

NOW WORK PROBLEMS 23 AND 27.

Two comments about computing the slope of a nonvertical line may prove helpful:

1. Any two distinct points on the line can be used to compute the slope of the line. (See Figure 13 for justification.)
2. The slope of a line may be computed from \( P = (x_1, y_1) \) to \( Q = (x_2, y_2) \) or from \( Q \) to \( P \) because
\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}
\]

EXAMPLE 7 Finding the Slopes of Various Lines Containing the Same Point \( P \)

Compute the slopes of the lines \( L_1, L_2, L_3, \) and \( L_4 \) containing the following pairs of points. Graph all four lines on the same set of coordinate axes.

\[
\begin{align*}
L_1: & \quad P = (2, 3) \quad Q_1 = (-1, -2) \\
L_2: & \quad P = (2, 3) \quad Q_2 = (3, -1) \\
L_3: & \quad P = (2, 3) \quad Q_3 = (5, 3) \\
L_4: & \quad P = (2, 3) \quad Q_4 = (2, 5)
\end{align*}
\]

SOLUTION Let \( m_1, m_2, m_3, \) and \( m_4 \) denote the slopes of the lines \( L_1, L_2, L_3, \) and \( L_4, \) respectively.
Then

\[ m_1 = \frac{-2 - 3}{1 - 2} = \frac{-5}{-3} = \frac{5}{3} \]  
A rise of 5 divided by a run of 3

\[ m_2 = \frac{-1 - 3}{3 - 2} = \frac{-4}{1} = -4 \]  
A rise of -4 divided by a run of 1

\[ m_3 = \frac{3 - 3}{5 - 2} = \frac{0}{3} = 0 \]  
A rise of 0 divided by a run of 3

\[ m_4 \text{ is undefined} \]  
The x-coordinates of \( P \) and \( Q_4 \) are equal (both \( = 2 \)).

The graphs of these lines are given in Figure 14.

As Figure 14 illustrates,

1. When the slope \( m \) of a line is positive, the line slants upward from left to right (\( L_1 \)).
2. When the slope \( m \) is negative, the line slants downward from left to right (\( L_2 \)).
3. When the slope \( m \) is 0, the line is horizontal (\( L_3 \)).
4. When the slope \( m \) is undefined, the line is vertical (\( L_4 \)).

**COMMENT** Now read Section C.3, Square Screens, in Appendix C.

**SEEING THE CONCEPT** On the same square screen, graph the following equations:

\[ Y_1 = 0 \quad \text{Slope of line is 0.} \]
\[ Y_2 = \frac{1}{4} x \quad \text{Slope of line is } \frac{1}{4}. \]
\[ Y_3 = \frac{1}{2} x \quad \text{Slope of line is } \frac{1}{2}. \]
\[ Y_4 = x \quad \text{Slope of line is 1.} \]
\[ Y_5 = 2x \quad \text{Slope of line is 2.} \]
\[ Y_6 = 6x \quad \text{Slope of line is 6.} \]

See Figure 15.

**SEEING THE CONCEPT** On the same square screen, graph the following equations:

\[ Y_1 = 0 \quad \text{Slope of line is 0.} \]
\[ Y_2 = -\frac{1}{4} x \quad \text{Slope of line is } -\frac{1}{4}. \]
\[ Y_3 = -\frac{1}{2} x \quad \text{Slope of line is } -\frac{1}{2}. \]
\[ Y_4 = -x \quad \text{Slope of line is } -1. \]
\[ Y_5 = -2x \quad \text{Slope of line is } -2. \]
\[ Y_6 = -6x \quad \text{Slope of line is } -6. \]

See Figure 16.

Figures 15 and 16 illustrate that the closer the line is to the vertical position, the greater the magnitude of the slope.

The next example illustrates how the slope of a line can be used to graph the line.
4 Graph a Line Given a Point on the Line and the Slope

EXAMPLE 8 Graphing a Line When Its Slope and a Point Are Given

Draw a graph of the line that contains the point (3, 2) and has a slope of
(a) \( \frac{3}{4} \)  
(b) \( -\frac{4}{5} \)

**SOLUTION**

(a) Slope = \( \frac{\text{rise}}{\text{run}} \). The fact that the slope is \( \frac{3}{4} \) means that for every horizontal movement (run) of 4 units to the right, there will be a vertical movement (rise) of 3 units. If we start at the given point (3, 2) and move 4 units to the right and 3 units up, we reach the point (7, 5). By drawing the line through this point and the point (3, 2), we have the graph. See Figure 17(a).

(b) The fact that the slope is \( -\frac{4}{5} = \frac{-4}{5} \) means that for every horizontal movement of 5 units to the right, there will be a corresponding vertical movement of \(-4\) units (a downward movement). If we start at the given point (3, 2) and move 5 units to the right and then 4 units down, we arrive at the point (8, \(-2\)). By drawing the line through these points, we have the graph. See Figure 17(b).

Alternatively, we can set \( -\frac{4}{5} = \frac{4}{-5} \) so that for every horizontal movement of \(-5\) units (a movement to the left), there will be a corresponding vertical movement of 4 units (upward). This approach brings us to the point \((-2, 6)\), which is also on the graph shown in Figure 17(b).

NOW WORK PROBLEM 35.

5 Use the Point–Slope Form of a Line

Let \( L \) be a nonvertical line with slope \( m \) and containing the point \((x_1, y_1)\). See Figure 18. Since any two distinct points on \( L \) can be used to compute slope, for any other point \((x, y)\) on \( L \), we have

\[
m = \frac{y - y_1}{x - x_1} \quad \text{or} \quad y - y_1 = m(x - x_1)
\]
Theorem

Point–Slope Form of an Equation of a Line
An equation of a nonvertical line with slope \( m \) that contains the point \((x_1, y_1)\) is

\[
y - y_1 = m(x - x_1)
\]  

(3)

EXAMPLE 9 Using the Point–Slope Form of a Line

FIGURE 19

An equation of the line with slope 4 and containing the point \((1, 2)\) can be found by using the point–slope form with \(m = 4, x_1 = 1, \) and \(y_1 = 2\):

\[
y - y_1 = m(x - x_1) \quad \text{Point–slope form}
\]

\[
y - 2 = 4(x - 1) \quad m = 4, x_1 = 1, y_1 = 2
\]

\[
y - 2 = 4x - 4 \quad \text{Simplify.}
\]

\[
4x - y = 2 \quad \text{General equation}
\]

See Figure 19.

\[\text{NOW WORK PROBLEMS 17(c) AND 47.}\]

EXAMPLE 10 Finding the Equation of a Horizontal Line

Find an equation of the horizontal line containing the point \((3, 2)\). Graph the line.

FIGURE 20

\[\text{SOLUTION}\]

The slope of a horizontal line is 0. To get an equation, we use the point–slope form with \(m = 0, x_1 = 3, \) and \(y_1 = 2\):

\[
y - y_1 = m(x - x_1) \quad \text{Point–slope form}
\]

\[
y - 2 = 0 \cdot (x - 3) \quad m = 0, x_1 = 3, y_1 = 2
\]

\[
y - 2 = 0
\]

\[
y = 2
\]

See Figure 20 for the graph.

As suggested by Example 10, we have the following result:

Theorem

Equation of a Horizontal Line
A horizontal line is given by an equation of the form

\[
y = b
\]

where \((0, b)\) is the \(y\)-intercept.

\[\text{NOW WORK PROBLEM 17(b).}\]
EXAMPLE 11 Finding an Equation of a Line Given Two Points

Find an equation of the line containing the points (2, 3) and (–4, 5). Graph the line.

**SOLUTION** Since two points are given, we first compute the slope of the line:

\[ m = \frac{5 - 3}{-4 - 2} = \frac{2}{-6} = \frac{1}{-3} = -\frac{1}{3} \]

We use the point (2, 3) and the fact that the slope \( m = -\frac{1}{3} \) to get the point–slope form of the equation of the line:

\[ y - 3 = -\frac{1}{3}(x - 2) \]

See Figure 21 for the graph.

**FIGURE 21**

In the solution to Example 11 we could have used the point (–4, 5) instead of the point (2, 3). The equation that results, although it looks different, is equivalent to the equation we obtained in the example. (Try it for yourself.)

The general form of the equation of the line in Example 11 can be obtained by multiplying both sides of the point–slope equation by 3 and collecting terms:

\[ y - 3 = -\frac{1}{3}(x - 2) \quad \text{Point–slope equation} \]

\[ 3(y - 3) = 3\left(-\frac{1}{3}\right)(x - 2) \quad \text{Multiply by 3.} \]

\[ 3y - 9 = -1(x - 2) \quad \text{Simplify.} \]

\[ 3y - 9 = -x + 2 \quad \text{Simplify.} \]

\[ x + 3y = 11 \quad \text{General equation} \]

This is the general form of the equation of the line.

**NOW WORK PROBLEM 51.**
1.1 Lines

\[ y - y_1 = m(x - x_1) \]  \hspace{1cm} \text{Point-slope form}

\[ y - b = m(x - 0) \quad \text{if} \quad x = 0, \ y = b \]

\[ y = mx + b \]  \hspace{1cm} \text{Simplify and solve for } y.

**Theorem** \hspace{1cm} **Slope–Intercept Form of an Equation of a Line**

An equation of a line with slope \( m \) and \( y \)-intercept \((0, b)\) is

\[ y = mx + b \] \hspace{1cm} (4)

**SEEING THE CONCEPT** To see the role that the slope \( m \) plays in the equation \( y = mx + b \), graph the following lines on the same square screen.

**FIGURE 22**

\[ \begin{align*}
Y_1 &= -3x + 2 \\
Y_2 &= 3x + 2 \\
Y_3 &= -x + 2 \\
Y_4 &= x + 2 \\
Y_5 &= x + 2
\end{align*} \]

See Figure 22. What do you conclude about the lines \( y = mx + 2 \)?

**FIGURE 23**

\[ \begin{align*}
Y_1 &= 2x \\
Y_2 &= 2x + 1 \\
Y_3 &= 2x - 1 \\
Y_4 &= 2x + 4 \\
Y_5 &= 2x - 4
\end{align*} \]

See Figure 23. What do you conclude about the lines \( y = 2x + b \)?

When an equation of a line is written in slope–intercept form, it is easy to find the slope \( m \) and \( y \)-intercept \((0, b)\) of the line. For example, suppose the equation of the line is

\[ y = -2x + 3 \]

Compare it to \( y = mx + b \):

\[ \begin{align*}
&\quad y = -2x + 3 \\
&\uparrow \quad \uparrow \\
&y = mx + b
\end{align*} \]

The slope of this line is \(-2\) and its \( y \)-intercept is \((0, 3)\).
EXAMPLE 12 Finding the Slope and $y$-Intercept of a Line

Find the slope $m$ and $y$-intercept $(0, b)$ of the line $2x + 4y = 8$. Graph the line.

**SOLUTION** To obtain the slope and $y$-intercept, we transform the equation into its slope–intercept form. To do this, we need to solve for $y$:

\[
2x + 4y = 8 \\
4y = -2x + 8 \\
y = -\frac{1}{2}x + 2
\]

The coefficient of $x$, $-\frac{1}{2}$, is the slope, and the $y$-intercept is $(0, 2)$.

We can graph the line in either of two ways:

1. Use the fact that the $y$-intercept is $(0, 2)$ and the slope is $-\frac{1}{2}$. Then, starting at the point $(0, 2)$, go to the right 2 units and then down 1 unit to the point $(2, 1)$. Plot these points and draw the line containing them. See Figure 24.

2. Locate the intercepts. The $y$-intercept is $(0, 2)$. To obtain the $x$-intercept, we let $y = 0$ in the equation $2x + 4y = 8$ and solve for $x$. When $y = 0$, we have

\[
2x + 4 \cdot 0 = 8 \\
2x = 8 \\
x = 4
\]

The intercepts are $(4, 0)$ and $(0, 2)$. Plot these points and draw the line containing them. Look again at Figure 24.

**NOTE** The second method, locating the intercepts, only produces one point when the line passes through the origin. In this case some other point on the line must be found in order to graph the line. Refer back to Example 3.

NOW WORK PROBLEM 69.

9 Solve Applied Problems Involving Linear Equations

EXAMPLE 13 Daily Cost of Production

A factory that manufactures microwave ovens has daily fixed overhead expenses of $2000. Each microwave oven produced costs $100. Find an equation that relates the daily cost $C$ to the number $x$ of microwaves produced each day.

The fixed overhead expense of $2000 represents the fixed cost, the cost incurred no matter how many microwaves are produced. Since each microwave produced costs $100, the variable cost of producing $x$ microwaves is $100x$. Then the total daily cost $C$ of production is

\[C = 100x + 2000\]

The graph of this equation is given by the line in Figure 25. Notice that the fixed cost $2000$ is represented by the $y$-intercept, while the $100$ cost of producing each microwave is the slope. Also notice that a different scale is used on each axis.
EXAMPLE 14  Predicting the Cost of a Home

In 2008 the cost of an average home in Chicago was $251,364. In 2009 the cost was $222,960.

(a) Assuming that the relationship between time and cost is linear, develop a formula for predicting the cost of an average home in 2010.

(b) Comment on whether you think this trend will continue.

Source: Sperling’s Best Places; www.bestplaces.net

SOLUTION

(a) We let $x$ represent the year and $y$ represent the cost. We seek a relationship between $x$ and $y$. The assumption is that the equation relating $x$ and $y$ is linear. Two points on the graph of the linear equation relating $x$ and $y$ are $(2008, 251,364)$ and $(2009, 222,960)$.

The slope of this line is

\[
\frac{222,960 - 251,364}{2009 - 2008} = -28,404
\]

Using this fact and the point $(2008, 251,364)$, the point–slope form of the equation of the line is

\[
y - 251,364 = -28,404(x - 2008)
\]

For $x = 2010$ we predict the cost of an average home to be

\[
\]

We predict the average cost of a home in Chicago in 2010 to be $194,556.

(b) This prediction of future cost is based on the assumption that annual changes in price remain the same. In this example, the assumption is that each year the cost of a house will go down $28,404 (the slope of the line). If this assumption is not correct, the predicted cost may be incorrect. Also, it is not reasonable to expect home prices to decline for an extended length of time.

<table>
<thead>
<tr>
<th>YEAR</th>
<th>COST</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008</td>
<td>251,364</td>
</tr>
<tr>
<td>2009</td>
<td>222,960</td>
</tr>
<tr>
<td>2010</td>
<td>194,556</td>
</tr>
</tbody>
</table>

FIGURE 26

NOW WORK PROBLEM 113.
SUMMARY

The graph of a linear equation, \( Ax + By = C \), where \( A \) and \( B \) are not both zero, is a line. In this form it is referred to as the general equation of a line.

1. Given the general equation of a line, information can be found about the line:
   (a) Place the equation in slope–intercept form \( y = mx + b \) to find the slope \( m \) and \( y \)-intercept \( (0, b) \).
   (b) Let \( x = 0 \) and solve for \( y \) to find the \( y \)-intercept.
   (c) Let \( y = 0 \) and solve for \( x \) to find the \( x \)-intercept.

2. Given information about a line, an equation of the line can be found. The form of the equation to use depends on the given information. See the table below.

<table>
<thead>
<tr>
<th>Given</th>
<th>Use</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point ((x_1, y_1)), slope (m)</td>
<td>Point–slope form</td>
<td>( y - y_1 = m(x - x_1) )</td>
</tr>
<tr>
<td>Two points ((x_1, y_1), (x_2, y_2))</td>
<td>If ( x_1 = x_2 ), the line is vertical</td>
<td>( x = x_1 )</td>
</tr>
<tr>
<td></td>
<td>If ( x_1 \neq x_2 ), find the slope (m):</td>
<td>( m = \frac{y_2 - y_1}{x_2 - x_1} )</td>
</tr>
<tr>
<td></td>
<td>Then use the point–slope form</td>
<td>( y - y_1 = m(x - x_1) )</td>
</tr>
<tr>
<td>Slope (m), (y)-intercept ((0, b))</td>
<td>Slope–intercept form</td>
<td>( y = mx + b )</td>
</tr>
</tbody>
</table>

EXERCISE 1.1 Answers Begin on Page AN–1.

‘Are You Prepared?’ Problems Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. True or False On the real number line, the coordinate of the origin is the number 0. (pp. A–16 to A–20)

2. Solve the equation: \( 2x + 6 = 10 \) (p. A–24)

Concepts and Vocabulary

3. True or False To find the \( y \)-intercept of a linear equation, let \( x = 0 \) and solve for \( y \).

4. The equation of a vertical line with \( x \)-intercept at \((-3, 0)\) is _______.

5. If the slope of a line is undefined, the line is _______.

6. The line \( y = -4x + 6 \) has slope _______ and \( y \)-intercept _______.

7. The point–slope form of the equation of a line with slope \( m \) containing the point \((x_1, y_1)\) is _______.

8. If the graph of a line slants downward from left to right, its slope \( m \) is (positive, negative, zero).

Skill Building

9. Give the coordinates of each point in the following figure. Assume each coordinate is an integer.

10. Plot each point in the \( xy \)-plane. Tell in which quadrant or on what coordinate axis each point lies.
   - (a) \( A = (-3, 2) \)
   - (b) \( B = (6, 0) \)
   - (c) \( C = (-2, -2) \)
   - (d) \( D = (6, 5) \)
   - (e) \( E = (0, -3) \)
   - (f) \( F = (6, -3) \)
11. Plot the points (2, 0), (2, -3), (2, 4), (2, 1), and (2, -1). Describe the collection of all points of the form (2, y), where y is a real number.

In Problems 13–16, use the given equation to fill in the missing values in each table. Use these points to graph the equation.

13. \( y = 2x + 4 \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>-2</th>
<th>4</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

14. \( y = -3x + 6 \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>-2</th>
<th>4</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

15. \( 2x - y = 6 \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>-2</th>
<th>4</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

16. \( x + 2y = 8 \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>-2</th>
<th>4</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In Problems 17–22 a point is given.
(a) Find the equation of the vertical line containing the given point.
(b) Find the equation of the horizontal line containing the given point.
(c) Find the general equation of a line with slope 5 containing the given point.

17. (2, -3)  18. (5, 4)  19. (-4, 1)  20. (-6, -3)  21. (0, 3)  22. (-6, 0)

In Problems 23–26, find the slope of the line. Give an interpretation of the slope.

23. (0, 0); (2, 1)
24. (-2, 1); (0, 0)
25. (-1, 3); (1, 1)
26. (-1, 1); (2, 2)

In Problems 27–34, plot each pair of points and find the slope of the line containing them. Interpret the slope and graph the line.

27. (2, 3); (1, 0)  28. (1, 2); (3, 4)  29. (-2, 3); (2, 1)  30. (-1, 1); (2, 3)  31. (-3, -1); (2, -1)  32. (4, 2); (-5, 2)  33. (-1, 2); (-1, -2)  34. (2, 0); (2, 2)

In Problems 35–42, graph the line containing the point P and having slope m.

35. \( P = (1, 2); \ m = 2 \)
36. \( P = (2, 1); \ m = 3 \)
37. \( P = (2, 4); \ m = -\frac{3}{4} \)
38. \( P = (1, 3); \ m = -2 \)
39. \( P = (-1, 3); \ m = 0 \)
40. \( P = (2, -4); \ m = 0 \)
41. \( P = (0, 3); \ \text{slope undefined} \)
42. \( P = (-2, 0); \ \text{slope undefined} \)

In Problems 43–64, find the general equation of each line; that is, write the equation in the form \( Ax + By = C \).

43. \( (0, 0); (2, 1) \)
44. \( (-2, 1); (0, 0) \)
45. \( (-1, 3); (1, 1) \)
46. \( (-1, 1); (2, 2) \)

47. Slope = 2; containing the point (-4, 1)
48. Slope = 3; containing the point (-3, 4)
49. Slope = -\( \frac{2}{3} \); containing the point (1, -1)
50. Slope = \( \frac{1}{2} \); containing the point (3, 1)
51. Containing the points (1, 3) and (-1, 2)
52. Containing the points (-3, 4) and (2, 5)
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53. Slope = -2; y-intercept = (0, 3)
55. Slope = 3; x-intercept = (-4, 0)
57. Slope = $\frac{4}{5}$; containing the point (0, 0)
59. x-intercept = (2, 0); y-intercept = (0, -1)
61. Slope undefined; containing the point (1, 4)
63. Slope = 0; containing the point (1, 4)

In Problems 65–80, find the slope and y-intercept of each line. Graph the line.

65. $y = 2x + 3$
66. $y = -3x + 4$
67. $\frac{1}{2}y = x - 1$
68. $\frac{1}{3}x + y = 2$
69. $2x - 3y = 6$
70. $3x + 2y = 6$
71. $x + y = 1$
72. $x - y = 2$
73. $x = -4$
74. $y = -1$
75. $y = 5$
76. $x = 2$
77. $y - x = 0$
78. $x + y = 0$
79. $2y - 3x = 0$
80. $3x + 2y = 0$

In Problems 81–88, use a graphing utility to graph each linear equation. Be sure to use a viewing rectangle that shows the intercepts. Then locate each intercept rounded to two decimal places.

81. $1.2x + 0.8y = 2$
82. $-1.3x + 2.7y = 8$
83. $21x - 15y = 53$
84. $5x - 3y = 82$
85. $\frac{4}{17}x + \frac{6}{23}y = \frac{2}{3}$
86. $\frac{9}{14}x - \frac{3}{8}y = \frac{2}{7}$
87. $11x - \sqrt{3}y = \sqrt{6}$
88. $x + \sqrt{15}y = \sqrt{15}$

In Problems 89–92, match each graph with the correct equation:
(a) $y = x$
(b) $y = 2x$
(c) $y = \frac{x}{2}$
(d) $y = 4x$

In Problems 93–96, write an equation of each line. Express your answer using either the general form or the slope-intercept form of the equation of a line, whichever you prefer.

93.
94.
95.
96.

Applications

97. Cost of Operating a Car  According to the American Automobile Association (AAA), the average annual cost of operating a standard-sized car, including gasoline, oil, tires, and maintenance decreased to $0.54 per mile in 2009.
(a) Find a linear equation that relates the average cost $C$ of operating a standard-sized car and the number $x$ of miles it is driven.
(b) What is the annual cost if you drive 15,000 miles?
(c) Graph the linear equation for $0 \leq x \leq 50,000$.
(d) Write a practical sentence expressing how $C$ changes for every unit increase in $x$.

Source: AAA.com
98. **Cost of Renting a Truck**  In January 2010, the cost of renting a truck in Naples, FL, was $224 per week plus a charge of $0.52 per mile driven.

(a) Find a linear equation that relates the cost \( C \) for a weekly rental in which the truck is driven \( x \) miles.
(b) What is the rental cost if you drive 500 miles in a week?
(c) Graph the linear equation for \( 0 \leq x \leq 2000 \).
(d) Write a practical sentence expressing how \( C \) changes for every unit increase in \( x \).
(e) Write a practical sentence about the \( y \)-intercept.

**Source:** Budget, Tamiami Trail, Naples

99. **Electricity Rates in Illinois**  Commonwealth Edison Company supplies electricity to residential customers for a monthly customer charge of $8.23 plus 10.438 cents per kilowatt-hour for up to 400 kilowatt-hours.

(a) Find a linear equation that relates the monthly charge \( C \), in dollars, to the number \( x \) of kilowatt-hours used in a month, \( 0 \leq x \leq 400 \).
(b) Graph this equation.
(c) What is the monthly charge for using 100 kilowatt-hours?
(d) What is the monthly charge for using 500 kilowatt-hours?
(e) Write a practical sentence expressing how \( C \) changes for every unit change in \( x \).

**Source:** Commonwealth Edison Company, January 2010.

100. **Electricity Rates in Florida**  Florida Power & Light Company supplies electricity to residential customers for a monthly customer charge of $5.69 plus 8.735 cents per kilowatt-hour for up to 1000 kilowatt-hours.

(a) Find a linear equation that relates the monthly charge \( C \), in dollars, to the number \( x \) of kilowatt-hours used in a month, \( 0 \leq x \leq 1000 \).
(b) Graph this equation.
(c) What is the monthly charge for using 200 kilowatt-hours?
(d) What is the monthly charge for using 500 kilowatt-hours?
(e) Write a practical sentence expressing how \( C \) changes for every unit change in \( x \).

**Source:** Florida Power & Light Company, January 2010.

101. **Temperature Conversion**  The relationship between Celsius (°C) and Fahrenheit (°F) degrees for measuring temperature is linear.

(a) Find a linear equation relating °C and °F if 0°C corresponds to 32°F and 100°C corresponds to 212°F.
(b) Use the equation to find the Celsius measure of 68°F.

102. **Temperature Conversion**  The Kelvin (K) scale for measuring temperature is obtained by adding 273 to the Celsius temperature.

(a) Find a linear equation relating K and °C.
(b) Find a linear equation relating K and °F (see Problem 101).

103. **Water Preservation**  At Harlan County Dam in Nebraska, the U.S. Bureau of Reclamation reports that the storage content of the reservoir increased from 315,000 acre-feet (102.7 billion gallons of water) on December 21, 2009 to 319,300 acre-feet (104.1 billion gallons of water) on January 20, 2010. Suppose that the rate of increase of water remains constant.

(a) Find a linear equation that relates the amount \( A \) of water, in billions of gallons, to the time \( t \), in days. Use \( t = 0 \) for December 21, \( t = 1 \) for December 22, and so on.
(b) How much water was in the reservoir on December 31 (\( t = 10 \))?
(c) Interpret the slope.
(d) How much water is predicted to be in the reservoir on January 31, 2010 (\( t = 41 \))?
(e) Write a practical sentence about the reservoir overflow and cause flooding?
(f) Comment on your answer to part (e).

**Source:** U.S. Bureau of Reclamation.

104. **Product Promotion**  A cereal company finds that the number of people who will buy one of its products the first month it is introduced is linearly related to the amount of money it spends on advertising. If it spends $4000 on advertising, 100,000 boxes of cereal will be sold, and if it spends $6000, then 300,000 boxes will be sold.

(a) Find a linear equation describing the relation between the amount \( A \) spent on advertising and the number \( N \) of boxes sold.
(b) How much advertising is needed to sell 200,000 boxes of cereal?
(c) Write a practical sentence expressing how \( A \) changes for each unit increase in \( N \).

105. **Predicting Sales**  Suppose the sales of a company are given by

\[ S = 5000x + 80,000 \]

where \( x \) is measured in years and \( x = 0 \) corresponds to the year 2006.

(a) Find \( S \) when \( x = 0 \).
(b) Find \( S \) when \( x = 3 \).
(c) Find the predicted sales in 2012, assuming this trend continues.
(d) Find the predicted sales in 2015, assuming this trend continues.

106. **Disease Propagation**  Research indicates that in a controlled environment, the number of diseased mice will increase linearly each day after one of the mice in the cage is infected with a particular type of disease-causing germ. There were 8 diseased mice 4 days after the first exposure and 14 diseased mice after 6 days. Write a linear equation that will give the number of diseased mice \( t \) days after the first exposure and \( 14 \) diseased mice 4 days after the first exposure and 14 diseased mice after 6 days. Write a linear equation that will give the number of diseased mice after any given number of days. If there were 40 mice in the cage, how long will it take until they are all infected?

107. **Wages of a Car Salesperson**  In 2008, median earnings, including commissions, for car salespersons was $21.43 per hour or $857.31 per week. Dan receives $400 per week for selling new and used cars. In addition, he receives 5% of the profit on any sales he generates.

(a) Find a linear equation that relates Dan’s weekly salary \( S \) when he has sales that generate a profit of \( x \) dollars.
(b) If Dan has sales that generate a profit of $4000, what are his weekly earnings?
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(c) To equal the median earnings of a car salesperson, Dan would have to have sales that generate a profit of how many dollars?


108. Oil Depletion  The Alaskan oil fields, in operation since 1977, had an estimated reserve of 4.9 billion barrels in 1999. In 2009, the fields had an estimated reserve of 3.5 billion barrels. Assume the rate of depletion is constant.

(a) Find a linear equation that relates the amount $A$, in millions of barrels, of oil left in the fields at any time $t$, where $t$ is the year.

(b) If the trend continues, when will the fields dry out?

(c) Write a practical sentence expressing how $A$ changes for every unit change in $t$.

(d) The Jack Field, discovered in the Gulf of Mexico off the coast of Louisiana in 2006, is estimated to contain up to 15 billion barrels of oil. At the same rate of depletion, how long will this field last?

Source: Energy Information Administration

109. SAT Scores  The average score on the mathematics portion of the SAT has been increasing over the past 10 years. In 1999 the average SAT mathematics score was 475, while in 2009 the average SAT mathematics score was 496. Assume the rate of increase is constant.

(a) Find a linear equation that relates the average SAT mathematics score $S$ at any time $t$, where $t$ is the year.

(b) If the trend continues, will the average SAT mathematics score be in 2011?

Source: The College Board

110. Financial Statement  In November 2005, SBC Communications acquired AT&T through a merger and formally adopted the AT&T name. After the merger, AT&T’s net income increased from $1.45 billion at the end of the first financial quarter in 2006 to $1.81 billion at the end of the second financial quarter in 2006. Assume the rate of increase is constant.

(a) Find a linear equation that relates the net income $I$, in billions of dollars, and the time $t$. Use $t = 1$ for the first quarter of 2006, $t = 2$ for the second quarter of 2006, and so on.

(b) What is the projected net income for the fourth quarter?

(c) Write a practical sentence expressing how $I$ changes for every unit change in $t$.

111. Percent of Population with Bachelor’s Degrees  In 1998 the percent of people over 25 years old who had a bachelor’s degree or higher was 24.4%. By 2008, the percent of people over 25 with a bachelor’s degree or higher was 29.4%. Assume the rate of increase is constant.

(a) Find a linear equation that relates the percent $P$ of people over 25 with a bachelor’s degree or higher at any time $t$, where $t$ is the year.

(b) If the trend continues, estimate the percentage of people over 25 who will have a bachelor’s degree or higher by 2011.

(c) Write a practical sentence expressing how $P$ changes for every unit change in $t$.


112. College Degrees  In 2000, 1,237,875 bachelor’s degrees were conferred by colleges and universities in the United States. In 2007, 1,524,092 bachelor’s degrees were awarded. Suppose we assume the relationship between time and degrees conferred is linear.

(a) Find a linear equation that relates the number $N$ of bachelor’s degrees awarded in the year $t$.

(b) If the trend continues, estimate the number of bachelor’s degrees that are to be awarded in 2011.

(c) Write a practical sentence expressing how $N$ changes for every unit change in $t$.


113. Predicting the Cost of a Home  In 2008, the average cost of a home in Memphis was $94,823. In 2009, the average cost was $88,280.

(a) Assuming that the relationship between time and cost is linear, develop a formula for predicting the average cost of a home.

(b) What will be the average cost of a home in 2011?

Source: National Association of Realtors

114. Weight–Height Relation in the U.S. Army  Assume the recommended weight $w$ of females aged 17–20 years in the U.S. Army is linearly related to their height $h$. If an Army female who is 67 inches tall should weigh 139 pounds and an Army female who is 70 inches tall should weigh 151 pounds, find a linear equation that expresses recommended weight in terms of height.

115. Predicting Sales  The total sales and other operating income for Chevron Corporation were $214,091 billion in 2007 and $264,908 billion in 2008.

(a) Assuming that the dollar amount of annual increase in total sales and other operating income will remain constant, find a linear equation expressing total sales and other operating income $S$ for Chevron Corporation in terms of the time $t$ in years.

(b) Use this equation to predict total sales and other operating income for Chevron Corporation in the year 2011.

Source: Chevron Corporation

116. Predicting Revenue  The net revenue for Dell, Inc. for the fiscal year ending in 2009 was $61.1 billion and its net revenue for the fiscal year ending in 2008 was $61.13 billion.

(a) Assuming that the dollar amount of annual increase in net revenue will remain constant, find a linear equation expressing net revenue $R$ for Dell, Inc. in terms of the time $t$ in years. Let $t = 0$ correspond to the fiscal year ending in 2008.

(b) Use the equation found in part (a) to predict the net revenue for Dell, Inc. for the fiscal year ending in 2011.

Source: Dell, Inc.
117. **Cost of Gasoline** The average price of a gallon of regular gasoline in the United States on January 18, 2010 was $2.739, and one year earlier, the average price of a gallon of regular gasoline was $1.847. Suppose that the Smith family drives a 2010 GMC Yukon 1500 4WD, which has a fuel usage of 17 miles per gallon (city and highway combined).

(a) Find a linear equation that gives this family’s annual fuel cost C as a function of the number x of miles driven in one year, using the price of a gallon of regular gasoline on January 18, 2010.

(b) Find a similar equation using the price of a gallon of regular gasoline one year earlier.

(c) Assuming the Smith family drives 15,000 miles annually, determine their annual fuel cost using the price of gasoline on January 18, 2010.

(d) Determine the Smith family’s annual fuel cost using the price of gasoline one year earlier.

(e) What is the difference between these annual fuel costs?

**Source:** Energy Information Administration, United States Department of Energy and www.fueleconomy.gov

Discussion and Writing

120. Which of the following equations might have the graph shown? (More than one answer is possible.)

(a) $2x + 3y = 6$
(b) $-2x + 3y = 6$
(c) $3x - 4y = -12$
(d) $x - y = 1$
(e) $x - y = -1$
(f) $y = 3x - 5$
(g) $y = 2x + 3$
(h) $y = -3x + 3$

121. Which of the following equations might have the graph shown. (More than one answer is possible.)

(a) $2x + 3y = 6$
(b) $2x - 3y = 6$
(c) $3x + 4y = 12$
(d) $x - y = 1$
(e) $x - y = -1$
(f) $y = -2x + 1$
(g) $y = -\frac{1}{2}x + 10$
(h) $y = x + 4$

118. **Cost of Gasoline** Repeat parts (a)–(e) of Problem 117 for the Jones family who drive a 2010 Ford Fusion FWD that has a fuel usage of 39 miles per gallon (city and highway combined).

119. **Credit and Debit Card Growth** At the end of the first quarter of 2009, there were $N = 1147.5$ million credit and signature debit cards in force on the four major U.S. credit card networks and at the end of the first quarter of 2006, there were 954.7 million credit and signature debit cards in force. Let $t = 0$ correspond to the end of the first quarter of 2006.

(a) Find an equation of the line containing the points $(0, 954.7)$ and $(3, 1147.5)$.

(b) Give an interpretation of the slope of this line as an average rate of change.

(c) Use the equation of the line found in part (a) to estimate the number of credit and signature debit cards that will be in force on the four major U.S. credit card networks at the end of the first quarter of 2011.

(d) Use the equation of the line found in part (a) to estimate the year when the number of credit and signature debit cards that will be in force on the four major U.S. credit card networks will first exceed 1.5 billion cards.

**Source:** CardData

‘Are You Prepared?’ Answers

1. True  
2. {2}
Suppose we look at two lines in the same plane. See Figure 27. Then exactly one of the following must hold:

1. They have no points in common, in which case the lines are **parallel**.
2. They have one point in common, in which case the lines **intersect**.
3. They have two points in common, in which case the lines are **coincident** or **identical**, and all the points on one of the lines are the same as the points on the other line.

**FIGURE 27**

Parallel Lines: No points in common  
Intersecting Lines: One point in common  
Coincident Lines: All points on each line are the same

---

1. Show That Two Lines Are Coincident

Figure 28 illustrates vertical coincident lines and nonvertical coincident lines.

**FIGURE 28**

Coincident lines that are vertical  
Coincident lines that are nonvertical

We are led to the following result.
Theorem

**Coincident Lines**

Coincident lines that are vertical have undefined slope and the same $x$-intercept. Coincident lines that are nonvertical have the same slope and the same intercepts.

To show that two nonvertical lines are coincident only requires that you show they have the same slope and the same $y$-intercept. Do you see why?

**EXAMPLE 1**  **Showing That Two Lines Are Coincident**

Show that the lines given by the following equations are coincident.

$L: 2x - y = 5$  
$M: -4x + 2y = -10$

**SOLUTION**

Write each equation in slope–intercept form:

$L: 2x - y = 5$  
$2x - y = 5$  
$-y = -2x + 5$  
$y = 2x - 5$

$M: -4x + 2y = -10$  
$-4x + 2y = -10$  
$2y = 4x - 10$  
$y = 2x - 5$

The lines $L$ and $M$ have the same slope 2 and the same $y$-intercept $(0, -5)$ so they are coincident. See Figure 29.

**NOW WORK PROBLEM 7.**

**2**  **Show That Two Lines Are Parallel**

Look at the parallel lines in Figure 30.

We see in Figure 30(a) that the two vertical parallel lines have different $x$-intercepts. For the two nonvertical parallel lines in Figure 30(b), equal runs result in equal rises. As a result, nonvertical parallel lines have the same slope. Since they also have no points in common, they will have different $x$- and $y$-intercepts.

**Theorem**  **Parallel Lines**

Parallel lines that are vertical have undefined slope and different $x$-intercepts. Parallel lines that are nonvertical have the same slope and different intercepts.
To show that two nonvertical lines are parallel only requires that you show they have the same slope and different \( y \)-intercepts. Do you see why?

**EXAMPLE 2** Showing That Two Lines Are Parallel

Show that the lines given by the following equations below are parallel.

\[ \begin{align*}
L: & \quad 2x + 3y = 6 \\
M: & \quad 4x + 6y = 0
\end{align*} \]

**SOLUTION** To see if these lines have equal slopes, write each equation in slope–intercept form:

\[ \begin{align*}
L: & \quad 2x + 3y = 6 \\
& \quad 3y = -2x + 6 \\
& \quad y = -\frac{2}{3}x + 2 \\
M: & \quad 4x + 6y = 0 \\
& \quad 6y = -4x \\
& \quad y = -\frac{2}{3}x \\
\end{align*} \]

Slope = \(-\frac{2}{3}\) \\
\( y \)-intercept = (0, 2) \\
\( y \)-intercept = (0, 0)

Since each has slope \(-\frac{2}{3}\) but different \( y \)-intercepts, the lines are parallel. See Figure 31.

**NOW WORK PROBLEM 3.**

**EXAMPLE 3** Finding the Equation of a Line Parallel to a Given Line

Given the line \( x - 4y = 8 \), find an equation for the line that contains the point (2, 1) and is parallel to the given line.

**SOLUTION** First find the slope of the line \( x - 4y = 8 \) by writing it in slope–intercept form \( y = mx + b \):

\[ \begin{align*}
x - 4y &= 8 \\
-4y &= -x + 8 \\
y &= \frac{1}{4}x - 2
\end{align*} \]

Proceed to solve for \( y \).

The slope of the line is \( \frac{1}{4} \).

We seek a line parallel to the given line that contains the point (2, 1). The slope of this line must be \( \frac{1}{4} \). (Do you know why?) Using the point–slope form of the equation of a line, we have

\[ \begin{align*}
y - y_1 &= m(x - x_1) \\
y - 1 &= \frac{1}{4}(x - 2) \\
&= \frac{1}{4}x - \frac{1}{2} \\
y &= \frac{1}{4}x + \frac{1}{2} \\
x - 4y &= -2
\end{align*} \]

Slope–intercept form

General form
Figure 32 illustrates the solution.

**FIGURE 32**

NOW WORK PROBLEM 29.

### 3 Show That Two Lines Intersect

If two lines have exactly one point in common, the common point is called the **point of intersection**. The slopes of intersecting lines are unequal. Do you know why?

**Theorem**

**Intersecting Lines**

Intersecting lines have different slopes.

### EXAMPLE 4 Showing That Two Lines Intersect

Show that the lines given by the following equations below intersect.

\[ L: 2x - y = 5 \]
\[ M: x + y = 4 \]

**SOLUTION** Write each equation in slope–intercept form.

\[ L: 2x - y = 5 \]
\[ -y = -2x + 5 \]
\[ y = 2x - 5 \]

\[ M: x + y = 4 \]
\[ y = -x + 4 \]

The slope of \( L \) is 2 and the slope of \( M \) is −1, so the lines intersect.

NOW WORK PROBLEM 5.

### 4 Find the Point of Intersection of Two Intersecting Lines

**EXAMPLE 5 Finding the Point of Intersection**

Find the point of intersection of the two lines

\[ L: 2x - y = 5 \]
\[ M: x + y = 4 \]

**SOLUTION** The slope–intercept form of each line, as found in Example 4, is

\[ L: y = 2x - 5 \]
\[ M: y = -x + 4 \]
If \((x_0, y_0)\) denotes the point of intersection, then \((x_0, y_0)\) is a point on both \(L\) and \(M\). As a result, we must have

\[
y_0 = 2x_0 - 5 \quad \text{and} \quad y_0 = -x_0 + 4
\]

Set these equal and solve for \(x_0\).

\[
2x_0 - 5 = -x_0 + 4
\]

\[
3x_0 = 9
\]

\[
x_0 = 3
\]

Substituting \(x_0 = 3\) in \(y_0 = 2x_0 - 5\) (or in \(y_0 = -x_0 + 4\)), we find

\[
y_0 = 2(3) - 5 = 6 - 5 = 1
\]

The point of intersection of \(L\) and \(M\) is \((3, 1)\). See Figure 33.

\[\checkmark\] **CHECK:** To verify that the point \((3, 1)\) is on both \(L\) and \(M\), we check to see if \(x = 3\), \(y = 1\) satisfies each equation.

\[
L:\ 2x - y = 6 - 1 = 5\quad M:\ x + y = 3 + 1 = 4
\]

**NOW WORK PROBLEM 15.**

---

**EXAMPLE 6** **Mixing Peanuts**

A store that specializes in selling nuts sells cashews for $5 per pound and peanuts for $2 per pound. At the end of the month the manager finds that the peanuts are not selling well. In order to sell 30 pounds of peanuts more quickly, the manager decides to mix the 30 pounds of peanuts with some cashews and sell the mixture of peanuts and cashews for $3 a pound. How many pounds of cashews should be mixed with the peanuts so that the revenue remains the same as it would be selling the nuts separately?

**SOLUTION**

There are two unknowns: the number of pounds of cashews (call this \(x\)) and the number of pounds of the mixture (call this \(y\)). Since we know that the number of pounds of cashews plus 30 pounds of peanuts equals the number of pounds of the mixture, we can write

\[x + 30 = y\]

Also, in order to keep revenue the same, we must have

\[
\left(\frac{\text{Price per pound}}{\text{of cashews}}\right) \cdot \left(\text{Pounds of cashews}\right) + \left(\frac{\text{Price per pound}}{\text{of peanuts}}\right) \cdot \left(\text{Pounds of peanuts}\right) = \left(\frac{\text{Price per pound}}{\text{of mixture}}\right) \cdot \left(\text{Pounds of mixture}\right)
\]

That is,

\[
5x + 2(30) = 3y
\]

\[
\frac{5}{3}x + 20 = y \quad \text{Divide both sides by 3.}
\]

We now have two equations

\[y = \frac{5}{3}x + 20 \quad \text{and} \quad y = x + 30\]
Since the number of pounds of the mixture, \(y\), is the same in each case, we have

\[
\frac{5}{3}x + 20 = x + 30
\]

\[
\frac{2}{3}x = 10
\]

\[
x = 15
\]

The manager should mix 15 pounds of cashews with 30 pounds of peanuts. See Figure 34. Notice that the point of intersection (15, 45) represents the pounds of cashews (15) in the mixture (45 pounds).

**NOW WORK PROBLEM 37.**

### Example 7: Financial Planning

Kathleen has $80,000 to invest and wants to earn $2400 from it. She can invest in a safe, government-insured Certificate of Deposit, but it only pays 1% per year ($800 per year.) To obtain a higher return, she agrees to invest some of her money in noninsured corporate bonds paying 6% per year. How much should be placed in each investment to achieve her goals?

**Solution**

To earn $2400 on an $80,000 investment requires a rate of interest of 3%. The question is how should Kathleen split her money between the two investments in order to realize her goal of earning a 3% rate of return on her money? We need two dollar amounts: the amount to invest in corporate bonds (call this \(x\)) and the amount to invest in the Certificate of Deposit (call this \(y\)). Since we know that the amount she invests in corporate bonds plus the amount she invests in the Certificate of Deposit equals $80,000, we can write

\[x + y = 80,000\]

Solving this for \(y\), we get

\[y = 80,000 - x\]

which is the amount that will be invested in the Certificate of Deposit. See Table 1.

<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>Amount $</th>
<th>Rate</th>
<th>Time yr</th>
<th>Interest $</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>(x)</td>
<td>6%</td>
<td>0.06</td>
<td>0.06x</td>
</tr>
<tr>
<td>Certificate</td>
<td>(80,000 - x)</td>
<td>1%</td>
<td>0.01</td>
<td>0.01(80,000 - x)</td>
</tr>
<tr>
<td>Total</td>
<td>80,000</td>
<td>3%</td>
<td>0.03</td>
<td>0.03(80,000) = 2400</td>
</tr>
</tbody>
</table>

Since the total interest from the investments is equal to 0.03($80,000) = $2400, the equation relating the interest earned on the accounts is given as

Interest earned on the bonds + Interest earned on the certificate = Total interest earned

\[0.06x + 0.01(80,000 - x) = 2400\]
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(Note that the units are consistent: The unit is dollars on both sides of the equation.)
Now solve the equation for $x$, the amount invested in corporate bonds.

$$
0.06x + 800 - 0.01x = 2400 \\
0.05x + 800 = 2400 \\
0.05x = 1600 \\
x = 32,000
$$

Kathleen should invest $32,000 in corporate bonds and $80,000 − $32,000 = $48,000 in the Certificate of Deposit.

NOW WORK PROBLEM 45.

### SUMMARY

<table>
<thead>
<tr>
<th>Pair of Lines</th>
<th>Conclusion: The lines are</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both vertical</td>
<td>(a) Coincident, if they have the same $x$-intercept</td>
</tr>
<tr>
<td>One vertical, one nonvertical</td>
<td>(b) Parallel, if they have different $x$-intercepts</td>
</tr>
<tr>
<td>Neither vertical</td>
<td>Intersecting</td>
</tr>
<tr>
<td>Write the equation of each line in slope–intercept form:</td>
<td></td>
</tr>
</tbody>
</table>

$$
y = m_1 x + b_1, \quad y = m_2 x + b_2
$$

(a) Coincident, if $m_1 = m_2, b_1 = b_2$
(b) Parallel, if $m_1 = m_2, b_1 \neq b_2$
(c) Intersecting, if $m_1 \neq m_2$

### EXERCISE 1.2  Answers Begin on Page AN–3.

#### Concepts and Vocabulary

1. If two lines have no points in common, they are ______.  
2. If two lines have different slopes, they will ______.

#### Skill Building

In Problems 3–14, determine whether the given pairs of lines are parallel, coincident, or intersecting.

<table>
<thead>
<tr>
<th>3. $L$: $x + y = 10$</th>
<th>4. $L$: $x - y = 5$</th>
<th>5. $L$: $2x + y = 4$</th>
<th>6. $L$: $2x + y = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$: $3x + 3y = 6$</td>
<td>$M$: $-2x + 2y = 8$</td>
<td>$M$: $2x - y = 8$</td>
<td>$M$: $2x - y = -4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$: $2x - 2y = -4$</td>
<td>$M$: $3x + 3y = -12$</td>
<td>$M$: $6x - 9y = -2$</td>
<td>$M$: $-2x + y = -2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$: $x - 2y = -4$</td>
<td>$M$: $2x - y = -1$</td>
<td>$M$: $y = -2$</td>
<td>$M$: $x = -2$</td>
</tr>
</tbody>
</table>

In Problems 15–26, the given pairs of lines intersect. Find the point of intersection. Graph each pair of lines.

<table>
<thead>
<tr>
<th>15. $L$: $x + y = 5$</th>
<th>16. $L$: $2x + y = 7$</th>
<th>17. $L$: $x - y = 2$</th>
<th>18. $L$: $2x - y = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$: $3x - y = 7$</td>
<td>$M$: $x - y = -4$</td>
<td>$M$: $2x + y = 7$</td>
<td>$M$: $x + y = 4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$: $4x - 2y = 4$</td>
<td>$M$: $6x + 3y = 0$</td>
<td>$M$: $x + 2y = 4$</td>
<td>$M$: $2x - y = 1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$: $3x + y = -2$</td>
<td>$M$: $4x - 2y = 0$</td>
<td>$M$: $y = -2$</td>
<td>$M$: $y = 0$</td>
</tr>
</tbody>
</table>
In Problems 27 and 28, find an equation for the line \( L \). Express the answer using the general form or the slope-intercept form, whichever you prefer.

27. \( L \) is parallel to \( y = 2x \) for the line containing the point \((3, 3)\).

28. \( L \) is parallel to \( y = -x \) for the line containing the point \((1, 2)\).

In Problems 29–34, find an equation for the line with the given properties. Express your answer using the general form or the slope-intercept form, whichever you prefer.

29. Parallel to the line \( y = 4x \); containing the point \((-1, 2)\)
30. Parallel to the line \( y = -3x \); containing the point \((-1, 2)\)
31. Parallel to the line \( 2x - y = -2 \); containing the point \((0, 0)\)
32. Parallel to the line \( x - 2y = -5 \); containing the point \((0, 0)\)
33. Parallel to the line \( x = 3 \); containing the point \((4, 2)\)
34. Parallel to the line \( y = 3 \); containing the point \((4, 2)\)
35. Find the equation of the line containing the point \((-2, -5)\) and parallel to the line containing the points \((-2, 9)\) and \((3, -10)\).
36. Find the equation of the line containing the point \((-2, -5)\) and parallel to the line containing the points \((-4, 5)\) and \((2, -1)\).

Applications

37. **Mixing Candy** Sweet Delight Candies sells boxes of candy consisting of creams and caramels. Each box sells for $8.00 and holds 50 pieces of candy (all pieces are the same size). If the caramels cost $0.10 to produce and the creams cost $0.20 to produce, how many caramels and creams should be in each box for no profit or loss? Would you increase or decrease the number of caramels in order to obtain a profit?

38. **Mixing Nuts** The manager of Nutt’s Nuts regularly sells cashews for $6.50 per pound, pecans for $7.50 per pound, and peanuts for $2.00 per pound. How many pounds of cashews and pecans should be mixed with 40 pounds of peanuts to obtain a mixture of 100 pounds that will sell for $4.89 a pound so that the revenue is unchanged?

39. **Financial Planning** Mr. Nicholson has just retired and needs $10,000 per year in supplementary income. He has $150,000 to invest and can invest in AA bonds at 10% annual interest or in Savings and Loan Certificates at 5% interest per year. How much money should be invested in each so that he realizes exactly $10,000 in extra income per year?

40. **Financial Planning** Mr. Nicholson finds after 2 years that because of inflation he now needs $12,000 per year in supplementary income. How should he transfer his funds to achieve this amount? (Use the data from Problem 39.)

41. **Mixing Coffee** California Coffee Roasters sells Kona coffee for $22.95 per pound and Colombian coffee for $6.75 per pound. Suppose they offer a blend of those two coffees for a price of $10.80 per pound. What amounts of Kona and Colombian coffees should be blended to obtain the desired mixture? Hint: Assume that the total weight of the blend is 100 pounds.

42. **Livestock Feed** Both cornmeal and soybean meal are popular feed for livestock. The protein content of cornmeal is 22% and the protein content of soybean meal is 44%. A farmer wants a 300 pound mixture that is 30% protein. How much of each feed should he use to obtain the desired mixture? (Round answer to the nearest integer.)

43. **Mixing Acid** One solution is 15% acid and another is 5% acid. How many cubic centimeters of each should be mixed to obtain 100 cubic centimeters of a solution that is 8% acid?

44. **Financial Planning** A bank loaned $10,000, some at an annual rate of 8% and some at an annual rate of 12%. If the income from these loans was $1000, how much was loaned at 8%? How much at 12%?

45. **Investing in Gold** Suppose an investor purchased \( x \) ounces of gold in January 2006 at the prevailing price of $549.86 per ounce and then sold this amount of gold in January 2010 for $1112.30 per ounce.

(a) Find an equation to express the realized gain, \( y \), in terms of \( x \).

(b) Find the point of intersection of the graph of this equation with the graph of \( y = 10,000 \) to determine the number of ounces of gold (rounded to the nearest tenth) that would result in a gain of $10,000.

Source: The Financial Forecast Center

46. **Comparing Gasoline Costs** On January 21, 2010, the average price of a gallon of regular gasoline was $2.775. The Environmental Protection Agency’s fuel economy estimate for a 2010 Honda Civic Hybrid is 42 miles per gallon (city and highway combined), while the EPA’s fuel economy estimate for a 2010 Ford Fusion Hybrid is 39 mpg (city and highway combined).
Chapter 1  Linear Equations

(a) Using the price of gasoline on January 21, 2010, find an equation to express the annual gasoline cost, \( y \), in terms of the total annual miles driven, \( x \), in a 2010 Honda Civic Hybrid.
(b) Find a similar equation for a 2010 Ford Fusion Hybrid.
(c) Graph each of these equations and find the point of intersection of these graphs with the graph of the line \( x = 15,000 \).
(d) Give an interpretation of the distance between these two points of intersection.

Source: US Energy Information Association and www.fueleconomy.gov

HD Radio On February 1, 2008, there were 1615 High Definition radio stations broadcasting a digital signal alongside their analog signal. On February 1, 2009, there were 1877 such stations. Assume the rate of increase is constant.
(a) Find an equation to express the number \( N \) of High Definition radio stations in terms of \( x \), the number of days after February 1, 2008. Use 365 days in a year.
(b) Find the point of intersection of the graph of this equation with the graph of \( y = 2000 \) to predict the date on which the 2000th HD radio station began broadcasting.

Source: This Week in Consumer Electronics, March 9, 2009

Discussion and Writing

48. The figure below shows the graph of two parallel lines. Which of the following pairs of equations might have such a graph?

(a) \( x - 2y = 3 \)
   \( x + 2y = 7 \)

(b) \( x + y = 2 \)
   \( x + y = -1 \)

(c) \( x - y = -2 \)
   \( x - y = 1 \)

(d) \( x - y = -2 \)
   \( 2x - 2y = -4 \)

(e) \( x + 2y = 2 \)
   \( x + 2y = -1 \)

1.3 Applications in Business and Economics

OBJECTIVES 1 Solve problems involving the break-even point (p. 30)
2 Solve problems involving supply and demand equations (p. 33)

Solve Problems Involving the Break-Even Point

In many businesses the cost \( C \) of production and the number \( x \) of items produced can be expressed as a linear equation \( C = mx + b \), where \( m \) represents the unit variable cost of producing each item and \( b \) is the fixed cost. Similarly, sometimes the revenue \( R \) obtained from sales and the number \( x \) of items produced can also be expressed as a linear equation of the form \( R = px \), where \( p \) is the price charged for each unit sold. When the cost \( C \) of production exceeds the revenue \( R \) from the sales, the business is operating at a loss; when the revenue \( R \) exceeds the cost \( C \), there is a profit; and when the revenue \( R \) and the cost \( C \) are equal, there is no profit or loss. The point at which \( R = C \), that is, the point of intersection of the two lines, is usually referred to as the break-even point.

EXAMPLE 1 Finding the Break-Even Point

Sweet Delight Candies, Inc., has daily fixed costs from salaries, rent, and other operations of $300. Each pound of candy produced costs $1 and is sold for $2.
1.3 Applications to Business and Economics

(a) Find the cost $C$ of production for $x$ pounds of candy.
(b) Find the revenue $R$ from selling $x$ pounds of candy.
(c) What is the break-even point? That is, how many pounds of candy must be produced and sold daily to guarantee no loss and no profit?
(d) Graph $C$ and $R$ and label the break-even point.

**SOLUTION**

(a) The cost $C$ of production is the fixed cost of $300 plus the variable cost of producing $x$ pounds of candy at $1 per pound. That is,

\[ C = 1 \cdot x + 300 = x + 300 \]

(b) The revenue $R$ realized from the sale of $x$ pounds of candy at the price of $2$ per pound is

\[ R = 2 \cdot x = 2x \]

(c) The break-even point is the point where $R = C$.

\[ 2x = x + 300 \]

Rearrange to find $x$:

\[ 2x = x + 300 \]
\[ 2x - x = 300 \]
\[ x = 300 \]

That is, 300 pounds of candy must be produced and sold daily to break even. The break-even point is $(300, 600)$.

(d) Figure 35 shows the graphs of $C$ and $R$ and the break-even point. Notice that for $x > 300$, the revenue $R$ always exceeds the cost $C$ so that a profit results. Similarly, for $x < 300$, the cost exceeds the revenue, resulting in a loss.

**EXAMPLE 2 Analyzing Break-Even Points**

After negotiations with employees of Sweet Delight Candies and an increase in the price of sugar, the daily cost $C$ of production for $x$ pounds of candy changed to

\[ C = 1.05x + 330 \]

(a) If each pound of candy is sold for $2.00, how many pounds must be produced and sold daily to make a profit?

(b) If the selling price is increased to $2.25 per pound, what is the break-even point?

(c) If it is known that 325 pounds of candy can be produced and sold daily, what price should be charged per pound to guarantee no loss?

**SOLUTION**

(a) If each pound is sold for $2.00, the revenue $R$ from sales is

\[ R = 2x \]

where $x$ represents the number of pounds produced and sold daily. Set $R = C$ to find the break-even point.

\[ R = C \]
\[ 2x = 1.05x + 330 \]
\[ 0.95x = 330 \]
\[ x = \frac{330}{0.95} = 347.37 \]

If 347 pounds or less of candy are produced and sold daily, a loss is incurred; if 348 pounds or more are sold daily, a profit results.
32  Chapter 1  Linear Equations

(b) If the selling price is increased to $2.25 per pound, the revenue \( R \) from sales is

\[
R = 2.25x
\]

The break-even point is the point where \( R = C \).

\[
2.25x = 1.05x + 330 \quad R = 2.25x, \quad C = 1.05x + 330
\]

\[
x = \frac{330}{1.2} = 275
\]

With the new selling price, the break-even point occurs when 275 pounds of candy are produced and sold daily.

(c) If we know that 325 pounds of candy will be produced and sold daily, the price per pound \( p \) needed to guarantee no loss (that is, to guarantee at worst a break-even point) is the solution of the equation

\[
R = C
\]

\[
xp = 1.05x + 330 \quad R = xp, \quad C = 1.05x + 330
\]

\[
325p = (1.05)(325) + 330 \quad x = 325
\]

\[
325p = 671.25 \quad \text{Simplify.}
\]

\[
p = \$2.07 \quad \text{Solve for } p.
\]

The company should charge $2.07 per pound to guarantee no loss, provided at least 325 pounds will be produced and sold daily.

\[\square\]

EXAMPLE 3  Analyzing Break-Even Points

A producer sells items for $0.30 each.

(a) Determine the revenue \( R \) from selling \( x \) items.

(b) If the cost for production is

\[
C_1 = 0.15x + 105
\]

where \( x \) is the number of items sold.

(c) If the cost can be changed to

\[
C_2 = 0.12x + 110
\]

would it be advantageous?

(d) Graph \( R, C_1 \), and \( C_2 \) together.

SOLUTION  (a) The revenue \( R \) from selling \( x \) items is

\[
R = 0.30x
\]

(b) If the cost for production is \( C_1 = 0.15x + 105 \), then the break-even point is the point where \( R = C_1 \).

\[
0.3x = 0.15x + 105 \quad R = 0.3x, \quad C_1 = 0.15x + 105
\]

\[
0.15x = 105 \quad \text{Simplify.}
\]

\[
x = 700 \quad \text{Solve for } x.
\]

The break-even point occurs when 700 items are sold.
(c) If the revenue received remains at \( R = 0.3x \), but the cost for production changes to \( C_2 = 0.12x + 110 \), then the break-even point is the point where \( R = C_2 \).

\[
R = C_2 \\
0.3x = 0.12x + 110 \\
0.18x = 110 \\
x = 611.11
\]

Simplify. Solve for \( x \).

The break-even point for the cost in (a) was 700 items. Since the cost in (b) will require fewer items to be sold in order to break even, management should probably change over to the new cost.

(d) Figure 36 shows the graphs of \( R \), \( C_1 \), and \( C_2 \).

2. Solve Problems Involving Supply and Demand Equations

The supply equation in economics is used to specify the amount of a particular commodity that sellers are willing to offer in the market at various prices. The demand equation specifies the amount of a particular commodity that buyers are willing to purchase at various prices.

An increase in price \( p \) usually causes an increase in the supply \( S \) and a decrease in demand \( D \). On the other hand, a decrease in price brings about a decrease in supply and an increase in demand. The market price is defined as the price at which supply and demand are equal (the point of intersection).

Figure 37 illustrates a typical supply/demand situation.

**EXAMPLE 4** Supply and Demand

The supply and demand for flour have been estimated as being given by the equations

\[
S = 0.8p + 0.5 \quad D = -0.4p + 1.5
\]

where \( p \) is measured in dollars and \( S \) and \( D \) are measured in pound units of flour. Find the market price and graph the supply and demand equations.
The market price \( p \) is the solution of the equation

\[
S = D
\]

\[
0.8p + 0.5 = -0.4p + 1.5
\]

\[
1.2p = 1
\]

\[
p = 0.83
\]

At a price of $0.83 per pound, supply and demand for flour are equal. The graphs are shown in Figure 38.

Now work problems 9 and 15.
17. **DVD Club** A DVD club offers four DVDs for $0.49 each plus shipping and handling and additional DVDs for $17.95 each plus shipping and handling. Shipping and handling is $2.31 per DVD. A discount retailer offers the same DVDs for $14.95 each plus 7% sales tax. What is the maximum number of DVDs that can be ordered from the club while keeping it a better deal than the discount retailer?

18. **Broadcasting Profits** The following table gives the operating expenses and revenues for radio and television broadcasting for the years 2004 and 2007.

<table>
<thead>
<tr>
<th></th>
<th>2004</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Radio</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Revenue (in billions)</td>
<td>$13.817</td>
<td>$13.624</td>
</tr>
<tr>
<td>Cost (in billions)</td>
<td>$9.920</td>
<td>$9.988</td>
</tr>
<tr>
<td>Profit</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Fill in the profit rows in each table.
(b) Find equations that relate the profit \( P_r \) for radio broadcasting and the profit \( P_t \) for television broadcasting in terms of the year \( t \). Assume the relationships are linear.
(c) Using the equations from part (a), determine when the two media had the same profit.

**Source:** U.S. Census Bureau, 2010 *Statistical Abstract*

### OBJECTIVES

1. Draw and interpret scatter diagrams (p. 35)
2. Distinguish between linear and nonlinear relations (p. 36)
3. Use a graphing utility to find the line of best fit (p. 38)

#### 1.4 Scatter Diagrams; Linear Curve Fitting

**EXAMPLE 1** Drawing a Scatter Diagram

The data listed in Table 2 represent the apparent temperature versus the relative humidity in a room whose actual temperature is 72° Fahrenheit.

<table>
<thead>
<tr>
<th>Relative Humidity (( % )), ( x )</th>
<th>Apparent Temperature, ( y )</th>
<th>(( x ), ( y ))</th>
<th>Relative Humidity (( % )), ( x )</th>
<th>Apparent Temperature, ( y )</th>
<th>(( x ), ( y ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>64</td>
<td>(0, 64)</td>
<td>60</td>
<td>72</td>
<td>(60, 72)</td>
</tr>
<tr>
<td>10</td>
<td>65</td>
<td>(10, 65)</td>
<td>70</td>
<td>73</td>
<td>(70, 73)</td>
</tr>
<tr>
<td>20</td>
<td>67</td>
<td>(20, 67)</td>
<td>80</td>
<td>74</td>
<td>(80, 74)</td>
</tr>
<tr>
<td>30</td>
<td>68</td>
<td>(30, 68)</td>
<td>90</td>
<td>75</td>
<td>(90, 75)</td>
</tr>
<tr>
<td>40</td>
<td>70</td>
<td>(40, 70)</td>
<td>100</td>
<td>76</td>
<td>(100, 76)</td>
</tr>
<tr>
<td>50</td>
<td>71</td>
<td>(50, 71)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(a) Draw a scatter diagram by hand.

(b) Use a graphing utility to draw a scatter diagram.*

(c) Describe what happens to the apparent temperature as the relative humidity increases.

SOLUTION

(a) To draw a scatter diagram by hand, we plot the ordered pairs listed in Table 2, with the relative humidity as the x-coordinate and the apparent temperature as the y-coordinate. See Figure 39(a). Notice that the points in a scatter diagram are not connected.

(b) Figure 39(b) shows the scatter diagram using a graphing utility.

(c) We see from the scatter diagrams that, as the relative humidity increases, the apparent temperature increases.

NOW WORK PROBLEMS 9(a) and 9(d).

2 Distinguish Between Linear and Nonlinear Relations

Scatter diagrams are used to help us see the type of relation that may exist between two variables. In this text, we concentrate on distinguishing between linear and nonlinear relations. See Figure 40.
EXAMPLE 2  Distinguishing Between Linear and Nonlinear Relations

Determine whether the relation between the two variables in Figure 41 is linear or nonlinear.

![Figure 41](image)

(a) Linear  (b) Nonlinear  (c) Nonlinear  (d) Nonlinear

**SOLUTION** (a) Linear  (b) Nonlinear  (c) Nonlinear  (d) Nonlinear

NOW WORK PROBLEM 3.

In this book we only study data whose scatter diagrams imply that a linear relation exists between the two variables.

Suppose that the scatter diagram of a set of data appears to be linearly related as in Figure 40(a) or (b). We might wish to find an equation of a line that relates the two variables. One way to obtain an equation for such data is to draw a line through two points on the scatter diagram and find the equation of the line.

EXAMPLE 3  Find an Equation for Linearly Related Data

Using the data in Table 2 from Example 1, select two points from the data and find an equation of the line containing the points.

(a) Graph the line on the scatter diagram obtained in Example 1(a).

(b) Graph the line on the scatter diagram obtained in Example 1(b).

**SOLUTION** Select two points, say (10, 65) and (70, 73). (You should select your own two points and complete the solution.) The slope of the line joining the points (10, 65) and (70, 73) is

\[
m = \frac{73 - 65}{70 - 10} = \frac{8}{60} = \frac{2}{15}
\]

The equation of the line with slope \(\frac{2}{15}\) and passing through (10, 65) is found using the point–slope form with \(m = \frac{2}{15}\), \(x_1 = 10\), and \(y_1 = 65\).

\[
y - y_1 = m(x - x_1)
\]

\[
y - 65 = \frac{2}{15}(x - 10)
\]

\[
y = \frac{2}{15}x + \frac{191}{3}
\]
Chapter 1  Linear Equations

38

(a) Figure 42(a) shows the scatter diagram with the graph of the line drawn by hand.

(b) Figure 42(b) shows the scatter diagram with the graph of the line using a graphing utility.

NOW WORK PROBLEMS 9(b) and (c).

Use a Graphing Utility to Find the Line of Best Fit

The line obtained in Example 3 depends on the selection of points, which will vary from person to person. So the line that we found might be different from the line that you found. Although the line that we found in Example 3 appears to “fit” the data well, there may be a line that “fits better.” Do you think your line fits the data better? Is there a line of best fit? As it turns out, there is a method for finding the line that best fits linearly related data (called the line of best fit).*

Example 4  Finding the Line of Best Fit

Using the data in Table 2 from Example 1,

(a) Find the line of best fit using a graphing utility.

(b) Graph the line of best fit on the scatter diagram obtained in Example 1(b).

(c) Interpret the slope of the line of best fit.

(d) Use the line of best fit to predict the apparent temperature of a room whose actual temperature is 72°F and relative humidity is 45%.

Solution

(a) Graphing utilities contain built-in programs that find the line of best fit for a collection of points in a scatter diagram. (Look in your owner’s manual under Linear Regression or Line of Best Fit for details on how to execute the program.) Upon executing the LINear REGression program on a TI-84 Plus, we obtain the results shown in Figure 43. The output the utility provides shows us the equation \( y = ax + b \), where \( a \) is the slope of the line and \( (0, b) \) is the \( y \)-intercept. The line of best fit that relates relative humidity to apparent temperature may be expressed as the line \( y = 0.121x + 64.409 \).

* We show how this line of best fit is found algebraically in Chapter 2. Here we use a graphing utility to find the line of best fit.
1.4 Scatter Diagrams; Linear Curve Fitting

(b) Figure 44 shows the graph of the line of best fit, along with the scatter diagram.
(c) The slope of the line of best fit is 0.121, which means that, for every 1% increase in the relative humidity, apparent room temperature increases 0.121°F.
(d) Letting \( x = 45 \) in the equation of the line of best fit, we obtain 
\[
y = 0.121(45) + 64.409 \approx 70°F
\]
which is the apparent temperature in the room.

NOW WORK PROBLEMS 9(d), (e), AND (f).

Does the line of best fit appear to be a good fit? In other words, does the line appear to accurately describe the relation between temperature and relative humidity?

And just how “good” is this line of best fit? The answers are given by what is called the correlation coefficient. Look again at Figure 43. The last line of output is \( r = 0.994 \). This number, called the correlation coefficient, \( r \), \(-1 \leq r \leq 1\), is a measure of the strength of the linear relation that exists between two variables. The closer that \(|r|\) is to 1, the more perfect the linear relationship is. If \( r \) is close to 0, there is little or no linear relationship between the variables. A negative value of \( r \), \( r < 0 \), indicates that as \( x \) increases \( y \) decreases; a positive value of \( r \), \( r > 0 \), indicates that as \( x \) increases \( y \) does also. The data given in Example 1, having a correlation coefficient of 0.994, are indicative of a strong linear relationship with positive slope.

Exercise 1.4 Answers Begin on Page AN–4.

Concepts and Vocabulary

1. **True or False** A scatter diagram is used to identify the type of relation that might exist between the independent variable and the dependent variable.

2. The mathematical measure of the goodness of fit of a line is called the______ ______.

Skill Building

In Problems 3–8, examine the scatter diagram and determine whether the type of relation that may exist is linear or nonlinear.
In Problems 9–16,
(a) Draw a scatter diagram by hand.
(b) Select two points from the scatter diagram and find the equation of the line containing the points selected.*
(c) Graph the line found in part (b) on the scatter diagram.
(d) Use a graphing utility to draw a scatter diagram.
(e) Use a graphing utility to find the line of best fit.
(f) Use a graphing utility to graph the line of best fit on the scatter diagram.

9. \[ \begin{array}{cccccccc}
   x & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
   y & 4 & 6 & 7 & 10 & 12 & 14 & 16 \\
\end{array} \]

10. \[ \begin{array}{cccccccc}
   x & 3 & 5 & 7 & 9 & 11 & 13 \\
   y & 0 & 2 & 3 & 6 & 9 & 11 \\
\end{array} \]

11. \[ \begin{array}{cccc}
   x & -2 & -1 & 0 & 1 & 2 \\
   y & -4 & 0 & 1 & 4 & 5 \\
\end{array} \]

12. \[ \begin{array}{cccc}
   x & -2 & -1 & 0 & 1 & 2 \\
   y & 7 & 6 & 3 & 2 & 0 \\
\end{array} \]

13. \[ \begin{array}{cccccccc}
   x & 20 & 30 & 40 & 50 & 60 \\
   y & 100 & 95 & 91 & 83 & 70 \\
\end{array} \]

14. \[ \begin{array}{cccccccc}
   x & 5 & 10 & 15 & 20 & 25 \\
   y & 2 & 4 & 7 & 9 & 11 \\
\end{array} \]

15. \[ \begin{array}{cccccccc}
   x & -20 & -17 & -15 & -14 & -10 \\
   y & 100 & 120 & 118 & 130 & 140 \\
\end{array} \]

16. \[ \begin{array}{cccccccc}
   x & -30 & -27 & -25 & -20 & -14 \\
   y & 10 & 12 & 13 & 13 & 18 \\
\end{array} \]

Applications

17. Consumption and Disposable Income  An economist wishes to estimate a line that relates personal consumption expenditures \( C \) and disposable income \( I \). Both \( C \) and \( I \) are in thousands of dollars. She interviews eight heads of households for families of size 3 and obtains the data below.

\[
\begin{array}{c|ccccc}
I \text{ (in thousands)} & 20 & 20 & 18 & 27 & 36 \\
C \text{ (in thousands)} & 16 & 18 & 13 & 21 & 27 \\
\end{array}
\]

Let \( I \) represent the independent variable and \( C \) the dependent variable.
(a) Draw a scatter diagram by hand.
(b) Find a line that fits the data.*
(c) Interpret the slope. The slope of this line is called the marginal propensity to consume.
(d) Predict the consumption of a family whose disposable income is $42,000.
(e) Use a graphing utility to find the line of best fit.

18. Marginal Propensity to Save  The same economist as in Problem 17 wants to estimate a line that relates savings \( S \) and disposable income \( I \). Let \( S = I - C \) be the dependent variable and \( I \) the independent variable.
(a) Draw a scatter diagram by hand.
(b) Find a line that fits the data.
(c) Interpret the slope. The slope of this line is called the marginal propensity to save.
(d) Predict the savings of a family whose disposable income is $42,000.
(e) Use a graphing utility to find the line of best fit.

19. Apparent Room Temperature  The data on page 41 represent the apparent temperature versus the relative humidity in a room whose actual temperature is 65°F. Let \( h \) represent the independent variable and \( T \) the dependent variable.
(a) Use a graphing utility to draw a scatter diagram of the data.
(b) Use a graphing utility to find the line of best fit to the data.
(c) Graph the line of best fit on the scatter diagram drawn in part (a).
(d) Interpret the slope of the line of best fit.
(e) Determine the apparent temperature of a room whose actual temperature is 65°F if the relative humidity is 75%.
1.4 Scatter Diagrams; Linear Curve Fitting

### 20. Apparent Room Temperature

The data in the table represent the apparent temperature versus the relative humidity in a room whose actual temperature is 75°F. Let \( h \) represent the independent variable and let \( T \) be the dependent variable.

<table>
<thead>
<tr>
<th>Relative Humidity, ( h )%</th>
<th>Apparent Temperature, ( T ) (°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>59</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>20</td>
<td>61</td>
</tr>
<tr>
<td>30</td>
<td>61</td>
</tr>
<tr>
<td>40</td>
<td>62</td>
</tr>
<tr>
<td>50</td>
<td>63</td>
</tr>
<tr>
<td>60</td>
<td>64</td>
</tr>
<tr>
<td>70</td>
<td>65</td>
</tr>
<tr>
<td>80</td>
<td>65</td>
</tr>
<tr>
<td>90</td>
<td>66</td>
</tr>
<tr>
<td>100</td>
<td>67</td>
</tr>
</tbody>
</table>

**Source:** National Oceanic and Atmospheric Administration

<table>
<thead>
<tr>
<th>Year</th>
<th>Annual Residential Energy Use in the United States from Petroleum (in quadrillion btu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>10.71</td>
</tr>
<tr>
<td>1998</td>
<td>10.28</td>
</tr>
<tr>
<td>1999</td>
<td>10.69</td>
</tr>
<tr>
<td>2000</td>
<td>11.24</td>
</tr>
<tr>
<td>2001</td>
<td>10.98</td>
</tr>
<tr>
<td>2002</td>
<td>11.25</td>
</tr>
<tr>
<td>2003</td>
<td>11.61</td>
</tr>
<tr>
<td>2004</td>
<td>11.43</td>
</tr>
<tr>
<td>2005</td>
<td>11.56</td>
</tr>
<tr>
<td>2006</td>
<td>10.8</td>
</tr>
<tr>
<td>2007</td>
<td>11.38</td>
</tr>
<tr>
<td>2008</td>
<td>11.48</td>
</tr>
<tr>
<td>2009</td>
<td>11.22</td>
</tr>
</tbody>
</table>

**Source:** U.S. Department of Energy, Energy Information Administration

Let \( E \) be the dependent variable representing annual energy use from petroleum and let \( t \) be the independent variable representing time. Let \( t = 0 \) correspond to the year 1995.

(a) Find the line of best fit for this data and then use this line to predict annual energy use from petroleum for the years 2015, 2030, and 2035.

(b) Compare your results with the following forecasts given by the U.S. Department of Energy. Can you explain any differences?

<table>
<thead>
<tr>
<th>Year</th>
<th>Forecasted Annual Residential Energy Use in the United States from Petroleum (in quadrillion btu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2015</td>
<td>11.07</td>
</tr>
<tr>
<td>2030</td>
<td>11.93</td>
</tr>
<tr>
<td>2035</td>
<td>12.12</td>
</tr>
</tbody>
</table>

(d) Interpret the slope of the line of best fit.

(e) Determine the apparent temperature of a room whose actual temperature is 75°F if the relative humidity is 75%.
CHAPTER 1 REVIEW

OBJECTIVES

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<th>You should be able to</th>
<th>Review Exercises</th>
</tr>
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<td>1–8</td>
</tr>
<tr>
<td></td>
<td>4, 5</td>
<td>2. Graph a vertical line (p. 6)</td>
<td>12, 13</td>
</tr>
<tr>
<td></td>
<td>6, 7</td>
<td>3. Find the slope of a line and interpret it (p. 7)</td>
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</tr>
<tr>
<td></td>
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<td>10</td>
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</tr>
<tr>
<td></td>
<td>11</td>
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<tr>
<td></td>
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<tr>
<td></td>
<td>13, 14</td>
<td>9. Solve applied problems involving linear equations (p. 14)</td>
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</tr>
<tr>
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<td>1</td>
<td>1. Show that two lines are coincident (p. 22)</td>
<td>28, 31</td>
</tr>
<tr>
<td></td>
<td>2, 3</td>
<td>2. Show that two lines are parallel (p. 23)</td>
<td>19, 20, 27, 32</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3. Show that two lines intersect (p. 25)</td>
<td>29, 30</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4. Find the point of intersection of two intersecting lines (p. 25)</td>
<td>33–38</td>
</tr>
<tr>
<td></td>
<td>6, 7</td>
<td>5. Solve applied problems involving pairs of lines (p. 26)</td>
<td>39, 40, 42</td>
</tr>
<tr>
<td>1.3</td>
<td>1, 2, 3</td>
<td>1. Solve problems involving the break even point (p. 30)</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2. Solve problems involving supply and demand equations (p. 33)</td>
<td>45</td>
</tr>
<tr>
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<td>1</td>
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<td>43, 44, 46(a), 47(a), 48(a), 49(a)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2. Distinguish between linear and nonlinear relations (p. 36)</td>
<td>43, 44, 48(b), 49(b)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3. Use a graphing utility to find the line of best fit (p. 38)</td>
<td>46(f), 47(f), 48(e), 49</td>
</tr>
</tbody>
</table>

THINGS TO KNOW

Linear Equation, General Form (pp. 3 and 4)

\[ Ax + By = C \]

A, B not both zero

Vertical Line (p. 6)

\[ x = a \]

(a, 0) is the x-intercept

Slope of a Line (p. 7)

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

if \( x_1 \neq x_2 \); undefined if \( x_1 = x_2 \)

Point–Slope Form of the Equation of a Line (p. 11)

\[ y - y_1 = m(x - x_1) \]

\( m \) is the slope of the line; \((x_1, y_1)\) is a point on the line

Horizontal Line (p. 11)

\[ y = b \]

\( (0, b) \) is the y-intercept

Slope–Intercept Form of the Equation of a Line (p. 13)

\[ y = mx + b \]

\( m \) is the slope of the line; \((0, b)\) is the y-intercept

Pair of Lines (pp. 22–25)

Conclusion: The lines are

Both vertical

(a) Coincident, if they have the same x-intercept

(b) Parallel, if they have different x-intercepts

One vertical, one nonvertical

Intersecting

Write the equation of each line in slope–intercept form:

\[ y = m_1x + b_1, y = m_2x + b_2 \]

(a) Coincident, if \( m_1 = m_2, b_1 = b_2 \)

(b) Parallel, if \( m_1 = m_2, b_1 \neq b_2 \)

(c) Intersecting, if \( m_1 \neq m_2 \)
REVIEW EXERCISES  Answers to odd-numbered problems begin on page AN–6.

Blue problem numbers indicate the author’s suggestions for use in a practice test.

In Problems 1–4, graph each equation.

1. \( y = -2x + 3 \)
2. \( y = 6x - 2 \)
3. \( 2y = 3x + 6 \)
4. \( 3y = 2x + 6 \)

In Problems 5–8, (a) find and interpret the slope of the line containing each pair of points; (b) find an equation for the line containing each pair of points. Write the equation using the general form or the slope–intercept form, whichever you prefer. (c) Graph each line.

5. \( P = (1, 2) \quad Q = (-3, 4) \)
6. \( P = (-1, 3) \quad Q = (1, 1) \)
7. \( P = (-1, 5) \quad Q = (-2, 3) \)
8. \( P = (-2, 3) \quad Q = (0, 0) \)

In Problems 9–20, find an equation of the line having the given characteristics. Write the equation using the general form or the slope–intercept form, whichever you prefer. Graph each line.

9. Slope = -3; containing the point (2, -1)
10. Slope = 4; containing the point (-1, -3)
11. Slope = 0; containing the point (-3, 4)
12. Slope undefined; containing the point (-3, 4)
13. Vertical; containing the point (8, 5)
14. Horizontal; containing the point (5, 8)
15. \( x \)-intercept = (2, 0); containing the point (4, -5)
16. \( y \)-intercept = (0, -2); containing the point (5, -3)
17. \( x \)-intercept = (-3, 0); \( y \)-intercept = (0, -4)
18. Containing the points (3, -4) and (2, 1)
19. Parallel to the line \( 2x + 3y = -4 \); containing the point (-5, 3)
20. Parallel to the line \( x + y = 2 \); containing the point (1, -3)

In Problems 21–26, find the slope and \( y \)-intercept of each line. Graph each line.

21. \( 9x + 2y = 18 \)
22. \( 4x + 5y = 20 \)
23. \( 4x + 2y = 9 \)
24. \( 3x + 2y = 8 \)
25. \( \frac{1}{2}x + \frac{1}{3}y = \frac{1}{6} \)
26. \( \frac{1}{4}x - \frac{1}{3}y = \frac{5}{12} \)

In Problems 27–32, determine whether the two lines are parallel, coincident, or intersecting.

27. \( 3x - 4y = -12 \)
28. \( 2x + 3y = -5 \)
29. \( x - y = -2 \)
28. \( 6x - 8y = -9 \)
30. \( 4x + 6y = -10 \)
31. \( 2x + 3y = -12 \)
32. \( -3x + y = 0 \)
27. \( x - y = -9 \)
30. \( x - 2y = -6 \)
31. \( 2x + 3y = -6 \)
32. \( 6x - 2y = -5 \)

In Problems 33–38, the given pair of lines intersect. Find the point of intersection. Graph the lines.

33. \( L: x - y = 4 \quad M: x + 2y = 7 \)
34. \( L: x + y = 4 \quad M: x - 2y = 1 \)
35. \( L: x - y = -2 \quad M: x + 2y = 7 \)
36. \( L: 2x + 4y = 4 \quad M: 2x - 4y = 8 \)
37. \( L: 2x - 4y = -8 \quad M: 3x + 6y = 0 \)
38. \( L: 3x + 4y = 2 \quad M: x - 2y = 1 \)

39. Financial Planning Karen has just retired and finds she needs an additional $10,000 per year to live on. Fortunately, she has a nest egg of $90,000, which she can invest in somewhat risky B-rated bonds at 12% interest per year or in a well-known bank at 5% per year. How much money should she invest in each so that she realizes exactly $10,000 in interest income each year?

40. Mixing Acid One solution is 20% HCl acid and another is 12% HCl acid. How many cubic centimeters of each solution should be mixed to obtain 100 cubic centimeters of a solution that is 15% HCl acid?

41. Attendance at a Dance A church group is planning a dance in the school auditorium to raise money for its school. The band they will hire charges $500; the advertising costs are estimated at $100; and food will be supplied at the rate of $5 per person. The church group would like to clear at least $900 after expenses.

(a) Determine how many people need to attend in order to achieve the desired profit if tickets are sold for $10 each.
(b) Determine how many people need to attend in order to achieve the desired profit if tickets are sold for $12 each.
(c) Answer the above two questions if the tickets are sold for $12 each.
42. **Mixing Coffee** A coffee manufacturer wants to market a new blend of coffee that will cost $6.00 per pound by mixing $5.00 per pound coffee and $7.50 per pound coffee. What amounts of the $5.00 per pound coffee and $7.50 per pound coffee should be blended to obtain the desired mixture?

**Hint:** Assume the total weight of the desired blend is 100 pounds.

In Problems 43 and 44, draw a scatter diagram for each set of data. Then determine whether the relation that may exist is linear or nonlinear.

43. 

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>90</td>
<td>45</td>
<td>21</td>
<td>12</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

44. 

<table>
<thead>
<tr>
<th>x</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>74</td>
<td>70</td>
<td>67</td>
<td>58</td>
<td>55</td>
<td>51</td>
</tr>
</tbody>
</table>

45. **Supply and Demand** The supply and demand equations for corn are estimated to be

\[ S = 0.8p + 0.2 \quad D = -0.4p + 1.8 \]

where \( p \) is the price in dollars and \( S \) and \( D \) are in millions of bushels.

(a) Find the market price.

(b) What quantity is supplied at this price?

(c) Graph both \( S \) and \( D \).

(d) Interpret the point of intersection.

46. **Concentration of Carbon Monoxide in the Air** The following data represent the average concentration of carbon monoxide in parts per million (ppm) in the air for 1992–2008.

<table>
<thead>
<tr>
<th>Year</th>
<th>Concentration of Carbon Monoxide (ppm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1992</td>
<td>5.47</td>
</tr>
<tr>
<td>1994</td>
<td>5.34</td>
</tr>
<tr>
<td>1996</td>
<td>4.44</td>
</tr>
<tr>
<td>1998</td>
<td>4.03</td>
</tr>
<tr>
<td>2000</td>
<td>3.51</td>
</tr>
<tr>
<td>2002</td>
<td>2.97</td>
</tr>
<tr>
<td>2004</td>
<td>2.57</td>
</tr>
<tr>
<td>2006</td>
<td>2.22</td>
</tr>
<tr>
<td>2008</td>
<td>1.88</td>
</tr>
</tbody>
</table>

**Source:** U.S. Environmental Protection Agency

(a) Treating the year as the \( x \)-coordinate and the average level of carbon monoxide as the \( y \)-coordinate, draw a scatter diagram of the data.

(b) What is the slope of the line joining the points (1992, 5.47) and (1998, 4.03)?

(c) Interpret this slope.

(d) What is the slope of the line joining the points (2000, 3.51) and (2008, 2.57)?

(e) Interpret this slope.

(f) Use a graphing utility to find the slope of the line of best fit for these data.

47. **Housing Costs** The following data represent the mean price of houses sold in the United States for 1998–2008.

<table>
<thead>
<tr>
<th>Year</th>
<th>Price (Dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>181,900</td>
</tr>
<tr>
<td>1999</td>
<td>195,600</td>
</tr>
<tr>
<td>2000</td>
<td>207,000</td>
</tr>
<tr>
<td>2001</td>
<td>213,200</td>
</tr>
<tr>
<td>2002</td>
<td>228,700</td>
</tr>
<tr>
<td>2003</td>
<td>246,300</td>
</tr>
<tr>
<td>2004</td>
<td>274,500</td>
</tr>
<tr>
<td>2005</td>
<td>297,000</td>
</tr>
<tr>
<td>2006</td>
<td>305,900</td>
</tr>
<tr>
<td>2007</td>
<td>313,600</td>
</tr>
<tr>
<td>2008</td>
<td>292,600</td>
</tr>
</tbody>
</table>

(a) Treating the year as the \( x \)-coordinate and the price of the houses as the \( y \)-coordinate, draw a scatter diagram of the data.

(b) What is the slope of the line joining the points (1998, 181900) and (2000, 207000)?

(c) Interpret this slope.

(d) What is the slope of the line joining the points (2002, 228700) and (2008, 292600)?

(e) Interpret this slope.

(f) Use a graphing utility to find the slope of the line of best fit for these data.
(g) Interpret this slope.

(h) How do you explain the differences among the three slopes obtained?

(i) What is the trend in the data? In other words, what is happening to the average price of a home in the United States? Why do you think this is happening?

Source: U.S. Census


<table>
<thead>
<tr>
<th>Year</th>
<th>Value per Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>2003</td>
<td>102.67</td>
</tr>
<tr>
<td>2004</td>
<td>111.64</td>
</tr>
<tr>
<td>2005</td>
<td>114.92</td>
</tr>
<tr>
<td>2006</td>
<td>130.59</td>
</tr>
<tr>
<td>2007</td>
<td>135.15</td>
</tr>
</tbody>
</table>

(a) Treating the year as the x-coordinate and the value of the Vanguard 500 Index Fund as the y-coordinate, draw a scatter diagram of the data.

(b) Do the data appear to be linearly related?

(c) What is the slope of the line connecting (2003, 102.67) and (2007, 135.15)?

(d) Interpret the slope.

(e) Use a graphing utility to find the line of best fit for these data.

(f) Assuming the line of best fit truly represents the trend in the data, predict the value of a share of Vanguard 500 Index Fund in the year 2008.

Source: The Vanguard Group, Inc.

49. Value of a Portfolio Investment ads typically contain statements such as “past performance is no guarantee of future performance.” Vanguard’s Web site shows the value of a share was $83.09 in 2008 and $102.67 in 2009.

(a) Add these data to the chart in Problem 48 and draw a revised scatter diagram representing the years 2003–2009.

(b) Do the data appear to be linearly related?

(c) What would you say about the prediction you made in Problem 48(f)?

(d) Statements like the one given in this problem are usually meant to warn investors that funds may not always do well. How does the statement apply in this problem?

50. Make up four problems that you might be asked to do given the two points and (6, 1). Each problem should involve a different concept. Be sure that your directions are clearly stated.

51. Describe each of the following graphs in the xy-plane. Give justification.

(a) $x = 0$

(b) $y = 0$

(c) $x + y = 0$

---

Chapter 1 Project

CHOOSING A CELLPHONE PLAN*

You have an important decision to make. You want to spend the least amount possible on your cell phone each month but still be able to talk to friends and family whenever you want. The plans found online have either unlimited minutes a month for a set price or they have a certain number of free minutes per month for a specific price and then a cost per minute if you exceed the allowable minutes. You decide to save money by not having texting or Web capabilities and will only compare plans for talk time. As it turns out, the equations involved are all linear, so the cell phone plans can be compared using techniques learned in this chapter.

You decide to compare two companies to start. Metro PCS offers unlimited minutes per month for a flat rate of $60. This means that you can talk as much as you want and the bill will never change. Verizon offers a plan that costs $39.99 per month for 450 free minutes and $0.45 per each additional minute. Of course, if you talk less than 450 minutes a month, Verizon’s plan is the better deal. However, if you talk more than 450 minutes a month, at some point Metro PCS becomes the better deal.

To analyze the situation, let $x$ denote the number of minutes you talk on the cell phone per month.

---

*All rates quoted have been taken from quotes in February 2010. There are other fees and taxes that are later added to each quote; these fees and taxes are ignored in our analysis.
1. Suppose $M$ is the cost per month for the Metro PCS plan. Find a linear equation involving $M$ and $x$ to represent the amount you will have to pay.

2. Now let $V$ be the cost per month for the Verizon plan. Find a linear equation involving $V$ and $x$ where $x \leq 450$. Also find a linear equation involving $V$ and $x$ where $x > 450$.

3. Graph the three linear equations found in part (1) and part (2) on the same set of coordinate axes. Be careful about the restrictions on $x$ for the equations found in part (2).

4. Find the number of minutes at which Metro PCS becomes the cheaper plan each month by finding the point of intersection of two graphs. Label this point on your graph. Round to the nearest minute.

5. Explain how you can use the solution in part (4) to see which plan is more economical.

6. You are still not convinced that you have the best plan. You look up prices for T-Mobile plans and see that it will charge you $49.99 a month for up to 650 minutes and $0.50 for each additional minute. Let $T$ be the cost per month for the T-Mobile plan. Find a linear equation that represents the cost if you talk less than 650 minutes a month and find a linear equation that represents the cost if you talk more than 650 minutes a month.

7. Graph the two equations from part (6) on the same graph you found in part (3).

8. It is now clear that if you talk less than 450 minutes a month, Verizon is the best plan to save money each month. Who has the cheapest plan if you talk 480 minutes a month?

9. For what range of minutes is Metro PCS or T-Mobile the cheapest?

10. If you only want to spend $55 a month and want to use T-Mobile, how many minutes are you limited to per month over the allowable 650 minutes?

11. If Metro PCS charges $10 a month extra to cover the cost of the phone for one year and Verizon’s phone is free, how many minutes can you talk a month for the year you are paying the extra charge with Metro PCS where Verizon is the better deal?

12. Suppose a new job requires you to be on the phone between 2000 and 2400 minutes a month. You will need to do a lot of traveling and, unfortunately, the Metro PCS service does not have the nationwide coverage you will need. Discuss the suitability of the T-Mobile and Verizon cell phone plans for your new job.

---

Mathematical Questions from Professional Exams

1. **CPA Exam** The Oliver Company plans to market a new product. Based on its market studies, Oliver estimates that it can sell 5500 units in 1992. The selling price will be $2 per unit. Variable costs are estimated to be 40% of the selling price. Fixed costs are estimated to be $6000. What is the break-even point?
   (a) 3750 units
   (b) 5000 units
   (c) 5500 units
   (d) 7500 units

2. **CPA Exam** The Breiden Company sells rodaks for $6 per unit. Variable costs are $2 per unit. Fixed costs are $37,500. How many rodaks must be sold to realize a profit before income taxes of 15% of sales?
   (a) 9375 units
   (b) 9740 units
   (c) 11,029 units
   (d) 12,097 units

3. **CPA Exam** Given the following notations, what is the break-even sales level in units?
   \[
   SP = \text{Selling price per unit} \\
   FC = \text{Total fixed cost} \\
   VC = \text{Variable cost per unit}
   \]
   (a) \( \frac{SP}{FC + VC} \)
   (b) \( \frac{FC}{VC + SP} \)
   (c) \( \frac{VC}{SP - FC} \)
   (d) \( \frac{FC}{SP - VC} \)

* Copyright © 1991–2010 by the American Institute of Certified Public Accountants, Inc. All rights reserved. Reprinted with permission.
4. **CPA Exam** At a break-even point of 400 units sold, the variable costs were $400 and the fixed costs were $200. What will the 401st unit sold contribute to profit before income taxes?
(a) $0
(b) $0.50
(c) $1.00
(d) $1.50

5. **CPA Exam** A graph is set up with “depreciation expense” on the vertical axis and “time” on the horizontal axis. Assuming linear relationships, how would the graphs for straight-line and sum-of-the-year’s-digits depreciation, respectively, be drawn?
(a) Vertically and sloping down to the right
(b) Vertically and sloping up to the right
(c) Horizontally and sloping down to the right
(d) Horizontally and sloping up to the right

The following statement applies to Questions 6–8:
In analyzing the relationship of total factory overhead with changes in direct labor hours, the following relationship was found to exist: $Y = 1000 + 2X$.

6. **CPA Exam** The relationship as shown above is
(a) Parabolic
(b) Curvilinear
(c) Linear
(d) Probabilistic
(e) None of the above

7. **CPA Exam** $Y$ in the above equation is an estimate of
(a) Total variable costs
(b) Total factory overhead
(c) Total fixed costs
(d) Total direct labor hours
(e) None of the above

8. **CPA Exam** The $2$ in the equation is an estimate of
(a) Total fixed costs
(b) Variable costs per direct labor hour
(c) Total variable costs
(d) Fixed costs per direct labor hour
(e) None of the above
When selecting a cell phone plan, a number of considerations will influence your decision. What are the additional costs for texting or Web access? Is a two-year service contract required? What extra features are needed, such as email capability or a family plan? Will you be making international calls? Which carrier offers the strongest cell phone signal in the geographical area where you will primarily use your phone? Some of these considerations can be quantified and included in a model to help compare cell phone plans. The Chapter Project at the end of this chapter develops such a model.

OUTLINE

2.1 Systems of Linear Equations: Substitution; Elimination
2.2 Systems of Linear Equations: Gaussian Elimination
2.3 Systems of \( m \) Linear Equations Containing \( n \) Variables

• Chapter Review
• Chapter Project
• Mathematical Questions from Professional Exams

A Look Back, A Look Forward

In Section 1.2 of Chapter 1, we discussed pairs of lines: coincident lines, parallel lines, and intersecting lines. Each line was given by a linear equation containing two variables. So a pair of lines is given by two linear equations containing two variables. We refer to this as a system of two linear equations containing two variables.

In this chapter we take up the problem of solving systems of linear equations containing two or more variables. As the section titles suggest, there are various ways to do this. The method of substitution for solving equations in several unknowns goes back to ancient times. The method of elimination, though it had existed for centuries, was put into systematic order by Karl Friedrich Gauss (1777–1855) and by Camille Jordan (1838–1922). This method led to the matrix method (Gauss–Jordan method) that is now used for solving large systems by computer.
2.1 Systems of Linear Equations: Substitution; Elimination

PREPARING FOR THIS SECTION Before getting started, review the following:
• Pairs of Lines (Section 1.2, pp. 22–26)
• Algebra Essentials (Appendix A, Section A.2, p. A–24)

NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 65

OBJECTIVES
1 Solve systems of equations by substitution (p. 53)
2 Solve systems of equations by elimination (p. 55)
3 Identify inconsistent systems of equations containing two variables (p. 57)
4 Express the solutions of a system of dependent equations containing two variables (p. 58)
5 Solve systems of three equations containing three variables (p. 59)
6 Identify inconsistent systems of equations containing three variables (p. 61)
7 Express the solutions of a system of dependent equations containing three variables (p. 62)
8 Solve applied problems involving systems of equations (p. 63)

EXAMPLE 1 Movie Theater Ticket Sales

A movie theater sells tickets for $8.00 each, with seniors receiving a discount of $2.00. One evening the theater took in $3580 in revenue. If \( x \) represents the number of tickets sold at $8.00 and \( y \) the number of tickets sold at the discounted price of $6.00, write an equation that relates these variables.

**SOLUTION**

Each nondiscounted ticket brings in $8.00, so \( x \) tickets will bring in \( 8x \) dollars. Similarly, \( y \) discounted tickets bring in \( 6y \) dollars. Since the total brought in is $3580, we must have

\[
8x + 6y = 3580
\]

This equation is an example of a linear equation containing two variables. Some other examples of linear equations are

\[
\begin{align*}
2x + 3y &= 2 \\
5x - 2y + 3z &= 10 \\
8x_1 + 8x_2 - 2x_3 + 5x_4 &= 0
\end{align*}
\]

2 variables \hspace{1cm} 3 variables \hspace{1cm} 4 variables

In general, an equation containing \( n \) variables is said to be linear if it can be written in the form

\[
a_1x_1 + a_2x_2 + \cdots + a_nx_n = b
\]

where \( x_1, x_2, \ldots, x_n \) are \( n \) distinct variables, \( a_1, a_2, \ldots, a_n, b \) are constants, and at least one of the \( a_i \)'s is not 0.

* The notation \( x_n \) is read as "x sub n." The number \( n \) is called a subscript and should not be confused with an exponent. We use subscripts to distinguish one variable from another when a large or undetermined number of variables is required.
In Example 1, suppose that we also know that 525 tickets were sold that evening. Then we have another equation relating the variables $x$ and $y$, namely,

$$x + y = 525$$

The two linear equations

$$8x + 6y = 3580$$
$$x + y = 525$$

form a *system* of linear equations.

In general, a *system of linear equations* is a collection of two or more linear equations, each containing one or more variables.

**EXAMPLE 2** Examples of Systems of Linear Equations

(a) \[
\begin{aligned}
2x + y &= 5 \\
-4x + 6y &= -2
\end{aligned}
\] (1) Two equations containing two variables, $x$ and $y$

(b) \[
\begin{aligned}
x + y + z &= 6 \\
3x - 2y + 4z &= 9 \\
x - y - z &= 0
\end{aligned}
\] (1) Three equations containing three variables, $x$, $y$, and $z$

(c) \[
\begin{aligned}
x + y + z &= 6 \\
x - y &= 2
\end{aligned}
\] (1) Two equations containing three variables, $x$, $y$, and $z$

(d) \[
\begin{aligned}
x + y + z &= 6 \\
2x + z &= 4 \\
y + z &= 2
\end{aligned}
\] (1) Five equations containing three variables, $x$, $y$, and $z$

(e) \[
\begin{aligned}
x_1 - 2x_2 + x_3 - x_4 &= 5 \\
3x_1 + x_2 - x_3 - 5x_4 &= 2
\end{aligned}
\] (1) Two equations containing four variables, $x_1$, $x_2$, $x_3$, and $x_4$

A brace, as shown above, to remind us that we are dealing with a system of linear equations. Notice that each equation in the system has been numbered for easy reference.

A *solution* of a system of linear equations consists of values of the variables that are solutions of each equation of the system. To *solve* a system of linear equations means to find all solutions of the system.

For example, $x = 2$, $y = 1$ is a solution of the system in Example 2(a) because

\[
\begin{aligned}
2x + y &= 5 \\
-4x + 6y &= -2
\end{aligned}
\]

\[
\begin{aligned}
2(2) + 1 &= 4 + 1 = 5 \\
-4(2) + 6(1) &= -8 + 6 = -2
\end{aligned}
\]

Furthermore, $x = 2$, $y = 1$ is the only solution of this system. Do you know why? The lines represented by the equations have different slopes and so they intersect in a single point.
A solution of the system in Example 2(b) is $x = 3, y = 2, z = 1$, because
\[
\begin{align*}
3x - 2y + 4z &= 9 \\
x - y - z &= 0
\end{align*}
\]
Note that $x = 3, y = 3, z = 0$ is not a solution of the system in Example 2(b).

Although these values satisfy Equations (1) and (3), they do not satisfy Equation (2).

Any solution of the system must satisfy each equation of the system.

The system of equations given in Example 2(c) has multiple solutions. For example, $x = 2, y = 0, z = 4; x = 3, y = 1, z = 2; x = 4, y = 2, z = 0; x = 5, y = 3, z = -2$ are each solutions, as you can verify. As it turns out, when a system of equations has multiple solutions, the number of solutions is infinite!

The system of equations given in Example 2(d) has no solution. Look at Equations (4) and (5), which tell us $x = 4$ and $y = 2$. Substitute these values into Equations (1) and (2). From Equation (1), we obtain $z = 0$ and from Equation (2), we obtain $z = -4$. This contradiction means the system has no solution.

**Definition**

If a system of equations has at least one solution, it is said to be **consistent**; if it has no solution, it is said to be **inconsistent**. If a consistent system of equations has exactly one solution, the equations of the system are said to be **independent**; if it has an infinite number of solutions, the equations are called **dependent**.

**NOW WORK PROBLEM 9.**

**Two Linear Equations Containing Two Variables**

Based on the discussion in Section 1.2, we can view the problem of solving a system of two linear equations containing two variables as a geometry problem. Because the graph of each equation in such a system is a line, a system of two linear equations containing two variables represents a pair of lines. The lines either (1) are parallel or (2) are intersecting or (3) are coincident (that is, identical).

1. If the lines are parallel, then the system of equations has no solution, because the lines never intersect. The system is inconsistent.
2. If the lines intersect, then the system of equations has one solution, given by the point of intersection. The system is consistent and the equations are independent.
3. If the lines are coincident, then the system of equations has infinitely many solutions, represented by the totality of points on the line. The system is consistent and the equations are dependent.

Based on this, a system of equations is either

(I) Inconsistent with no solution

or

(II) Consistent with

(a) One solution (equations are independent)

or

(b) Infinitely many solutions (equations are dependent)
2.1 Systems of Linear Equations: Substitution, Elimination

Figure 1 illustrates these conclusions.

**FIGURE 1**
(a) Parallel lines; system has no solution and is inconsistent
(b) Intersecting lines; system has one solution and is consistent; the equations are independent
(c) Coincident lines; system has infinitely many solutions and is consistent; the equations are dependent

---

**EXAMPLE 3** Graphing a System of Linear Equations

Graph the system:

\[
\begin{align*}
2x + y &= 5 \\
-4x + 6y &= 12
\end{align*}
\]

**SOLUTION**

Equation (1) is a line with \( x \)-intercept \( \left( \frac{5}{2}, 0 \right) \) and \( y \)-intercept \( (0, 5) \). Equation (2) is a line with \( x \)-intercept \( (-3, 0) \) and \( y \)-intercept \( (0, 2) \).

Figure 2 shows their graphs.

From the graph in Figure 2 we see that the lines intersect, so the system is consistent and the equations are independent. We can also use the graph as a means of approximating the solution. For this system the solution would appear to be close to the point \( (1, 3) \). The actual solution, which you should verify, is \( \left( \frac{9}{8}, \frac{11}{4} \right) \).

To obtain the exact solution, we use algebraic methods. The first algebraic method we take up is the method of substitution.

---

**EXAMPLE 4** Solving a System of Equations Using Substitution

Solve:

\[
\begin{align*}
2x + y &= 5 \\
-4x + 6y &= 12
\end{align*}
\]

**SOLUTION**

Solve the first equation for \( y \), obtaining

\[
y = -2x + 5 \tag{1}
\]

Subtract \( 2x \) from each side.

Substitute this value of \( y \) in the second equation. This results in an equation containing one variable, which we can solve.

\[
-4x + 6y = 12 \tag{2}
\]

\[
-4x + 6(-2x + 5) = 12 \quad \text{Substitute } y = -2x + 5 \text{ in (2)}
\]

\[
-4x - 12x + 30 = 12 \quad \text{Remove parentheses.}
\]

\[
-16x = -18 \quad \text{Combine like terms; subtract 30 from each side.}
\]

\[
x = \frac{-18}{-16} = \frac{9}{8} \quad \text{Divide each side by } -16.
\]
Once we know that \( x = \frac{9}{8} \), we can find the value of \( y \) by **back-substitution**, that is, by substituting \( \frac{9}{8} \) for \( x \) in one of the original equations.

Use the first equation.

\[
2x + y = 5 \quad (1)
\]

\[
2 \left( \frac{9}{8} \right) + y = 5 \\
\frac{9}{4} + y = 5 \\
y = 5 - \frac{9}{4} \\
y = \frac{20}{4} - \frac{9}{4} = \frac{11}{4}
\]

The solution of the system is \( x = \frac{9}{8} = 1.125, y = \frac{11}{4} = 2.75 \). We can also write the solution as the ordered pair \( \left( \frac{9}{8}, \frac{11}{4} \right) \).

**CHECK:**

\[
\begin{align*}
2x + y &= 5: \\
2 \left( \frac{9}{8} \right) + \frac{11}{4} &= \frac{9}{4} + \frac{11}{4} = \frac{20}{4} = 5 \\
-4x + 6y &= 12: \\
-4 \left( \frac{9}{8} \right) + 6 \left( \frac{11}{4} \right) &= -\frac{9}{2} + \frac{33}{2} = \frac{24}{2} = 12
\end{align*}
\]

**COMMENT** We can also verify our algebraic solution in Example 4 using a graphing utility.

First, we solve each equation for \( y \). This is equivalent to writing each equation in slope–intercept form. Equation (1) in slope–intercept form is \( Y_1 = -2x + 5 \). Equation (2) in slope–intercept form is \( Y_2 = \frac{2}{3}x + 2 \). Figure 3 shows the graphs using a graphing utility.

From the graph in Figure 3, we see that the lines intersect, so the system is consistent and the equations are independent. Using INTERSECT, we obtain the solution (1.125, 2.75), which is equivalent to \( \left( \frac{9}{8}, \frac{11}{4} \right) \).

The method used to solve the system in Example 4 is called **substitution**.

---

**Steps for Solving by Substitution**

**STEP 1** Pick one of the equations and solve for one of the variables in terms of the remaining variables.

**STEP 2** Substitute the result in the remaining equations.

**STEP 3** If one equation in one variable results, solve this equation. Otherwise, repeat Steps 1 and 2 until a single equation with one variable remains.

**STEP 4** Find the values of the remaining variables by back-substitution.

**STEP 5** Check the solution found.
EXAMPLE 5 Solving a System of Equations Using Substitution

Solve: \[
\begin{align*}
3x - 2y &= 5 \\
5x - y &= 6
\end{align*}
\] (1)
(2)

SOLUTION

STEP 1 After looking at the two equations, we conclude that it is easiest to solve for the variable \( y \) in Equation (2):

\[
5x - y = 6 \quad (2)
\]
\[
5x - 6 = y \quad \text{Add } y \text{ and subtract 6 from each side.}
\]

STEP 2 Substitute this result into Equation (1) and simplify:

\[
3x - 2y = 5 \quad (2)
\]
\[
3x - 2(5x - 6) = 5 \quad y = 5x - 6
\]
\[
-7x + 12 = 5 \quad \text{Simplify.}
\]
\[
-7x = -7 \quad \text{Simplify.}
\]
\[
x = 1 \quad \text{Solve for } x.
\]

STEP 3 Knowing \( x = 1 \), we can find \( y \) from the equation

\[
y = 5x - 6 = 5(1) - 6 = -1 \quad x = 1
\]

STEP 4 CHECK: \[
\begin{align*}
3(1) - 2(-1) &= 3 + 2 = 5 \\
5(1) - (-1) &= 5 + 1 = 6
\end{align*}
\]
The solution of the system is \( x = 1, y = -1 \) or, using ordered pairs, \((1, -1)\).

NOW WORK PROBLEM 15 USING SUBSTITUTION.

2 Solve Systems of Equations by Elimination

A second method for solving a system of linear equations is the method of elimination. This method is usually preferred over substitution if substitution leads to fractions or if the system contains more than two variables. Elimination also provides the necessary motivation for solving systems using matrices, the subject of the next section.

The idea behind the method of elimination is to replace the original system of equations by an equivalent system so that adding two of the equations eliminates a variable. The rules for obtaining equivalent equations are the same as those studied earlier. However, we may also interchange any two equations of the system and/or replace any equation in the system by the sum (or difference) of that equation and a nonzero multiple of any other equation in the system.

Rules for Obtaining an Equivalent System of Equations

1 Interchange any two equations in the system.
2 Multiply (or divide) each side of an equation by the same nonzero constant.
3 Replace any equation in the system by the sum (or difference) of that equation and a nonzero multiple of any other equation in the system.
An example will give you the idea. As you work through the example, pay particular attention to the pattern being followed.

**EXAMPLE 6 Solving a System of Linear Equations Using Elimination**

Solve: \[
\begin{align*}
2x + 3y &= 1 \quad (1) \\
-x + y &= -3 \quad (2)
\end{align*}
\]

**SOLUTION** Multiply each side of Equation (2) by 2 so that the coefficients of \(x\) in the two equations are opposites of one another. The result is the equivalent system

\[
\begin{align*}
2x + 3y &= 1 \quad (1) \\
-2x + 2y &= -6 \quad (2)
\end{align*}
\]

Now replace Equation (2) of this system by the sum of the two equations, to obtain an equation containing just the variable \(y\), which we can solve.

\[
\begin{align*}
2x + 3y &= 1 \quad (1) \\
-2x + 2y &= -6 \quad (2)
\end{align*}
\]

\[5y = -5 \quad \text{Add (1) and (2).} \]

\[y = -1 \quad \text{Solve for } y. \]

Back-substitute this value for \(y\) in Equation (1) and simplify to get

\[
\begin{align*}
2x + 3y &= 1 \quad (1) \\
2x + 3(-1) &= 1 \quad \text{Substitute } y = -1 \text{ in (1).} \\
2x &= 4 \quad \text{Simplify.} \\
x &= 2 \quad \text{Solve for } x.
\end{align*}
\]

The solution of the original system is \(x = 2, y = -1\), or using ordered pairs, \((2, -1)\). We leave it to you to check the solution. ■

The procedure used in Example 6 is called the **method of elimination**. Notice the pattern of the solution. First, we eliminated the variable \(x\) from the second equation. Then we back-substituted; that is, we substituted the value found for \(y\) back into the first equation to find \(x\).

**Steps for Solving by Elimination**

**STEP 1** Select two equations from the system and replace them by two equivalent equations that, when added, eliminate at least one variable.

**STEP 2** If there are additional equations in the original system, pair off each one with one of the equations selected in Step 1 and eliminate the same variable from them.

**STEP 3** Continue Steps 1 and 2 on successive systems until one equation containing one variable remains.

**STEP 4** Solve for this variable and back-substitute in previous equations until all the variables have been found.

**STEP 5** Check the solution found.

**NOW WORK PROBLEM 15 USING ELIMINATION.**
Let’s return to the movie theater example (Example 1).

**EXAMPLE 7 Movie Theater Ticket Sales**

A movie theater sells tickets for $8.00 each, with seniors receiving a discount of $2.00. One evening the theater sold 525 tickets and took in $3580 in revenue. How many of each type of ticket were sold?

**SOLUTION** If \(x\) represents the number of tickets sold at $8.00 and \(y\) the number of tickets sold at the discounted price of $6.00, then the given information results in the system of equations

\[
\begin{align*}
8x + 6y &= 3580 \quad (1) \\
x + y &= 525 \quad (2)
\end{align*}
\]

Use elimination and multiply Equation (2) by \(-6\); then add the equations.

\[
\begin{align*}
8x + 6y &= 3580 \quad (1) \\
-6x - 6y &= -3150 \quad (2)
\end{align*}
\]

\[
\begin{align*}
2x &= 430 \\
x &= 215
\end{align*}
\]

Since \(x + y = 525\), then \(y = 525 - x = 525 - 215 = 310\). We conclude that 215 nondiscounted tickets and 310 senior discount tickets were sold.

**NOW WORK PROBLEM 47.**

### 3 Identify Inconsistent Systems of Equations Containing Two Variables

The previous examples dealt with consistent systems of equations that had one solution. The next two examples deal with two other possibilities that may occur, the first being a system that has no solution.

**EXAMPLE 8 An Inconsistent System of Linear Equations**

Solve: \[
\begin{align*}
2x + y &= 5 \quad (1) \\
4x + 2y &= 8 \quad (2)
\end{align*}
\]

**SOLUTION** We choose to use the method of substitution and solve Equation (1) for \(y\).

\[
2x + y = 5 \quad (1)
\]

\[
y = -2x + 5 \quad \text{Subtract } 2x \text{ from each side.}
\]

Now substitute \(y = -2x + 5\) for \(y\) in Equation (2) and solve for \(x\).

\[
4x + 2y = 8 \quad (2)
\]

\[
4x + 2(-2x + 5) = 8 \quad \text{Subtract } -2x + 5 \text{ in (2).}
\]

\[
4x - 4x + 10 = 8 \quad \text{Remove parentheses.}
\]

\[
0 \cdot x = -2 \quad \text{Subtract 10 from each side.}
\]

This equation has no solution. We conclude that the system itself has no solution and is therefore inconsistent.
Chapter 2  Systems of Linear Equations

Figure 4 illustrates the pair of lines whose equations form the system in Example 8. Notice that the graphs of the two equations are lines, each with slope $-2$; one line has $y$-intercept $(0, 5)$, the other has $y$-intercept $(0, 4)$. The lines are parallel and have no point of intersection. This geometric statement is equivalent to the algebraic statement that the system is inconsistent and has no solution.

NOW WORK PROBLEM 19.

4 Express the Solutions of a System of Dependent Equations Containing Two Variables

EXAMPLE 9  Solving a System of Linear Equations with Infinitely Many Solutions

Solve: \[
\begin{aligned}
2x + y &= 4 \\
-6x - 3y &= -12
\end{aligned}
\]

SOLUTION  Use the method of elimination:

\[
\begin{aligned}
2x + y &= 4 \\
-6x - 3y &= -12
\end{aligned}
\]

Multiply each side of Equation (1) by 3.

\[
\begin{aligned}
6x + 3y &= 12 \\
-6x - 3y &= -12
\end{aligned}
\]

Replace Equation (2) by the sum of Equations (1) and (2).

\[
\begin{aligned}
6x + 3y &= 12 \\
0 &= 0
\end{aligned}
\]

The original system is equivalent to a system containing one equation, so the equations are dependent. This means that any values of $x$ and $y$ for which $6x + 3y = 12$ (or, equivalently, $2x + y = 4$) are solutions. For example, $x = 2, y = 0; x = 0, y = 4; x = -2, y = 8; x = 4, y = -4$; and so on, are solutions. There are, in fact, infinitely many values of $x$ and $y$ for which $2x + y = 4$, so the original system has infinitely many solutions. We will write the solutions of the original system either as

\[y = 4 - 2x\]

where $x$ can be any real number, or as

\[x = 2 - \frac{1}{2}y\]

where $y$ can be any real number.
2.1 Systems of Linear Equations: Substitution; Elimination

Using ordered pairs, we write the solution as \( (x, y) \mid y = -2x + 4, x \text{ any real number} \) or as \( (x, y) \mid x = 2 - \frac{1}{2}y, y \text{ any real number} \).

Figure 5 illustrates the situation presented in Example 9. Notice that the graphs of the two equations are lines, each with slope \(-2\) and each with \(y\)-intercept \((0, 4)\). The lines are coincident. Notice also that Equation (2) in the original system is \(-3\) times Equation (1), indicating that the two equations are dependent.

For the system in Example 9 we can find some of the infinite number of solutions by assigning values to \(x\) and then finding \(y = 4 - 2x\). When we express the solution in this way, we call \(x\) a parameter.

Alternatively, if we express the solution in the form \(x = 2 - \frac{1}{2}y\), then \(y\) is the parameter and we can assign values to \(y\) in order to find \(x\).

\[
\begin{align*}
\text{If } x &= 4, \text{ then } y = -4. & \text{This is the point } (4, -4) \text{ on the graph.} \\
\text{If } x &= 0, \text{ then } y = 4. & \text{This is the point } (0, 4) \text{ on the graph.} \\
\text{If } x &= \frac{1}{2}, \text{ then } y = 3. & \text{This is the point } \left(\frac{1}{2}, 3\right) \text{ on the graph.}
\end{align*}
\]

NOW WORK PROBLEM 21.

5 Solve Systems of Three Equations Containing Three Variables

Just as with a system of two linear equations containing two variables, a system of three linear equations containing three variables has either (1) exactly one solution (a consistent system with independent equations), or (2) no solution (an inconsistent system), or (3) infinitely many solutions (a consistent system with dependent equations).

We can view the problem of solving a system of three linear equations containing three variables as a geometry problem. The graph of each equation in such a system is a plane in space. A system of three linear equations containing three variables represents three planes in space. Figure 6 illustrates some of the possibilities.
Typically, when solving a system of three linear equations containing three variables, we use the method of elimination. Recall that the idea behind the method of elimination is to form equivalent equations so that adding two of the equations eliminates a variable.

**EXAMPLE 10 Solving a System of Three Linear Equations with Three Variables Using Elimination**

Use the method of elimination to solve the system of equations.

**SOLUTION**

For a system of three equations, we attempt to eliminate one variable at a time, using pairs of equations, until an equation with a single variable remains. Our plan of attack on this system will be to use Equation (1) to eliminate the variable \(x\) from Equations (2) and (3). Alternatively, we could use Equation (1) to eliminate either \(y\) or \(z\) from Equations (2) and (3). Try one of these for yourself.

Begin by multiplying each side of Equation (1) by \(-2\) and adding the result to Equation (2). (Do you see why? The coefficients of \(x\) are now opposites of each other.)

Now multiply Equation (1) by \(-2\) and add the result to Equation (3). Notice that these two procedures result in the removal of the \(x\)-variable from Equations (2) and (3).

Now concentrate on Equations (2) and (3), treating them as a system of two equations containing two variables. It is easier to eliminate \(z\). Multiply each side of Equation (3) by 6 and add Equations (2) and (3). The result is the new Equation (3).

Now solve Equation (3) for \(y\) by dividing both sides of the equation by \(-31\).
Back-substitute $y = -2$ in Equation (2) and solve for $z$.

$$-7y + 6z = 20 \quad (2)$$

Substitute $y = -2$ in (2).

$$-7(-2) + 6z = 20$$

$$6z = 6$$

Subtract 14 from each side.

$$z = 1$$

Divide each side by 6.

Finally, back-substitute $y = -2$ and $z = 1$ in Equation (1) and solve for $x$.

$$x + y - z = -1 \quad (1)$$

Substitute $y = -2$ and $z = 1$ in (1).

$$x + (-2) - 1 = -1$$

Simplify.

$$x = 2$$

Add 3 to each side.

The solution of the original system is $x = 2$, $y = -2$, $z = 1$ or, using ordered triplets, $(2, -2, 1)$. You should verify this solution.

Look back over the solution given in Example 10. Note the pattern of removing one of the variables from two of the equations, followed by solving this system of two equations and two unknowns. Although which variables to remove is your choice, the methodology remains the same for all systems.

**NOW WORK PROBLEM 31.**

The previous example was a consistent system that had a unique solution. The next two examples deal with the two other possibilities that may occur.

## Identify Inconsistent Systems of Equations Containing Three Variables

**EXAMPLE 11 An Inconsistent System of Linear Equations**

Solve:

\[
\begin{align*}
2x + y - z &= -2 \\ x + 2y - z &= -9 \\ x - 4y + z &= 1
\end{align*}
\]

**SOLUTION**

Our plan of attack is the same as in Example 10. However, in this system, it seems easiest to eliminate the variable $z$ first. Do you see why?

Add Equations (1) and (3). Add Equations (2) and (3).

$$\begin{align*}
2x + y - z &= -2 \\ x - 4y + z &= -1
\end{align*} \quad (1)$$

Add.

$$3x - 3y = -1$$

Now concentrate on Equations (2) and (3), treating them as a system of two equations containing two variables. Multiply each side of Equation (2) by $\frac{1}{3}$ and each side of Equation (3) by $\frac{1}{2}$.
The contradiction tells us the system is inconsistent.

NOW WORK PROBLEM 33.

Express the Solutions of a System of Dependent Equations Containing Three Variables

**EXAMPLE 12 Solving a System of Dependent Equations**

Solve: 
\[
\begin{align*}
3x - 3y &= -1 \\
2x - 2y &= -8
\end{align*}
\]

**SOLUTION**

Multiply each side of Equation (1) by \( \frac{1}{3} \) and add the result to Equation (2). Also, multiply each side of Equation (1) by \( \frac{1}{2} \) and add the result to Equation (3).

\[
\begin{align*}
-2x + 4y + 2z &= -16 \\
x - y &= -\frac{1}{3} \\
2x - 3y + z &= 23 \\
x - y &= -4 \\
4x - 5y + 5z &= 53 \\
\end{align*}
\]

Add.

\[
\begin{align*}
x - 2y - z &= 8 \\
x + 6z - 14 - z &= 8 \\
x + 5z &= 22
\end{align*}
\]

The original system is equivalent to a system containing two equations, so the original equations are dependent and the original system has infinitely many solutions. If we let \( z \) represent any real number, then, solving Equation (2) for \( y \), we determine that \( y = -3z + 7 \). Substitute this expression into Equation (1) to determine \( x \) in terms of \( z \).

\[
\begin{align*}
x - 2y - z &= 8 \\
x - 2(-3z + 7) - z &= 8 \\
x + 6z - 14 - z &= 8 \\
x + 5z &= 22 \\
x &= -5z + 22 \quad \text{Solve for } x.
\end{align*}
\]
2.1 Systems of Linear Equations: Substitution; Elimination

Write the solution to the system as

\[
\begin{align*}
  x &= -5z + 22 \\
  y &= -3z + 7
\end{align*}
\]

where \( z \), the parameter, can be any real number.

To find specific solutions to the system, choose any value of \( z \) and use the equations \( x = -5z + 22 \) and \( y = -3z + 7 \) to determine \( x \) and \( y \). For example, if \( z = 0 \), then \( x = 22 \) and \( y = 7 \), and if \( z = 1 \), then \( x = 17 \) and \( y = 4 \).

**NOW WORK PROBLEM 35.**

8 Solve Applied Problems Involving Systems of Equations

In economics, the supply equation is used to determine the amount of a product that a company is willing to make available for sale at a given price. The demand equation is the amount of a product that consumers are willing to purchase at a given price.

**EXAMPLE 13 Supply and Demand Equations**

Suppose that the quantity supplied \( S \) and the quantity demanded \( D \) of an mp3 player are given by the equations

\[
\begin{align*}
  S &= 60p - 900 \\
  D &= -15p + 2850
\end{align*}
\]

where \( p \) is the price, in dollars, of the mp3 player.

(a) The equilibrium price or market price of a product is the price at which quantity supplied equals quantity demanded. That is, the equilibrium price is the price for which \( S = D \). Find the equilibrium price of the mp3 player.

(b) What is the equilibrium quantity, the amount demanded (or supplied) at the equilibrium price?

(c) Graph the supply equation and the demand equation. Label the equilibrium point.

(d) What happens when the price is less than the equilibrium price?

(e) What happens if the price exceeds the equilibrium price?

**SOLUTION**

(a) To find the equilibrium price, solve the system of equations.

\[
\begin{align*}
  S &= 60p - 900 \\
  D &= -15p + 2850 \\
  S &= D
\end{align*}
\]

Using substitution, we are led to a single equation involving the price \( p \).

\[
\begin{align*}
  60p - 900 &= -15p + 2850 \\
  60p &= -15p + 3750 \\
  75p &= 3750 \\
  p &= 50
\end{align*}
\]

Add 900 to each side. 
Add 15p to each side. 
Divide each side by 75.

The equilibrium price is $50 per mp3 player.

(b) To find the equilibrium quantity, we evaluate either \( S \) or \( D \) at \( p = 50 \).

\[
S = 60(50) - 900 = 2100 \quad S = 60p - 900; \quad p = 50
\]

The equilibrium quantity is 2100 mp3 players. At a price of $50, the company will produce and sell 2100 mp3 players and have no shortages or excess inventory.
Figure 7 shows the graphs of the supply and demand equations with the equilibrium point labeled.

Look at Figure 7. If the price is less than $50, the demand is greater than the supply, which means the mp3 player would be hard to find.

Look at Figure 7. If the price exceeds $50, supply is greater than demand, which means there is excess inventory (unsold mp3 players).

**FIGURE 7**

Erica wants to invest long term the $30,000 she inherited from her great-aunt. Over the next 10 years she hopes to have earnings averaging $2000 per year. Based on the yearly average return for 10 years ending December 31, 2009, her financial advisor recommended three funds: Franklin Total Return at 5%, Franklin Strategic Income at 6%, and Franklin High Income at 7% that would allow Erica to achieve her goal while providing diversification. Prepare a table showing the various ways Erica can achieve her goal.

**Source:** www.franklintempleton.com.

**SOLUTION**

Begin by naming the variables: Let $x$ be the amount invested in Total Return, $y$ the amount invested in Strategic Incomes, and $z$ the amount invested in High Income. Then the average annual amount earned by each investment choice is expected to be

- Total Return: $0.05x$
- Strategic Income: $0.06y$
- High Income: $0.07z$

Since Erica’s goal is to average $2000 each year from these investments, we have the equation

$$0.05x + 0.06y + 0.07z = 2000$$

The total amount available to invest is $30,000, which leads to the equation

$$x + y + z = 30,000$$

These two equations form the system

$$\begin{align*}
0.05x + 0.06y + 0.07z &= 2000 \\
x + y + z &= 30,000
\end{align*}$$
Solve the system using elimination. We choose to eliminate \( z \) by multiplying Equation (2) by \(-0.07\) and adding the equations.

\[
\begin{align*}
0.05x + 0.06y + 0.07z &= 2000 \\
-0.07x - 0.07y - 0.07z &= -2100 \\
-0.02x - 0.01y &= -100
\end{align*}
\]

Add.

Solve for \( y \).

\[
0.01y = -0.02x + 100 \quad \text{Multiply by 100.}
\]

Back-substituting into the equation \( x + y + z = 30,000 \) and solving for \( z \), we have

\[
x + (-2x + 10,000) + z = 30,000 \quad \text{Simplify.}
\]

\[
-x + z = 20,000
\]

\[
z = x + 20,000 \quad \text{Solve for } z.
\]

The solution of the system is

\[
\begin{align*}
y &= -2x + 10,000 \\
z &= x + 20,000
\end{align*}
\]

where \( x \) is the parameter.

Now, the variable \( x \) must be nonnegative and cannot exceed $5000, since any value of \( x > 5000 \) makes the variable \( y \) negative.

Set up a table as shown in Table 1.

**TABLE 1**

<table>
<thead>
<tr>
<th>Total Return, ( x )</th>
<th>Strategic Income, ( y )</th>
<th>High Income, ( z )</th>
<th>Amount Invested</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10,000</td>
<td>20,000</td>
<td>30,000</td>
</tr>
<tr>
<td>1000</td>
<td>8000</td>
<td>21,000</td>
<td>30,000</td>
</tr>
<tr>
<td>2000</td>
<td>6000</td>
<td>22,000</td>
<td>30,000</td>
</tr>
<tr>
<td>3000</td>
<td>4000</td>
<td>23,000</td>
<td>30,000</td>
</tr>
<tr>
<td>4000</td>
<td>2000</td>
<td>24,000</td>
<td>30,000</td>
</tr>
<tr>
<td>5000</td>
<td>0</td>
<td>25,000</td>
<td>30,000</td>
</tr>
</tbody>
</table>

Table 1 shows six plans that Erica could follow to achieve her objective.

**NOW WORK PROBLEM 65.**

**EXERCISE 2.1 Answers Begin on Page AN–9.**

**‘Are You Prepared?’ Problems** Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. **True or False** The lines \( 3x - 4y = 24 \) and \( 3x - 4y = 12 \) are parallel. (pp. 22–26)
2. Solve: \( 2x - 6 = 6 - 4x \) (p. A–24)
3. Find the point of intersection of the lines \( y = 60x - 900 \) and \( y = -15x + 2850 \). (pp. 22–26)
4. Solve for \( x \) if \( 2x + 3z = 6 \). (p. A–24)

**Concepts and Vocabulary**

5. **True or False** A system of two linear equations containing two variables always has at least one solution.
6. A system of equations that has no solution is called ______.
Applications

43. **Dimensions of a Floor** The perimeter of a rectangular floor is 90 feet. Find the dimensions of the floor if the length is twice the width.

44. **Dimensions of a Field** The length of fence required to enclose a rectangular field is 3000 meters. What are the dimensions of the field if the difference between its length and width is 50 meters?

45. **Agriculture** According to the Illinois Farm Business Management Association, in 2008 in the state of Illinois, the nonland production cost for planting corn was $512 per acre and the...
nonland cost for planting soybeans was $336 per acre. The average Illinois farm used 368 acres of land to raise corn and soybeans and budgeted $146,671 for planting these crops. If all the land and all the money budgeted is used, how many acres of each crop should be planted?

Source: Illinois Farm Business Management Association

46. Movie Theater Tickets The Coral movie theater charges $9.00 for adults and $7.00 for senior citizens. On a day when 325 people paid an admission, the total receipts were $2495. How many who paid were adults? How many were seniors?

47. Mixing Nuts A store sells cashews for $5.00 per pound and peanuts for $1.50 per pound. The manager decides to mix 30 pounds of peanuts with some cashews and sell the mixture for $3.00 per pound. How many pounds of cashews should be mixed with the peanuts so that the mixture will produce the same revenue as would selling the nuts separately?

48. Financial Planning A recently retired couple needs $6000 per year to supplement their Social Security. They have $150,000 to invest to obtain this income. They have decided on two investment options: Corporate bonds yielding 6% per annum and a Bank Certificate yielding 3% per annum.

(a) How much should be invested in each to realize exactly $6000?

(b) If, after two years, the couple requires $7000 per year in income, how should they reallocate their investment to achieve the new amount?

49. Cost of Food in Japan In Osaka, Japan, the cost of three 1-liter cartons of whole milk and two 330-gram packages of tofu is 1002 yen. Three packages of tofu cost 8 yen more than two cartons of whole milk. What is the cost of a carton of whole milk? What is the cost for a package of tofu?

Source: www.tanutech.com/japan/jprice.html (July 2006)

50. Cost of Fast Food Four large cheeseburgers and two chocolate shakes cost a total of $7.90. Two shakes cost 15¢ more than one cheeseburger. What is the cost of a cheeseburger? What is the cost of a shake?

51. Computing a Refund The grocery store we use does not mark prices on its goods. My wife went to this store, bought three 1-pound packages of bacon and two cartons of eggs, and paid a total of $7.45. Not knowing that she went to the store, I also went to the same store, purchased two 1-pound packages of bacon and three cartons of eggs, and paid a total of $6.45. Now we want to return two 1-pound packages of bacon and two cartons of eggs. How much will be refunded?

52. Blending Coffees A coffee manufacturer wants to market a new blend of coffee that will cost $5 per pound by mixing $3.75-per-pound coffee and $8-per-pound coffee. What amounts of the $3.75-per-pound coffee and $8-per-pound coffee should be blended to obtain the desired mixture? [Hint: Assume the total weight of the desired blend is 100 pounds.]

53. Pharmacy A doctor’s prescription calls for a daily intake of liquid containing 40 mg of vitamin C and 30 mg of vitamin D. Your pharmacy stocks two liquids that can be used: one contains 20% vitamin C and 30% vitamin D, the other 40% vitamin C and 20% vitamin D. How many milligrams of each liquid should be mixed to fill the prescription?

54. Pharmacy A doctor’s prescription calls for the creation of pills that contain 12 units of vitamin B12 and 12 units of vitamin E. Your pharmacy stocks two powders that can be used to make these pills: one contains 20% vitamin B12 and 30% vitamin E, the other 40% vitamin B12 and 20% vitamin E. How many units of each powder should be mixed in each pill?

55. Diet Preparation A 600- to 700-pound yearling horse needs 33.0 grams of calcium and 21.0 grams of phosphorus per day for a healthy diet. A farmer provides a combination of rolled oats and molasses to provide those nutrients. Rolled oats provide 0.41 grams of calcium per pound and 0.36 grams of phosphorus per pound, while molasses provides 3.35 grams of calcium per pound and 0.36 grams of phosphorus per pound. How many pounds each of rolled oats and molasses should the farmer feed the yearling in order to meet the daily requirements?

Source: Balancing Rations for Horses, R. D. Setzler, Washington State University

56. Restaurant Management A restaurant manager wants to purchase 200 sets of dishes. One design costs $25 per set, while another costs $45 per set. If she only has $7400 to spend, how many of each design should be ordered?

57. Theater Revenues A Broadway theater has 500 seats, divided into orchestra, main, and balcony seating. Orchestra seats sell for $50, main seats for $35, and balcony seats for $25. If all the seats are sold, the gross revenue to the theater is $17,100. If all the main and balcony seats are sold, but only half the orchestra seats are sold, the gross revenue is $14,600. How many are there of each kind of seat?

58. Theater Revenues The Coral movie theater charges $9.00 for adults, $4.50 for children, and $6.00 for senior citizens. One day the theater sold 405 tickets and collected $2320 in receipts. There were twice as many children’s tickets sold as adult tickets. How many adults, children, and senior citizens went to the theater that day?

59. Prices of Fast Food One group of customers bought 8 deluxe hamburgers, 6 orders of large fries, and 6 large colas for $26.10.
A second group ordered 10 deluxe hamburgers, 6 large fries, and 8 large colas and paid $31.60. Is there sufficient information to determine the price of each food item? If not, construct a table showing the various possibilities. Assume that the hamburgers cost between $1.75 and $2.25, the fries between $0.75 and $1.00, and the colas between $0.60 and $0.90.

**60. Prices of Fast Food** Use the information given in Problem 59. Suppose that a third group purchased 3 deluxe hamburgers, 2 large fries, and 4 large colas for $10.95. Now is there sufficient information to determine the price of each food item? If so, determine each price.

**61. Supply and Demand** Suppose that the quantity supplied \( S \) and quantity demanded \( D \) of T-shirts at a concert are given by the following equations:

\[
S = -200 + 50p \\
D = 1000 - 25p
\]

where \( p \) is the price.

(a) Find the equilibrium price for T-shirts at this concert.
(b) What is the equilibrium quantity?
(c) Graph each equation and label the equilibrium point.
(d) What do you think will eventually happen to the price of T-shirts if quantity demanded is greater than quantity supplied?

**62. Supply and Demand** Suppose that the quantity supplied \( S \) and quantity demanded \( D \) of hot dogs at a baseball game are given by the following equations:

\[
S = -2000 + 3000p \\
D = 10,000 - 1000p
\]

where \( p \) is the price.

(a) Find the equilibrium price for hot dogs at the baseball game.
(b) What is the equilibrium quantity?
(c) Graph each equation and label the equilibrium point.
(d) What do you think will eventually happen to the price of hot dogs if quantity demanded is less than quantity supplied?

**63. IS-LM Model in Economics** In economics, the IS curve is a linear equation that represents all combinations of income \( Y \) and interest rates \( r \) that maintain an equilibrium in the market for goods in the economy. The LM curve is a linear equation that represents all combinations of income \( Y \) and interest rates \( r \) that maintain an equilibrium in the market for money in the economy. In an economy, suppose the equilibrium level of income (in millions of dollars) and interest rates satisfy the system of equations

\[
\begin{align*}
0.06Y - 5000r &= 240 \\
0.06Y + 6000r &= 900
\end{align*}
\]

Find the equilibrium level of income and the interest rate.

**64. IS-LM Model in Economics** In economics, the IS curve is a linear equation that represents all combinations of income \( Y \) and interest rates \( r \) that maintain an equilibrium in the market for goods in the economy. The LM curve is a linear equation that represents all combinations of income \( Y \) and interest rates \( r \) that maintain an equilibrium in the market for money in the economy. In an economy, suppose the equilibrium level of income (in millions of dollars) and interest rates satisfy the system of equations

\[
\begin{align*}
0.05Y - 1000r &= 10 \\
0.05Y + 800r &= 100
\end{align*}
\]

Find the equilibrium level of income and the interest rate.

**65. Financial Planning** Nathan wants to invest long term the $70,000 profit he made on the sale of his house. Over 10 years he hopes to have earnings averaging $5000 per year. Based on the yearly average return for 10 years ending December 31, 2009, his financial advisor recommended three funds: Mutual Shares at 4%, Franklin Strategic Income at 6%, and Franklin Small Cap Value at 8% that would allow Nathan to achieve his goal while providing diversification.

(a) Prepare a table showing the various ways Nathan can achieve his goal.
(b) What advice would you give him regarding the amount to invest and the choices available?

**Source:** Franklin Templeton

**66. Financial Planning** A married couple wants to invest $60,000 long term to supplement their retirement income. They hope to have earnings averaging $4000 per year. Based on the yearly average return for 10 years ending December 31, 2009, their financial advisor recommended three funds: Franklin Total Return at 5%, Franklin Strategic Income at 6%, and Franklin Small Cap Value at 8% that would allow them to achieve their goal while providing diversification.

(a) Prepare a table showing the various ways they can achieve their goal.
(b) After seeing the table, the couple asked that the amount invested in the Total Return Fund be twice that invested in the Small Cap Value Fund, how does that change their investment program?

**Source:** Franklin Templeton

**Discussion and Writing**

**67. Make up three systems of two linear equations containing two variables, one that has no solution, one that has exactly one solution, and one that has infinitely many solutions. Give the three systems to a friend to solve and critique.**

**68. Write a brief paragraph outlining your strategy for solving a system of two linear equations containing two variables.**
2.2 Systems of Linear Equations: Gaussian Elimination

69. Do you prefer the method of substitution or the method of elimination for solving a system of two linear equations containing two variables? Give reasons.

70. Look at the table obtained in Example 14. What advice would you give this couple? Provide reasons.

‘Are You Prepared?’ Answers
1. True  2. {2}  3. (50, 2100)  4. \( x = \frac{3}{2}z + 3 \)

2.2 Systems of Linear Equations: Gaussian Elimination

The systematic approach of the method of elimination for solving a system of linear equations provides another method of solution that involves a simplified notation using a matrix.

A **matrix** is defined as a rectangular array of numbers, enclosed by brackets. The numbers are referred to as the **entries** of the matrix. A matrix is further identified by naming its **rows** and **columns**. Some examples of matrices are

<table>
<thead>
<tr>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Row 2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Row 3</td>
<td>−2</td>
<td>4</td>
</tr>
</tbody>
</table>

**Write the Augmented Matrix of a System of Linear Equations**

Consider the following systems of two linear equations containing two variables

\[
\begin{align*}
x + 4y &= 14 \\
3x - 2y &= 0
\end{align*}
\]

We observe that, except for the symbols used to represent the variables, these two systems are identical. As a result, we can dispense altogether with the letters used to symbolize the variables, provided we have some means of keeping track of them. A matrix serves us well in this regard.

When a matrix is used to represent a system of linear equations, it is called the **augmented matrix** of the system. For example,

<table>
<thead>
<tr>
<th>System of Equations</th>
<th>Augmented Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 4y = 14 )</td>
<td>\begin{bmatrix} 1 &amp; 4 &amp; 14 \ 3 &amp; -2 &amp; 0 \end{bmatrix} )</td>
</tr>
<tr>
<td>( 3x - 2y = 0 )</td>
<td>\begin{bmatrix} 1 &amp; 4 &amp; 14 \ 3 &amp; -2 &amp; 0 \end{bmatrix} )</td>
</tr>
</tbody>
</table>

\[ \text{Row 1 [Equation (1)]} \]

\[ \text{Row 2 [Equation (2)]} \]
Here it is understood that column 1 contains the coefficients of the variable $x$, column 2 contains the coefficients of the variable $y$, and column 3 contains the numbers to the right of the equal sign. Each row of the matrix represents an equation of the system. Although not required, it has become customary to place a vertical bar in the matrix as a reminder of the equal sign.

In this book we shall follow the practice of using $x$ and $y$ to denote the variables for systems containing two variables. We will use $x$, $y$, and $z$ for systems containing three variables; we will use subscripted variables ($x_1, x_2, x_3, x_4$, etc.) for systems containing four or more variables.

In writing the augmented matrix of a system, the variables of each equation must be on the left side of the equal sign and the constants on the right side. A variable that does not appear in an equation has a coefficient of 0.

## EXAMPLE 1 Writing the Augmented Matrix of a System of Linear Equations

Write the augmented matrix of each system of equations.

(a) \[
\begin{cases}
3x - 4y = -6 & (1) \\
2x - 3y = -5 & (2)
\end{cases}
\]

(b) \[
\begin{cases}
2x - y + z = 0 & (1) \\
x + z - 1 = 0 & (2) \\
x + 2y - 8 = 0 & (3)
\end{cases}
\]

(c) \[
\begin{cases}
3x_1 - x_2 + x_3 + x_4 = 5 & (1) \\
2x_1 + 6x_3 = 2 & (2)
\end{cases}
\]

### SOLUTION

(a) The augmented matrix is

\[
\begin{bmatrix}
3 & -4 & -6 \\
2 & -3 & -5
\end{bmatrix}
\]

(b) Care must be taken that the system be written so that the coefficients of all variables are present (if any variable is missing, its coefficient is 0). Also, all constants must be to the right of the equal sign. We need to rearrange the given system as follows:

\[
\begin{cases}
2x - y + z = 0 & (1) \\
x + z - 1 = 0 & (2) \\
x + 2y - 8 = 0 & (3)
\end{cases}
\]

The augmented matrix is

\[
\begin{bmatrix}
2 & -1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 2 & 0 & 8
\end{bmatrix}
\]
2.2 Systems of Linear Equations: Gaussian Elimination

(c) The augmented matrix is
\[
\begin{bmatrix}
3 & -1 & 1 & 1 & 5 \\
2 & 0 & 6 & 0 & 2
\end{bmatrix}
\]

NOW WORK PROBLEM 5.

2 Write a System of Linear Equations from an Augmented Matrix

EXAMPLE 2 Writing the System of Linear Equations from the Augmented Matrix

Write the system of linear equations corresponding to each augmented matrix.

(a) \[
\begin{bmatrix}
5 & 2 & 13 \\
-3 & 1 & -10
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
3 & -1 & -1 & 7 \\
4 & 0 & 2 & 8 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

SOLUTION

(a) The matrix has two rows and so represents a system of two equations. The two columns to the left of the vertical bar indicate that the system has two variables. If \(x\) and \(y\) are used to denote these variables, the system of equations is
\[
\begin{align*}
5x + 2y &= 13 \quad (1) \\
-3x + y &= -10 \quad (2)
\end{align*}
\]

(b) Since the augmented matrix has three rows, it represents a system of three equations. Since there are three columns to the left of the vertical bar, the system contains three variables. If \(x\), \(y\), and \(z\) are the three variables, the system of equations is
\[
\begin{align*}
3x - y - z &= 7 \quad (1) \\
4x + 2z &= 8 \quad (2) \\
y + z &= 0 \quad (3)
\end{align*}
\]

3 Perform Row Operations on a Matrix

Row operations on a matrix are used to solve systems of equations when the system is written as an augmented matrix. There are three basic row operations.

Row Operations

1. Interchange any two rows.
2. Replace a row by a nonzero multiple of that row.
3. Replace a row by the sum of that row and a constant nonzero multiple of some other row.

These three row operations correspond to the three rules given earlier for obtaining an equivalent system of equations. When a row operation is performed on a matrix, the resulting matrix represents a system of equations equivalent to the system represented by the original matrix.
For example, consider the augmented matrix
\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & -1 & 2
\end{bmatrix}
\]
Suppose that we want to apply a row operation to this matrix that results in a matrix whose entry in row 2, column 1 is a 0. The row operation to use is
\[
\text{Multiply each entry in row 1 by } -4 \text{ and add the result to the corresponding entry in row 2.}
\]
If we use \( r_2 \) to represent the new entries in row 2 and we use \( r_1 \) and \( r_2 \) to represent the original entries in rows 1 and 2, respectively, then we can represent the row operation in statement (1) by
\[
R_2 = -4r_1 + r_2
\]
Then
\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & -1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
-4(1) + 4 & -4(2) + (-1) & -4(3) + 2
\end{bmatrix}
= \begin{bmatrix}
1 & 2 & 3 \\
0 & -9 & -10
\end{bmatrix}
\]
As desired, we now have the entry 0 in row 2, column 1.

**EXAMPLE 3 Applying a Row Operation to an Augmented Matrix**

Apply the row operation \( R_2 = -3r_1 + r_2 \) to the augmented matrix
\[
\begin{bmatrix}
1 & -2 & 2 \\
3 & -5 & 9
\end{bmatrix}
\]
**SOLUTION** The row operation \( R_2 = -3r_1 + r_2 \) tells us that the entries in row 2 are to be replaced by the entries obtained after multiplying each entry in row 1 by \(-3\) and adding the result to the corresponding entry in row 2. Thus,
\[
\begin{bmatrix}
1 & -2 & 2 \\
3 & -5 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 2 \\
-3(1) + 3 & (-3)(-2) + (-5) & -3(2) + 9
\end{bmatrix}
= \begin{bmatrix}
1 & -2 & 2 \\
0 & 1 & 3
\end{bmatrix}
\]

**NOW WORK PROBLEM 17.**

**EXAMPLE 4 Finding a Particular Row Operation**

Using the augmented matrix
\[
\begin{bmatrix}
1 & -2 & 2 \\
0 & 1 & 3
\end{bmatrix}
\]
find a row operation that will result in this augmented matrix having a 0 in row 1, column 2.

**SOLUTION** We want a 0 in row 1, column 2. This result can be accomplished by multiplying row 2 by 2 and adding the result to row 1. That is, apply the row operation \( R_1 = 2r_2 + r_1 \).
\[
\begin{bmatrix}
1 & -2 & 2 \\
0 & 1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2(0) + 1 & 2(1) + (-2) & 2(3) + 2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 8 \\
0 & 1 & 3
\end{bmatrix}
\]
A word about the notation just introduced. A row operation such as \( R_1 = 2r_2 + r_1 \) changes the entries in row 1. Note also that for this type of row operation the entries in a given row are changed by multiplying the entries in some other row by an appropriate nonzero number and adding the results to the original entries of the row to be changed.

4 Solve a System of Linear Equations Using Gaussian Elimination

To solve a system of linear equations using matrices, we use row operations on the augmented matrix of the system to obtain a matrix that is in row echelon form.

Definition

A matrix is in **row echelon form** when

1. The entry in row 1, column 1 is a 1, and 0s appear below it.
2. The first nonzero entry in each row after the first row is a 1, 0s appear below it, and it appears to the right of the first nonzero entry in any row above.
3. Any rows that contain all 0s to the left of the vertical bar appear at the bottom.

For example, for a system of two linear equations containing two variables the augmented matrix is in row echelon form if it is in one of the forms

\[
\begin{bmatrix}
1 & a & b \\
0 & 0 & c
\end{bmatrix}
\text{ or }
\begin{bmatrix}
1 & a & b \\
0 & 0 & 1 & c
\end{bmatrix}
\]

where \( a, b, \) and \( c \) are real numbers.

Look at the augmented matrix (I). The entry \( c \) in row 2 is either 0 or 1. If \( c = 1 \), then the system is inconsistent. (Do you see why? The second row gives rise to the equation \( 0x + 0y = 1 \).) If \( c = 0 \), the system is consistent and has infinitely many solutions given by \( x = b - ay \), where \( y \) is any real number.

Look at the augmented matrix (II). The second row represents the equation \( y = c \). Then \( x \) can be found by back-substituting \( y = c \) into the equation represented by the first row. The system is consistent and the solution is unique.

For a system of three equations containing three variables, the augmented matrix is in row echelon form if it is in one of the forms

\[
\begin{bmatrix}
1 & a & b & d \\
0 & 1 & c & e \\
0 & 0 & 1 & f
\end{bmatrix}
\text{ or }
\begin{bmatrix}
1 & a & b & d \\
0 & 0 & 0 & e \\
0 & 0 & 0 & f
\end{bmatrix}
\text{ or }
\begin{bmatrix}
1 & a & b \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where \( a, b, c, d, e, \) and \( f \) are real numbers.

Look at the augmented matrix (I). We see that \( z = f \) and that back-substitution using row 2 will get \( y \) and another back-substitution using row 1 will get \( x \). The system is consistent and the solution is unique.

The remaining augmented matrices are consistent or inconsistent depending on the numbers \( a, b, c, d, e, \) and \( f \). In practice, looking at the equations represented by the rows provides the necessary information.
The process of writing the augmented matrix of a system is called **Gaussian elimination**.

Two advantages of solving a system of equations by writing the augmented matrix in row echelon form are the following:

1. The process is algorithmic; that is, it consists of repetitive steps that can be programmed on a computer.
2. The process works on any system of linear equations, no matter how many equations or variables are present.

Let’s see how Gaussian elimination is used to solve a system of linear equations. To see what is happening, we’ll write the corresponding system of equations next to the matrix obtained after a row operation is performed.

**EXAMPLE 5 Solving a System of Linear Equations Using Gaussian Elimination**

Solve: \[
\begin{align*}
4x + 3y &= 11 \\
x - 3y &= -1
\end{align*}
\]

**SOLUTION**

Write the augmented matrix that represents this system:

\[
\begin{bmatrix}
4 & 3 & 11 \\
1 & -3 & -1
\end{bmatrix}
\]

The next step is to place a 1 in row 1, column 1. An interchange of rows 1 and 2 is the easiest way to do this.

\[
\begin{bmatrix}
4 & 3 & 11 \\
1 & -3 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -3 & -1 \\
4 & 3 & 11
\end{bmatrix}
\]

Now we want a 0 under the entry 1 in column 1. (This eliminates the variable x from the second equation.) Use the row operation \(R_2 = -4R_1 + R_2\).

\[
\begin{bmatrix}
1 & -3 & -1 \\
4 & 3 & 11
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -3 & -1 \\
0 & 15 & 15
\end{bmatrix}
\]

Now we want the entry 1 in row 2, column 2. (This makes it easy to solve for y.) Use \(R_2 = \frac{1}{15}R_2\).

\[
\begin{bmatrix}
1 & -3 & -1 \\
0 & 15 & 15
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -3 & -1 \\
0 & 1 & 1
\end{bmatrix}
\]

This matrix is the row echelon form of the augmented matrix. The second row of the matrix on the right represents the equation \(y = 1\). Using \(y = 1\), back-substitute into the equation \(x - 3y = -1\) (from the first row) to get

\[
\begin{align*}
x - 3y &= -1 \\
x - 3(1) &= -1 \\
x &= 2
\end{align*}
\]

The solution of the system is \(x = 2, y = 1\) or, using ordered pairs, \((2, 1)\). □

**NOW WORK PROBLEM 39.**
The steps used to solve the system of linear equations in Example 5 can be summarized as follows:

**Solving a System of Linear Equations Using Gaussian Elimination**

**STEP 1** Write the augmented matrix that represents the system.

**STEP 2** Perform row operations that place the entry 1 in row 1, column 1.

**STEP 3** Perform row operations that leave the entry 1 in row 1, column 1 unchanged, while causing 0s to appear below it in column 1.

**STEP 4** Perform row operations that place the entry 1 in row 2, column 2, but leave the entries in column 1 unchanged. If it is impossible to place a 1 in row 2, column 2, then proceed to place a 1 in row 2, column 3. Once a 1 is in place, perform row operations to place 0s below it.

**STEP 5** Now repeat Step 4, placing a 1 in the next row, but one column to the right, while leaving all entries in columns to the left unchanged. Continue until the bottom row or the vertical bar is reached. (If any rows are obtained that contain only 0’s on the left side of the vertical bar, place such rows at the bottom of the matrix.)

**STEP 6** The matrix that results is the row echelon form of the augmented matrix. Analyze the system of equations corresponding to it to solve the original system.

---

**EXAMPLE 6** Solving a System of Linear Equations Using Gaussian Elimination

Solve:

\[
\begin{align*}
  x - y + z &= 8 \quad (1) \\
  2x + 3y - z &= -2 \quad (2) \\
  3x - 2y - 9z &= 9 \quad (3)
\end{align*}
\]

**SOLUTION**

**STEP 1** The augmented matrix of the system is

\[
\begin{bmatrix}
  1 & -1 & 1 & 8 \\
  2 & 3 & -1 & -2 \\
  3 & -2 & -9 & 9
\end{bmatrix}
\]

**STEP 2** Because the entry 1 is already present in row 1, column 1, we can go to Step 3.

**STEP 3** Perform the row operations \( R_2 = -2r_1 + r_2 \) and \( R_3 = -3r_1 + r_3 \). Each of these leaves the entry 1 in row 1, column 1 unchanged, while causing 0s to appear under it.*

\[
\begin{bmatrix}
  1 & -1 & 1 & 8 \\
  2 & 3 & -1 & -2 \\
  3 & -2 & -9 & 9
\end{bmatrix} \quad \begin{bmatrix}
  1 & -1 & 1 & 8 \\
  0 & 5 & -3 & -18 \\
  0 & 1 & -12 & -15
\end{bmatrix}
\]

* You should convince yourself that doing both of these simultaneously is the same as doing the first followed by the second.
STEP 4  The easiest way to obtain the entry 1 in row 2, column 2 without altering column 1 is to interchange rows 2 and 3 (another way would be to multiply row 2 by \( \frac{1}{5} \), but this introduces fractions).

\[
\begin{pmatrix}
1 & -1 & 1 \\
0 & 1 & -12 \\
0 & 5 & -3
\end{pmatrix}
\]

To get a 0 under the 1 in row 2, column 2, perform the row operation

\[
R_3 = -5r_2 + r_3.
\]

STEP 5  Continuing, obtain a 1 in row 3, column 3 by using \( R_3 = \frac{1}{57} r_3 \).

STEP 6  The matrix on the right is the row echelon form of the augmented matrix. The system of equations represented by the matrix in row echelon form is

\[
\begin{align*}
x - y + z &= 8 \\
y - 12z &= -15 \\
z &= 1
\end{align*}
\]

Using \( z = 1 \), back-substitute to get

\[
\begin{align*}
x - y + 1 &= 8 \\
y - 12(1) &= -15
\end{align*}
\]

Simplify to get \( x - y = 7 \) \( y = -3 \).

So \( y = -3 \), and back-substituting into \( x - y = 7 \), we find that \( x = 4 \). The solution of the system is \( x = 4, y = -3, z = 1 \) or, using ordered triplets, \((4, -3, 1)\).

USING TECHNOLOGY

EXAMPLE 7  Solving a System of Linear Equations Using a Graphing Utility

Rework Example 6 using a graphing utility.

SOLUTION  A graphing utility can be used to obtain the row echelon form of the augmented matrix. The augmented matrix of the system given in Example 6 is

\[
\begin{pmatrix}
1 & -1 & 1 & 8 \\
2 & 3 & -1 & -2 \\
3 & -2 & -9 & 9
\end{pmatrix}
\]

Enter this matrix into a graphing utility and name it \( A \). See Figure 8(a). Use the \text{REF (Row Echelon Form)} command on matrix \( A \) to obtain the results shown in Figure 8(b). Since the entire matrix does not fit on the screen, you will need to scroll right to see the rest of it. See Figure 8(c).
The system of equations represented by the matrix in row echelon form is

\[
\begin{align*}
  x - \frac{2}{3}y - 3z &= 3 \quad &\text{(1)} \\
  y + \frac{15}{13}z &= -\frac{24}{13} \quad &\text{(2)} \\
  z &= 1 \quad &\text{(3)}
\end{align*}
\]

Using \( z = 1 \), back-substitute to get

\[
\begin{align*}
  x - \frac{2}{3}y - 3(1) &= 3 \quad &\text{(1)} \\
  y + \frac{15}{13}(1) &= -\frac{24}{13} \quad &\text{(2)} \quad \text{Simplify.}
\end{align*}
\]

Solve the second equation for \( y \), to obtain \( y = -3 \). Back-substitute \( y = -3 \) into \( x - \frac{2}{3}y = 6 \), to find that \( x = 4 \). The solution of the system is \( x = 4, y = -3, z = 1 \) or, using ordered triplets, \( (4, -3, 1) \).

Notice that the row echelon form of the augmented matrix using the graphing utility differs from the row echelon form in our algebraic solution, yet both matrices provide the same solution! This is because the two solutions used different row operations to obtain the row echelon form. In all likelihood, the two solutions parted ways in Step 4 of the algebraic solution, where we avoided introducing fractions by interchanging rows 2 and 3.

Sometimes it is advantageous to write a matrix in \textbf{reduced row echelon form}. In this form, row operations are used to obtain entries that are 0 above (as well as below) the leading 1 in a row. For example, the row echelon form obtained in the algebraic solution to Example 6 is

\[
\begin{bmatrix}
  1 & -1 & 1 & 8 \\
  0 & 1 & -12 & -15 \\
  0 & 0 & 1 & 1
\end{bmatrix}
\]

To write this matrix in reduced row echelon form, proceed as follows:

\[
\begin{bmatrix}
  1 & -1 & 1 & 8 \\
  0 & 1 & -12 & -15 \\
  0 & 0 & 1 & 1
\end{bmatrix} \xrightarrow{R_1 = r_2 + r_1} \begin{bmatrix}
  1 & 0 & -11 & -7 \\
  0 & 1 & -12 & -15 \\
  0 & 0 & 1 & 1
\end{bmatrix} \xrightarrow{R_2 = 12r_3 + r_2} \begin{bmatrix}
  1 & 0 & 0 & 4 \\
  0 & 1 & 0 & -3 \\
  0 & 0 & 1 & 1
\end{bmatrix}
\]

The matrix is now written in reduced row echelon form. The advantage of writing the matrix in this form is that the solution to the system, \( x = 4, y = -3, z = 1 \) is readily found without the need to back-substitute.

The process of writing the augmented matrix of a system of equations in reduced row echelon form is called \textbf{Gauss–Jordan elimination}. 

\[ \text{FIGURE 8} \]

(a) \hspace{1cm} (b) \hspace{1cm} (c)
Most graphing utilities also have the ability to put a matrix in reduced row echelon form. Figure 9 shows the reduced row echelon form of the augmented matrix from Example 7 using the RREF command on a TI-84 Plus graphing calculator.

NOW WORK PROBLEMS 45 AND 55.

Express the Solution of a System with an Infinite Number of Solutions

EXAMPLE 8 Solving a System of Linear Equations Using Gaussian Elimination (Infinitely Many Solutions)

Solve: \[\begin{align*}
2x - 3y &= 5 \\
4x - 6y &= 10
\end{align*}\]

**SOLUTION** The augmented matrix representing this system is

\[
\begin{bmatrix}
2 & -3 & 5 \\
4 & -6 & 10
\end{bmatrix}
\]

To place a 1 in row 1, column 1, use \(R_1 = \frac{1}{2}r_1:\)

\[
\begin{bmatrix}
2 & -3 & 5 \\
4 & -6 & 10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{3}{2} & \frac{5}{2} \\
4 & -6 & 10
\end{bmatrix}
\]

To place a 0 in row 2, column 1, use \(R_2 = -4r_1 + r_2:\)

\[
\begin{bmatrix}
1 & -\frac{3}{2} & \frac{5}{2} \\
4 & -6 & 10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{3}{2} & \frac{5}{2} \\
0 & 0 & 0
\end{bmatrix}
\]

The system of equations looks like

\[
\begin{align*}
x - \frac{3}{2}y &= \frac{5}{2} \\
0x + 0y &= 0
\end{align*}
\]

The second equation is true for any choice of \(x\) and \(y\), so all numbers \(x\) and \(y\) that obey the first equation are solutions of the system. Since any point on the line \(x - \frac{3}{2}y = \frac{5}{2}\) is a solution, there are an infinite number of solutions.

Using \(y\) as parameter, we can list some of these solutions by assigning values to \(y\) and then calculating \(x\) from the equation \(x = \frac{3}{2}y + \frac{5}{2}\).

If \(y = 0\), then \(x = \frac{5}{2}\), so \(x = \frac{5}{2}, y = 0\) is a solution.

If \(y = 1\), then \(x = 4\), so \(x = 4, y = 1\) is a solution.

If \(y = 5\), then \(x = 10\), so \(x = 10, y = 5\) is a solution.

If \(y = -3\), then \(x = -2\), so \(x = -2, y = -3\) is a solution.
And so on.

We write the solution as

\[ x = \frac{3}{2}y + \frac{5}{2}, \quad y \text{ is any real number} \]

or, using ordered pairs, as \( \{(x, y) \mid x = \frac{3}{2}y + \frac{5}{2}, y \text{ is any real number}\} \).

Now work Problem 41.

**EXAMPLE 9  Solving a System of Linear Equations Using Gaussian Elimination (Infinitely Many Solutions)**

Solve:

\[
\begin{align*}
6x - y - z &= 4 \quad (1) \\
-12x + 2y + 2z &= -8 \quad (2) \\
5x + y - z &= 3 \quad (3)
\end{align*}
\]

**SOLUTION** Start with the augmented matrix of the system. Then use row operations to obtain a 1 in row 1, column 1 and 0s in the remainder of column 1.

\[
\begin{bmatrix}
6 & -1 & -1 & | & 4 \\
-12 & 2 & 2 & | & -8 \\
5 & 1 & -1 & | & 3
\end{bmatrix}
\]

\( R_1 = -r_3 + r_1 \)

\[
\begin{bmatrix}
1 & -2 & 0 & | & 1 \\
-12 & 2 & 2 & | & -8 \\
5 & 1 & -1 & | & 3
\end{bmatrix}
\]

\( R_2 = 12r_1 + r_2 \)

\[
\begin{bmatrix}
1 & -2 & 0 & | & 1 \\
0 & 2 & 2 & | & -2 \\
5 & 1 & -1 & | & 3
\end{bmatrix}
\]

\( R_3 = -5r_1 + r_3 \)

\[
\begin{bmatrix}
1 & -2 & 0 & | & 1 \\
0 & 2 & 2 & | & -2 \\
0 & 11 & -1 & | & -2
\end{bmatrix}
\]

Obtaining a 1 in row 2, column 2 without altering column 1 can be accomplished by \( R_2 = -\frac{1}{22}r_2 \), or by \( R_3 = \frac{1}{11}r_3 \) and interchanging rows 2 and 3, or by \( R_2 = \frac{23}{11}r_3 + r_2 \).

We shall use the first of these.

\[
\begin{bmatrix}
1 & -2 & 0 & | & 1 \\
0 & 2 & 2 & | & 4 \\
0 & 11 & -1 & | & 2
\end{bmatrix}
\]

\( R_1 = -\frac{1}{22}r_2 \)

\[
\begin{bmatrix}
1 & -2 & 0 & | & 1 \\
0 & 1 & -\frac{1}{11} & | & -\frac{2}{11} \\
0 & 11 & -1 & | & -2
\end{bmatrix}
\]

\( R_2 = -11r_1 + r_3 \)

\[
\begin{bmatrix}
1 & -2 & 0 & | & 1 \\
0 & 1 & -\frac{1}{11} & | & -\frac{2}{11} \\
0 & 0 & -\frac{1}{11} & | & -\frac{2}{11}
\end{bmatrix}
\]

This matrix is in row echelon form. Because the bottom row consists entirely of 0s, the system actually consists of only two equations.

\[
\begin{align*}
x - 2y &= 1 \quad (1) \\
y - \frac{1}{11}z &= -\frac{2}{11} \quad (2)
\end{align*}
\]

From the second equation we get \( y = \frac{1}{11}z - \frac{2}{11} \). Then back-substitute this solution for \( y \) into the first equation to get

\[
x = 2y + 1 = 2\left(\frac{1}{11}z - \frac{2}{11}\right) + 1 = \frac{2}{11}z + \frac{7}{11}
\]
The original system is equivalent to the system
\[
\begin{align*}
\begin{cases}
  x &= \frac{2}{11}z + \frac{7}{11} \\
  y &= \frac{1}{11}z - \frac{2}{11}
\end{cases}
\end{align*}
\]
where \( z \), the parameter, can be any real number.

Let's look at the situation. The original system of three equations is equivalent to a system containing two equations. This means that any values of \( x, y, z \) that satisfy both
\[
\begin{align*}
  x &= \frac{2}{11}z + \frac{7}{11} \\
  y &= \frac{1}{11}z - \frac{2}{11}
\end{align*}
\]
will be solutions. For example, \( z = 0, x = \frac{7}{11}, y = -\frac{2}{11} ; z = 1, x = \frac{9}{11}, y = -\frac{1}{11} ; \) and \( z = -1, x = \frac{5}{11}, y = -\frac{3}{11} \) are some of the solutions of the original system. There are, in fact, infinitely many values of \( x, y, \) and \( z \) for which the two equations are satisfied. That is, the original system has infinitely many solutions. We will write the solution of the original system as
\[
\begin{align*}
\begin{cases}
  x &= \frac{2}{11}z + \frac{7}{11} \\
  y &= \frac{1}{11}z - \frac{2}{11}
\end{cases}
\end{align*}
\]
where \( z \), the parameter, can be any real number.

We can also use Gauss–Jordan elimination to find the solution by writing the augmented matrix in reduced row echelon form. Starting with the row echelon form, we have
\[
\begin{bmatrix}
  1 & -2 & 0 & -\frac{1}{11} \\
  0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

\( R_1 = 2r_2 + r_1 \)

The matrix on the right is in reduced row echelon form. The corresponding system of equations is
\[
\begin{align*}
\begin{cases}
  x &= \frac{2}{11}z + \frac{7}{11} \\
  y &= \frac{1}{11}z - \frac{2}{11}
\end{cases}
\end{align*}
\]

or, equivalently,
\[
\begin{align*}
\begin{cases}
  x &= \frac{2}{11}z + \frac{7}{11} \\
  y &= \frac{1}{11}z - \frac{2}{11}
\end{cases}
\end{align*}
\]

where \( z \), the parameter, can be any real number.

\( \text{NOW WORK PROBLEM 47.} \)
Use Gaussian Elimination to Identify an Inconsistent System

**EXAMPLE 10** Identifying an Inconsistent System of Linear Equations

Solve: \[
\begin{align*}
x + y + z &= 6 \\
2x &- y - z = 3 \\
x + 2y + 2z &= 0
\end{align*}
\]

**SOLUTION** Proceed as follows, beginning with the augmented matrix.

\[
\begin{bmatrix}
1 & 1 & 1 & | & 6 \\
2 & -1 & -1 & | & 3 \\
1 & 2 & 2 & | & 0
\end{bmatrix}
\]

Interchange rows 2 and 3.

\[
\begin{bmatrix}
1 & 1 & 1 & | & 6 \\
0 & -3 & -3 & | & -9 \\
0 & 1 & 1 & | & -6
\end{bmatrix}
\]

Subtract \( r_1 \) from \( r_3 \).

\[
\begin{bmatrix}
1 & 1 & 1 & | & 6 \\
0 & -3 & -3 & | & -9 \\
0 & 0 & 0 & | & -27
\end{bmatrix}
\]

This matrix is in row echelon form. The bottom row is equivalent to the equation \( 0x + 0y + 0z = -27 \) which has no solution. The original system is inconsistent.

**NOW WORK PROBLEM 27.**

Solve Applied Problems Involving Systems of Equations

**EXAMPLE 11** Calculating Production Output

FoodPerfect Corporation manufactures three models of the Perfect Foodprocessor. Each Model X processor requires 30 minutes of electrical assembly, 40 minutes of mechanical assembly, and 30 minutes of testing; each Model Y requires 20 minutes of electrical assembly, 50 minutes of mechanical assembly, and 30 minutes of testing; and each Model Z requires 30 minutes of electrical assembly, 30 minutes of mechanical assembly, and 20 minutes of testing. If 2500 minutes of electrical assembly, 3500 minutes of mechanical assembly, and 2400 minutes of testing are used in one day, how many of each model will be produced?

**SOLUTION** The table that follows summarizes the given information:

<table>
<thead>
<tr>
<th>Model</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Time Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>30</td>
<td>20</td>
<td>30</td>
<td>2500</td>
</tr>
<tr>
<td>Mechanical</td>
<td>40</td>
<td>50</td>
<td>30</td>
<td>3500</td>
</tr>
<tr>
<td>Testing</td>
<td>30</td>
<td>30</td>
<td>20</td>
<td>2400</td>
</tr>
</tbody>
</table>

Assign variables to represent the unknowns:

\[
x = \text{Number of Model X produced} \\
y = \text{Number of Model Y produced} \\
z = \text{Number of Model Z produced}
\]

Based on the table, we obtain the following system of equations:

\[
\begin{align*}
30x + 20y + 30z &= 2500 \\
40x + 50y + 30z &= 3500 \\
30x + 30y + 20z &= 2400
\end{align*}
\]

Divide each equation by 10.

\[
\begin{align*}
3x + 2y + 3z &= 250 \quad (1) \\
4x + 5y + 3z &= 350 \quad (2) \\
3x + 3y + 2z &= 240 \quad (3)
\end{align*}
\]
The augmented matrix of this system is
\[
\begin{bmatrix}
3 & 2 & 3 & 250 \\
4 & 5 & 3 & 350 \\
3 & 3 & 2 & 240 \\
\end{bmatrix}
\]

We could obtain a 1 in row 1, column 1 by using the row operation \(R_1 = \frac{1}{3}r_1\), but the introduction of fractions is best avoided. Instead, use
\[
R_2 = -r_1 + r_2
\]
to place a 1 in row 2, column 1. The result is
\[
\begin{bmatrix}
3 & 2 & 3 & 250 \\
1 & 3 & 0 & 100 \\
3 & 3 & 2 & 240 \\
\end{bmatrix}
\]

Next, interchange row 1 and row 2:
\[
\begin{bmatrix}
1 & 3 & 0 & 100 \\
3 & 2 & 3 & 250 \\
3 & 3 & 2 & 240 \\
\end{bmatrix}
\]

Use \(R_2 = -3r_1 + r_2\) and \(R_3 = -3r_1 + r_3\) to obtain
\[
\begin{bmatrix}
1 & 3 & 0 & 100 \\
0 & -7 & 3 & -50 \\
0 & -6 & 2 & -60 \\
\end{bmatrix}
\]

Use \(R_2 = -r_1\) followed by \(R_2 = r_3 + r_2\):
\[
\begin{bmatrix}
1 & 3 & 0 & 100 \\
0 & 7 & -3 & 50 \\
0 & -6 & 2 & -60 \\
\end{bmatrix}
\]

Next, use \(R_3 = 6r_2 + r_3\) to obtain
\[
\begin{bmatrix}
1 & 3 & 0 & 100 \\
0 & 1 & -1 & -10 \\
0 & -6 & 2 & -60 \\
\end{bmatrix}
\]

Next, use \(R_3 = -r_2\), The result is
\[
\begin{bmatrix}
1 & 3 & 0 & 100 \\
0 & 1 & -1 & -10 \\
0 & 0 & -4 & -120 \\
\end{bmatrix}
\]

The matrix is now in row echelon form. We find \(z = 30\). From row 2, we have \(y - z = -10\) so that \(y = z - 10 = 30 - 10 = 20\). Finally, from row 1, we have \(x + 3y = 100\) so \(x = -3y + 100 = -60 + 100 = 40\). The solution of the system is \(x = 40, y = 20, z = 30\). In one day 40 Model X, 20 Model Y, and 30 Model Z processors are produced.

**NOW WORK PROBLEM 65.**
EXAMPLE 12 Nutrition

A dietitian at Cook County Hospital wants a patient to have a meal that has 65 grams of protein, 95 grams of carbohydrates, and 905 milligrams of calcium. The hospital food service tells the dietitian that the dinner for today is chicken à la king, baked potatoes, and 2% milk. Each serving of chicken à la king has 30 grams of protein, 35 grams of carbohydrates, and 200 milligrams of calcium. Each serving of baked potatoes contains 4 grams of protein, 33 grams of carbohydrates, and 10 milligrams of calcium. Each glass of 2% milk contains 9 grams of protein, 13 grams of carbohydrates, and 300 milligrams of calcium. How many servings of each food should the dietitian provide for the patient?

**SOLUTION**

Let $c$, $p$, and $m$ represent the number of servings of chicken à la king, baked potatoes, and milk, respectively. The dietitian wants the patient to have 65 grams of protein. Each serving of chicken à la king has 30 grams of protein, so $c$ servings will have $30c$ grams of protein. Each serving of baked potatoes contains 4 grams of protein, so $p$ potatoes will have $4p$ grams of protein. Finally, each glass of milk has 9 grams of protein, so $m$ glasses of milk will have $9m$ grams of protein. The same logic will result in equations for carbohydrates and calcium, and we have the following system of equations:

$$
\begin{align*}
30c + 4p + 9m &= 65 \\
35c + 33p + 13m &= 95 \\
200c + 10p + 300m &= 905
\end{align*}
$$

Begin with the augmented matrix and proceed as follows:

$\begin{bmatrix}
30 & 4 & 9 & 65 \\
35 & 33 & 13 & 95 \\
200 & 10 & 300 & 905
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 2 & 3 & 13 \\
15 & 10 & 6 \\
35 & 33 & 13 & 95 \\
200 & 10 & 300 & 905
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 2 & 3 & 13 \\
0 & 5 & 2 & 115 \\
0 & 50 & 240 & 1415 \\
0 & 23 & 34 & 23
\end{bmatrix}$

$R_1 = \left(\frac{1}{30}\right)r_1$

$R_2 = -35r_1 + r_2$

$R_3 = -200r_1 + r_3$

$\begin{bmatrix}
1 & 2 & 3 & 13 \\
0 & 5 & 2 & 115 \\
0 & 50 & 240 & 1415 \\
0 & 23 & 34 & 23
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 2 & 3 & 13 \\
0 & 1 & 34 & 34 \\
0 & 50 & 240 & 1415 \\
0 & 0 & 910 & 8210
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 2 & 3 & 13 \\
0 & 1 & 34 & 34 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}$

$R_2 = \left(\frac{3}{86}\right)r_2$

$R_3 = \left(\frac{50}{3}\right)r_2 + r_3$

$R_3 = \left(\frac{17}{4105}\right)r_3$

The matrix is now in echelon form. The final matrix represents the system

$$
\begin{align*}
c + \frac{2}{15}p + \frac{3}{10}m &= \frac{13}{6} \\
p + \frac{3}{34}m &= \frac{23}{34} \\
m &= 2
\end{align*}
$$

The matrix is now in echelon form. The final matrix represents the system

$$
\begin{align*}
c + \frac{2}{15}p + \frac{3}{10}m &= \frac{13}{6} \\
p + \frac{3}{34}m &= \frac{23}{34} \\
m &= 2
\end{align*}
$$
From (3), we determine that 2 glasses of milk should be served. Back-substitute $m = 2$ into Equation (2) to find that $p = \frac{1}{2}$, so $\frac{1}{2}$ of a baked potato should be served. Back-substitute these values into Equation (1) and find that $c = \frac{3}{2}$, so 1.5 servings of chicken à la king should be given to the patient to meet the dietary requirements.

**EXERCISE 2.2** Answers Begin on Page AN–10.

### Concepts and Vocabulary

1. An $m$ by $n$ rectangular array of numbers is called a(n) _____ matrix.

2. The matrix used to represent a system of linear equations is called a(n) _____ matrix.

3. **True or False**: The augmented matrix of a system of two equations containing three variables has two rows and four columns.

4. **True or False**: The matrix $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ is in row echelon form.

### Skill Building

**In Problems 5–16, write the augmented matrix of each system of equations.**

5. \[
\begin{align*}
2x - 3y &= 5 \\
x - y &= 3
\end{align*}
\]

6. \[
\begin{align*}
4x + y &= 5 \\
2x + y &= 5
\end{align*}
\]

7. \[
\begin{align*}
2x + y + 6 &= 0 \\
3x + y &= -1
\end{align*}
\]

8. \[
\begin{align*}
-3x - y &= -3 \\
4x - y + 2 &= 0
\end{align*}
\]

9. \[
\begin{align*}
2x - y - z &= 0 \\
x - y + z &= 1 \\
3x - y &= 2
\end{align*}
\]

10. \[
\begin{align*}
x + y + z &= 3 \\
2x + z &= 0 \\
3x - y - z &= 1
\end{align*}
\]

11. \[
\begin{align*}
2x - 3y + z - 7 &= 0 \\
x + y - z &= 1 \\
2x + 2y - 3z + 4 &= 0
\end{align*}
\]

12. \[
\begin{align*}
5x - 3y + 6z + 1 &= 0 \\
x - y + z &= 1 \\
2x + 3y + 5 &= 0
\end{align*}
\]

13. \[
\begin{align*}
4x_1 - x_2 + 2x_3 - x_4 &= 4 \\
x_1 + x_2 + 6 &= 0 \\
2x_2 - x_3 + x_4 &= 5
\end{align*}
\]

14. \[
\begin{align*}
3x_1 - 5x_2 + x_3 &= 2 \\
x_1 - x_2 + x_3 &= 6 \\
2x_1 + x_3 + 4 &= 0
\end{align*}
\]

15. \[
\begin{align*}
x_1 - x_2 + x_3 - x_4 &= 0 \\
2x_1 + 3x_2 - x_3 + 4x_4 &= 5
\end{align*}
\]

16. \[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 4 \\
x_1 - 2x_2 + 3x_3 - 4x_4 &= 5
\end{align*}
\]

**In Problems 17–24, perform each row operation on the given augmented matrix.**

17. \[
\begin{bmatrix}
1 & -3 & -2 \\
2 & -5 & 5
\end{bmatrix} \quad R_2 = -2r_1 + r_2
\]

18. \[
\begin{bmatrix}
1 & -3 & -3 \\
2 & -5 & -4
\end{bmatrix} \quad R_2 = -2r_1 + r_2
\]

19. \[
\begin{bmatrix}
1 & -3 & 4 & 3 \\
2 & -5 & 6 & 6 \\
-3 & 3 & 4 & 6
\end{bmatrix}
\]

(a) $R_2 = -2r_1 + r_2$

(b) $R_3 = 3r_1 + r_3$

20. \[
\begin{bmatrix}
1 & -3 & 3 & -5 \\
2 & -5 & -3 & -5 \\
-3 & -2 & 4 & 6
\end{bmatrix}
\]

(a) $R_2 = -2r_1 + r_2$

(b) $R_3 = 3r_1 + r_3$

21. \[
\begin{bmatrix}
1 & -3 & 2 & -6 \\
2 & -5 & 3 & -4 \\
-3 & -6 & 2 & 6
\end{bmatrix}
\]

(a) $R_2 = -2r_1 + r_2$

(b) $R_3 = 3r_1 + r_3$

22. \[
\begin{bmatrix}
1 & -3 & -4 & -6 \\
2 & -5 & 6 & -6 \\
-3 & 1 & 4 & 6
\end{bmatrix}
\]

(a) $R_2 = -2r_1 + r_2$

(b) $R_3 = 3r_1 + r_3$
2.2 Systems of Linear Equations: Gaussian Elimination

23. \[
\begin{bmatrix}
1 & -3 & 1 & -2 \\
2 & -5 & 6 & -2 \\
-3 & 1 & 4 & 6 \\
\end{bmatrix}
\]
(a) \( R_2 = -2r_1 + r_2 \)
(b) \( R_3 = 3r_1 + r_3 \)

24. \[
\begin{bmatrix}
1 & -3 & 1 & -2 \\
2 & -5 & 2 & 6 \\
-3 & -6 & 4 & 6 \\
\end{bmatrix}
\]
(a) \( R_2 = -2r_1 + r_2 \)
(b) \( R_3 = 3r_1 + r_3 \)

In Problems 25–36, the row echelon form of a system of linear equations is given.
(a) Write the system of equations corresponding to the given matrix. Use \( x, y; \) or \( x, y, z; \) or \( x, y, z, x_1, x_2, x_3, x_4 \) as variables.
(b) Determine whether the system is consistent or inconsistent. If it is consistent, give the solution.

25. \[
\begin{bmatrix}
1 & 2 & 5 \\
0 & 1 & -1 \\
\end{bmatrix}
\]
26. \[
\begin{bmatrix}
1 & -3 & -4 \\
0 & 1 & 0 \\
\end{bmatrix}
\]
27. \[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & 1 & 4 & 2 \\
0 & 0 & 0 & 3 \\
\end{bmatrix}
\]
28. \[
\begin{bmatrix}
1 & 2 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 2 \\
\end{bmatrix}
\]
29. \[
\begin{bmatrix}
1 & 0 & 2 & -1 \\
0 & 1 & -4 & -2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
30. \[
\begin{bmatrix}
1 & 0 & 4 & 4 \\
0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
31. \[
\begin{bmatrix}
1 & 2 & -1 & 1 \\
0 & 1 & 4 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
32. \[
\begin{bmatrix}
1 & 2 & 4 & 0 & 1 \\
0 & 1 & -1 & 2 & 2 \\
0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 1 & -2 \\
\end{bmatrix}
\]
33. \[
\begin{bmatrix}
1 & 2 & 0 & 4 & 2 \\
0 & 1 & 1 & 3 & 3 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
34. \[
\begin{bmatrix}
1 & 0 & 3 & 0 & 1 \\
0 & 1 & 4 & 3 & 2 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
35. \[
\begin{bmatrix}
1 & -2 & 0 & 1 & -2 \\
0 & 1 & -3 & 2 & 2 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
36. \[
\begin{bmatrix}
1 & 3 & 0 & 4 & 1 \\
0 & 1 & 2 & -1 & 2 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

In Problems 37–54, solve each system of equations using Gaussian elimination or Gauss–Jordan elimination. If the system has no solution, say it is inconsistent.

37. \[
\begin{align*}
2x - 3y &= 6 \\
6x - 9y &= 10
\end{align*}
\]
38. \[
\begin{align*}
x + y &= 1 \\
3x - 2y &= \frac{4}{3}
\end{align*}
\]
39. \[
\begin{align*}
2x + 6y &= 1 \\
5x + 15y &= 10
\end{align*}
\]
40. \[
\begin{align*}
3x - 4y &= 3 \\
6x + 2y &= 1
\end{align*}
\]
41. \[
\begin{align*}
4x - y &= \frac{11}{4} \\
3x + y &= \frac{5}{2}
\end{align*}
\]
42. \[
\begin{align*}
3x + 5y &= 5 \\
6x + 10y &= 10
\end{align*}
\]
43. \[
\begin{align*}
x + y &= 1 \\
2x - y + z &= 5 \\
x + y + z &= 5
\end{align*}
\]
44. \[
\begin{align*}
2x - 2y - z &= 2 \\
2x + 3y + z &= 2 \\
3x + 2y &= 0
\end{align*}
\]
45. \[
\begin{align*}
2x - y + z &= 6 \\
x - y - z &= -3 \\
x + y + 2z &= 7
\end{align*}
\]
46. \[
\begin{align*}
x + y &= 1 \\
x + 3y + 2z &= 1 \\
x + y + z &= 2
\end{align*}
\]
47. \[
\begin{align*}
2x + 2y + z &= 6 \\
x - y - z &= -2 \\
x - 2y - 2z &= -5
\end{align*}
\]
48. \[
\begin{align*}
2x - y - z &= -5 \\
x + y + z &= 2 \\
x + 2y + 2z &= 5
\end{align*}
\]
49. \[
\begin{align*}
x + y - z &= 2 \\
x + 3y + 2z &= 1 \\
x + y + z &= 2
\end{align*}
\]
50. \[
\begin{align*}
x - y - z &= 6 \\
x - 2y - 2z &= -5
\end{align*}
\]
51. \[
\begin{align*}
4x + 4y - 4z &= -1 \\
2x + y + z &= 2
\end{align*}
\]
52. \[
\begin{align*}
x + y - z &= 0 \\
4x + 2y - 4z &= 0 \\
x + 2y + z &= 0
\end{align*}
\]
53. \[
\begin{align*}
3x + y - z &= \frac{2}{3} \\
2x - y + z &= 1 \\
4x + 2y &= \frac{8}{3}
\end{align*}
\]
54. \[
\begin{align*}
x - 3y &= 1 \\
2x - y + z &= 1 \\
x + 2y + z &= \frac{8}{3}
\end{align*}
\]
86 Chapter 2 Systems of Linear Equations

In Problems 55–60, use a graphing utility to find the row echelon form (REF) and the reduced row echelon form (RREF) of the augmented matrix of each of the following systems. Solve each system. If the system has no solution, say it is inconsistent.

55. \[
\begin{align*}
2x - 2y + z &= 2 \\
x - \frac{1}{2}y + 2z &= 1 \\
2x + \frac{1}{3}y - z &= 0
\end{align*}
\]

56. \[
\begin{align*}
x + y &= -1 \\
x - z &= 0 \\
y - z &= 1
\end{align*}
\]

57. \[
\begin{align*}
x + y &= 4 \\
x - y - z &= 0 \\
y - z &= -4
\end{align*}
\]

58. \[
\begin{align*}
2x + y + z &= 6 \\
x - y - z &= -3 \\
3x + y + 2z &= 7
\end{align*}
\]

59. \[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 20 \\
x_2 + x_3 + x_4 &= 0 \\
x_3 + x_4 &= 13 \\
x_2 - 2x_4 &= -5
\end{align*}
\]

60. \[
\begin{align*}
x_1 - 2x_2 + 3x_3 - 4x_4 &= 40 \\
4x_2 + 6x_4 &= -10 \\
x_3 - x_4 &= 12 \\
x_2 + 2x_4 &= -10
\end{align*}
\]

Applications

61. **Financial Planning** Carla has $120,000 to invest and wants to have an annual income of $5000. As her financial consultant, you recommend that she invest in Treasury Bills that yield 2%, bank CDs that yield 4%, and corporate bonds that yield 6%. Being conservative, Carla wants the amount invested in corporate bonds to be half that invested in bank CDs. Find the amount she should place in each investment.

62. **Financial Planning** Kurt has $20,000 to invest. As his financial planner, you recommend that he diversify into three mutual funds based on their average annual returns over the past 10 years ending December 31, 2009: Total Return at 5%, High Income at 7%, and Global Bond at 9%. Kurt wants to have an average annual return of $1280 over the next 10 years. Since Kurt is worried that interest rates may go up resulting in bonds declining, he wants the amount invested in the Total Return Fund to be two times that invested in the Global Bond Fund. Find the amount in each investment.

63. **Diet Preparation** A hospital dietician is planning a meal consisting of three foods whose ingredients are summarized as follows:

<table>
<thead>
<tr>
<th>Food</th>
<th>Chicken Breast (3-oz boneless, skinless)</th>
<th>Potato (1 average potato, 5.3 oz)</th>
<th>Spinach (1 cup, boiled &amp; drained)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grams of Protein</td>
<td>24</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Grams of Carbohydrates</td>
<td>0</td>
<td>26</td>
<td>7</td>
</tr>
<tr>
<td>Grams of Fat</td>
<td>1.5</td>
<td>0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Determine the number of servings of each food needed to create a meal containing 38 grams of protein, 40 grams of carbohydrates, and 2.5 grams of fat.

**Source:** www.dietfacts.com

64. **Ordering Cookies** Sally’s Girl Scout troop is selling cookies for the Christmas season. There are three different kinds of cookies in three different containers: bags that hold 1 dozen chocolate chip and 1 dozen oatmeal; gift boxes that hold 2 dozen chocolate chip, 1 dozen mint, and 1 dozen oatmeal; and cookie tins that hold 3 dozen mint and 2 dozen chocolate chip. Sally’s mother is having a Christmas party and wants 6 dozen oatmeal; 10 dozen mint, and 14 dozen chocolate chip cookies. How can Sally fill her mother’s order?

65. **Production** A juice company completes the preparation of its products by cleaning, filling, and labeling bottles. Each case of orange juice requires 10 minutes in the cleaning machine, 4 minutes in the filling machine, and 2 minutes in the labeling machine. For each case of tomato juice, the times are 15 minutes of cleaning, 4 minutes of filling, and 1 minute of labeling. Pineapple juice requires 9 minutes of cleaning, 6 minutes of filling, and 1 minute of labeling per case. If the company runs the cleaning machine for 398 minutes, the filling machine for 164 minutes, and the labeling machine for 58 minutes, how many cases of each type of juice are prepared?

66. **Production** The finishing stage of the manufacture of an automobile requires painting, drying, and polishing. The Rome Motor Company produces three types of cars: the Centurion, the Tribune, and the Senator. Each Centurion requires 8 hours for painting, 2 hours for drying, and 1 hour for polishing. A Tribune needs 10 hours for painting, 3 hours for drying, and 2 hours for polishing. It takes 16 hours of painting, 5 hours of drying, and 3 hours of polishing to prepare a Senator. If the company uses 240 hours for painting, 69 hours for drying, and 41 hours for polishing in a given month, how many of each type of car are produced?

67. **Theater Seating** The Fabulous Fox Theater in St. Louis, Missouri, offers several levels of seating, among them mezzanine, lower balcony, and middle balcony. One group of patrons buys 4 mezzanine tickets and 6 lower balcony tickets for $558. Another group spends $787 for 2 mezzanine tickets, 7 lower balcony tickets, and 8 middle balcony tickets. A third group purchases 3 lower balcony tickets and 12 middle balcony tickets for $615. What is the individual price of a mezzanine ticket, a lower balcony ticket, and a middle balcony ticket?
Source: Fox Theater, St. Louis, Missouri. Based on seat pricing for the production of MAMA MIA, Feb., 2010.

68. Cost of Fast Food One group of people purchased 10 hot dogs and 5 soft drinks at a cost of $37.50. A second group bought 7 hot dogs and 4 soft drinks at a cost of $27. What is the cost of a single hot dog? A single soft drink?

69. Inventory Control An art teacher finds that colored paper can be bought in three different packages. The first package has 20 sheets of white paper, 15 sheets of blue paper, and 1 sheet of red paper. The second package has 3 sheets of blue paper and 1 sheet of red paper. The last package has 40 sheets of white paper and 30 sheets of blue paper. If he needs 200 sheets of white paper, 180 sheets of blue paper, and 12 sheets of red paper, how many of each type of package should he order?

70. Inventory Control An interior decorator has ordered 12 cans of sunset paint, 35 cans of brown paint, and 18 cans of fuchsia paint. The paint store has special pair packs, containing 1 can each of sunset and fuchsia; darkening packs, containing 2 cans of sunset, 5 cans of brown, and 2 cans of fuchsia; and economy packs, containing 3 cans of sunset, 15 cans of brown, and 6 cans of fuchsia. How many of each type of pack should the paint store send to the interior decorator?

71. Packaging A recreation center wants to purchase compact discs (CDs) to be used in the center. There is no requirement as to the artists. The only requirement is that they purchase 40 rock CDs, 32 western CDs, and 14 blues CDs. There are three different shipping packages offered by the company. They are an assorted carton containing 2 rock CDs, 4 western CDs, and 1 blues CD; a mixed carton containing 4 rock and 2 western CDs; and a single carton containing 2 blues CDs. What combination of these packages is needed to fill the center's order?

72. Production A luggage manufacturer produces three types of luggage: economy, standard, and deluxe. The company produces 1000 pieces of luggage at a cost of $20, $25, and $30 for the economy, standard, and deluxe luggage, respectively. The manufacturer has a budget of $20,700. Each economy luggage requires 6 hours of labor, each standard luggage requires 10 hours of labor, and each deluxe model requires 20 hours of labor. The manufacturer has a maximum of 6800 hours of labor available. If the manufacturer sells all the luggage, consumes the entire budget, and uses all the available labor, how many of each type of luggage should be produced?

73. Mixing Nuts Suppose that a store has three sizes of cans of nuts. The large size contains 2 pounds of peanuts and 1 pound of cashews. The mammoth size contains 1 pound of walnuts, 4 pounds of peanuts, and 2 pounds of cashews. The giant size contains 1 pound of walnuts, 4 pounds of peanuts, and 2 pounds of cashews. Suppose that the store receives an order for 5 pounds of walnuts, 26 pounds of peanuts, and 12 pounds of cashews. How can it fill this order with the given sizes of cans?

74. Mixing Nuts Suppose that the store in Problem 73 receives a new order for 6 pounds of walnuts, 34 pounds of peanuts, and 15 pounds of cashews. How can this order be filled with the given cans?

75. Buying Stock Karen has $200,000 to invest. She will purchase shares in National City Bank preferred (NCC.B), IBM (IBM), and Express Scripts (ESRX). The table that follows shows the price she pays per share, the expected annual dividend, and the expected annual growth rate. Over the next 12 months, Karen wants these stocks to provide $8000 in dividends with an average growth rate of 4%. How many shares of each stock should be purchased to achieve this goal?

<table>
<thead>
<tr>
<th></th>
<th>Price per Share</th>
<th>Expected Annual Dividend per Share</th>
<th>Expected Annual Growth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>NCC Pfd</td>
<td>25</td>
<td>2</td>
<td>0%</td>
</tr>
<tr>
<td>IBM</td>
<td>100</td>
<td>2</td>
<td>5%</td>
</tr>
<tr>
<td>Express Scripts</td>
<td>50</td>
<td>0</td>
<td>10%</td>
</tr>
</tbody>
</table>

Discussion and Writing

76. Write a brief paragraph or two that outlines your strategy for solving a system of linear equations using matrices.

77. When solving a system of linear equations using matrices, do you prefer to place the augmented matrix in row echelon form or in reduced row echelon form? Give reasons for your choice.
2.3 Systems of \( m \) Linear Equations Containing \( n \) Variables

**OBJECTIVES**

1. Analyze the reduced row echelon form of an augmented matrix (p. 89)
2. Solve a system of \( m \) linear equations containing \( n \) variables (p. 90)
3. Express the solution of a system with an infinite number of solutions (p. 92)
4. Solve applied problems involving systems of equations (p. 94)

Below is an example of a system of three linear equations containing four variables

\[
\begin{align*}
  x_1 + 3x_2 + 5x_3 + x_4 &= 2 \\
  2x_1 + 3x_2 + 4x_3 + 2x_4 &= 1 \\
  x_1 + 2x_2 + 3x_3 + x_4 &= 1 
\end{align*}
\]

A general definition of a system of \( m \) linear equations containing \( n \) variables is given next.

**Definition**

A system of \( m \) linear equations containing \( n \) variables \( x_1, x_2, \ldots, x_n \) is of the form

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\
  &\vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \\
  a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i \\
  &\vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

where \( a_{ij} \) and \( b_i \) are real numbers, \( i = 1, 2, \ldots, m; j = 1, 2, \ldots, n \).

A **solution** of a system of \( m \) linear equations containing \( n \) variables \( x_1, x_2, \ldots, x_n \) is any ordered set \( (x_1, x_2, \ldots, x_n) \) of real numbers for which **each** of the \( m \) linear equations of the system is satisfied.

**Reduced Row Echelon Form**

A system of \( m \) linear equations containing \( n \) variables will have either no solution, one solution, or infinitely many solutions. We can determine which of these possibilities occurs and, if solutions exist, find them, by performing row operations on the augmented matrix of the system until we arrive at the reduced row echelon form of the augmented matrix.
Let’s review the conditions required for the reduced row echelon form:

**Conditions for the Reduced Row Echelon Form of a Matrix**

1. The first nonzero entry in each row is 1 and it has 0s above it and below it.
2. The leftmost 1 in any row is to the right of the leftmost 1 in the row above.
3. Any rows that contain all 0s to the left of the vertical bar appear at the bottom.

**EXAMPLE 1** Examples of Matrices That Are in Reduced Row Echelon Form

(a) \[
\begin{bmatrix}
1 & 0 & -3 & 4 \\
0 & 1 & -2 & 2 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & -2 & 0 & 1 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 5 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & 4 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

**EXAMPLE 2** Examples of Matrices That Are Not in Reduced Row Echelon Form

(a) \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 0 & 2 & 4 \\
0 & 2 & 4 & 3 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 3 & 0 \\
\end{bmatrix}
\]

The second row contains all 0s and the third does not—this violates the rule that states that any rows with all 0s are at the bottom.

The first nonzero entry in row 2 is not a 1.

The leftmost 1 in the third row is not to the right of the leftmost 1 in the row above it.

NOW WORK PROBLEM 3.

1. Analyze the Reduced Row Echelon Form of an Augmented Matrix

**EXAMPLE 3** Analyzing the Reduced Row Echelon Form of an Augmented Matrix

The matrix

\[
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

is the reduced row echelon form of the augmented matrix of a system of four equations containing three variables. If the variables are \(x, y, z\), the equation represented by the third row is

\[0 \cdot x + 0 \cdot y + 0 \cdot z = 1 \quad \text{or} \quad 0 = 1\]

Since 0 = 1 is a contradiction, we conclude the system is inconsistent.
EXAMPLE 4 Analyzing the Reduced Row Echelon Form of an Augmented Matrix

The matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 2 & 5 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 3 & 4
\end{bmatrix}
\]

is the reduced row echelon form of a system of three equations containing four variables. If \( x_1, x_2, x_3, x_4 \) are the variables, the system of equations is

\[
\begin{align*}
\begin{cases}
 x_1 + 2x_4 &= 5 \\
x_2 + x_4 &= 2 \\
x_3 + 3x_4 &= 4
\end{cases}
\begin{cases}
 x_1 &= -2x_4 + 5 \\
x_2 &= -x_4 + 2 \\
x_3 &= -3x_4 + 4
\end{cases}
\end{align*}
\]

The system has infinitely many solutions. In this form, the variable \( x_4 \) is the parameter. We assign values to the parameter \( x_4 \) from which the variables \( x_1, x_2, x_3 \) can be calculated. Some of the possibilities are

- If \( x_4 = 0 \), then \( x_1 = 5, x_2 = 2, x_3 = 4 \).
- If \( x_4 = 1 \), then \( x_1 = 3, x_2 = 1, x_3 = 1 \).
- If \( x_4 = 2 \), then \( x_1 = 1, x_2 = 0, x_3 = -2 \).

And so on.

NOW WORK PROBLEM 9.

EXAMPLE 5 Solving a System of Three Linear Equations Containing Two Variables

Solve:

\[
\begin{align*}
\begin{cases}
 x - y &= 2 \\
2x - 3y &= 2 \\
3x - 5y &= 2
\end{cases}
\end{align*}
\]

SOLUTION The augmented matrix of this system is

\[
\begin{bmatrix}
1 & -1 & 2 \\
2 & -3 & 2 \\
3 & -5 & 2
\end{bmatrix}
\]

We will use Gauss–Jordan elimination to find the reduced row echelon form of the augmented matrix. The entry in row 1, column 1 is 1. Proceed to obtain a matrix in which all the remaining entries in column 1 are 0s by performing the row operations

\[
\begin{align*}
R_2 &= -2r_1 + r_2 \\
R_3 &= -3r_1 + r_3
\end{align*}
\]

The new matrix is

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & -1 & -2 \\
0 & -2 & -4
\end{bmatrix}
\]
2.3 Systems of \( m \) Linear Equations Containing \( n \) Variables

We want the entry in row 2, column 2 (now \( -1 \)) to be 1. After using the row operation \( R_2 = (-1)r_2 \), we obtain

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 1 & 2 \\
0 & -2 & -4
\end{bmatrix}
\]

Now we want the entry in row 1, column 2 and in row 3, column 2 to be 0. This can be accomplished by applying the row operations

\[
R_1 = r_2 + r_1 \\
R_3 = 2r_2 + r_3
\]

The new matrix is

\[
\begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

This is the reduced row echelon form of the matrix. We conclude that the system has the solution \( x = 4, y = 2 \) or, using ordered pairs, \( (4, 2) \).

**COMMENT** A graphing utility can be used to solve systems of \( m \) linear equations containing \( n \) variables. Check the solution to Example 5 using the RREF feature of your graphing utility.

---

**EXAMPLE 6** Solving a System of Four Linear Equations Containing Three Variables

Solve:

\[
\begin{aligned}
x - y + 2z &= 2 \\
2x - 3y + 2z &= 1 \\
3x - 5y + 2z &= -3 \\
-4x + 12y + 8z &= 10
\end{aligned}
\]

**SOLUTION** We find the reduced row echelon form of the augmented matrix of this system, namely,

\[
\begin{bmatrix}
1 & -1 & 2 & 2 \\
2 & -3 & 2 & 1 \\
3 & -5 & 2 & -3 \\
-4 & 12 & 8 & 10
\end{bmatrix}
\]

The entry 1 is already present in row 1, column 1. To obtain 0s elsewhere in column 1, use the row operations

\[
R_2 = -2r_1 + r_2 \\
R_3 = -3r_1 + r_3 \\
R_4 = 4r_1 + r_4
\]

The new matrix is

\[
\begin{bmatrix}
1 & -1 & 2 & 2 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -9 \\
0 & 8 & 16 & 18
\end{bmatrix}
\]
To obtain the entry 1 in row 2, column 2, use obtaining

\[
\begin{bmatrix}
1 & -1 & 2 & 2 \\
0 & 1 & 2 & 3 \\
0 & -2 & -4 & -9 \\
0 & 8 & 16 & 18 \\
\end{bmatrix}
\]

To obtain 0s elsewhere in column 2, use

\[ R_1 = r_2 + r_1 \quad R_3 = 2r_2 + r_3 \quad R_4 = -8r_2 + r_4 \]

The new matrix is

\[
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & -3 \\
0 & 0 & 0 & -6 \\
\end{bmatrix}
\]

We can stop here because the third row yields the equation

\[ 0 \cdot x + 0 \cdot y + 0 \cdot z = -3 \]

We conclude the system is inconsistent.

\[ \text{NOW WORK PROBLEM 17.} \]

3 Express the Solution of a System with an Infinite Number of Solutions

**EXAMPLE 7** Solving a System of Two Linear Equations Containing Three Variables

Solve:

\[
\begin{align*}
x + y + z &= 7 \\
x - y - 3z &= 1 \\
\end{align*}
\]

**SOLUTION** The augmented matrix of the system is

\[
\begin{bmatrix}
1 & 1 & 1 & 7 \\
1 & -1 & -3 & 1 \\
\end{bmatrix}
\]

The reduced row echelon form (as you should verify) is

\[
\begin{bmatrix}
1 & 0 & -1 & 4 \\
0 & 1 & 2 & 3 \\
\end{bmatrix}
\]

The system of equations represented by this matrix is

\[
\begin{align*}
x - z &= 4 \\
y + 2z &= 3
\end{align*}
\]

or

\[
\begin{align*}
x &= z + 4 \\
y &= -2z + 3
\end{align*}
\]

(1)

The system has infinitely many solutions. In the form (1), the variable \( z \) is the parameter. We can assign any value to \( z \) and use it to compute values of \( x \) and \( y \).

**CHECK:** Check the solution to Example 7 as follows:

\[
\begin{align*}
x + y + z &= (z + 4) + (-2z + 3) + z = 4 + 3 + z - 2z + z = 7 \\
x - y - 3z &= (z + 4) - (-2z + 3) - 3z = 4 - 3 + z + 2z - 3z = 1
\end{align*}
\]

The solution is verified.
2.3 Systems of \( m \) Linear Equations Containing \( n \) Variables

The next example illustrates a system having an infinite number of solutions with two parameters.

EXAMPLE 8  Solving a System of Three Linear Equations Containing Four Variables

Solve:

\[
\begin{align*}
\begin{cases}
x_1 + x_2 + 2x_3 + 2x_4 &= 2 \\
x_1 + x_3 + x_4 &= 0 \\
x_2 + x_3 + x_4 &= 2
\end{cases}
\end{align*}
\]

SOLUTION  The augmented matrix of the system is

\[
\begin{bmatrix}
1 & 1 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 2
\end{bmatrix}
\]

The reduced row echelon form (as you should verify) is

\[
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The equations represented by this system are

\[
\begin{align*}
x_1 + x_3 + x_4 &= 0 \\
x_2 + x_3 + x_4 &= 2
\end{align*}
\]

Rewrite this system in the form

\[
\begin{align*}
x_1 &= -x_3 - x_4 \\
x_2 &= -x_3 - x_4 + 2
\end{align*}
\]

The system has infinitely many solutions. In the form (2), the system has two parameters \( x_3 \) and \( x_4 \). Solutions are obtained by assigning the two parameters \( x_3 \) and \( x_4 \) arbitrary values. Some choices are shown in Table 2.

<table>
<thead>
<tr>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( (x_1, x_2, x_3, x_4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>((0, 2, 0, 0))</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>((-1, 1, 1, 0))</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>((-2, 0, 0, 2))</td>
</tr>
</tbody>
</table>

The variables used as parameters are not unique. We could have chosen \( x_1 \) and \( x_4 \) as parameters by rewriting the system of equations (2) in the following manner.

From the first equation

\[
x_3 = -x_1 - x_4
\]

replace the parameter \( x_3 \) in the second equation to produce

\[
x_2 = 2 - x_3 - x_4 = 2 + x_1 + x_4 - x_4 = 2 + x_1
\]

We then obtain the system

\[
\begin{align*}
x_2 &= x_1 + 2 \\
x_3 &= -x_1 - x_4
\end{align*}
\]

showing the solution with \( x_1 \) and \( x_4 \) as parameters.

NOW WORK PROBLEM 23.
A couple has $60,000 to invest. They wish to earn an average of $5000 per year on the investment over a 5 year period. Based on the yearly average return on mutual funds for 5 years ending December 31, 2009, they are considering the following funds: Franklin High Income at 5%, Royce 100K at 6%, TCW Small Capital Growth at 7%, and Franklin Natural Resources at 10%.

(a) As their financial advisor, prepare a table showing the various ways the couple can achieve their goal.

(b) Comment on the various possibilities and the overall plan.

**SOLUTION**

Begin by naming the variables: Let \( x_1 \) be the amount invested in the High Income Fund, \( x_2 \) the amount invested in the Royce 100 K Fund, \( x_3 \) the amount invested in the Small Capital Growth Fund, and \( x_4 \) the amount in the Natural Resources Fund. Then the average annual amount earned by each investment choice would be

- High Income: 0.05 \( x_1 \)
- Royce 100 K: 0.06 \( x_2 \)
- Small Capital Growth: 0.07 \( x_3 \)
- Natural Resources: 0.01 \( x_4 \)

Since the couple requires $5000 from these investments, we have the equation

\[ 0.05x_1 + 0.06x_2 + 0.07x_3 + 0.10x_4 = 5000 \]

The total amount available to invest is $60,000, which leads to the equation

\[ x_1 + x_2 + x_3 + x_4 = 60,000 \]

These two equations form the system

\[
\begin{align*}
0.05x_1 + 0.06x_2 + 0.07x_3 + 0.10x_4 &= 5000 \\
x_1 + x_2 + x_3 + x_4 &= 60,000
\end{align*}
\]

Write the augmented matrix of this system and proceed to row reduce.

\[
\begin{bmatrix}
0.05 & 0.06 & 0.07 & 0.10 & 5000 \\
1 & 1 & 1 & 1 & 60,000
\end{bmatrix}
\]

Interchange rows.

\[
\begin{bmatrix}
0.05 & 0.06 & 0.07 & 0.10 & 5000 \\
1 & 1 & 1 & 1 & 60,000
\end{bmatrix}
\]

\[ R_2 = -0.05r_1 + r_2 \]

\[
\begin{bmatrix}
1 & 0.01 & 0.02 & 0.05 & 2000 \\
1 & 1 & 1 & 1 & 60,000
\end{bmatrix}
\]

\[ R_2 = 100r_2 \]

\[
\begin{bmatrix}
1 & 1 & 1 & 60,000 \\
0 & 1 & 2 & 5 & 200,000
\end{bmatrix}
\]

\[ R_2 = -r_2 + r_1 \]

This matrix is in reduced row echelon form. The solution of the system is

\[ x_1 = x_3 + 4x_4 - 140,000 \]
\[ x_2 = -2x_3 + 5x_4 + 200,000 \]

where \( x_3 \) and \( x_4 \) are parameters.

Now, each of the variables must be nonnegative and each must be less than or equal to 60,000. Set up the table shown in Table 3.
Table 3 shows five plans that the couple could follow to achieve their financial goals.

(b) Of the five plans, the one that provides for some money to be invested in each of the four mutual funds is preferable as this gives more diversification to the investment. There are other plans that would also provide diversification across all four funds. A difficulty with the overall plan is that one fund, Franklin Natural Resources, must have more than half the money invested in it. The reason is that the expected yield in this fund is very high. The couple should probably set their required income level a little lower so that more diversification can be achieved in the investment.

NOW WORK PROBLEM 35.

SUMMARY

Steps for Solving a System of \( m \) Linear Equations Containing \( n \) Variables

STEP 1 Write the augmented matrix.
STEP 2 Find the reduced row echelon form of the augmented matrix.
STEP 3 Analyze this matrix to determine if the system has no solution, one solution, or infinitely many solutions.

EXERCISE 2.3 Answers Begin on Page AN–11.

Concepts and Vocabulary

1. **True or False** A system of three linear equations containing two variables is always inconsistent.

2. **True or False** A system of three linear equations containing four variables can have infinitely many solutions.

Skill Building

In Problems 3–8, tell whether the given matrix is in reduced row echelon form. If it is not, tell why.

3. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
In Problems 9–16, the reduced row echelon form of the augmented matrix of a system of linear equations is given. Tell whether the system has one solution, no solution, or infinitely many solutions. Write the solutions or, if there is no solution, say the system is inconsistent.

9.\[
\begin{bmatrix}
1 & 0 & -2 & 6 \\
0 & 1 & 3 & 1
\end{bmatrix}
\]
10.\[
\begin{bmatrix}
1 & 0 & 1 & -1 & 0 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
11.\[
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 3 & 0 \\
0 & 0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
12.\[
\begin{bmatrix}
1 & 2 & 0 & 0 & -4 \\
0 & 0 & 1 & 0 & -3 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

In Problems 17–28, solve each system of equations by finding the reduced row echelon form of the augmented matrix. If there is no solution, say the system is inconsistent.

17.\[
\begin{cases}
3x - 3y = 12 \\
3x + 2y = -3 \\
x + y = 4
\end{cases}
\]
18.\[
\begin{cases}
6x + y = 8 \\
x - 3y = -5 \\
x + 2y = 2
\end{cases}
\]
19.\[
\begin{cases}
2x - 4y = 8 \\
x - 2y = 4 \\
x + y = 2
\end{cases}
\]
20.\[
\begin{cases}
x + y = 8 \\
6x + 2y = 16 \\
-9x - 3y = -24
\end{cases}
\]
21.\[
\begin{cases}
x + 2y + 3z = -1 \\
x - y + 3z = 8
\end{cases}
\]
22.\[
\begin{cases}
x + 2y + 3z = 5 \\
x - 2x + 6y + 4z = 0
\end{cases}
\]
23.\[
\begin{cases}
x + x_2 = 7 \\
x_2 - x_3 + x_4 = 5 \\
x_1 - x_2 + x_3 + x_4 = 6 \\
x_2 - x_4 = 10
\end{cases}
\]
24.\[
\begin{cases}
x_1 + 2x_2 + 3x_3 - x_4 = 0 \\
x_2 - x_3 - x_4 = 2
\end{cases}
\]
25.\[
\begin{cases}
x_1 + x_2 + x_3 + x_4 = 4 \\
2x_1 - x_2 + x_3 = 0 \\
3x_1 + 2x_2 + x_3 - x_4 = 6 \\
x_1 - 2x_2 - 2x_3 + 2x_4 = -1
\end{cases}
\]
26.\[
\begin{cases}
x - y + z = 5 \\
2x - 2y + 2z = 8
\end{cases}
\]
27.\[
\begin{cases}
x_1 + x_2 + x_3 + x_4 = 4 \\
x_1 - 2x_2 + x_3 + x_4 = 0 \\
x_2 + x_3 + x_4 = 0
\end{cases}
\]

Applications

29. Pharmacy A doctor’s prescription calls for a daily intake of a supplement containing 40 mg of vitamin C and 30 mg of vitamin D. Your pharmacy stocks three supplements that can be used: one contains 20% vitamin C and 30% vitamin D; a second, 40% vitamin C and 20% vitamin D; and a third, 30% vitamin C and 50% vitamin D. Create a table showing the possible combinations that could be used to fill the prescription.

30. Pharmacy A doctor’s prescription calls for the creation of pills that contain 12 units of vitamin B12 and 12 units of vitamin E. Your pharmacy stocks three powders that can be used to make these pills: one contains 20% vitamin B12 and 30% vitamin E; a second, 40% vitamin B12 and 20% vitamin E; and a third, 30% vitamin B12 and 40% vitamin E. Create a table showing the possible combinations of each powder that could be mixed in each pill.

31. Weight Control To control his weight, Steven watches what he eats, paying particular attention to his intake of calories, carbohydrates, and fat. Each workweek (5 days) he consumes 700 calories, 20 grams of fat, and 100 grams of carbohydrates for lunch. He can choose from turkey bologna, bananas, low-fat cottage cheese, and low-fat chocolate milk. The nutritional information per serving for these items is provided in the table.

<table>
<thead>
<tr>
<th>Food Item</th>
<th>Calories (kcal)</th>
<th>Fat (g)</th>
<th>Carbohydrates (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turkey Bologna</td>
<td>184</td>
<td>13.20</td>
<td>4.85</td>
</tr>
<tr>
<td>Banana</td>
<td>92</td>
<td>0.48</td>
<td>23.43</td>
</tr>
<tr>
<td>Low-Fat Cottage Cheese</td>
<td>90</td>
<td>1.93</td>
<td>0.01</td>
</tr>
<tr>
<td>Low-Fat Chocolate Milk</td>
<td>72</td>
<td>2.00</td>
<td>10.40</td>
</tr>
</tbody>
</table>

Source: Nutri-facts.com

(a) Write a system of equations that describes Steven’s intake of calories, carbohydrates, and fat. Be sure to name all variables.
(b) Solve the system by writing its augmented matrix in reduced row echelon form.
(c) Prepare a table showing various lunch options.
32. **Making Ends Meet**  
Newlyweds Nick and Dana each work two part-time jobs in order to make ends meet as they finish college. Nick’s college schedule allows him to work a total of 30 hours per week both for a department store that pays him $9.50 per hour and for a fast food restaurant that pays him $8.50 per hour. Dana’s college schedule allows her to work a total of 25 hours per week for a grocery store that pays her $10.00 per hour and for a novelty shop that pays her $8.00 per hour. The couple must earn $500 per week in order to eat and pay bills.

(a) Write a system of equations that describes Nick and Dana’s earnings opportunities. Be sure to name all variables.

(b) Solve the system by writing its augmented matrix in reduced row echelon form.

(c) Prepare a table showing various work options for Nick and Dana.

(d) What would you recommend to them?

33. **Mutual Funds and College Tuition**  
Bill and Colleen recently inherited $50,000. They plan to invest the money in order to pay for their son Timmy’s tuition. They anticipate they will need an average annual return of $6000 in order to protect their capital and have enough to cover the tuition (based on the average 2010–2011 tuition for a 4-year public college). They will invest in four mutual funds with the following average annual returns over the 5-year period prior to December 31, 2009, rounded to the nearest whole percent: John Hancock Large Cap Equity Growth at 9%, T Rowe Price Emerging Markets at 13%, Templeton China World Fund at 14%, and TCW Small Cap Growth at 7%.

(a) Write a system of equations that describes Bill and Colleen’s financial needs and restrictions. Be sure to name all variables.

(b) Solve the system by writing its augmented matrix in reduced row echelon form.

(c) Prepare a table showing various investment options.

(d) What investment advice would you give to Bill and Colleen?

**Source:** collegeboard.com, Franklin Templeton

35. **Investments**  
Three couples want to go into partnership to finance their own retail business in 2015. They wish to invest monies that would earn an average of $2000 per annum for 5 years. As their financial consultant, you recommended they invest in certain mutual funds based on their performance over the past 5 years as a December 31, 2009: some money in the Franklin High Income fund that yielded 5%, some in Mutual Global Discovery that yielded 6%; some money in Templeton Global Bond that yielded 8%; and some money in Franklin International Small Cap Growth that yielded 12%. Prepare a table for each couple showing various ways that their goals can be achieved:

(a) If the first couple has $20,000 to invest

(b) If the second couple has $25,000 to invest

(c) If the third couple has $30,000 to invest

(d) What advice would you give each couple? Give reasons.

36. **Financial Planning**  
A young couple has $25,000 to invest. As their financial consultant, based on an average yearly return over a 5-year period ending on December 31, 2009, you recommend that they invest some money in Templeton Global Opportunity Trust that yielded 4%, some in the Franklin Mutual European Fund that yielded 6%; the John Hancock Large Cap Equity Growth Fund that yielded 9%; and Eastern European Equity A Fund that yielded 11%. Prepare a table showing various ways this couple can achieve the following goals:

(a) The couple wants $1500 per year in income.

(b) The couple wants $2000 per year in income.

(c) The couple wants $2500 per year in income.

(d) What advice would you give this couple regarding the income that they require and the choices available? Give reasons.

37. **Bacteria Control**  
Three species of bacteria will be kept in one test tube and will feed on three resources. Each member of the first species consumes 3 units of the first resource and 1 unit of the third. Each bacterium of the second type consumes...
Discussion and Writing

38. Make up a system of three linear equations containing four variables that has infinitely many solutions. How many parameters will be in the solution to this system? Solve the system and create a table showing various solutions to the system of equations.

39. Make up a system of two linear equations containing four variables that has infinitely many solutions. How many parameters will be in the solution to this system? Solve the system and create a table showing various solutions to the system of equations.

1 unit of the first resource and 2 units each of the second and third. Each bacterium of the third type consumes 2 units of the first resource and 4 each of the second and third. If the test tube is supplied daily with 12,000 units of the first resource, 12,000 units of the second, and 14,000 units of the third, how many of each species can coexist in equilibrium in the test tube so that all of the supplied resources are consumed? Prepare a table that shows some of the possibilities.

CHAPTER 2 REVIEW OBJECTIVES

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<th>Examples</th>
<th>You should be able to</th>
<th>Review Exercises</th>
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</thead>
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<tr>
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<td></td>
<td>2. Solve systems of equations by elimination (p. 55)</td>
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<td>5. Solve systems of three equations containing three variables (p. 59)</td>
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<td>15–32</td>
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<td>11–14</td>
</tr>
<tr>
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<td>15–32</td>
</tr>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
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</tr>
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<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
**THINGS TO KNOW**

**Systems of Linear Equations** (pp. 51–52)  
Systems with a solution are consistent and either have a unique solution or have infinitely many solutions  
Systems with no solution are inconsistent

**Solving Systems of Linear Equations**

- Using substitution (pp. 53–55)
- Using elimination (pp. 55–56)
- Using Gaussian elimination (pp. 73–76)
- Using Gauss–Jordan elimination (p. 77)

**Matrix**

- Augmented matrix of a system of linear equations (p. 69)
- Row operations (p. 69)

**REVIEW EXERCISES**  
Answers to odd-numbered problems begin on page AN-13.

Blue problem numbers represent the author's suggestions for a Practice Test.

In Problems 1–10, solve each system of equations algebraically using the method of substitution or the method of elimination. If the system has no solution, say it is inconsistent.

1. \[
\begin{align*}
2x - y &= 5 \\
5x + 2y &= 8
\end{align*}
\]

2. \[
\begin{align*}
2x + 3y &= 2 \\
7x - y &= 3
\end{align*}
\]

3. \[
\begin{align*}
x - 2y - 4 &= 0 \\
x + 3y + 2y &= 4
\end{align*}
\]

4. \[
\begin{align*}
1 - x + 3y + 4 &= 0 \\
\frac{3}{2} - 2y + \frac{4}{3} &= 0
\end{align*}
\]

5. \[
\begin{align*}
3x - 2y &= 8 \\
x - \frac{2}{3}y &= 12
\end{align*}
\]

6. \[
\begin{align*}
2x + 5y &= 10 \\
4x + 10y &= 20
\end{align*}
\]

7. \[
\begin{align*}
x + 2y - z &= 6 \\
2x - y + 3z &= -13 \\
x - 2y + 3z &= -16
\end{align*}
\]

8. \[
\begin{align*}
x - 5y - z &= 2 \\
3x + 2y + z &= 7 \\
x - y + 2z &= 11
\end{align*}
\]

In Problems 11–14, write the system of equations corresponding to the given augmented matrix. In Problems 13–14, also write the solution.

9. \[
\begin{bmatrix}
2x - 4y + z &= -15 \\
x + 2y - 4z &= 27 \\
x - 6y - 2z &= -3
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
x - 4y + 3z &= 15 \\
-3x + y - 5z &= -5 \\
-7x - 5y - 9z &= 10
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
3 & 2 & 8 \\
1 & 4 & -1
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
1 & 2 & 5 & -2 \\
5 & 0 & -3 & 8 \\
2 & -1 & 0 & 0
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 6 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
1 & 0 & 3 & 5 \\
0 & 1 & 2 & 8
\end{bmatrix}
\]

In Problems 15–32, use matrices to find the solution, if it exists, of each system of linear equations. If the system has infinitely many solutions, write the solution using parameters and then list at least three solutions. If the system has no solution, say it is inconsistent.

15. \[
\begin{align*}
-5x + 2y &= -2 \\
-3x + 3y &= 4
\end{align*}
\]

16. \[
\begin{align*}
-3x + 2y &= 3 \\
-3x + 4y &= 4
\end{align*}
\]

17. \[
\begin{align*}
x + 2y + 5z &= 6 \\
x + 7y + 12z &= 23 \\
x + 4y &= 25
\end{align*}
\]

18. \[
\begin{align*}
x + 2y - z &= -4 \\
x + 7y - z &= -21 \\
x + 4y - 6z &= -17
\end{align*}
\]

19. \[
\begin{align*}
x + 2y + 7z &= 2 \\
3x + 7y + 18z &= -1 \\
x + 4y + 2z &= -13
\end{align*}
\]

20. \[
\begin{align*}
x + 2y - 7z &= -1 \\
3x + 7y - 24z &= 6 \\
x + 4y - 12z &= 26
\end{align*}
\]

21. \[
\begin{align*}
x - y + z &= 2 \\
3x - y + z &= 0 \\
3x &= 3y + 3z = 3
\end{align*}
\]

22. \[
\begin{align*}
x + 3y - z &= 5 \\
x - y + z &= 1 \\
x - 3x + 3z &= 3
\end{align*}
\]

23. \[
\begin{align*}
y - 2z &= 6 \\
3x + 2y - z &= 2 \\
4x + 3z &= -1
\end{align*}
\]

24. \[
\begin{align*}
2x - y + 3z &= 5 \\
x + 2z &= 0 \\
3x + 2y + z &= -3
\end{align*}
\]

25. \[
\begin{align*}
x - 3y &= 5 \\
3y + z &= 0 \\
2x - y + 2z &= 2
\end{align*}
\]

26. \[
\begin{align*}
x - z &= 2 \\
2x - y &= 4 \\
x + y + z &= 6
\end{align*}
\]
In Problems 33–36, analyze the solution, if any, of the system of equations having the given augmented matrix.

33. 34. 35. 36.

37. **Mixture** Sweet Delight Candies, Inc., sells boxes of candy consisting of creams and caramels. Each box sells for $4 and holds 50 pieces of candy (all pieces are the same size). If the caramels cost $0.05 to produce and the creams cost $0.10 to produce, how many caramels and creams should be in each box for no profit and no loss? Would you increase or decrease the number of caramels in order to obtain a profit?

38. **Cookie Orders** A cookie company makes three kinds of cookies—oatmeal raisin, chocolate chip, and shortbread—packaged in small, medium, and large boxes. The small box contains 1 dozen oatmeal raisin and 1 dozen chocolate chip; the medium box has 2 dozen oatmeal raisin, 1 dozen chocolate chip, and 1 dozen shortbread; the large box contains 2 dozen oatmeal raisin, 2 dozen chocolate chip, and 3 dozen shortbread. If you require exactly 15 dozen oatmeal raisin, 10 dozen chocolate chip, and 11 dozen shortbread cookies, how many of each size box should you buy?

39. **Mixture Problem** A store sells almonds for $6 per pound, cashews for $5 per pound, and peanuts for $2 per pound. One week the manager decides to prepare 100 16-ounce packages of nuts by mixing the peanuts, almonds, and cashews. Each package will be sold for $4. The mixture is to produce the same revenue as selling the nuts separately. Prepare a table that shows some of the possible ways the manager can prepare the mixture.

40. **Investments** Three couples wish to invest monies that would earn an average of $1800 per annum for a period of 5 years to finance their dream trip to Europe. As their financial consultant, you recommended they invest in certain mutual funds based on their performance over the past 5 years as of December 31, 2009: some money in Global Discovery at 6%, some money in Templeton Global Bond that yielded 8%, and some money in Franklin Natural Resource that yielded 10%. Prepare a table for each couple showing the various ways their goal can be achieved.

(a) If the first couple has $20,000 to invest
(b) If the second couple has $25,000 to invest
(c) If the third couple has $30,000 to invest

**Source:** Franklin Templeton

41. **Investments** The Barkers want to retire in 10 years and buy a home in the Southwest. They have $40,000 to invest over that period and plan to use their returns together with their initial investment and the profit on the sale of their existing home toward this purchase. As their financial consultant, you recommend they invest in mutual funds that had the following average returns per annum over the past 10 years as of December 31, 2009: some money in Franklin Strategic Income at 6%, some money in Franklin Small Cap Value that yielded 8%, and some money in Franklin Micro Cap Value that yielded 10%. Prepare a table showing the various ways this couple can achieve the following goals:

(a) They want $2500 per year in income.
(b) They want $3000 per year in income.
(c) They want $3500 per year in income.

**Source:** Franklin Funds

42. **Investments** Kelly has $20,000 to invest for a period of 10 years. As her financial planner, you recommend that she diversify into three Franklin Mutual Funds based on their average annual returns over the past 10 years ending December 31, 2009: Total Return at 5%, High Income at 7%, and Global Bond at 10%. Kelly wishes to earn an average of $1390 per year in income. Also, Kelly wants her investment in Total Return to be $3000 more than her investment in the Global Bond Fund. How much money should Kelly place in each investment?

**Source:** Franklin Funds
You decide to compare the basic charges for two cell phone plans that include texting and Web access. Here are the monthly costs for two plans.* Plan A includes a flat rate of $39.99 for 450 anytime minutes plus $.02 for each text message (sent or received) and $1.99 per megabyte of data usage (50 mobile Web pages will incur data usage charges for about .3 megabytes). For Plan A, additional minutes above 450 minutes cost $.45 per minute. Plan B includes a flat rate of $44.99 for 500 anytime minutes plus $.04 for each text message and $2.29 per megabyte of data usage. For Plan B, additional minutes above 500 minutes cost $.50 per minute.

The variables of the model are:
- \(x\): the number of anytime minutes used per month.
- \(y\): the number of text messages sent or received per month.
- \(z\): the number of megabytes of data usage per month.
- \(A\): the monthly cost for Plan A
- \(B\): the monthly cost for Plan B

1. Assume that \(x \geq 450\) and express \(A\) as a linear expression involving \(x, y,\) and \(z\).
2. Assume that \(x \geq 500\) and express \(B\) as a linear expression involving \(x, y,\) and \(z\).
3. Suppose that 600 anytime minutes were used for a given month. Find the values of \(y\) and \(z\) so that \(A = B\). Let \(z\) be a parameter.
4. Suppose that 600 anytime minutes and 5 megabytes of data usage were billed for a given month. Let \(y_0\) represent the number of text messages sent or received so that \(A = B\) for that month. Find \(y_0\). Determine which plan is less expensive for that month if \(y > y_0\).
5. You estimate that you will send or receive about 350 text messages per month and use 8 megabytes of data per month. Assuming that \(x \geq 500\), find the values of \(x\) for which \(A \leq B\).

6. You decide that you will spend about $100 per month for your cell phone plan. Set \(A = B = 100\) and find valid solutions for the resulting system of equations. Let \(z\) be a parameter. Note that solutions are valid only when \(x \geq 500\), \(y \geq 0\), and \(z \geq 0\).

7. You estimate that you will use about 600 anytime minutes and require 3 megabytes of data usage in a given month. For what numbers of text messages would the cost of Plan B be less than or equal to the cost of Plan A for that month?

8. For what values of \(x\) will Plan A be less expensive than Plan B, if \(y = z = 0\)?

9. Suppose you learn of a third cell phone plan, Plan C, which includes unlimited text messaging and an unlimited data plan. The cost of Plan C is $59.99 for 500 anytime minutes. Each additional minute costs $.55 cents per minute. Let \(C\) represent the cost of Plan C. Find a linear expression for \(C\) in terms of the variables \(x, y,\) and \(z\), assuming that \(x \geq 500\).

10. If you expect to send or receive 450 text messages per month and use 10 megabytes of data per month, find the values of \(x\) for which \(A \leq C\) and \(C \leq B\), assuming that \(x \geq 500\).

11. If you expect to send or receive 800 text messages per month and do not expect to use your cell phone to access the Web, find the values of \(x\) for which \(C \leq A\) and \(C \leq B\), assuming \(x \geq 500\). Then find the values of \(x\) for which \(A \leq C\) and \(A \leq B\), assuming that \(x \geq 500\).

12. Suppose you expect to need between 600 and 1000 anytime minutes per month, an expect to send 400 text messages a month, and to have a data usage of about 8 megabytes per month. Determine which of the three plans (A, B, or C) would be the most economical.

* Go to http://www.myrateplan.com/wireless plans/ to compare cell phone plans available in your area.
Mathematical Questions from Professional Exams*

Use the following information to answer Problems 1–4:
Akron, Inc. owns 80% of the capital stock of Benson Company and 70% of the capital stock of Cashin, Inc. Benson Company owns 15% of the capital stock of Cashin, Inc. Cashin, Inc., in turn, owns 25% of the capital stock of Akron, Inc. These ownership interrelationships are illustrated in the diagram.

Net income before adjusting for interests in intercompany net income for each corporation follows:
- Akron, Inc. $190,000
- Benson Co. $170,000
- Cashin, Inc. $230,000

Ignore all income tax considerations.

1. CPA Exam The equation, in a set of simultaneous equations, which computes $A_e$ is
   (a) $A_e = .75(190,000 + .8B_e + .7C_e)$
   (b) $A_e = 190,000 + .8B_e + .7C_e$
   (c) $A_e = .75(190,000) + .8(170,000) + .7(230,000)$
   (d) $A_e = .75(190,000) + .8B_e + .7C_e$

2. CPA Exam The equation, in a set of simultaneous equations, which computes $B_e$ is
   (a) $B_e = 170,000 + .15C_e - .75A_e$
   (b) $B_e = 170,000 + .15C_e$
   (c) $B_e = .2(170,000) + .15(230,000)$
   (d) $B_e = .2(170,000) + .15C_e$

3. CPA Exam Cashin's minority interest in consolidated net income is
   (a) $.15(230,000)$
   (b) $230,000 + .25A_e$
   (c) $.15(230,000) + .25A_e$
   (d) $.15C_e$

4. CPA Exam Benson's minority interest in consolidated net income is
   (a) $34,316$
   (b) $25,500$
   (c) $45,755$
   (d) $30,675

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Economists are always talking about the influence of consumer spending on the economy. What do they mean? Suppose you have $5000 in disposable income and could spend it on a new bathroom for your house or on a plasma TV or on a dream vacation in Fiji. Would your decision impact the economy? What if you decided to spend the $5000 on the construction of the new bathroom? This would help the construction industry but would not help the consumer electronics industry or the travel industry. What if 200 people were in the same position as you and each of them made the decision to spend $5000 on construction, resulting in increased demand for construction of $1,000,000? How would this impact other segments of the economy? Which ones does it help? Which ones does it hurt? A famous economic model, the Leontief model, was constructed to answer such questions. We study this model in Section 3.4 and answer some of the questions listed here in the Chapter Project at the end of the chapter.

A Look Back, A Look Forward

In Chapter 2 we used a matrix to solve a system of linear equations. As it turns out, matrices have a life of their own. The theory of matrices was developed in 1857 by Arthur Cayley (1821–1895), though only later were matrices used as we use them in this chapter. Matrices have become a very flexible instrument, invaluable in almost all areas of mathematics.
3.1 Matrix Algebra

PREPARING FOR THIS SECTION  Before getting started, review the following:

• Properties of Real Numbers (Appendix A, Section A.1, pp. A–8 to A–12)

NOW WORK THE ‘ARE YOU PREPARED’ PROBLEMS ON PAGE 117

OBJECTIVES  1 Find the dimension of a matrix (p. 105)
2 Find the sum of two matrices (p. 107)
3 Work with properties of matrices (p. 108)
4 Find the difference of two matrices (p. 110)
5 Find scalar multiples of a matrix (p. 113)
6 Solve applied problems involving matrices (p. 116)

Matrices can be added, subtracted, and multiplied. They also possess many of the algebraic properties of real numbers. Matrix algebra is the study of these properties. Its importance lies in the fact that many situations in both pure and applied mathematics involve rectangular arrays of numbers. In fact, in many branches of business and the biological and social sciences, it is necessary to express and use data in a rectangular array.

Let’s begin with an example that illustrates how matrices can be used to conveniently represent information.

**EXAMPLE 1** Arranging Data in a Matrix

In a survey of 900 people, the following information was obtained:

- 200 Males Thought federal defense spending was too high
- 150 Males Thought federal defense spending was too low
- 45 Males Had no opinion
- 315 Females Thought federal defense spending was too high
- 125 Females Thought federal defense spending was too low
- 65 Females Had no opinion

We can arrange the above data in a rectangular array as follows:

<table>
<thead>
<tr>
<th></th>
<th>Too High</th>
<th>Too Low</th>
<th>No Opinion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Male</strong></td>
<td>200</td>
<td>150</td>
<td>45</td>
</tr>
<tr>
<td><strong>Female</strong></td>
<td>315</td>
<td>125</td>
<td>65</td>
</tr>
</tbody>
</table>

or as the matrix

\[
\begin{bmatrix}
200 & 150 & 45 \\
315 & 125 & 65
\end{bmatrix}
\]

This matrix has two rows (representing males and females) and three columns (representing “too high,” “too low,” and “no opinion”).
A matrix is defined as a rectangular array of the form:

\[
\begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{im} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn}
\end{bmatrix}
\]  

(1)

The symbols \(a_{ij}\) of a matrix are referred to as the entries (or elements) of the matrix. Each entry \(a_{ij}\) of the matrix has two indices: the row index, \(i\), and the column index, \(j\). The symbols \(a_{11}, a_{12}, \ldots, a_{1n}\) represent the entries in the 1st row, and the symbols \(a_{ij}, a_{2j}, \ldots, a_{mj}\) represent the entries in the \(j\)th column.

We shall use capital letters to denote matrices. If we denote the matrix in display (1) above by \(A\), then we can abbreviate \(A\) by

\[A = [a_{ij}] \quad i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, n\]

The matrix \(A\) has \(m\) rows and \(n\) columns.

Definition

The dimension of a matrix \(A\) is determined by the number of rows and the number of columns in the matrix. If a matrix \(A\) has \(m\) rows and \(n\) columns, we denote the dimension of \(A\) by \(m \times n\), read as “\(m\) by \(n\).”

For a \(2 \times 3\) matrix, remember that the first number 2 denotes the number of rows and the second number 3 is the number of columns. A matrix with 3 rows and 2 columns is of dimension \(3 \times 2\).

Definition

If a matrix \(A\) has the same number of rows as it has columns, it is called a square matrix.

In a square matrix \(A = [a_{ij}]\) the entries for which \(i = j\), namely \(a_{11}, a_{22}, a_{33}, a_{44}\), and so on, are the diagonal entries of \(A\).

A row matrix has one row; a column matrix has one column.

1 Find the Dimension of a Matrix

**EXAMPLE 2** Finding the Dimension of a Matrix

Find the dimension of each matrix. Indicate if the matrix is a square matrix, a row matrix, or a column matrix.

(a) \[
\begin{bmatrix}
    5 & -1 \\
    2 & 5
\end{bmatrix}
\]  
(b) \[
\begin{bmatrix}
    2 & 3 & 0 \\
    1 & -2 & 5
\end{bmatrix}
\]  
(c) \(\begin{bmatrix}
    1 & 0 & 4
\end{bmatrix}\)  
(d) \(\begin{bmatrix}
    2 \\
    1
\end{bmatrix}\)  
(e) \(\begin{bmatrix}
    9
\end{bmatrix}\)
SOLUTION

(a) $2 \times 2$, a square matrix
(b) $2 \times 3$
(c) $1 \times 3$, a row matrix
(d) $2 \times 1$, a column matrix
(e) $1 \times 1$, a square matrix, a row matrix, and a column matrix

Now work Problem 9.

EXAMPLE 3 Arranging Data as a Matrix

In the recent U.S. census, the following figures were obtained with regard to the city of Oak Lawn. Each year 7% of city residents move to the suburbs and 1% of the people in the suburbs move to the city. This situation can be represented by the matrix

$$
P = \begin{bmatrix}
0.93 & 0.07 \\
0.01 & 0.99
\end{bmatrix}
$$

Here, the entry in row 1, column 2—0.07—indicates that 7% of city residents move to the suburbs. The matrix $P$ is a square matrix and its dimension is $2 \times 2$. The diagonal entries are 0.93 and 0.99.

Now work Problem 59.

Equality of Matrices

In an algebra system, it is important to know when two quantities are equal. So, we ask, “When, if at all, are two matrices equal?”

Let’s try to arrive at a sound definition for equality of matrices by requiring equal matrices to have certain desirable properties. First, it would seem necessary that two equal matrices have the same dimension—that is, that they both be $m \times n$ matrices. Next, it would seem necessary that their entries be identical numbers.

Definition

Equality of Matrices

Two matrices $A$ and $B$ are equal if they are of the same dimension and if corresponding entries are equal. In this case we write $A = B$, read as “matrix $A$ is equal to matrix $B$.”

EXAMPLE 4 Determining Equality of Matrices

In order for the two matrices

$$
\begin{bmatrix}
p & q \\
1 & 0
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
2 & 4 \\
n & 0
\end{bmatrix}
$$

to be equal, we must have $p = 2$, $q = 4$, and $n = 1$.
EXAMPLE 5 Determining Equality of Matrices

Let $A$ and $B$ be two matrices given by

$$A = \begin{bmatrix} x + y & 6 \\ 2x - 3 & 2 - y \end{bmatrix} \quad B = \begin{bmatrix} 5 & 5x + 2 \\ y & x - y \end{bmatrix}$$

Determine if there are values of $x$ and $y$ so that $A$ and $B$ are equal.

**SOLUTION** Both $A$ and $B$ are $2 \times 2$ matrices so $A = B$ if

$$x + y = 5 \quad (1) \quad 6 = 5x + 2 \quad (2)$$
$$2x - 3 = y \quad (3) \quad 2 - y = x - y \quad (4)$$

Here we have four equations containing the two variables $x$ and $y$. From Equation (4) we see that $x = 2$. Using this value in Equation (1), we obtain $y = 3$. But $x = 2, y = 3$ do not satisfy either Equation (2) or Equation (3). There are no values for $x$ and $y$ satisfying all four equations. This means $A$ and $B$ can never be equal.

NOW WORK PROBLEM 15.

2 Find the Sum of Two Matrices

In an algebra system there are operations, like addition and subtraction. Can two matrices be added? And, if so, what is the rule for addition of matrices?

EXAMPLE 6 Adding Two Matrices

Motors, Inc., produces three models of cars: a sedan, a convertible, and an SUV. If the company wishes to compare the units of raw material and the units of labor involved in 1 month’s production of each of these models, the rectangular array displayed in Table 1 may be used to present the data:

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Sedan Model</th>
<th>Convertible Model</th>
<th>SUV Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units of Material</td>
<td>23</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td>Units of Labor</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>

The same information may be written concisely as the matrix

$$A = \begin{bmatrix} 23 & 16 & 10 \\ 7 & 9 & 11 \end{bmatrix}$$

where the rows represent units of material and labor and the columns represent sedans, convertibles, and SUVs.

Suppose the next month’s production is

$$B = \begin{bmatrix} 18 & 12 & 9 \\ 14 & 6 & 8 \end{bmatrix}$$

in which the pattern of recording units and models remains the same.
The total production for the 2 months can be displayed by the matrix

\[
C = \begin{bmatrix}
41 & 28 & 19 \\
21 & 15 & 19
\end{bmatrix}
\]

since the number of units of material for sedan models is \(41 = 23 + 18\), the number of units of material for convertible models is \(28 = 16 + 12\), and so on.

This leads to the following definition.

**Definition**

**Addition of Matrices**

We define the sum \(A + B\) of two matrices \(A\) and \(B\) with the same dimension as the matrix consisting of the sum of corresponding entries from \(A\) and \(B\). That is, if \(A = [a_{ij}]\) and \(B = [b_{ij}]\) are two \(m \times n\) matrices, the sum \(A + B\) is the \(m \times n\) matrix \([a_{ij} + b_{ij}]\).

**EXAMPLE 7** **Adding Two Matrices**

\[
\begin{align*}
(a) \quad \begin{bmatrix}
23 & 16 & 10 \\
7 & 9 & 11
\end{bmatrix} + \begin{bmatrix}
18 & 12 & 9 \\
14 & 6 & 8
\end{bmatrix} &= \begin{bmatrix}
23 + 18 & 16 + 12 & 10 + 9 \\
7 + 14 & 9 + 6 & 11 + 8
\end{bmatrix} \\
&= \begin{bmatrix}
41 & 28 & 19 \\
21 & 15 & 19
\end{bmatrix}

(b) \quad \begin{bmatrix}
0.6 & 0.4 \\
0.1 & 0.9
\end{bmatrix} + \begin{bmatrix}
2.3 & 0.6 \\
1.8 & 5.2
\end{bmatrix} &= \begin{bmatrix}
0.6 + 2.3 & 0.4 + 0.6 \\
0.1 + 1.8 & 0.9 + 5.2
\end{bmatrix} \\
&= \begin{bmatrix}
2.9 & 1.0 \\
1.9 & 6.1
\end{bmatrix}
\end{align*}
\]

Notice that it is possible to add two matrices only if their dimensions are the same. Also, the dimension of the sum of two matrices is the same as that of the two original matrices.

The matrices \(A\) and \(B\) below cannot be added since they are of different dimensions:

(I) \(A = \begin{bmatrix} 1 & 2 \\ 7 & 2 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}\)

(II) \(A = \begin{bmatrix} 2 & 3 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\)

(III) \(A = \begin{bmatrix} -1 & 7 & 0 \\ 2 & 1 & 0 \end{bmatrix}\) and \(B = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 1 & 5 \end{bmatrix}\)

**NOW WORK PROBLEM 27.**

**Work with Properties of Matrices**

It turns out that the properties of addition of real numbers (such as the commutative property and associative property) are also valid for matrix addition.
EXAMPLE 8  **Demonstrating the Commutative Property**

Let \( A = \begin{bmatrix} 1 & 5 \\ 7 & -3 \end{bmatrix} \) and \( B = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} \)

Then

\[
A + B = \begin{bmatrix} 1 & 5 \\ 7 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 + 3 & 5 + (-2) \\ 7 + 4 & -3 + 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 11 & -2 \end{bmatrix}
\]

\[
B + A = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 11 & -2 \end{bmatrix}
\]

**Commutative Property for Addition**

If \( A \) and \( B \) are two matrices of the same dimension, then

\[
A + B = B + A
\]

**Associative Property for Addition**

If \( A \), \( B \), and \( C \) are three matrices of the same dimension, then

\[
A + (B + C) = (A + B) + C
\]

The fact that addition of matrices is associative means that the notation \( A + B + C \) is not ambiguous, since \((A + B) + C = A + (B + C)\).

**NOW WORK PROBLEM 43.**

A matrix in which all entries are 0 is called a **zero matrix**. We use the symbol 0 to represent a zero matrix of any dimension.

For real numbers, 0 has the property that \( x + 0 = x \) for any \( x \). A property of a zero matrix is that \( A + 0 = A \), provided the dimension of 0 is the same as that of \( A \).

EXAMPLE 9  **Demonstrating a Property of the Zero Matrix**

Let

\[
\begin{bmatrix} 3 & 4 & -1 \\ \sqrt{2} & 0 & 3 \end{bmatrix}
\]

For real numbers, 0 has the property that \( x + 0 = x \) for any \( x \). A property of a zero matrix is that \( A + 0 = A \), provided the dimension of 0 is the same as that of \( A \).
Then

\[
A + 0 = \begin{bmatrix}
3 & 4 & \frac{-1}{2} \\
\sqrt{2} & 0 & 3
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
3 + 0 & 4 + 0 & \frac{-1}{2} + 0 \\
\sqrt{2} + 0 & 0 + 0 & 3 + 0
\end{bmatrix} = \begin{bmatrix}
3 & 4 & \frac{-1}{2} \\
\sqrt{2} & 0 & 3
\end{bmatrix} = A
\]

If \( A \) is any matrix, the additive inverse of \( A \), denoted by \( -A \), is the matrix obtained by replacing each number in \( A \) by its additive inverse.

**EXAMPLE 10 Finding the Additive Inverse of a Matrix**

If

\[
A = \begin{bmatrix}
-3 & 0 \\
5 & -2 \\
1 & 3
\end{bmatrix}
\]

then

\[
-A = \begin{bmatrix}
3 & 0 \\
-5 & 2 \\
-1 & -3
\end{bmatrix}
\]

**Additive Inverse Property**

For any matrix \( A \), we have the property that

\[
A + (-A) = 0
\]

**NOW WORK PROBLEM 45.**

**4 Find the Difference of Two Matrices**

Now that we have defined the sum of two matrices and the additive inverse of a matrix, we can ask about the difference of two matrices. As you will see, subtracting matrices and subtracting real numbers are much the same kind of process.

**Definition Subtraction of Matrices**

We define the difference \( A - B \) of two matrices \( A \) and \( B \) with the same dimension as the matrix consisting of the difference of corresponding entries from \( A \) and \( B \). That is, if \( A = [a_{ij}] \) and \( B = [b_{ij}] \) are two \( m \times n \) matrices, the difference \( A - B \) is the \( m \times n \) matrix \( [a_{ij} - b_{ij}] \).

The matrix \( A - B \) is nothing more than the matrix formed by subtracting the entries in \( B \) from the corresponding entries in \( A \).
EXAMPLE 11  Adding and Subtracting Matrices

Suppose that

\[
A = \begin{bmatrix} 2 & 4 & 8 & -3 \\ 0 & 1 & 2 & 3 \\ \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \\ \end{bmatrix}
\]

Find:  \(a\)  \(A + B\)  \(b\)  \(A - B\)

**SOLUTION**

\(a\)  \(A + B = \begin{bmatrix} 2 & 4 & 8 & -3 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}
\]
\(= \begin{bmatrix} 2 + (-3) & 4 + 4 & 8 + 0 & -3 + 1 \\ 0 + 6 & 1 + 8 & 2 + 2 & 3 + 0 \end{bmatrix}
\]
\(= \begin{bmatrix} -1 & 8 & 8 & -2 \\ 6 & 9 & 4 & 3 \end{bmatrix}
\)

(b)  \(A - B = \begin{bmatrix} 2 & 4 & 8 & -3 \end{bmatrix} - \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}
\]
\(= \begin{bmatrix} 2 - (-3) & 4 - 4 & 8 - 0 & -3 - 1 \\ 0 - 6 & 1 - 8 & 2 - 2 & 3 - 0 \end{bmatrix}
\]
\(= \begin{bmatrix} 5 & 0 & 8 & -4 \\ -6 & -7 & 0 & 3 \end{bmatrix}
\)

Notice that \(A - B = -(B - A)\). Also notice that \(A - B \neq B - A\) unless \(A = B\), illustrating that matrix subtraction, like subtraction of real numbers, is not commutative.

_Now work Problem 35._

**COMMENT**  Graphing utilities make the sometimes tedious process of matrix algebra easy. Let’s see how a graphing utility adds and subtracts matrices by solving Example 11 using a graphing utility.

**USING TECHNOLOGY**

EXAMPLE 12  Adding and Subtracting Matrices Using a Graphing Utility

Rework Example 11 using a graphing utility.

**SOLUTION**  Enter the matrices into a graphing utility. Name them \([A]\) and \([B]\). Figure 1 shows the results of adding and subtracting \([A]\) and \([B]\).

**FIGURE 1**

\[
\begin{align*}
\text{[A]+[B]} & \quad \begin{bmatrix} 2 & 4 & 8 & -3 \\ 0 & 1 & 2 & 3 \\ \end{bmatrix} + \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \\ \end{bmatrix} \\
\text{[A]-[B]} & \quad \begin{bmatrix} 2 & 4 & 8 & -3 \end{bmatrix} - \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}
\end{align*}
\]

_Now work Problem 35 using a graphing utility._
EXAMPLE 13 Applying Matrix Addition and Subtraction

In 2008, the number of U.S. doctoral degrees awarded in the physical sciences was 8127, of which 2271 were awarded to females. The same year, 7852 U.S. doctoral degrees were awarded in engineering, of which 6164 were awarded to males, and 11,074 doctoral degrees were awarded in life sciences, of which 5854 were awarded to females. In 2007, a total of 8020 doctoral degrees were awarded in physical sciences, of which 2254 were awarded to females; 7733 doctoral degrees were awarded in engineering, of which 6131 were awarded to males; and 10,615 doctoral degrees were awarded in life sciences, of which 5457 were awarded to females.

(a) Write $2 \times 3$ matrices for each year, showing the three fields of doctoral degrees awarded by gender.

(b) Use matrix addition to find the total number of doctoral degrees awarded over the two years.

(c) Use matrix subtraction to find the difference in the number of doctoral degrees awarded from 2007 to 2008.


SOLUTION (a) For 2008:

8127 doctoral degrees in the physical sciences, of which 2271 went to females, leaving $8127 - 2271 = 5856$ going to males;
7852 doctoral degrees in engineering, of which 6164 went to males, leaving $7852 - 6164 = 1688$ going to females;
11,074 doctoral degrees in the life sciences, of which 5854 went to females, leaving $11,074 - 5854 = 5220$ going to males.

A $2 \times 3$ matrix representing these data is

\[
\begin{bmatrix}
5856 & 6164 & 5220 \\
2271 & 1688 & 5854
\end{bmatrix}
\]

For 2007:

8020 doctoral degrees in the physical sciences, of which 2254 went to females leaving $8020 - 2254 = 5766$ going to males;
7733 doctoral degrees in engineering, of which 6131 went to males, leaving $7733 - 6131 = 1602$ going to females;
10,615 doctoral degrees in the life sciences, of which 5457 went to females, leaving $10,615 - 5457 = 5158$ going to males.

A $2 \times 3$ matrix representing these data is

\[
\begin{bmatrix}
5766 & 6131 & 5158 \\
2254 & 1602 & 5457
\end{bmatrix}
\]
(b) The total number of doctoral degrees in the three fields over the two-year period is the sum of the matrices

\[
\begin{bmatrix}
5856 & 6164 & 5220 \\
2271 & 1688 & 5854
\end{bmatrix}
\begin{bmatrix}
5766 & 6131 & 5158 \\
2254 & 1602 & 5457
\end{bmatrix}
= \text{Male}
\begin{bmatrix}
11,622 & 12,295 & 10,378 \\
4525 & 3290 & 11,311
\end{bmatrix}
\text{Female}
\]

For example, over the two-year period 2007–2008, 12,295 males received a doctorate in engineering.

(c) The difference in the number of doctoral degrees awarded from 2007 to 2008 is

\[
\begin{bmatrix}
5856 & 6164 & 5220 \\
2271 & 1688 & 5854
\end{bmatrix}
\begin{bmatrix}
5766 & 6131 & 5158 \\
2254 & 1602 & 5457
\end{bmatrix}
= \text{Male}
\begin{bmatrix}
90 & 33 & 62 \\
17 & 86 & 397
\end{bmatrix}
\text{Female}
\]

For example, from 2007 to 2008, there was an increase of 397 in the number of females receiving a doctorate in the life sciences.

5 Find Scalar Multiples of a Matrix

Let's return to the production of Motors, Inc., during the month specified in Example 6. The matrix \( A \) describing this production is

\[
A = \begin{bmatrix}
23 & 16 & 10 \\
7 & 9 & 11
\end{bmatrix}
\]

Let's assume that for 3 consecutive months, the monthly production remained the same. Then the total production for the 3 months is simply the sum of the matrix \( A \) taken 3 times. If we represent the total production by the matrix \( T \), then

\[
T = \begin{bmatrix}
23 & 16 & 10 \\
7 & 9 & 11
\end{bmatrix}
+ \begin{bmatrix}
23 & 16 & 10 \\
7 & 9 & 11
\end{bmatrix}
+ \begin{bmatrix}
23 & 16 & 10 \\
7 & 9 & 11
\end{bmatrix}
= \begin{bmatrix}
23 + 23 + 23 & 16 + 16 + 16 & 10 + 10 + 10 \\
7 + 7 + 7 & 9 + 9 + 9 & 11 + 11 + 11
\end{bmatrix}
= \begin{bmatrix}
3 \cdot 23 & 3 \cdot 16 & 3 \cdot 10 \\
3 \cdot 7 & 3 \cdot 9 & 3 \cdot 11
\end{bmatrix}
= \begin{bmatrix}
69 & 48 & 30 \\
21 & 27 & 33
\end{bmatrix}
\]

In other words, when we add the matrix \( A \) three times, we multiply each entry of \( A \) by 3. This leads to the following definition.
Definition Scalar Multiplication

Let \( A \) be an \( m \times n \) matrix and let \( c \) be a real number, called a scalar. The product of the matrix \( A \) by the scalar \( c \), called scalar multiplication, is the \( m \times n \) matrix \( cA \), whose entries are the product of \( c \) and the corresponding entries of \( A \). That is, if \( A = [a_{ij}] \), then \( cA = [ca_{ij}] \).

When multiplying a matrix by a real number, each entry of the matrix is multiplied by the number. Notice that the dimension of \( A \) and the dimension of the product \( cA \) are the same.

EXAMPLE 14 Finding Scalar Multiples of a Matrix

Suppose that

\[
A = \begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix}
\]

Find: (a) \( 4A \) \hspace{1cm} (b) \( \frac{1}{3} C \) \hspace{1cm} (c) \( 3A - 2B \)

**SOLUTION**

(a) \( 4A = 4 \begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 & 4 \cdot 1 & 4 \cdot 5 \\ 4(-2) & 4 \cdot 0 & 4 \cdot 6 \end{bmatrix} = \begin{bmatrix} 12 & 4 & 20 \\ -8 & 0 & 24 \end{bmatrix} \)

(b) \( \frac{1}{3} C = \frac{1}{3} \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cdot 9 & \frac{1}{3} \cdot 0 \\ \frac{1}{3} \cdot (-3) & \frac{1}{3} \cdot 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \)

(c) \( 3A - 2B = 3 \begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} - 2 \begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 3 & 3 \cdot 1 & 3 \cdot 5 \\ 3(-2) & 3 \cdot 0 & 3 \cdot 6 \end{bmatrix} - \begin{bmatrix} 2 \cdot 4 & 2 \cdot 1 & 2 \cdot 0 \\ 2 \cdot 8 & 2 \cdot 1 & 2(-3) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 15 \\ -6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 8 & 2 & 0 \\ 16 & 2 & -6 \end{bmatrix} = \begin{bmatrix} -7 & 1 & 15 \\ -22 & -2 & 24 \end{bmatrix} \)

NOW WORK PROBLEM 37.
USING TECHNOLOGY

EXAMPLE 15 Finding Scalar Multiples of a Matrix Using a Graphing Utility

Rework Example 14 using a graphing utility.

SOLUTION Enter the matrices \([A], [B],\) and \([C]\) into a graphing utility. Figure 2 shows the required computations.

![Figure 2](image)

(a) \(\begin{bmatrix} 12 & 4 & 20 \\ -8 & 0 & 24 \end{bmatrix}\)
(b) \(\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}\)
(c) \(\begin{bmatrix} 12 & 20 \\ -22 & -2 & 24 \end{bmatrix}\)

NOW WORK PROBLEM 37 USING A GRAPHING UTILITY.

Properties of Scalar Multiplication

Let \(k\) and \(h\) be two real numbers and let \(A\) and \(B\) be two matrices of dimension \(m \times n\). Then

\[
k(hA) = (kh)A \quad (2)
\]

\[
(k + h)A = kA + hA \quad (3)
\]

\[
k(A + B) = kA + kB \quad (4)
\]

Properties (2), (3), and (4) are illustrated in the following example.

EXAMPLE 16 Using Properties of Scalar Multiplication

For \(A = \begin{bmatrix} 2 & -3 & -1 \\ 5 & 6 & 4 \end{bmatrix}\) and \(B = \begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 5 \end{bmatrix}\),

show that

(a) \(5(2A) = 10A\) \quad (b) \((4 + 3)A = 4A + 3A\) \quad (c) \(3(A + B) = 3A + 3B\)

SOLUTION (a) \(5(2A) = \begin{bmatrix} 4 & -6 & -2 \\ 10 & 12 & 8 \end{bmatrix} = \begin{bmatrix} 20 & -30 & -10 \\ 50 & 60 & 40 \end{bmatrix}\)

\[
10A = \begin{bmatrix} 20 & -30 & -10 \\ 50 & 60 & 40 \end{bmatrix}
\]
EXAMPLE 17 Applying Scalar Multiplication

According to information from the American Red Cross, the distribution of blood types in the United States is as shown in the accompanying table.

(a) Write a $2 \times 4$ matrix depicting the distribution of blood types by Rh factor, using decimals as entries.

(b) On any given day in the United States, 38,000 units of blood are needed. Use scalar multiplication to determine the number of units required each day in the United States by blood type, assuming daily distributions match the percentages above.

(c) State in words what this means about the number of units of $O^+$ ($O, Rh^+$) required.

Source: American Red Cross

SOLUTION

(a) A $2 \times 4$ matrix representing the distribution of blood types is

\[
\begin{bmatrix}
A & B & AB & O \\
\text{Rh}^+ & 0.34 & 0.09 & 0.03 & 0.38 \\
\text{Rh}^- & 0.06 & 0.02 & 0.01 & 0.07 \\
\end{bmatrix}
\]

(b) If 38,000 units of blood are required and the distribution by type is the matrix found in part (a), the scalar multiple of the matrix by the scalar 38,000 will yield a matrix that gives the number of units required by blood type.

\[
38,000 \begin{bmatrix}
0.34 & 0.09 & 0.03 & 0.38 \\
0.06 & 0.02 & 0.01 & 0.07 \\
\end{bmatrix} = \begin{bmatrix}
38,000 \cdot 0.34 & 38,000 \cdot 0.09 & 38,000 \cdot 0.03 & 38,000 \cdot 0.38 \\
38,000 \cdot 0.06 & 38,000 \cdot 0.02 & 38,000 \cdot 0.01 & 38,000 \cdot 0.07 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
12,920 & 3420 & 1140 & 14,440 \\
2280 & 760 & 380 & 2660 \\
\end{bmatrix}
\]

(c) The entry in row $\text{Rh}^+$ and column $O$ tells us that 14,440 units of $O^+$ would be required.
EXERCISE 3.1 Answers Begin on Page AN–14.

`Are You Prepared?` Problems Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. **True or False** For real numbers, addition and subtraction are commutative. (pp. A–8 to A–12)

2. The Associative Property for Addition of real numbers states that ____. (pp. A–8 to A–12)

**Concepts and Vocabulary**

3. The dimension of the matrix \[
\begin{bmatrix}
1 & 4 & \sqrt{2} \\
0 & 0 & 0
\end{bmatrix}
\] is ____.

4. **True or False** \[
\begin{bmatrix}
1 & \sqrt{4} & 8 + 2 \\
-2 & 5 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 10 \\
-2 & 5 & 1 - 1
\end{bmatrix}
\]

5. **True or False** There are examples of matrices for which \[A + B \neq B + A\].

6. **True or False** Two matrices can always be added as long as the entries are real numbers.

**Skill Building**

In Problems 7–14, find the dimension of each matrix. Indicate if the matrix is a square matrix, a row matrix, or a column matrix.

7. \[
\begin{bmatrix}
3 & 2 \\
-1 & 3
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
2 & 1 & -3 \\
1 & 0 & -1
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
0 & -3
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
0 & -3 & 6 \\
0 & 5 & 2 \\
1 & 8 & 7
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
1 & 4 \\
-2 & 8 \\
0 & 0
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
8 & 1 & 0 & 0 \\
3 & -4 & 0 & 0
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
4 \\
1
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
2 & 1 & -3
\end{bmatrix}
\]

In Problems 15–26, determine whether the given statements are true or false. If false, tell why.

15. \[
\begin{bmatrix}
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 & 1
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
3 & 2 \\
-1 & 0
\end{bmatrix} = \begin{bmatrix}
3 & 2 \\
-1 & 4
\end{bmatrix}
\]

17. \[
\begin{bmatrix}
5 & 0 \\
0 & 1
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
3 & 2 & 1 \\
4 & -1 & 0
\end{bmatrix}
\]

is \(3 \times 2\)

19. \[
\begin{bmatrix}
x & 2 \\
4 & 0
\end{bmatrix} = \begin{bmatrix}
3 & 2 \\
4 & 0
\end{bmatrix}
\]

if \(x = 3\)

20. \[
\begin{bmatrix}
x & y \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
x & y
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
5 & 0 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
2 + 3 & 0 \\
1 & 1
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
3 - 2 & 3 - 3 \\
3 - 3 & 3 - 2
\end{bmatrix}
\]

23. \[
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
0 & 4
\end{bmatrix}
\]

24. \[
\begin{bmatrix}
8 & 0 \\
5 & -1
\end{bmatrix} = \begin{bmatrix}
-8 & 0 \\
5 & 1
\end{bmatrix}
\]

25. \[
\begin{bmatrix}
8 \\
1
\end{bmatrix} + \begin{bmatrix}
2 \\
9
\end{bmatrix} = \begin{bmatrix}
10
\end{bmatrix}
\]

26. \[
\begin{bmatrix}
6 & 0 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
7 & 1
\end{bmatrix}
\]

In Problems 27–34, perform the indicated operations. Express your answer as a single matrix.

27. \[
\begin{bmatrix}
3 & -1 \\
4 & 2
\end{bmatrix} + \begin{bmatrix}
-2 & 2 \\
2 & 5
\end{bmatrix}
\]

28. \[
\begin{bmatrix}
2 & 4 \\
3 & -4
\end{bmatrix} - \begin{bmatrix}
8 & 9 \\
0 & 7
\end{bmatrix}
\]

29. \[
\begin{bmatrix}
2 & 6 & 0 \\
4 & -2 & 1
\end{bmatrix}
\]

30. \[
\begin{bmatrix}
2 & 1 \\
-2 & 1 \\
0 & 3
\end{bmatrix}
\]

31. \[
\begin{bmatrix}
1 & -1 & 8 \\
2 & 4 & 1
\end{bmatrix} - \begin{bmatrix}
0 & -2 & 8 \\
1 & 4 & 1
\end{bmatrix}
\]

32. \[
\begin{bmatrix}
2 & 1 \\
3 & 1 \\
-1 & 0
\end{bmatrix} + \begin{bmatrix}
6 & -4 \\
-2 & -3 \\
0 & 1
\end{bmatrix}
\]

33. \[
\begin{bmatrix}
a & 8 \\
b & 1 \\
c & -2
\end{bmatrix} + 5 \begin{bmatrix}
2a & 6 \\
-b & -2 \\
-c & 0
\end{bmatrix}
\]

34. \[
\begin{bmatrix}
2x & y & z \\
2 & -4 & 8
\end{bmatrix} - \begin{bmatrix}
-3x & 4y & 2z \\
6 & -1 & 4
\end{bmatrix}
\]
Applications

59. Voter Survey Use a matrix to display the information given below, which was obtained in a survey of voters. Label the rows UNDER $25,000 and OVER $25,000 and label the columns DEMOCRATS, REPUBLICANS, INDEPENDENTS.

<table>
<thead>
<tr>
<th></th>
<th>Democrats earning under $25,000</th>
<th>Republicans earning under $25,000</th>
<th>Independents earning under $25,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>351</td>
<td>Democrats earning under $25,000</td>
<td>Republicans earning under $25,000</td>
<td>Independents earning under $25,000</td>
</tr>
<tr>
<td>271</td>
<td>Democrats earning under $25,000</td>
<td>Republicans earning under $25,000</td>
<td>Independents earning under $25,000</td>
</tr>
<tr>
<td>73</td>
<td>Democrats earning under $25,000</td>
<td>Republicans earning under $25,000</td>
<td>Independents earning under $25,000</td>
</tr>
<tr>
<td>203</td>
<td>Democrats earning $25,000 or more</td>
<td>Republicans earning under $25,000</td>
<td>Independents earning under $25,000</td>
</tr>
<tr>
<td>215</td>
<td>Democrats earning $25,000 or more</td>
<td>Republicans earning under $25,000</td>
<td>Independents earning under $25,000</td>
</tr>
<tr>
<td>55</td>
<td>Democrats earning $25,000 or more</td>
<td>Republicans earning under $25,000</td>
<td>Independents earning under $25,000</td>
</tr>
</tbody>
</table>

60. Listing Stocks One day on the New York Stock Exchange, 800 issues went up and 600 went down. Of the 800 up issues, 200 went up more than $1 per share. Of the 600 down issues, 50 went down more than $1 per share. Express this information in a $2 \times 2$ matrix. Label the rows UP and DOWN and the columns MORE THAN $1 and LESS THAN $1.

61. Surveys In a survey of 1000 college students, the following information was obtained: 500 were liberal arts and sciences (LAS) majors, of which 50% were female; 300 were engineering (ENG) majors, of which 75% were male; and the remaining were education (EDUC) majors, of which 60% were female. Express this information using a $2 \times 3$ matrix. Label the rows MALE and FEMALE and the columns LAS, ENG, and EDUC.
62. **Candy Store Purchases**  Katy, Mike, and Danny go to the candy store. Katy buys 5 sticks of gum, 2 ice cream cones, and 20 jelly beans. Mike buys 2 sticks of gum, 15 jelly beans, and 3 candy bars. Danny buys 1 stick of gum, 1 ice cream cone, and 4 candy bars. Write a $3 \times 4$ matrix depicting this situation. Label the rows KATY, MIKE, DANNY and the columns GUM, ICE CREAM, JELLY BEANS, CANDY BARS.

63. **College Degrees by Gender**  Postsecondary degrees include associate, bachelor’s, master’s, and doctoral degrees. Projections for 2013–2014 are as follows: 735,000 associate degrees, of which 275,000 will be awarded to men; 1,582,000 bachelor’s degrees, of which 949,000 will be awarded to women; 693,000 master’s degrees, of which 418,000 will be awarded to women; and 54,900 doctoral degrees, of which 27,600 will be awarded to men.

(a) Write a $2 \times 3$ matrix representing this information.
(b) What does each row and each column represent?
(c) Write a $2 \times 3$ matrix representing this information.
(d) Calculate the percent increase in prison population for each category discussed and write this as a $1 \times 3$ matrix.

64. **Nail Production**  Nails Unlimited produces steel and aluminum nails. One week 25 gross of $\frac{1}{2}$-inch steel nails and 45 gross of 1-inch steel nails were produced. Suppose 13 gross of $\frac{1}{2}$-inch aluminum nails, 20 gross of 1-inch aluminum nails, 35 gross of 2-inch steel nails, and 23 gross of 2-inch aluminum nails were also made.

(a) Write a $2 \times 3$ matrix representing this information.
(b) What does each row and each column represent?
(c) Write a $2 \times 3$ matrix representing this information.
(d) What does each row and each column of this matrix represent?

65. **Prison Populations**  In 2008, local jails and state and federal prisons contained approximately 2.3 million people, as follows: 785,556 were in local jails, of which about 12.7% were female; 1,409,166 prisoners were in state prisons, of which about 92.8% were male; and 201,280 were in federal prisons, of which about 6.6% were female. Express this information using a $2 \times 3$ matrix. Label the rows MALE and FEMALE and the columns LOCAL, STATE, FED.

Source: United States Department of Justice, 2009

66. **Prison Populations**  In 2000, local jails and state and federal prisons contained nearly 2 million people, as follows: 613,534 prisoners were in local jails, of which 11.5% were female; 1,236,476 prisoners were in state prisons, of which 93.4% were male; and 145,416 were in federal prisons, of which 7.0% were female.

(a) Express this information using a $2 \times 3$ matrix. Label the rows MALE and FEMALE and the columns LOCAL, STATE, FED.
(b) Refer to Problem 65. Find the difference of the matrix obtained in Problem 65 and the one found in part (a) above.
(c) Interpret the entries in the matrix found above in part (b).
(d) Calculate the percent increase in prison population for each category discussed and write this as a $1 \times 3$ matrix.

Source: United States Department of Justice, 2001

67. **Sales of Cars**  Quality Dodge has two locations, one in the city and the other in the suburbs. In January, the city location sold 350 subcompacts, 225 intermediate-size cars, and 80 SUVs. In February, it sold 300 subcompacts, 175 intermediates, and 40 SUVs. In January, the suburban location sold 375 subcompacts, 200 intermediates, and 75 SUVs. In February, it sold 325 subcompacts, 150 intermediates, and 50 SUVs.

(a) Write matrices that summarize the sales data for each location for January and February (one matrix for each month).
(b) Use matrix addition to obtain total sales for the two-month period for each location.
(c) Write matrices that summarize the sales data for each month by location (one matrix for each location).
(d) Use matrix addition to obtain total sales for the two locations for each month.

68. **Sales of Cars**  The sales figures for two car dealers during June showed that a Honda dealer sold 100 compacts, 50 intermediates, and 40 full-size cars, while a Toyota dealer sold 120 compacts, 40 intermediates, and 35 full-size cars. During July, the Honda dealer sold 80 compacts, 30 intermediates, and 10 full-size cars, while the Toyota dealer sold 70 compacts, 40 intermediates, and 20 full-size cars. Total sales over the 3-month period of June–August revealed that the Honda dealer sold 300 compacts, 120 intermediates, and 65 full-size cars. In the same 3-month period, the Toyota dealer sold 250 compacts, 100 intermediates, and 80 full-size cars.

(a) Write $2 \times 3$ matrices summarizing sales data for June, July, and the 3-month period for each dealer.
(b) Use matrix addition to find the sales over the 2-month period for June and July for each dealer.
(c) Use matrix subtraction to find the sales in August for each dealer.

69. **Doctoral Degrees** In 2008 in the United States, there were 7508 doctoral degrees awarded in the social sciences, of which 4387 were awarded to females; 4721 doctoral degrees awarded in humanities, of which 2256 were awarded to males; and 6577 doctoral degrees awarded in education, of which 4414 were awarded to females. In 2007, there were 7193 doctoral degrees awarded in the social sciences, of which 4214 were awarded to females; 5106 doctoral degrees awarded in humanities, of which 2588 were awarded to males; and 6443 doctoral degrees awarded in education, of which 4340 were awarded to females.

(a) Write 2 × 3 matrices for each year, depicting the three fields of doctoral degrees awarded by gender.
(b) Use matrix addition to find the total number of doctoral degrees awarded by gender over the two years.
(c) Use matrix subtraction to find the difference in the number of doctoral degrees awarded by gender from 2007 to 2008.

**Sources:** NSF/NIH/USED/NEH/USDA/NASA, 2007 Survey of Earned Doctorates; 2008 Survey of Earned Doctorates

70. **Ordering Produce** Reginald owns three produce stores. The table shows an order, in pounds, that was placed for the three stores for the first week of June.

<table>
<thead>
<tr>
<th>Tomatoes</th>
<th>Potatoes</th>
<th>Onions</th>
<th>Carrots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Store 1</td>
<td>350</td>
<td>450</td>
<td>150</td>
</tr>
<tr>
<td>Store 2</td>
<td>300</td>
<td>350</td>
<td>300</td>
</tr>
<tr>
<td>Store 3</td>
<td>450</td>
<td>300</td>
<td>250</td>
</tr>
</tbody>
</table>

(a) Write the information in the table as a 3 × 4 matrix A.
(b) In anticipation of a sale, the order was increased by 20% for the following week. Write a matrix B that represents the order for the second week of June.
(c) Find A + B and state what the entries of A + B represent.

71. **Home Internet Users** A survey conducted in late 2009 by the Pew Internet and American Life Project studied the relationship between home Internet users and community type. The distribution of the American adults included in the study follows:

<table>
<thead>
<tr>
<th>Community Type</th>
<th>Rural</th>
<th>Suburban</th>
<th>Urban</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internet Users</td>
<td>74%</td>
<td>77%</td>
<td>70%</td>
</tr>
</tbody>
</table>

**Source:** Pew Internet & American Life Project (2009).

Write a 1 × 3 matrix A depicting the distribution of internet access by community type, written in decimal form.

72. **Soccer Park Concessions** Beatrice runs the concession stands for a city soccer park. The park has four concession stands that are referred to by their directional location within the park. Each stand offers a limited menu of hotdogs, nachos, pretzels, and bottled water. The table shows the number of items sold at each stand on the first Saturday in October.

<table>
<thead>
<tr>
<th>Bottled</th>
<th>HotDogs</th>
<th>Nachos</th>
<th>Pretzels</th>
<th>Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>North Stand</td>
<td>50</td>
<td>26</td>
<td>18</td>
<td>71</td>
</tr>
<tr>
<td>South Stand</td>
<td>30</td>
<td>20</td>
<td>12</td>
<td>50</td>
</tr>
<tr>
<td>East Stand</td>
<td>26</td>
<td>22</td>
<td>14</td>
<td>48</td>
</tr>
<tr>
<td>West Stand</td>
<td>35</td>
<td>38</td>
<td>21</td>
<td>76</td>
</tr>
</tbody>
</table>

(a) Write the information in the table as a 4 × 4 matrix A.
(b) A tournament will be held at the park on the second Saturday in October, so Beatrice anticipates that the number of items sold at each stand that day will increase by 40%. Write a matrix B that represents the number of items by stand that Beatrice anticipates needing.
(c) Find B − A and state what the entries of B − A represent.

‘Are You Prepared?’ Answers

1. False

2. \( a + (b + c) = (a + b) + c \)
3.2 Multiplication of Matrices

**PREPARING FOR THIS SECTION** Before getting started, review the following:

- Properties of Real Numbers (Appendix A, Section A.1, pp. A-8 to A-12)

Now work the ’Are You Prepared?’ problems on page 131

**OBJECTIVES**

1. Find the product of two matrices (p. 122)
2. Work with properties of matrix multiplication (p. 126)
3. Solve applied problems involving matrices (p. 129)

While addition and subtraction of matrices and the product of a scalar and a matrix are fairly straightforward, defining the *product* of two matrices requires a bit more detail.

We explain first what we mean by the product of a row matrix (a matrix with one row) with a column matrix (a matrix with one column).

Let’s look at a simple example. In a given month suppose 23 units of material and 7 units of labor were required to manufacture 4-door sedans. We can represent this by the column matrix

\[
\begin{bmatrix}
23 \\
7
\end{bmatrix}
\]

Also, suppose the cost per unit of material is $450 and the cost per unit of labor is $600. We represent these costs by the row matrix

\[
\begin{bmatrix}
450 & 600
\end{bmatrix}
\]

The total cost of producing the sedans is calculated as follows:

\[
\text{Total cost} = (\text{Cost per unit of material}) \times (\text{Units of material}) + (\text{Cost per unit of labor}) \times (\text{Units of labor})
\]

\[
= 450 \times 23 + 600 \times 7 = 14,550
\]

In terms of the matrix representations,

\[
\text{Total cost} = \begin{bmatrix} 450 & 600 \end{bmatrix} \begin{bmatrix} 23 \\ 7 \end{bmatrix} = 450 \times 23 + 600 \times 7 = 14,550
\]

This leads to the following definition for multiplying a row matrix times a column matrix:

**Definition**

If \( R = [r_1r_2 \ldots r_n] \) is a row matrix of dimension \( 1 \times n \) and \( C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \) is a column matrix of dimension \( n \times 1 \), then the *product of \( R \) and \( C \)* we mean the number

\[
RC = r_1c_1 + r_2c_2 + r_3c_3 + \cdots + r_nc_n
\]
EXAMPLE 1 Finding the Product of a 1 × 3 Row Matrix and a 3 × 1 Column Matrix

If \( R = \begin{bmatrix} 1 & 5 & 3 \end{bmatrix} \) and \( C = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \)

then the product of \( R \) and \( C \) is

\[
RC = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \end{bmatrix} = 1 \cdot 2 + 5 \cdot (-1) + 3 \cdot 4 = 9
\]

Notice that for the product of a row matrix \( R \) and a column matrix \( C \) to be defined, if \( R \) is a 1 × \( n \) row matrix, then \( C \) must be of dimension \( n \times 1 \).

EXAMPLE 2 Finding the Product of a 1 × 4 Row Matrix and a 4 × 1 Column Matrix

Let \( R = \begin{bmatrix} 1 & 0 & 1 & 5 \end{bmatrix} \) and \( C = \begin{bmatrix} 0 \\ -11 \\ 0 \\ 8 \end{bmatrix} \)

Then the product of \( R \) and \( C \) is

\[
RC = \begin{bmatrix} 1 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ -11 \\ 0 \\ 8 \end{bmatrix} = 1 \cdot 0 + 0 \cdot (-11) + 1 \cdot 0 + 5 \cdot 8 = 40
\]

NOW WORK PROBLEM 7.

Given two matrices \( A \) and \( B \), the rows of \( A \) can be thought of as row matrices, while the columns of \( B \) can be thought of as column matrices. This observation will be used in the following definition.

Definition Multiplication of Matrices

Let \( A \) denote an \( m \times r \) matrix, and let \( B \) denote an \( r \times n \) matrix. The product \( AB \) is defined as the \( m \times n \) matrix whose entry in row \( i \), column \( j \) is the product of the \( i \)th row of \( A \) and the \( j \)th column of \( B \).

EXAMPLE 3 Finding the Product of Two Matrices

Find the product \( AB \) if

\[
A = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}
\]
**SOLUTION** First, we note that $A$ is $2 \times 3$ and $B$ is $3 \times 4$, so the product $AB$ is defined and will be a $2 \times 4$ matrix.

Suppose that we want the entry in row 2, column 3 of $AB$. To find it, we find the product of the row vector from row 2 of $A$ and the column vector from column 3 of $B$.

\[
\begin{align*}
\text{Column 3 of } B \\
\text{Row 2 of } A
\end{align*}
\]

So far we have

\[
AB = \begin{bmatrix}
\phantom{2} & \phantom{2} & \phantom{5} & \phantom{5} \\
\phantom{2} & \phantom{2} & \phantom{5} & \phantom{5} \\
\phantom{2} & \phantom{2} & \phantom{5} & \phantom{5}
\end{bmatrix}
\]

Now, to find the entry in row 1, column 4 of $AB$, we find the product of row 1 of $A$ and column 4 of $B$.

\[
\begin{align*}
\text{Column 4 of } B \\
\text{Row 1 of } A
\end{align*}
\]

Continuing in this fashion, we find $AB$.

\[
AB = \begin{bmatrix}
2 & 4 & -1 \\
5 & 8 & 0
\end{bmatrix}
\begin{bmatrix}
2 & 5 & 1 & 4 \\
4 & 8 & 0 & 6 \\
-3 & 1 & -2 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 \cdot 2 + 4 \cdot 4 + (-1)(-3) & 2 \cdot 5 + 4 \cdot 8 + (-1)(-1) & 2 \cdot 1 + 4 \cdot 0 + (-1)(-2) & 33 \text{ (from earlier)} \\
5 \cdot 2 + 8 \cdot 4 + 0(-3) & 5 \cdot 5 + 8 \cdot 8 + 0 \cdot 1 & 5 \text{ (from earlier)} & 5 \cdot 4 + 8 \cdot 6 + 0(-1)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
23 & 41 & 4 & 33 \\
42 & 89 & 5 & 68
\end{bmatrix}
\]

NOW WORK PROBLEM 33.
**EXAMPLE 4** Finding the Product of Two Matrices Using a Graphing Utility

Rework Example 3 using a graphing utility.

**SOLUTION**

Enter the matrices $A$ and $B$ into a graphing utility. Figure 3 shows the product $AB$.

![Figure 3](image)

**NOW WORK PROBLEM 33 USING A GRAPHING UTILITY.**

**EXAMPLE 5** Manufacturing Cost

One month’s production at Motors, Inc., may be given in matrix form as

\[
A = \begin{bmatrix}
23 & 16 & 10 \\
7 & 9 & 11
\end{bmatrix}
\]

Suppose that in this month’s production, the cost for each unit of material is $450 and the cost for each unit of labor is $600. What is the total cost to manufacture the sedans, the convertibles, and the SUVs?

**SOLUTION** For sedans, the cost is 23 units of material at $450 each, plus 7 units of labor at $600 each, for a total cost of

\[
23 \cdot \$450 + 7 \cdot \$600 = 10,350 + 4200 = 14,550
\]

Similarly, for convertibles, the total cost is

\[
16 \cdot \$450 + 9 \cdot \$600 = 7200 + 5400 = 12,600
\]

Finally, for SUVs, the total cost is

\[
10 \cdot \$450 + 11 \cdot \$600 = 4500 + 6600 = 11,100
\]

If we represent the cost of units of material and units of labor by the $1 \times 2$ matrix

\[
U = \begin{bmatrix}
450 \\ 600
\end{bmatrix}
\]

then the total cost is the product $UA$.

\[
UA = \begin{bmatrix}
450 & 600
\end{bmatrix} \begin{bmatrix}
23 & 16 & 10 \\
7 & 9 & 11
\end{bmatrix} = \begin{bmatrix}
450 \cdot 23 + 600 \cdot 7 & 450 \cdot 16 + 600 \cdot 9 & 450 \cdot 10 + 600 \cdot 11
\end{bmatrix} = \begin{bmatrix}
14,550 & 12,600 & 11,100
\end{bmatrix}
\]

We conclude that the total cost to manufacture the sedans is $14,550, to manufacture the convertibles is $12,600, and to manufacture the SUVs is $11,100.

**NOW WORK PROBLEM 65.**
USING TECHNOLOGY

EXAMPLE 6 Using Excel

Spreadsheets, such as Excel, can also be used to multiply matrices. Such utilities are especially useful when the dimensions of the matrices are large. Use Excel to do Example 5.

SOLUTION

STEP 1 Enter the matrices for $A$ and $U$ into an Excel spreadsheet as follows:

```
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
</tbody>
</table>
```

STEP 2 Highlight the cells that are to contain the product matrix. Since $U$ is a $1 \times 2$ matrix and $A$ is a $2 \times 3$ matrix, the product must be a $1 \times 3$ matrix.

STEP 3 Type: =MMULT(highlight U, highlight A)

STEP 4 Press Ctrl-Shift-Enter at the same time.
NOW WORK PROBLEM 65 USING EXCEL.

2 Work with Properties of Matrix Multiplication

If $A$ is a matrix of dimension $m \times r$ (which has $r$ columns) and $B$ is a matrix of dimension $p \times n$ (which has $p$ rows) and if $r \neq p$, the product $AB$ is not defined.

Multiplication of matrices is possible only if the number of columns of the matrix on the left equals the number of rows of the matrix on the right. If $A$ is of dimension $m \times r$ and $B$ is of dimension $r \times n$, then the product $AB$ is of dimension $m \times n$.

Figure 4 illustrates the result stated above.
For example, if $A$ is a matrix of dimension $2 \times 3$, and $B$ is a matrix of dimension $3 \times 4$, the product $AB$ is defined, but the product $BA$ is not defined.

EXAMPLE 7 Multiplying Two Matrices

If

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

find: (a) $AB$ (b) $BA$
3.2 Multiplication of Matrices

**SOLUTION**

(a) \( AB = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 7 \\ -1 & -1 \end{bmatrix} \)

\( 2 \times 3 \times 3 \times 2 = 2 \times 2 \)

(b) \( BA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 1 & 6 \\ 8 & 1 & 9 \end{bmatrix} \)

\( 3 \times 2 \times 2 \times 3 = 3 \times 3 \)

Notice in Example 7 that \( AB \) is \( 2 \times 2 \) and \( BA \) is \( 3 \times 3 \). It is possible for both \( AB \) and \( BA \) to be defined yet be unequal. In fact, even if \( A \) and \( B \) are both matrices, so that \( AB \) and \( BA \) are each defined and each are \( n \times n \) matrices, \( AB \) and \( BA \) will usually be unequal.

**EXAMPLE 8** Multiplying Two Square Matrices

If

\[ A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} \]

find:  (a) \( AB \)  (b) \( BA \)

**SOLUTION**

(a) \( AB = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & 8 \end{bmatrix} \)

(b) \( BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 1 \\ 2 & 9 \end{bmatrix} \)

The preceding examples demonstrate that an important property of real numbers, the commutative property of multiplication, is not shared by matrices.

**Theorem**

Matrix multiplication is not commutative. That is, in general,

\[ AB \text{ is not always equal to } BA \]

Matrix multiplication is associative.

**Associative Property of Matrix Multiplication**

Let \( A \) be a matrix of dimension \( m \times r \), let \( B \) be a matrix of dimension \( r \times p \), and let \( C \) be a matrix of dimension \( p \times n \), so that \( A(BC) \) and \( (AB)C \) are defined. Then matrix multiplication is **associative**. That is,

\[ A(BC) = (AB)C \]

The resulting matrix \( ABC \) is of dimension \( m \times n \).

**NOW WORK PROBLEM 47.**
**Distributive Property**

Let $A$ be a matrix of dimension $m \times r$. Let $B$ and $C$ be matrices of dimension $r \times n$ so that $A(B + C)$ and $AB + AC$ are defined. Then the **distributive property** states that

$$A(B + C) = AB + AC$$

The resulting matrix $AB + AC$ is of dimension $m \times n$.

**The Identity Matrix**

For an $n \times n$ square matrix, the entries located in row $i$, column $i$, $1 \leq i \leq n$, are called the **diagonal entries**. An $n \times n$ square matrix whose diagonal entries are 1s, while all other entries are 0s, is called the **identity matrix** $I_n$. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so on.

**Example 9** Finding the Product of a Matrix and the Identity Matrix $I_2$

For $A = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$ find

(a) $AI_2$  
(b) $I_2A$

**Solution**

(a) $AI_2 = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix} = A$

(b) $I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix} = A$

Example 9 demonstrates a more general result.

**Theorem**

If $A$ is a square matrix of dimension $n \times n$, then $AI_n = I_nA = A$. 
For square matrices the identity matrix plays the role that the number 1 plays for multiplication in the set of real numbers.

When a matrix $A$ is not square, care must be taken when forming the products $AI$ and $IA$. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$$

then $A$ is of dimension $3 \times 2$ and

$$AI_2 = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} = A$$

Although the product $I_2A$ is not defined, we can calculate the product $I_3A$ as follows:

$$I_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} = A$$

The above observations can be generalized as follows:

**Identity Property**

If $A$ is a matrix of dimension $m \times n$ and if $I_n$ denotes the identity matrix of dimension $n \times n$, and $I_m$ denotes the identity matrix of dimension $m \times m$, then

$$I_mA = A \quad \text{and} \quad AI_n = A$$

### 3 Solve Applied Problems Involving Matrices

**EXAMPLE 10** Applying Matrix Multiplication

The price per share for Wal-Mart (WMT), Target (TGT), and Costco (Cost) common stock at the close of trading on November 1 of the years 2007, 2008, and 2009 are shown in matrix $A$. Over these years, Kathleen and Shannon have each kept constant numbers of shares of the three stocks in their portfolios as shown in matrix $B$.

$$A = \begin{bmatrix} 2007 & WMT & 42.16 & 56.58 & 63.17 \\ 2008 & TGT & 54.54 & 39.24 & 56.03 \\ 2009 & COST & 50.03 & 49.17 & 57.75 \end{bmatrix} \quad B = \begin{bmatrix} 2007 & WMT & 150 \\ 2008 & TGT & 100 \\ 2009 & COST & 50 \end{bmatrix}$$

Compute $AB$ and interpret the result.

**Source**: Yahoo Finance

**SOLUTION** Notice that matrix $A$ is $3 \times 3$ and matrix $B$ is $3 \times 2$, so the matrix $AB$ is defined and will be a $3 \times 2$ matrix. Also notice that the columns of matrix $A$ are named WMT, TGT,
Chapter 3 Matrices

and COST, the same names given the rows of matrix \( B \). This means the rows of the product matrix \( AB \) will be named the same as the rows of \( A \) and the columns of \( AB \) will be named the same as the columns of \( B \). Now,

\[
AB = \begin{bmatrix}
42.16 & 56.58 & 63.17 & 150 & 125 \\
54.54 & 39.24 & 56.03 & 100 & 75 \\
50.03 & 49.17 & 57.75 & 50 & 100
\end{bmatrix}
\]

\[
= \begin{bmatrix}
42.16 \cdot 150 + 56.58 \cdot 100 + 63.17 \cdot 50 & 42.16 \cdot 125 + 56.58 \cdot 75 + 63.17 \cdot 100 \\
54.54 \cdot 150 + 39.24 \cdot 100 + 56.03 \cdot 50 & 54.54 \cdot 125 + 39.24 \cdot 75 + 56.03 \cdot 100 \\
50.03 \cdot 150 + 49.17 \cdot 100 + 57.75 \cdot 50 & 50.03 \cdot 125 + 49.17 \cdot 75 + 57.75 \cdot 100
\end{bmatrix}
\]

The entries in the product matrix indicate the total value of the stocks in the portfolios on November 1 of each year. The value of Kathleen’s portfolio was $15,140.50 in 2007, $14,906.50 in 2008, and $15,309.00 in 2009. The value of Shannon’s portfolio was $15,830.50 in 2007, $15,363.50 in 2008, and $15,716.50 in 2009. □

### EXAMPLE 11 Applying Matrix Multiplication

Margareta and Emilio are taking classes at both Joliet Junior College and Chicago State University. The number of credit hours taken and the cost per credit hour are as follows:

<table>
<thead>
<tr>
<th></th>
<th>JJC</th>
<th>CSU</th>
</tr>
</thead>
<tbody>
<tr>
<td>Margareta</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>Emilio</td>
<td>9</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cost per Credit Hour</th>
<th>JJC</th>
<th>CSU</th>
</tr>
</thead>
<tbody>
<tr>
<td>JJC</td>
<td>$73</td>
<td></td>
</tr>
<tr>
<td>CSU</td>
<td>$189</td>
<td></td>
</tr>
</tbody>
</table>

(a) Write a matrix \( A \) for the credit hours taken by each student and a matrix \( B \) for the cost per credit hour.

(b) Compute \( AB \).

(c) Interpret the results.

Sources: www.jjc.edu, www.csu.edu, for the 2006–2007 academic year

SOLUTION

(a) If we name the rows as Margareta and Emilio and the columns as JJC and CSU, then we can write the matrix \( A \) as the \( 2 \times 2 \) matrix

\[
A = \begin{bmatrix}
12 & 3 \\
9  & 6
\end{bmatrix}
\]

For the matrix \( B \), if we name the rows as JJC and CSU and the column as Cost per Credit Hour, then we can write the matrix \( B \) as the \( 2 \times 1 \) matrix

\[
B = \begin{bmatrix}
73 \\
189
\end{bmatrix}
\]
3.2 Multiplication of Matrices

(b) Since $A$ is $2 \times 2$ and $B$ is $2 \times 1$, the product $AB$ is defined and will be a $2 \times 1$ matrix. Then

$$AB = \begin{bmatrix} 12 & 3 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} 73 \\ 189 \end{bmatrix} = \begin{bmatrix} 1443 \\ 1791 \end{bmatrix}$$

(c) Because the columns of $A$ are named the same as the rows of $B$, JJC and CSU, the product $AB$ will give us the cost for each student to attend classes. That is, it costs Margareta $1443 to take her 15 credit hours (12 at JJC and 3 at CSU) and it costs Emilio $1791 to take his 15 credit hours (9 at JJC and 6 at CSU).

EXERCISE 3.2 Answer Begin on Page AN–15.

'Are You Prepared?' Problems Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. True or False For real numbers, multiplication is commutative. (pp. A–8 to A–12)

2. The Associative Property for Multiplication of real numbers states that _____ . (pp. A–8 to A–12)

Concepts and Vocabulary

3. If $A$ is of dimension $3 \times 5$ and $B$ is of dimension $5 \times 7$, the dimension of the matrix $AB$ is _____.

4. True or False

$$\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ -5 & 8 \end{bmatrix}$$

5. True or False There are examples of matrices for which $AB \neq BA$.

6. True or False Two matrices can always be multiplied as long as they are each of the same dimension.

Skill Building

In Problems 7–22, find each product.

7. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
8. $\begin{bmatrix} -1 & 4 \\ 2 & 5 \end{bmatrix}$
9. $\begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 \end{bmatrix}$
10. $\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 \end{bmatrix}$
11. $\begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 4 & -2 \end{bmatrix}$
12. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 4 & -2 \end{bmatrix}$
13. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 4 & -2 \end{bmatrix}$
14. $\begin{bmatrix} 1 & 4 & 3 \\ 1 & 2 \end{bmatrix}$
15. $\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$
16. $\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$
17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 \end{bmatrix}$
18. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 \end{bmatrix}$
19. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 4 & -2 \end{bmatrix}$
20. $\begin{bmatrix} 2 & 0 & 6 \\ 1 & -1 \\ 4 & 1 \end{bmatrix}$
21. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$
22. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

In Problems 23–32, let $A$ be of dimension $3 \times 4$, $B$ be of dimension $3 \times 3$, $C$ be of dimension $2 \times 3$, and $D$ be of dimension $3 \times 2$. Determine which of the following expressions are defined and, for those that are, give the dimension.

23. $BA$
24. $CD$
25. $AB$
26. $DC$
27. $(BA)C$
28. $A(CD)$
29. $BA + A$
30. $CD + BA$
31. $CB - A$
32. $DC + B$

In Problems 33–48, use the matrices given below.

For Problems 33–46, perform the indicated operation(s); for Problems 47 and 48, verify the indicated property.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ -2 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$$

33. $AB$
34. $DC$
35. $BC$
36. $AA$
37. $(D + I_3)C$
48. Verify the distributive property by finding and 
\(AB + EB\). 
\((A + E)B\)

47. Verify the associative property of matrix multiplication by 
finding 
\(D(CB)\) and 
\((DC)B\).

In Problems 51–58, use a graphing utility or EXCEL and the matrices given below to perform the indicated operations.

\(\begin{pmatrix} -1 & 1 & 3 \\ 2 & 6 & 2 \\ -4 & 2 & 3 \\ 7 & 0 & 5 \end{pmatrix}\) 
\(\begin{pmatrix} -1 & 2 & 4 \\ 0 & 5 & 3 \\ 0.5 & 6 & -7 \\ 5 & -1 & 2 \end{pmatrix}\) 
\(\begin{pmatrix} 13 & -8 & 7 \\ 0 & 5 & 0 \\ 5 & 0 & 7 \end{pmatrix}\)

51. \(AB\) 
52. \(BA\) 
53. \((AB)C\) 
54. \((A + C)B\) 
55. \(A(A + C)\) 
56. \((A + C)B\) 
57. \(A(2B - 3C)\) 
58. \((A + B)(A - B)\)

Applications and Extensions

59. **Stock Purchases** The price per share at the close of trading on July 7, 2009, for stock in Chevron, ConocoPhillips, and Exxon Mobil was:

- Chevron Corporation—$62.70
- ConocoPhillips—$39.99
- Exxon Mobil Corporation—$66.56

Let \(A = [62.70\ 39.99\ 66.56]\)

Let \(B\) be the following \(3 \times 3\) matrix showing the number of shares of stock purchased at the close of trading on July 7 by Kevin, Timmy, and Nolan.

\[
\begin{pmatrix}
100 & 415 & 175 \\
250 & 90 & 200 \\
325 & 225 & 130
\end{pmatrix}
\]

(a) What does the matrix \(A\) represent? What is the dimension of \(A\)?

(b) Find the matrix product \(AB\) and give an interpretation of the entries in this product.

(c) Let \(C\) be the \(3 \times 1\) matrix having a 1 as each entry. Find the matrix product \(BC\) and give an interpretation of the entries in this product.

60. **Gasoline Prices** Average gasoline prices in Los Angeles on January 4, 2010, were:

- Regular—$3.035 per gallon
- Midgrade—$3.144 per gallon
- Premium—$3.243 per gallon

Suppose a company made gasoline purchases (in gallons) at the prices given above for vehicles based at three different retail stores as shown in the matrix \(C\) below.

\[
\begin{pmatrix}
900 & 350 & 225 \\
750 & 500 & 150 \\
1050 & 730 & 75
\end{pmatrix}
\]
61. **Cost of College** John and Marsha take classes in Fall 2010 at both Joliet Junior College (JJC) and Chicago State University (CSU). John takes 12 credit hours at JJC and 4 credit hours at CSU, while Marsha takes 8 credit hours at each school. The cost per credit hour at JJC is $103 and at CSU is $249.

(a) Write a matrix $A$ that gives the credit hours each student is taking at each school and a matrix $B$ that gives the cost per credit hour at each school. Be sure to name the rows and columns and be sure that the columns of $A$ are named the same as the rows of $B$.

(b) What is the dimension of $A$?

(c) What is the dimension of $B$?

(d) Find the product $AB$.

(e) What is the dimension of $AB$? What are the names of its rows and columns?

(f) Interpret the entries in $AB$.

**Source:** Each school’s website.

62. **Cost of College** Sean, Kevin, and Bill take classes in Fall 2010 at both Joliet Junior College (JJC) and Chicago State University (CSU). Sean takes 8 credit hours at JJC and 4 credit hours at CSU; Kevin takes 10 credit hours at JJC and 6 at CSU, and Bill takes 6 credit hours at JJC and 4 at CSU. The cost per credit hour at JJC is $103 and at CSU is $249.

(a) Write a matrix $A$ that gives the credit hours each student is taking at each school and a matrix $B$ that gives the cost per credit hour at each school. Be sure to name the rows and columns and be sure that the columns of $A$ are named the same as the rows of $B$.

(b) What is the dimension of $A$?

(c) What is the dimension of $B$?

(d) Find the product $AB$.

(e) What is the dimension of $AB$? What are the names of its rows and columns?

(f) Interpret the entries in $AB$.

(g) Does $BA$ have any meaning? Explain.

63. **Portfolio Values** The price per share for General Electric (GE), Toyota Motors (TM), and Johnson & Johnson (JNJ) common stock in October 2008, April 2009, and October 2009 are shown in the table below. Over this period of time, Colleen, Mary, and Kathleen have owned shares in these companies, as shown in the table.

<table>
<thead>
<tr>
<th>Price per Share</th>
<th>Shares Owned</th>
</tr>
</thead>
<tbody>
<tr>
<td>George</td>
<td>Total Shares</td>
</tr>
<tr>
<td>October 2008</td>
<td>GE</td>
</tr>
<tr>
<td>Oct 2008</td>
<td>22</td>
</tr>
<tr>
<td>April 2009</td>
<td>11</td>
</tr>
<tr>
<td>Oct 2009</td>
<td>17</td>
</tr>
</tbody>
</table>

(a) Write a matrix $A$ that gives the price per share of the three stocks on the three dates and a matrix $B$ that gives the shares owned by Colleen, Mary, and Kathleen. Be sure to name the rows and columns and be sure that the columns of $A$ are named the same as the rows of $B$.

(b) What is the dimension of $A$?

(c) What is the dimension of $B$?

(d) Find the product $AB$.

(e) What is the dimension of $AB$? What are the names of its rows and columns?

(f) Interpret the entries in $AB$.

64. **Portfolio Values** The price per share for Boeing (BA), Verizon (VZ), and Exxon Mobil (XOM) common stock in October 2008, April 2009, and October 2009 are shown in the table below. Over this period of time, Colleen, Mary, and Kathleen have owned shares in these companies, as shown in the table.

<table>
<thead>
<tr>
<th>Price per Share</th>
<th>Shares Owned</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 2008</td>
<td>BA</td>
</tr>
<tr>
<td>Oct 2008</td>
<td>50.5</td>
</tr>
<tr>
<td>Apr 2009</td>
<td>40.5</td>
</tr>
<tr>
<td>Oct 2009</td>
<td>50.5</td>
</tr>
</tbody>
</table>

(a) Write a matrix $A$ that gives the price per share of the three stocks on the three dates and a matrix $B$ that gives the shares owned by Colleen, Mary, and Kathleen. Be sure to name the rows and columns and be sure that the columns of $A$ are named the same as the rows of $B$.

(b) What is the dimension of $A$?

(c) What is the dimension of $B$?

(d) Find the product $AB$.

(e) What is the dimension of $AB$? What are the names of its rows and columns?

(f) Interpret the entries in $AB$.

65. **Department Store Purchases** Lee went to a department store and purchased 6 pairs of pants, 8 shirts, and 2 jackets. Chan purchased 2 pairs of pants, 5 shirts, and 3 jackets. If pants are $25 each, shirts are $18 each, and jackets are $59 each, use matrix multiplication to find the amounts spent by Lee and Chan.

66. **Factory Production** Suppose a factory is asked to produce three types of products, which we will call $P_1$, $P_2$, and $P_3$. Suppose the cost for each of the materials and labor is $15, $20, $22, $25, respectively. The cost vector $C$ is

$$C = \begin{bmatrix} 10 \\ 12 \\ 15 \\ 20 \end{bmatrix}$$

Compute each of the following and interpret each one:

(a) $PQ$

(b) $QC$

(c) $PQC$
67. For what numbers $x$ will the following be true?

\[
\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0
\]

68. Let

\[
A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix}
\]

Show that $A \cdot A = A^2 = 0$. This shows the rule in the real number system that if $a^2 = 0$, then $a = 0$ does not hold for matrices.

69. What must be true about $a$, $b$, $c$, and $d$ if we demand that $AB = BA$ for the following matrices?

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

Assume that

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

70. Let

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

Find $a$ and $b$ such that $A^2 + A = 0$, where $A^2 = A \cdot A$.

71. For the matrix

\[
A = \begin{bmatrix} a & 1 - a \\ 1 + a & -a \end{bmatrix}
\]

show that $A^2 = A \cdot A = I_2$.

72. Find the row vector $[x_1 \ x_2]$ for which

\[
\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 4/4 \end{bmatrix} = [x_1 \ x_2]
\]

under the condition that $x_1 + x_2 = 1$. Here, the row vector $[x_1 \ x_2]$ is called a fixed vector of the matrix

73. Show that, for all values $a$, $b$, $c$, and $d$, the matrices

\[
A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}
\]

are commutative; that is, $AB = BA$.

Problems 74–81 require the following definition:

**Powers of Matrices** For a square matrix $A$ it is possible to find $A^2$. We can also compute

\[
A^n = A \cdot A \cdot \cdots \cdot A
\]

$n$ factors

74. $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

75. $A = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$

76. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

77. $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$

78. Can you guess what $A^n$ looks like for the matrix given in Problem 76?

79. Can you guess what $A^n$ looks like for the matrix given in Problem 77?

In Problems 80 and 81 use a graphing utility to compute $B^2$, $B^{10}$, and $B^{15}$.

80. $B = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.0 & 0.4 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.0 \end{bmatrix}$

81. $B = \begin{bmatrix} 0.74 & 0.02 & 0.24 & 0 \\ 0.3 & 0.2 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.2 & 0.2 \\ 0.0 & 0.1 & 0.2 & 0.7 \end{bmatrix}$

Discussion and Writing

82. Make up two matrices $A$ and $B$ for which $AB = BA$.

83. Make up two square matrices $A$ and $B$ for which $AB$ and $BA$ are both defined, but $AB \neq BA$.

**’Are You Prepared?’ Answers**

1. True
2. $a(bc) = (ab)c$
3.3 The Inverse of a Matrix

PREPARING FOR THIS SECTION
Before getting started, review the following:
• Properties of Real Numbers (Appendix A, Section A.1, pp. A-8 to A-12)

NOW WORK THE ’ARE YOU PREPARED?’ PROBLEMS ON PAGE 148

OBJECTIVES 1 Find the inverse of a matrix (p. 136)
2 Use the inverse of a matrix to solve a system of \( n \) linear equations containing \( n \) variables (p. 141)
3 Solve applied problems involving matrices (p. 143)
4 Application: Use matrices in cryptography (p. 145)

The inverse of a matrix, when it exists, plays the role in matrix algebra that the reciprocal of a number plays in the set of real numbers. For example, the product of 2 and its reciprocal, \( \frac{1}{2} \), equals 1, the multiplicative identity. The product of a matrix and its inverse equals the identity matrix.

Definition

Inverse of a Matrix

Let \( A \) be a matrix of dimension \( n \times n \). A matrix \( B \) of dimension \( n \times n \) is called the inverse of \( A \) if \( AB = BA = I_n \). We denote the inverse of a matrix \( A \), if it exists, by \( A^{-1} \).

EXAMPLE 1 Verifying That One Matrix Is the Inverse of Another Matrix

Show that \( \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \) is the inverse of \( \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \).

SOLUTION

Since

\[
\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

and

\[
\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

the required condition is met.

NOW WORK PROBLEM 5.
The definition of an inverse matrix requires that the matrix is square. What about nonsquare matrices? Can they have inverses? The answer is “No.” By definition, whenever a matrix has an inverse, it will commute with its inverse under multiplication. So if the nonsquare matrix \( A \) had the alleged inverse \( B \), then \( AB \) would have to be equal to \( BA \). But the fact that \( A \) is not square causes \( AB \) and \( BA \) to have different dimensions and prevents them from being equal. So such a \( B \) could not exist.

A nonsquare matrix has no inverse.

**EXAMPLE 2** Finding the Inverse of a Matrix

Find the inverse of the matrix

\[
A = \begin{bmatrix}
4 & 2 \\
3 & 1 \\
\end{bmatrix}
\]

**SOLUTION** Assuming \( A \) has an inverse, denote it by

\[
X = \begin{bmatrix}
x_1 & x_2 \\
x_3 & x_4 \\
\end{bmatrix}
\]

Then the product of \( A \) and \( X \) is the identity matrix of dimension \( 2 \times 2 \). That is,

\[
AX = I_2
\]

\[
\begin{bmatrix}
4 & 2 \\
3 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 & x_2 \\
x_3 & x_4 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

Performing the multiplication on the left yields

\[
\begin{bmatrix}
4x_1 + 2x_3 & 4x_2 + 2x_4 \\
3x_1 + x_3 & 3x_2 + x_4 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

This matrix equation can be written as the following system of four equations containing four variables:

\[
\begin{align*}
4x_1 + 2x_3 &= 1 \\
4x_2 + 2x_4 &= 0 \\
3x_1 + x_3 &= 0 \\
3x_2 + x_4 &= 1
\end{align*}
\]

We find the solution to be

\[
x_1 = \frac{1}{2} \quad x_2 = 1 \quad x_3 = \frac{3}{2} \quad x_4 = -2
\]

The inverse of \( A \) is

\[
A^{-1} = \begin{bmatrix}
-\frac{1}{2} & 1 \\
\frac{3}{2} & -2 \\
\end{bmatrix}
\]
3.3 The Inverse of a Matrix

Let’s look at this example more closely. The system of four equations containing four variables can be written as two systems:

\[(a) \begin{cases} 4x_1 + 2x_3 = 1 \\ 3x_1 + x_3 = 0 \end{cases} \quad (b) \begin{cases} 4x_2 + 2x_4 = 0 \\ 3x_2 + x_4 = 1 \end{cases}\]

Their augmented matrices are

\[(a) \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 4 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}\]

Since the matrix \(A\) appears in both (a) and (b), any row operation performed on (a) and (b) can be performed on the single augmented matrix that combines the two right-hand columns. Denote this matrix by \([A|I_2]\) and write

\[ [A|I_2] = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \]

Now perform the row operations on \([A|I_2]\) needed to obtain the reduced row echelon form of \(A\). The result is

\[ \begin{bmatrix} 1 & 0 & \frac{-1}{2} & 1 \\ 0 & 1 & \frac{3}{2} & -2 \end{bmatrix} \]

The \(2 \times 2\) matrix on the right-hand side of the vertical bar is \(A^{-1}\).

**Steps for Finding the Inverse of a Matrix of Dimension \(n \times n\)**

**STEP 1** Form the matrix \([A|I_n]\).

**STEP 2** Using row operations, write \([A|I_n]\) in reduced row echelon form.

**STEP 3** If the resulting matrix is of the form \([I_n|B]\), that is, if the identity matrix appears on the left side of the bar, then \(B\) is the inverse of \(A\). Otherwise, \(A\) has no inverse.

**EXAMPLE 3 Finding the Inverse of a Matrix**

Find the inverse of

\[ A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \]
**SOLUTION**

**STEP 1** Since \( A \) is of dimension \( 3 \times 3 \), use the identity matrix \( I_3 \). The matrix \([A|I_3]\) is

\[
\begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 1 & 0 \\
1 & 2 & 2 & 0 & 0 & 1
\end{bmatrix}
\]

**STEP 2** Proceed to obtain the reduced row echelon form of this matrix:

Use \( R_2 = -2r_1 + r_2 \) to obtain

\[
\begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 \\
0 & -1 & -4 & -2 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1
\end{bmatrix}
\]

Use \( R_2 = -1r_2 \) to obtain

\[
\begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 \\
0 & 1 & 4 & 2 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1
\end{bmatrix}
\]

Use \( R_1 = -1r_2 + r_1 \) to obtain

\[
\begin{bmatrix}
1 & 0 & -2 & -1 & 1 & 0 \\
0 & 1 & 4 & 2 & -1 & 0 \\
0 & 0 & -4 & -3 & 1 & 1
\end{bmatrix}
\]

Use \( R_3 = -\frac{1}{4}r_3 \) to obtain

\[
\begin{bmatrix}
1 & 0 & -2 & -1 & 1 & 0 \\
0 & 1 & 4 & 2 & -1 & 0 \\
0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}
\end{bmatrix}
\]

Use \( R_1 = 2r_3 + r_1 \) to obtain

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}
\end{bmatrix}
\]

The matrix \([A|I_3]\) is in reduced row echelon form.

**STEP 3** Since the identity matrix \( I_3 \) appears on the left side, the matrix appearing on the right is the inverse. That is,

\[A^{-1} = \begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-1 & 0 & 1 \\
\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}
\end{bmatrix}\]

You should verify that \( AA^{-1} = A^{-1}A = I_3 \).

**NOW WORK PROBLEM 19.**
EXAMPLE 4 Finding the Inverse of a Matrix Using a Graphing Utility

Rework Example 3 using a graphing utility.

**SOLUTION**

Enter the matrix $A$ into a graphing utility and use the $x^{-1}$ key to obtain $[A]^{-1}$. Figure 5 shows $A^{-1}$.

![Figure 5](image)

NOW WORK PROBLEM 51 USING A GRAPHING UTILITY.

EXAMPLE 5 Finding the Inverse of a Matrix Using Excel

Rework Example 3 using EXCEL.

**SOLUTION**

**STEP 1** Enter the matrix $A$.

**STEP 2** Highlight the cells that will contain $A^{-1}$; $A^{-1}$ is a $3 \times 3$ matrix.

**STEP 3** Type: = MINVERSE (highlight matrix $A$). The Excel screen should look like the one below.

![Excel Screen](image)

**STEP 4** Press Ctrl-Shift-Enter all at the same time. The inverse will be displayed in the highlighted cells, as in the screen below.

![Excel Screen](image)
CHECK: Multiply \( A^*A^{-1} \) using the steps below. This product should be the identity matrix.

1. Highlight the cells that are to contain the product matrix.
2. Type: = MMULT (highlight \( A \), highlight \( A^{-1} \))
3. Press Ctrl-Shift-Enter at the same time.

![Image of Excel spreadsheet]

Notice that \( a_{13} = 1.11E-16 \) is in scientific notation. In decimal notation it is 0.000000000000000111, very close to zero. So, \( A^*A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), as required.

NOW WORK PROBLEM 51 USING EXCEL.

EXAMPLE 6 Showing That a Matrix Has No Inverse

Show that the matrix given below has no inverse.

\[
\begin{bmatrix}
3 & 2 \\
6 & 4
\end{bmatrix}
\]

**SOLUTION** Set up the matrix

\[
\begin{bmatrix}
3 & 2 & 1 & 0 \\
6 & 4 & 0 & 1
\end{bmatrix}
\]

Use \( R_1 = \frac{1}{3}r_1 \) to obtain

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
\frac{2}{3} & \frac{3}{3} & 0 & 1 \\
6 & 4 & 0 & 1
\end{bmatrix}
\]

Use \( R_2 = -6r_1 + r_2 \) to obtain

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
\frac{2}{3} & \frac{3}{3} & 0 & 1 \\
0 & 0 & -2 & 1
\end{bmatrix}
\]

The 0s in row 2 tell us we cannot get the identity matrix. This, in turn, tells us the original matrix has no inverse.
COMMENT If a matrix has no inverse, a graphing utility will display an Error message. See Figure 6 for the result of doing Example 6.

FIGURE 6

NOW WORK PROBLEM 25.

2 Use the Inverse of a Matrix to Solve a System of 
\( n \) Linear Equations Containing \( n \) Variables

The inverse of a matrix can be used to solve a system of \( n \) linear equations containing \( n \) variables. We begin with a system of three linear equations containing three variables:

\[
\begin{align*}
    x + y + 2z &= 1 \\
    2x + y &= 2 \\
    x + 2y + 2z &= 3
\end{align*}
\]  \hspace{1cm} (1)

Let

\[
A = \begin{bmatrix}
    1 & 1 & 2 \\
    2 & 1 & 0 \\
    1 & 2 & 2
\end{bmatrix}, \quad X = \begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}, \quad B = \begin{bmatrix}
    1 \\
    2 \\
    3
\end{bmatrix}
\]

Then, since

\[
AX = \begin{bmatrix}
    1 & 1 & 2 \\
    2 & 1 & 0 \\
    1 & 2 & 2
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix} = \begin{bmatrix}
    x + y + 2z \\
    2x + y \\
    x + 2y + 2z
\end{bmatrix}
\]

the system of equations (1) can be written compactly as the matrix equation

\[
AX = B
\]

In general, any system of \( n \) linear equations containing \( n \) variables \( x_1, x_2, \ldots, x_n \), can be written in the form

\[
AX = B
\]

where \( A \) is the \( n \times n \) matrix of the coefficients of the variables, \( B \) is an \( n \times 1 \) column matrix whose entries are the numbers appearing to the right of each equal sign in the system, and \( X \) is an \( n \times 1 \) column matrix containing the \( n \) variables.

To find \( X \), start with the matrix equation \( AX = B \) and use properties of matrices. Assume that the \( n \times n \) matrix \( A \) has an inverse \( A^{-1} \).

\[
A^{-1}(AX) = A^{-1}B \quad \text{Multiply both sides by} \ A^{-1}. \\
(A^{-1}A)X = A^{-1}B \quad \text{Apply the Associative Property on the left side.} \\
I_nX = A^{-1}B \quad \text{Apply the Inverse Property:} \ A^{-1}A = I_n. \\
X = A^{-1}B \quad \text{Apply the Identity Property:} \ I_nX = X.
\]

This leads to the following result:
Chapter 3  Matrices

**Theorem**

A system of $n$ linear equations containing $n$ variables

\[ AX = B \]

for which $A$ is a square matrix and $A^{-1}$ exists, always has a unique solution that is given by

\[ X = A^{-1}B \]

**EXAMPLE 7 Solving a System of Equations Using Inverses**

Solve the system of equations:

\[
\begin{align*}
  x + y + 2z &= 1 \\
  2x + y &= 2 \\
  x + 2y + 2z &= 3
\end{align*}
\]

**SOLUTION**

Here

\[
A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\]

From Example 5 we know the inverse of $A$ is

\[
A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}
\]

Based on the result just stated, the solution $X$ of the system is

\[
X = A^{-1}B
\]

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -\frac{1}{2} \end{bmatrix}
\]

The solution is $x = 0, y = 2, z = -\frac{1}{2}$.

**NOW WORK PROBLEMS 39 AND 45.**

The method used to solve the system in Example 7 requires that $A$ have an inverse. If, for a system of equations $AX = B$, the matrix $A$ has no inverse, then the system must be analyzed using the methods discussed in Sections 2.2 and 2.3.
The method used in Example 7 for solving a system of equations is particularly useful for applications in which the constants appearing to the right of the equal sign change while the coefficients of the variables on the left side do not. See Problems 39–50 for an illustration. See also the discussion of Leontief models in Section 3.4.

**USING TECHNOLOGY**

**EXAMPLE 8 Solving a System of Equations Using EXCEL**

Rework Example 7 using EXCEL.

**SOLUTION** From Example 5, we know the inverse of \( A \) is

\[
A^{-1} = \begin{bmatrix}
0.5 & 0.5 & -0.5 \\
-1 & 0 & 1 \\
0.75 & -0.25 & -0.25
\end{bmatrix}.
\]

To find the solution to the system of equations, multiply \( A^{-1}B \), where \( B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \).

**STEP 1** Highlight the cells that are to contain the product matrix. This will be a 3 × 1 matrix.

**STEP 2** Type: \( \text{MMULT (highlight } A^{-1}, \text{highlight } B) \)

**STEP 3** Press Ctrl-Shift-Enter at the same time, to get \( A^{-1}B \).

So, \( x = 0 \), \( y = 2 \), and \( z = -0.5 \).

Now work Problems 39 and 45 using Excel.

**EXAMPLE 9 Mutual Fund Investing**

The average annual return (over the 5-year period prior to January 12, 2010) of three mutual funds offered by Franklin Templeton is shown in the table:

<table>
<thead>
<tr>
<th>Mutual Fund</th>
<th>Average Annual Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Income</td>
<td>13.82</td>
</tr>
<tr>
<td>Conservative Target</td>
<td>5.36</td>
</tr>
<tr>
<td>Mutual Shares</td>
<td>4.05</td>
</tr>
</tbody>
</table>

*Source: Franklin Templeton*
Suppose you have $6000 to invest in these three funds. You will invest $200 more in the High Income fund than you will in the Conservative Target fund.

(a) Assuming the accounts will earn the annual rates shown, how much should you invest for a year in each fund if you want your average return to be 7%?

(b) How much should you invest in each fund if you want your average return to be 8%?

(c) How much if you want 9%?

(d) Comment on the results found in (a), (b), and (c).

**SOLUTION**

Begin by naming the variables. Since parts (a), (b), (c) are repetitive, let $i$ be the annual rate of return (in decimal form) we want. Let $x$ represent the amount invested in the High Income fund, $y$ represent the amount invested in the Conservative Target fund, and $z$ represent the amount invested in the Mutual Shares fund. The amount earned on the investment of $6000 for one year is $6000i$. We have the following system of equations:

\[
\begin{align*}
&x + y + z = 6000 \\
&x - y = 200 \\
&0.1382x + 0.0536y + 0.0978z = 6000i
\end{align*}
\]

If

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 0 \\
0.1382 & 0.0536 & 0.0978
\end{bmatrix}
\]

the system of equations is given by

\[
AX = B
\]

The solution of the system is

\[
X = A^{-1}B
\]

Proceed to find $A^{-1}$ using a graphing utility or EXCEL:

\[
A^{-1} = \begin{bmatrix}
-0.365 & 0.12 & 9.025 \\
-0.365 & -0.88 & 9.025 \\
1.73 & 0.76 & -18.05
\end{bmatrix}
\]

Then

\[
X = A^{-1}B = \begin{bmatrix}
-0.365 & 0.12 & 9.025 \\
-0.365 & -0.88 & 9.025 \\
1.73 & 0.76 & -18.05
\end{bmatrix}
\begin{bmatrix}
6000 \\
200 \\
6000(0.07)
\end{bmatrix}
\]

(a) If the average annual rate of return is to be $i = 0.07$, then

\[
B = \begin{bmatrix}
6000 \\
200 \\
6000(0.07)
\end{bmatrix} = \begin{bmatrix}
6000 \\
200 \\
420
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
-0.365 & 0.12 & 9.025 \\
-0.365 & -0.88 & 9.025 \\
1.73 & 0.76 & -18.05
\end{bmatrix} \begin{bmatrix}
6000 \\
200 \\
420
\end{bmatrix} \approx \begin{bmatrix}
1624.50 \\
1424.50 \\
2951.00
\end{bmatrix}
\]
(b) If the average annual rate of return is to be \( i = 0.08 \), then
\[
B = \begin{bmatrix}
6000 \\
200 \\
480
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
-0.365 & 0.12 & 9.025 \\
-0.365 & -0.88 & 9.025 \\
1.73 & 0.76 & -18.05
\end{bmatrix}
\begin{bmatrix}
6000 \\
200 \\
480
\end{bmatrix} \approx \begin{bmatrix}
2166.00 \\
1966.00 \\
1868.00
\end{bmatrix}
\]

(c) If the average annual rate of return is to be \( i = 0.09 \), then
\[
B = \begin{bmatrix}
6000 \\
200 \\
540
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
-0.365 & 0.12 & 9.025 \\
-0.365 & -0.88 & 9.025 \\
1.73 & 0.76 & -18.05
\end{bmatrix}
\begin{bmatrix}
6000 \\
200 \\
540
\end{bmatrix} \approx \begin{bmatrix}
2707.50 \\
2507.50 \\
785.00
\end{bmatrix}
\]

The amount to invest in each fund to achieve the three rates of return is summarized in the following table.

<table>
<thead>
<tr>
<th>Expected Rate of Return</th>
<th>7%</th>
<th>8%</th>
<th>9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Income</td>
<td>1624.50</td>
<td>2166.00</td>
<td>2707.50</td>
</tr>
<tr>
<td>Conservative Target</td>
<td>1424.50</td>
<td>1966.00</td>
<td>2507.50</td>
</tr>
<tr>
<td>Mutual Shares</td>
<td>2951.00</td>
<td>1868.00</td>
<td>785.00</td>
</tr>
</tbody>
</table>

(d) Notice that to achieve higher rates of return, you need to invest less and less in the Mutual Shares fund and more in the other two funds. Setting your expected rate of return at 7% allows for a more level diversification among the three funds.

\[ \text{NOW WORK PROBLEM 63.} \]

4 Application: Use Matrices in Cryptography

Cryptography is the art of writing or deciphering secret codes. We begin by giving examples of elementary codes.

**EXAMPLE 10** Encoding a Message

A message can be encoded by associating each letter of the alphabet with some other letter of the alphabet according to a prescribed pattern. For example, we might have

\[
\begin{align*}
A & \rightarrow B \\
B & \rightarrow C \\
C & \rightarrow D \\
D & \rightarrow E \\
E & \rightarrow F \\
F & \rightarrow G \\
G & \rightarrow H \\
H & \rightarrow I \\
I & \rightarrow J \\
J & \rightarrow K \\
K & \rightarrow L \\
L & \rightarrow M \\
M & \rightarrow N \\
N & \rightarrow O \\
O & \rightarrow P \\
P & \rightarrow Q \\
Q & \rightarrow R \\
R & \rightarrow S \\
S & \rightarrow T \\
T & \rightarrow U \\
U & \rightarrow V \\
V & \rightarrow W \\
W & \rightarrow X \\
X & \rightarrow Y \\
Y & \rightarrow Z \\
Z & \rightarrow A \\
A & \rightarrow B \\
B & \rightarrow C \\
C & \rightarrow D \\
D & \rightarrow E \\
E & \rightarrow F \\
F & \rightarrow G \\
G & \rightarrow H \\
H & \rightarrow I \\
I & \rightarrow J \\
J & \rightarrow K \\
K & \rightarrow L \\
L & \rightarrow M \\
M & \rightarrow N \\
N & \rightarrow O \\
O & \rightarrow P \\
P & \rightarrow Q \\
Q & \rightarrow R \\
R & \rightarrow S \\
S & \rightarrow T \\
T & \rightarrow U \\
U & \rightarrow V \\
V & \rightarrow W \\
W & \rightarrow X \\
X & \rightarrow Y \\
Y & \rightarrow Z \\
Z & \rightarrow A \\
\end{align*}
\]

With the above code the word MESSAGE would become OGUUCIG.

**EXAMPLE 11** Encoding a Message

Another code may associate numbers with the letters of the alphabet. For example, we might have

\[
\begin{align*}
A & \rightarrow 1 \\
B & \rightarrow 2 \\
C & \rightarrow 3 \\
D & \rightarrow 4 \\
E & \rightarrow 5 \\
F & \rightarrow 6 \\
G & \rightarrow 7 \\
H & \rightarrow 8 \\
I & \rightarrow 9 \\
J & \rightarrow 10 \\
K & \rightarrow 11 \\
L & \rightarrow 12 \\
M & \rightarrow 13 \\
N & \rightarrow 14 \\
O & \rightarrow 15 \\
P & \rightarrow 16 \\
Q & \rightarrow 17 \\
R & \rightarrow 18 \\
S & \rightarrow 19 \\
T & \rightarrow 20 \\
U & \rightarrow 21 \\
V & \rightarrow 22 \\
W & \rightarrow 23 \\
X & \rightarrow 24 \\
Y & \rightarrow 25 \\
Z & \rightarrow 26 \\
\end{align*}
\]

In this code the word PEACE looks like 11 22 26 24 22.
Both of the above codes have one important feature in common. The association of letters with the coding symbols is made using a one-to-one correspondence so that no possible ambiguities can arise.

Suppose we want to encode the following message:

TOP SECURITY CLEARANCE

If we decide to divide the message into pairs of letters, the message becomes

TO PS EC UR IT YC LE AR AN CE

(If there is a letter left over, arbitrarily assign Z to the last position.) Using the correspondence of letters to numbers given in Example 12, and writing each pair of letters as a column vector, we obtain

\[
\begin{bmatrix}
T \\
O
\end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}, \quad
\begin{bmatrix}
P \\
S
\end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}, \quad
\begin{bmatrix}
E \\
C
\end{bmatrix} = \begin{bmatrix} 22 \\ 24 \end{bmatrix}, \quad
\begin{bmatrix}
U \\
R
\end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \quad
\begin{bmatrix}
I \\
T
\end{bmatrix} = \begin{bmatrix} 18 \\ 7 \end{bmatrix},
\]

\[
\begin{bmatrix}
Y \\
C
\end{bmatrix} = \begin{bmatrix} 2 \\ 24 \end{bmatrix}, \quad
\begin{bmatrix}
L \\
E
\end{bmatrix} = \begin{bmatrix} 15 \\ 22 \end{bmatrix}, \quad
\begin{bmatrix}
A \\
R
\end{bmatrix} = \begin{bmatrix} 26 \\ 9 \end{bmatrix}, \quad
\begin{bmatrix}
A \\
N
\end{bmatrix} = \begin{bmatrix} 26 \\ 13 \end{bmatrix}, \quad
\begin{bmatrix}
C \\
E
\end{bmatrix} = \begin{bmatrix} 24 \\ 22 \end{bmatrix}
\]

Next, arbitrarily choose a $2 \times 2$ matrix $A$, which has an inverse $A^{-1}$ (we'll see why later). Let's choose

\[
A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}
\]

The inverse $A^{-1}$ of $A$ is

\[
A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}
\]

Now transform the column vectors representing the message by multiplying each of them on the left by matrix $A$:

\[
A \begin{bmatrix}
T \\
O
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 50 \\ 31 \end{bmatrix},
\]

\[
A \begin{bmatrix}
P \\
S
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ 8 \end{bmatrix} = \begin{bmatrix} 46 \\ 27 \end{bmatrix},
\]

\[
A \begin{bmatrix}
E \\
C
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 22 \\ 24 \end{bmatrix} = \begin{bmatrix} 116 \\ 70 \end{bmatrix},
\]

\[
A \begin{bmatrix}
U \\
R
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 39 \\ 24 \end{bmatrix},
\]

\[
A \begin{bmatrix}
I \\
T
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 18 \\ 7 \end{bmatrix} = \begin{bmatrix} 57 \\ 32 \end{bmatrix},
\]

\[
A \begin{bmatrix}
Y \\
C
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 24 \end{bmatrix} = \begin{bmatrix} 76 \\ 50 \end{bmatrix},
\]

\[
A \begin{bmatrix}
L \\
E
\end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 22 \end{bmatrix} = \begin{bmatrix} 96 \\ 59 \end{bmatrix}
\]
The coded message is

\[ \begin{bmatrix} 50 & 31 & 46 & 27 & 116 & 70 & 39 & 24 & 57 & 32 & 76 & 50 & 96 & 59 & 79 & 44 & 91 & 52 & 114 & 68 \end{bmatrix} \]

To decode or unscramble the above message, pair the numbers in \( 2 \times 1 \) column vectors. Then on the left multiply each column vector by \( A^{-1} \). For example, the first two column vectors then become

\[ A^{-1} \begin{bmatrix} 50 \\ 31 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 50 \\ 31 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} T \\ O \end{bmatrix} \]

\[ A^{-1} \begin{bmatrix} 46 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 46 \\ 27 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix} = \begin{bmatrix} P \\ S \end{bmatrix} \text{ etc.} \]

Continuing in this way, the original message is obtained.

NOW WORK PROBLEM 67.

**EXAMPLE 12 Encoding a Message**

The message to be encoded is

SEPTEMBER IS OKAY

The encoded message is to be formed using triplets of numbers and the \( 3 \times 3 \) matrix

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 5 \\ -2 & 0 & 1 \end{bmatrix} \text{ whose inverse is } A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -13 & 1 & -5 \\ 2 & 0 & 1 \end{bmatrix}. \]

Use the following association of letters to numbers:

| A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |

**SOLUTION** Divide the message into triplets of letters as follows:

SEP TEM BER ISO KAY

Now multiply the matrix \( A \) times each column vector of the message.

[Note: If additional letters had been required to complete a triplet, we would have used Z or YZ.]
The coded message is

\[
\begin{bmatrix}
\end{bmatrix}
\]

To decode the message in Example 12, form column vectors of the numbers in the coded message and multiply on the left by \(A^{-1}\).

The above are elementary examples of encoding and decoding. Modern-day cryptography uses sophisticated computer-implemented codes that depend on higher-level mathematics.

**EXERCISE 3.3 Answers Begin on Page AN–16.**

**'Are You Prepared?' Problems** Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. **True or False** For every nonzero number \(a\), there is a nonzero number \(b\) so that \(ab = 1\). (pp. A–8 to A–12)

2. The number 1 is called the ______ ____. (pp. A–8 to A–12)

**Concepts and Vocabulary**

3. If \(A\) and \(B\) are matrices of dimension \(n \times n\) and if \(AB = BA = I_n\), the identity matrix, then \(B\) is called the ______ of \(A\).

4. **True or False** Every square matrix has an inverse.

**Skill Building**

In Problems 5–10, show that the given matrices are inverses of each other.

5. \[
\begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix}
\begin{bmatrix}
-3 & 2 \\
2 & -1
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & 5 \\
2 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
-1 & -2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
-3 & -2
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{7} & \frac{3}{7} \\
\frac{2}{7} & -\frac{1}{7}
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
-\frac{5}{2} & 2 & -\frac{1}{2} \\
\frac{2}{2} & 1 & 1 \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
1 & 3 & 3 \\
1 & 4 & 3 \\
1 & 3 & 4
\end{bmatrix}
\begin{bmatrix}
7 & -3 & -3 \\
-1 & 1 & 1 \\
-1 & 0 & 1
\end{bmatrix}
\]
3.3 The Inverse of a Matrix

In Problems 11–24, find the inverse of each matrix using the reduced row echelon technique.

11. \[
\begin{bmatrix}
3 & 7 \\
2 & 5
\end{bmatrix}
\]
12. \[
\begin{bmatrix}
4 & 1 \\
3 & 1
\end{bmatrix}
\]
13. \[
\begin{bmatrix}
1 & -1 \\
3 & -4
\end{bmatrix}
\]
14. \[
\begin{bmatrix}
5 & 3 \\
3 & 2
\end{bmatrix}
\]
15. \[
\begin{bmatrix}
2 & 1 \\
4 & 3
\end{bmatrix}
\]
16. \[
\begin{bmatrix}
2 & 3 \\
2 & -2
\end{bmatrix}
\]
17. \[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]
18. \[
\begin{bmatrix}
-1 & 1 & 0 \\
1 & 0 & 2 \\
3 & 1 & 0
\end{bmatrix}
\]
19. \[
\begin{bmatrix}
1 & 1 & -1 \\
3 & -1 & 0 \\
2 & -3 & 4
\end{bmatrix}
\]
20. \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]
21. \[
\begin{bmatrix}
1 & 1 & -1 \\
2 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]
22. \[
\begin{bmatrix}
2 & 3 & -1 \\
1 & 1 & 1 \\
0 & 2 & -1
\end{bmatrix}
\]
23. \[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
0 & 1 & 0 & -1
\end{bmatrix}
\]
24. \[
\begin{bmatrix}
1 & 2 & -3 & -2 \\
0 & 1 & 4 & -2 \\
3 & -1 & 4 & 0 \\
2 & 1 & 0 & 3
\end{bmatrix}
\]

In Problems 25–30, show that each matrix has no inverse.

25. \[
\begin{bmatrix}
4 & 6 \\
2 & 3
\end{bmatrix}
\]
26. \[
\begin{bmatrix}
-1 & 2 \\
3 & -6
\end{bmatrix}
\]
27. \[
\begin{bmatrix}
-8 & 4 \\
-4 & 2
\end{bmatrix}
\]
28. \[
\begin{bmatrix}
2 & 10 \\
1 & 5
\end{bmatrix}
\]
29. \[
\begin{bmatrix}
1 & 1 & 1 \\
3 & -4 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]
30. \[
\begin{bmatrix}
-1 & 2 & 3 \\
5 & 2 & 0 \\
2 & -4 & -6
\end{bmatrix}
\]

In Problems 31–34, find the inverse, if it exists, of each matrix.

31. \[
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}
\]
32. \[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]
33. \[
\begin{bmatrix}
3 & 2 \\
6 & 4
\end{bmatrix}
\]
34. \[
\begin{bmatrix}
4 & 2 \\
2 & 1
\end{bmatrix}
\]

35. For the matrices
\[
A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}
\]
find \( A^{-1} - B^{-1} \).

36. For the matrices
\[
A = \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}
\]
find \( A^{-1} - B^{-1} \).

37. Write the matrix product \( A^{-1}B \) used to find the solution to the system \( AX = B \).
\[
\begin{align*}
x + 3y + 2z &= 2 \\
2x + 7y + 3z &= 1 \\
x + 6z &= 3
\end{align*}
\]

38. Write the matrix product \( A^{-1}B \) used to find the solution to the system \( AX = B \).
\[
\begin{align*}
x + 2y + 2z &= 3 \\
2x + 5y + 7z &= 2 \\
2x + y - 4z &= 4
\end{align*}
\]

In Problems 39–50, solve each system of equations by the method of Example 7. For Problems 39–44, use the inverse found in Problem 11. For Problems 45–50, use the inverse found in Problem 19.

39. \[
\begin{align*}
3x + 7y &= 10 \\
2x + 5y &= 2
\end{align*}
\]
40. \[
\begin{align*}
3x + 7y &= -4 \\
2x + 5y &= -3
\end{align*}
\]
41. \[
\begin{align*}
3x + 7y &= 13 \\
2x + 5y &= 9
\end{align*}
\]
42. \[
\begin{align*}
3x + 7y &= 0 \\
2x + 5y &= 14
\end{align*}
\]
43. \[
\begin{align*}
3x + 7y &= 12 \\
2x + 5y &= -4
\end{align*}
\]
44. \[
\begin{align*}
3x + 7y &= -2 \\
2x + 5y &= 10
\end{align*}
\]
45. \[
\begin{align*}
x + y - z &= 3 \\
3x - y &= -4 \\
2x - 3y + 4z &= 6
\end{align*}
\]
46. \[
\begin{align*}
x + y - z &= 6 \\
3x - y &= 8 \\
2x - 3y + 4z &= -3
\end{align*}
\]
47. \[
\begin{align*}
x + y - z &= 12 \\
3x - y &= -4 \\
2x - 3y + 4z &= 16
\end{align*}
\]
48. \[
\begin{align*}
x + y - z &= -8 \\
x - y &= 12 \\
2x - 3y + 4z &= -2
\end{align*}
\]
49. \[
\begin{align*}
x + y - z &= 0 \\
x - y &= -8 \\
2x - 3y + 4z &= -6
\end{align*}
\]
50. \[
\begin{align*}
x + y - z &= 21 \\
x - y &= 12 \\
2x - 3y + 4z &= 14
\end{align*}
\]
150  Chapter 3  Matrices

In Problems 51–56, use a graphing utility or EXCEL to find the inverse, if it exists, of each matrix.

51.  
52.  
53.  

In Problems 57–60, use the idea behind Example 7 and either a graphing utility or EXCEL to solve each system of equations.

57.  
58.  
59.  
60.  

Applications and Extensions

61.  Investing  The average annual return (over the 5-year period prior to December 31, 2009) of two mutual funds is shown in the table:

<table>
<thead>
<tr>
<th>Mutual Fund</th>
<th>Average Annual Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mid Cap Growth Fund</td>
<td>4.57%</td>
</tr>
<tr>
<td>Bond Fund</td>
<td>6.39%</td>
</tr>
</tbody>
</table>

Source: Vanguard

You have $3000 to invest in these two funds. Assuming the accounts will continue to earn the annual rates shown, how much should you invest for a year in each fund if you want your average return to be
(a) 5%?  (b) 5.5%?  (c) 6%?

62.  Theater Seating  The University of Wisconsin–Milwaukee’s Main Stage at the Peck School of the Arts can seat 500 people. For an upcoming production of Our Town, the price of admission for students is $10 per ticket and admission for the general public is $15 per ticket. If the theater sells out, how many of each type of ticket must be sold if the total box office receipts are to be
(a) $6000?  (b) $6500?  (c) $7000?

Source: www4.uwm.edu/psoa

63.  Investing  The average annual return (over the 5-year period prior to December 31, 2009) of three mutual funds is shown in the table:

<table>
<thead>
<tr>
<th>Mutual Fund</th>
<th>Average Annual Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small Cap Growth Fund</td>
<td>2.56%</td>
</tr>
<tr>
<td>Large Cap Value Fund</td>
<td>4.79%</td>
</tr>
<tr>
<td>International Fund</td>
<td>8.73%</td>
</tr>
</tbody>
</table>

Source: Vanguard.

Suppose you have $8000 to invest in these three funds. You will invest $1000 more in the Small Cap Growth fund than you will in the Large Cap Value fund. Assuming the accounts will continue to earn the annual rates shown, how much should you invest for a year in each fund if you want your average return to be
(a) 4%?  (b) 4.5%?  (c) 5%?

64.  Annual Bonuses  A small company has three executives, a president, a chief financial officer (CFO), and a vice president, each of whom it rewards with annual bonuses. The bonus each executive receives is calculated as a percentage of the profits that remain after the bonuses of all three executives have been deducted from the profits. The president receives 5%, the CFO receives 4%, and the vice president receives 2.5%. What will be the bonus amount for each executive if the profits are
(a) $200,000?  (b) $250,000?  (c) $300,000?

65.  Annual Bonuses Revisited  Refer to Problem 64. Because business has been good, the company’s board of directors decided to increase the bonuses for the three executives as follows: The president receives 8%, the CFO receives 5%, and the vice president receives 4%. What will be the bonus amount for each executive if the profits are
(a) $300,000?  (b) $500,000?  (c) $750,000?
In Problems 66–71, use the correspondence

| A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |

66. The matrix

\[
A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}
\]

was used to code the following messages:
(a) 64 36 75 47 75 47 49 29 60 36 60 37 53 33 71 39 35 18
(b) 76 49 64 41 95 55 43 24 69 42 59 32 77 45 27 15 37 21 128 77

Use the inverse to decode each message.

67. Use the inverse of matrix

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 5 \\ -2 & 0 & 1 \end{bmatrix}
\]

to decode the message

4 152 17 15 106 -22 1 50 3 1 52 7 19 195 -12

68. The matrix \( A \) used to encode a message has as its inverse the matrix

\[
A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}
\]

Decode the message

70 39 62 41 113 69 93 57 51 28 29 15 54 33 14 9 62 40 67 43 116 71

69. Use the matrix

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 5 \\ -2 & 0 & 1 \end{bmatrix}
\]

to encode the message: I HAVE TICKETS TO THE FINAL FOUR.

70. Use the matrix \( A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \) to encode each of the following messages:
(a) I AM GOING TO DISNEY
(b) WE SURF THE NET
(c) LETS GO CLUBBING

71. Use the matrix

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 5 \\ -2 & 0 & 1 \end{bmatrix}
\]

to encode the messages given in Problem 70.

In Problems 75–78, use the result of Problem 74 to find the inverse of each matrix.

75. \[
\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}
\]

76. \[
\begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix}
\]

77. \[
\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}
\]

78. \[
\begin{bmatrix} 1 & 2 \\ 8 & 15 \end{bmatrix}
\]

Discussion and Writing

79. Refer to Example 9. Discuss the risk/reward of trying to achieve an 11% rate of return.

‘Are You Prepared?’ Answers

1. True
2. Multiplicative identity

72. Pin Codes
A four-digit pin code is required for a bank’s ATM card. For security reasons, the bank does not store a card’s pin code with the bank account. An encrypted code is stored with the account instead. The code is placed within a matrix \( C \).

For example, the pin code 1234 forms the code matrix

\[
C = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}
\]

The code matrix \( C \) is then multiplied by a key matrix \( K \) to obtain the encrypted matrix \( E \). That is,

\[
E = C \cdot K
\]

To determine an original pin code, the bank uses

(a) If the bank uses the key matrix known as the Fibonacci Q-matrix, encrypt the pin code 7, 3, 4, 1.

(b) Determine the pin code if the encrypted code is 2, 0, 13, 4.

Source: goldenmuseum.com

73. Student IDs
A university assigns a unique 12-digit identification number to each of its students upon admission. For security, the university encrypts the ID by placing the digits into a matrix \( D \) and multiplying by a key matrix \( K \) obtaining the encrypted matrix \( E \). That is,

\[
E = D \cdot K
\]

To determine an original ID, the university uses

(a) If the university uses the key matrix known as the Tribonacci matrix, encrypt the following student ID number: 8, 3, 0, 1, 9, 6, 8, 7, 0, 2, 4, 9.

(b) Determine the original ID number if the encrypted ID is 6, 14, 10, 2, 6, 2, 6, 7, 15, 9, 14, 17.

Source: http://webpages.charter.net/jtlien/padovan.powers

74. Show that the inverse of \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is given by the formula

\[
A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

where \( \Delta = ad - bc \neq 0 \).

The number \( \Delta \) is called the determinant of \( A \).
3.4 Applications in Economics (the Leontief Model), Accounting, and Statistics (the Method of Least Squares)*

Application in Economics: The Leontief Model

The Leontief models in economics are named after Wassily Leontief, who received the Nobel Prize in economics in 1973. These models can be characterized as a description of an economy in which input equals output or, in other words, consumption equals production. That is, the models assume that whatever is produced is always consumed.

Leontief models are of two types: closed, in which the entire production is consumed by those participating in the production; and open, in which some of the production is consumed by those who produce it and the rest of the production is consumed by external bodies.

In the closed model we seek the relative income of each participant in the system. In the open model we seek the amount of production needed to achieve a forecast demand when the amount of production needed to achieve current demand is known.

The Closed Model

EXAMPLE 1 Determining Relative Income Using a Closed Leontief Model

Three homeowners, Juan, Luis, and Carlos, each with certain skills, agreed to pool their talents to make repairs on their houses. As it turned out, Juan spent 20% of his time on his own house, 40% of his time on Luis’s house, and 40% on Carlos’s house. Luis spent 10% of his time on Juan’s house, 50% of his time on his own house, and 40% on Carlos’s house. Of Carlos’s time, 60% was spent on Juan’s house, 10% on Luis’s, and 30% on his own. Now that the projects are finished, they need to figure out how much money each should get for his work, including the work performed on his own house, so that the amount paid by each person equals the amount received by each one. They agreed in advance that the payment to each one should be approximately $3000.00.

SOLUTION

Place the information given in the problem in a $3 \times 3$ matrix, as follows:

\[
\begin{bmatrix}
0.2 & 0.1 & 0.6 \\
0.4 & 0.5 & 0.1 \\
0.4 & 0.4 & 0.3
\end{bmatrix}
\]

Next, define the variables:

\[
x = \text{Juan’s wages} \\
y = \text{Luis’s wages} \\
z = \text{Carlos’s wages}
\]

* Each application is optional, and the applications given are independent of one another.
We require that the amount paid out by each one equals the amount received by each one. Let's analyze this requirement by looking just at the work done on Juan's house. Juan's wages are $x$. Juan's expenditures for work done on his house are $0.2x + 0.1y + 0.6z$. Juan's wages and expenditures are required to be equal, so

$$x = 0.2x + 0.1y + 0.6z$$

Similarly,

$$y = 0.4x + 0.5y + 0.1z$$
$$z = 0.4x + 0.4y + 0.3z$$

These three equations can be written compactly as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0.6 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Some computation reduces the system to

$$\begin{cases} 0.8x - 0.1y - 0.6z = 0 \\ -0.4x + 0.5y - 0.1z = 0 \\ -0.4x - 0.4y + 0.7z = 0 \end{cases}$$

Solving for $x, y, z$, we find that

$$x = \frac{31}{36}z \quad y = \frac{32}{36}z$$

where $z$ is the parameter. To get solutions that fall close to $3000$, we set $z = 3600$.

The wages to be paid out are therefore

$$x = 3100 \quad y = 3200 \quad z = 3600$$

The matrix

$$\begin{bmatrix} 0.2 & 0.1 & 0.6 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.4 & 0.3 \end{bmatrix}$$

in Example 1 is called an **input–output matrix**.

In the general closed model we have an economy consisting of $n$ components. Each component produces an output of some goods or services, which, in turn, is completely used up by the $n$ components. The proportionate use of each component’s

* Other choices for $z$ are, of course, possible. The choice of which value to use is up to the homeowners. No matter what choice is made, wages and expenditures are equal.
output by the economy makes up the input–output matrix of the economy. The problem is to find suitable pricing levels for each component so that income equals expenditure.

### Theorem

**Closed Leontief Model**

In general, an input–output matrix for a closed Leontief model is of the form

\[ A = [a_{ij}] \quad i, j = 1, 2, \ldots, n \]

where the \( a_{ij} \) represent the fractional amount of goods or services used by \( i \) and produced by \( j \). For a closed model the sum of each column equals 1 (this is the condition that all production is consumed internally) and \( 0 \leq a_{ij} \leq 1 \) for all entries (this is the restriction that each entry is a fraction).

If \( A \) is the input–output matrix of a closed system with \( n \) components and \( X \) is a column vector representing the price of each output of the system, then

\[ X = AX \]

represents the requirement that income equal expenditure.

For example, the first entry of the matrix equality \( X = AX \) requires that

\[ x_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \]

The right side represents the price paid by component 1 for the goods it uses, while \( x_1 \) represents the income of component 1; we are requiring they be equal.

We can rewrite the equation \( X = AX \) as

\[ X - AX = 0 \]
\[ (I_n - A)X = 0 \]

This matrix equation, which represents a system of equations in which the right-hand side is always 0, is called a **homogeneous system of equations**. It can be shown that if the entries in the input–output matrix \( A \) are positive and if the sum of each column of \( A \) equals 1, then this system has a one-parameter solution; that is, we can solve for \( n - 1 \) of the variables in terms of the remaining one, which is the parameter. This parameter serves as a “scale factor.”

**NOW WORK PROBLEM 1.**

### The Open Model

For the open model, in addition to internal consumption of goods produced, there is an outside demand for the goods produced. This outside demand may take the form of exportation of goods or may be the goods needed to support consumer demand. Again, however, we make the assumption that whatever is produced is also consumed.

For example, suppose an economy consists of three industries \( R, S, \) and \( T \), and suppose each one produces a single product. We assume that a portion of \( R \)'s production is used by each of the three industries, while the remainder is used by consumers.
The same is true of the production of $S$ and $T$. To organize our thoughts, we construct a table that describes the interaction of the use of $R$, $S$, and $T$’s production over some fixed period of time. See Table 2.

<table>
<thead>
<tr>
<th>Amounts Produced</th>
<th>Amounts Consumed</th>
<th>R</th>
<th>S</th>
<th>T</th>
<th>Consumer</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td></td>
<td>50</td>
<td>20</td>
<td>40</td>
<td>70</td>
<td>180</td>
</tr>
<tr>
<td>$S$</td>
<td></td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>90</td>
<td>160</td>
</tr>
<tr>
<td>$T$</td>
<td></td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>50</td>
<td>120</td>
</tr>
</tbody>
</table>

All entries in the table are in appropriate units, say, in dollars. The first row (row $R$) represents the production in dollars of industry $R$ (input). Out of the total of $180$ worth of goods produced, $R$, $S$, and $T$ use $50$, $20$, and $40$, respectively, for the production of their goods, while consumers purchase the remaining $70$ for their consumption (output).

The second and third rows are interpreted in the same way.

An important observation is that the goal of $R$’s production is to produce $70$ worth of goods, since this is the demand of consumers. In order to meet this demand, $R$ must produce a total of $180$, since the difference, $110$, is required internally by $R$, $S$, and $T$.

Suppose, however, that consumer demand is expected to change. To effect this change, how much should each industry now produce? For example, in Table 2, current demand for $R$, $S$, and $T$ can be represented by a demand vector:

$$D_0 = \begin{bmatrix} 70 \\ 90 \\ 50 \end{bmatrix}$$

But suppose marketing forecasts predict that in 3 years the demand vector will be

$$D_3 = \begin{bmatrix} 60 \\ 110 \\ 60 \end{bmatrix}$$

Here, the demand for item $R$ has decreased; the demand for item $S$ has significantly increased, and the demand for item $T$ is higher. Given the current total output of $R$, $S$, and $T$ at $180$, $160$, and $120$, respectively, what must it be in 3 years to meet this projected demand?

The solution of this type of forecasting problem is derived from the open Leontief model using input–output analysis. In using input–output analysis to obtain a solution to such a forecasting problem, we take into account the fact that the output of any one of these industries is affected by changes in the other two, since the total demand for, say, $R$ in 3 years depends not only on consumer demand for $R$, but also on consumer demand for $S$ and $T$. That is, the industries are interrelated.

To obtain the solution, we need to determine how much of each of the three products $R$, $S$, and $T$ is required to produce 1 unit of $R$. For example, to obtain 180 units of $R$ requires the use of 50 units of $R$, 20 units of $S$, and 30 units of $T$ (the entries in column 1).
Forming the ratios, we find that to produce 1 unit of \( R \) requires \( \frac{50}{180} = 0.278 \) of \( R \), \( \frac{20}{180} = 0.111 \) of \( S \), and \( \frac{30}{180} = 0.167 \) of \( T \). If we want, say, \( x \) units of \( R \), we will require 0.278\( x \) units of \( R \), 0.111\( x \) units of \( S \), and 0.167\( x \) units of \( T \).

Continuing in this way, we can construct the matrix

\[
A = \begin{bmatrix}
0.278 & 0.125 & 0.333 \\
0.111 & 0.188 & 0.167 \\
0.167 & 0.125 & 0.167
\end{bmatrix}
\]

Observe that column 1 represents the amounts of \( R \), \( S \), \( T \) required for 1 unit of \( R \); column 2 represents the amounts of \( R \), \( S \), and \( T \) required for 1 unit of \( S \); and column 3 represents the amounts of \( R \), \( S \), and \( T \) required for 1 unit of \( T \). For example, the entry in row 3, column 2 (0.125), represents the amount of \( T \) needed to produce 1 unit of \( S \).

As a result of placing the entries this way, if

\[
X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

represents the total output required to obtain a given demand, the product \( AX \) represents the amount of \( R \), \( S \), and \( T \) required for internal consumption. The condition that production = consumption requires that

\[
\text{Internal consumption} + \text{Consumer demand} = \text{Total output}
\]

In terms of the matrix \( A \), the total output \( X \), and the demand vector \( D \), this requirement is equivalent to the equation

\[
AX + D = X
\]

In this equation we seek to find \( X \) for a prescribed demand \( D \) where the matrix \( A \) is calculated as we did above for some initial production process.*

**EXAMPLE 2** Finding Production Levels to Meet Future Demand: The Open Leontief Model

For the data given in Table 2, find the total output \( X \) required to achieve a future demand of

\[
D_3 = \begin{bmatrix}
60 \\
110 \\
60
\end{bmatrix}
\]

**SOLUTION** We need to solve for \( X \) in

\[
AX + D_3 = X
\]

* The entries in \( A \) can be checked by using the requirement that \( AX + D_0 = X \), for \( D_0 = \text{Initial demand and } X = \text{Total output}. Then

\[
\begin{bmatrix}
0.278 & 0.125 & 0.333 \\
0.111 & 0.188 & 0.167 \\
0.167 & 0.125 & 0.167
\end{bmatrix}
\begin{bmatrix}
180 \\
160 \\
120
\end{bmatrix}
+ \begin{bmatrix}
70 \\
90 \\
50
\end{bmatrix}
= \begin{bmatrix}
180 \\
160 \\
120
\end{bmatrix}
\]
Simplifying, we have

\[ [I_3 - A]X = D_3 \]

Solving for \( X \), we have

\[
X = [I_3 - A]^{-1} \cdot D_3
\]

\[
= \begin{bmatrix}
0.722 & -0.125 & -0.333 \\
-0.111 & 0.812 & -0.167 \\
-0.167 & -0.125 & 0.833
\end{bmatrix}^{-1}
\begin{bmatrix}
60 \\
110 \\
60
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1.6048 & 0.3568 & 0.7131 \\
0.2946 & 1.3363 & 0.3857 \\
0.3660 & 0.2721 & 1.4013
\end{bmatrix}
\begin{bmatrix}
60 \\
110 \\
60
\end{bmatrix}
\]

\[
= \begin{bmatrix}
178.33 \\
187.81 \\
135.96
\end{bmatrix}
\]

The total output of \( R, S, \) and \( T \) required for the forecast demand \( D_3 \) is

\[
x = 178.33 \quad y = 187.81 \quad z = 135.96
\]

NOW WORK PROBLEM 9.

**USING TECHNOLOGY**

**EXAMPLE 3 Using Excel**

Use EXCEL to do Example 2.

**SOLUTION** We need to solve \( AX + D_3 = X \) for \( X \). The solution to this equation is

\[
X = (I_3 - A)^{-1} \cdot D_3
\]

We begin by using EXCEL to compute \((I_3 - A)^{-1} \cdot D_3\).

**STEP 1** Enter matrices \( A, I_3, \) and \( D_3 \) into an EXCEL spreadsheet.
STEP 2 Compute $I_3 - A$.
(a) Highlight the cells where the difference is to go. This must be a $3 \times 3$ matrix.
(b) Type $(highlight \ I) - (highlight \ A)$ in the formula bar and press Ctrl-Shift-Enter.

STEP 3 Compute $(I_3 - A)^{-1}$. 
(a) Highlight the cells that contain $(I - A)^{-1}$; $(I - A)^{-1}$ is a $3 \times 3$ matrix.
(b) Type $= MINVERSE(highlight \ matrix \ (I - A))$ in the formula bar, and press Ctrl-Shift-Enter.

STEP 4 Compute $(I_3 - A)^{-1} D_3$.
(a) Highlight the cells that are to contain the product matrix. This will be a $3 \times 1$ matrix.
(b) Type $= MMULT(highlight (I - A)^{-1}, \ highlight \ D_3)$, and press Ctrl-Shift-Enter.
The results are given below.

So $x = 178.33$, $y = 187.81$, $z = 135.96$. 

NOW WORK PROBLEM 9 USING EXCEL.
The general open model can be described as follows: Suppose there are \( n \) industries in the economy. Each industry produces some goods or services, which are partially consumed by the \( n \) industries, while the rest are used to meet a prescribed current consumer demand. Given the output required of each industry to meet current demand, what should the output of each industry be to meet some different future demand?

**Theorem: Open Leontief Model**

The matrix of the open model is defined to consist of entries \( a_{ij} \), where \( a_{ij} \) is the amount of output of industry \( j \) required for one unit of output of industry \( i \). If \( X \) is a column vector representing the production of each industry in the system and \( D \) is a column vector representing future demand for goods produced in the system, then

\[
X = AX + D
\]

From the equation above we find

\[
[I_n - A]X = D
\]

It can be shown that the matrix \( I_n - A \) has an inverse, provided each entry in \( A \) is positive and the sum of each column in \( A \) is less than 1. Under these conditions we may solve for \( X \) to get

\[
X = [I_n - A]^{-1} \cdot D
\]

This form of the solution is particularly useful since it allows us to find \( X \) for a variety of demands \( D \) by doing one calculation: \( [I_n - A]^{-1} \).

We conclude by noting that the use of an input–output matrix to solve forecasting problems assumes that each industry produces a single commodity and that no technological advances take place in the period of time under investigation (in other words, the proportions found in the matrix \( A \) are fixed).

**EXERCISE 3.4 APPLICATION IN ECONOMICS**

Answers Begin on Page AN–17.

**Skill Building**

In Problems 1–4, find the relative wages of each person for the given closed input–output matrix. In each case take the wages of \( C \) to be the parameter and use \( z = C \)’s wages = $30,000.

1. \[
A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \end{bmatrix}
\]

2. \[
A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 6 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 6 & 4 \end{bmatrix}
\]

3. \[
A = \begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 0.3 & 0.7 \\ 0.4 & 0.4 & 0.5 \end{bmatrix}
\]

4. \[
A = \begin{bmatrix} 0.4 & 0.3 & 0.2 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}
\]

5. For the three industries \( R, S, \) and \( T \) in the open Leontief model of Table 2 on page 155, compute the total output vector \( X \) if the forecast demand vector is

\[
D_2 = \begin{bmatrix} 80 \\ 90 \\ 60 \end{bmatrix}
\]

6. Rework Problem 5 if the forecast demand vector is

\[
D_4 = \begin{bmatrix} 100 \\ 80 \\ 60 \end{bmatrix}
\]
Applications

7. Closed Leontief Model A society consists of four individuals: a farmer, a builder, a tailor, and a rancher (who produces meat products). Of the food produced by the farmer, \(\frac{3}{10}\) is used by the farmer, \(\frac{2}{10}\) by the builder, \(\frac{2}{10}\) by the tailor, and \(\frac{3}{10}\) by the rancher. The builder’s production is utilized 30% by the farmer, 30% by the builder, 10% by the tailor, and 30% by the rancher. The tailor’s production is used in the ratios \(\frac{3}{10}\), \(\frac{3}{10}\), \(\frac{1}{10}\), and \(\frac{3}{10}\) by the farmer, builder, tailor, and rancher, respectively. Finally, meat products are used 20% by each of the farmer, builder, and tailor, and 40% by the rancher. What is the relative income of each if the rancher’s income is scaled at $25,000?

8. Closed Leontief Model If in Problem 7 the meat production utilization changes so that it is used equally by all four individuals, while everyone else’s production utilization remains the same, what are the relative incomes?

9. Open Leontief Model Suppose the interrelationships between the production of two industries \(R\) and \(S\) in a given year are given in the table:

<table>
<thead>
<tr>
<th></th>
<th>(R)</th>
<th>(S)</th>
<th>Demand</th>
<th>Total Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R)</td>
<td>30</td>
<td>40</td>
<td>60</td>
<td>130</td>
</tr>
<tr>
<td>(S)</td>
<td>20</td>
<td>10</td>
<td>40</td>
<td>70</td>
</tr>
</tbody>
</table>

If the forecast demand in 2 years is

\[D_2 = \begin{bmatrix} 80 \\ 40 \end{bmatrix}\]

what should the total output \(X\) be?

10. Open Leontief Model Suppose the interrelationships between the production of five industries (manufacturing, electric power, petroleum, transportation, and textiles) in a given year are as listed in the table:

<table>
<thead>
<tr>
<th></th>
<th>Manu.</th>
<th>E.P.</th>
<th>Petr.</th>
<th>Tran.</th>
<th>Text.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manu.</td>
<td>0.2</td>
<td>0.12</td>
<td>0.15</td>
<td>0.18</td>
<td>0.1</td>
</tr>
<tr>
<td>E.P.</td>
<td>0.17</td>
<td>0.11</td>
<td>0</td>
<td>0.19</td>
<td>0.28</td>
</tr>
<tr>
<td>Petr.</td>
<td>0.11</td>
<td>0.11</td>
<td>0.12</td>
<td>0.46</td>
<td>0.12</td>
</tr>
<tr>
<td>Tran.</td>
<td>0.1</td>
<td>0.14</td>
<td>0.02</td>
<td>0.17</td>
<td>0.19</td>
</tr>
<tr>
<td>Text.</td>
<td>0.16</td>
<td>0.18</td>
<td>0.02</td>
<td>0.1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

If the demand vector is given by

\[D = \begin{bmatrix} 100 \\ 100 \\ 200 \\ 100 \\ 100 \end{bmatrix}\]

what should the production vector \(X\) be?

11. Closed Leontief Model In a professional cooperative, a physician, an attorney, and a financial planner trade services. They agree that the value of each one’s work is approximately $20,000. At the end of the period the physician had spent 30% of his time on his own care, 40% on the attorney’s care, and 30% on the financial planner’s care. The attorney had spent 50% of her time on the physician’s affairs, 20% on her own affairs, and 30% on the financial planner’s affairs. Finally, the financial planner spent 20% of his time managing the physician’s portfolio, 20% managing the attorney’s portfolio, and 60% managing his own portfolio. How should each of these three professionals be paid for their work?

12. Open Leontief Model Suppose three corporations each manufacture a different product, although each needs the others’ goods to produce its own. Moreover, suppose some of each manufacturer’s production is consumed by individuals outside the system. Current internal and external consumptions are given in the table below, but it has been forecast that consumer demand will change in the next several years and that in 5 years consumers will demand 80 units of product \(A\), 40 units of product \(B\), and 80 units of product \(C\). Find the total output needed to be made by each corporation to meet the predicted demand.

<table>
<thead>
<tr>
<th></th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>Consumer</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>100</td>
<td>50</td>
<td>40</td>
<td>60</td>
<td>250</td>
</tr>
<tr>
<td>(B)</td>
<td>20</td>
<td>10</td>
<td>30</td>
<td>40</td>
<td>100</td>
</tr>
<tr>
<td>(C)</td>
<td>30</td>
<td>40</td>
<td>30</td>
<td>100</td>
<td>200</td>
</tr>
</tbody>
</table>

Application in Accounting

Consider a firm that has two types of departments, production and service. The production departments produce goods that can be sold in the market, and the service departments provide services to the production departments. A major objective of the cost accounting process is the determination of the full cost of manufactured products on a per unit basis. This requires an allocation of indirect costs, first, from the service department
(where they are incurred) to the producing department in which the goods are manufactured and, second, to the specific goods themselves. For example, an accounting department usually provides accounting services for service departments as well as for the production departments. The indirect costs of service rendered by a service department must be determined in order to correctly assess the production departments. The total costs of a service department consist of its direct costs (salaries, wages, and materials) and its indirect costs (charges for the services it receives from other service departments). The nature of the problem and its solution are illustrated by the following example.

**EXAMPLE 1 Finding Total Monthly Costs**

Consider a firm with two production departments, $P_1$ and $P_2$, and three service departments, $S_1, S_2,$ and $S_3$. These five departments are listed in the leftmost column of Table 3.

<table>
<thead>
<tr>
<th>Department</th>
<th>Total Costs</th>
<th>Direct Costs, Dollars</th>
<th>Indirect Costs for Services from Departments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$x_1$</td>
<td>600</td>
<td>$0.25x_1$ $0.15x_2$ $0.15x_3$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$x_2$</td>
<td>1100</td>
<td>$0.35x_1$ $0.20x_2$ $0.25x_3$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$x_3$</td>
<td>600</td>
<td>$0.10x_1$ $0.10x_2$ $0.35x_3$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>$x_4$</td>
<td>2100</td>
<td>$0.15x_1$ $0.25x_2$ $0.15x_3$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$x_5$</td>
<td>1500</td>
<td>$0.15x_1$ $0.30x_2$ $0.10x_3$</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td></td>
<td>5900</td>
<td>$x_1$ $x_2$ $x_3$</td>
</tr>
</tbody>
</table>

The total monthly costs of these departments are unknown and are denoted by $x_1, x_2, x_3, x_4, x_5$. The direct monthly costs of the five departments are shown in the third column of the table. The fourth, fifth, and sixth columns show the allocation of charges for the services of $S_1, S_2,$ and $S_3$ to the various departments. Since the total cost for each department is its direct costs plus its indirect costs, the first three rows of the table yield the total costs for the three service departments:

\[
x_1 = 600 + 0.25x_1 + 0.15x_2 + 0.15x_3 \\
x_2 = 1100 + 0.35x_1 + 0.20x_2 + 0.25x_3 \\
x_3 = 600 + 0.10x_1 + 0.10x_2 + 0.35x_3
\]

Let $X, C,$ and $D$ denote the following matrices:

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad C = \begin{bmatrix} 0.25 & 0.15 & 0.15 \\ 0.35 & 0.20 & 0.25 \\ 0.10 & 0.10 & 0.35 \end{bmatrix}, \quad D = \begin{bmatrix} 600 \\ 1100 \\ 600 \end{bmatrix}
\]

Then the system of equations above can be written in matrix notation as

\[
X = D + CX
\]
which is equivalent to

\[ [I_3 - C]X = D \]

The total costs of the three service departments can be obtained by solving this matrix equation for \( X \):

\[ X = [I_3 - C]^{-1}D \]

Now

\[
[I_3 - C] = \begin{bmatrix}
0.75 & -0.15 & -0.15 \\
-0.35 & 0.80 & -0.25 \\
-0.10 & -0.10 & 0.65
\end{bmatrix}
\]

from which it can be shown that

\[
[I_3 - C]^{-1} = \begin{bmatrix}
1.57 & 0.36 & 0.50 \\
0.80 & 1.49 & 0.76 \\
0.36 & 0.28 & 1.73
\end{bmatrix}
\]

It is significant that the inverse of \([I_3 - C]\) exists, and that all of its entries are nonnegative. Because of this and the fact that the matrix \(D\) contains only nonnegative entries, the matrix \(X\) will also have only nonnegative entries. This means there is a meaningful solution to the accounting problem:

\[ X = [I_3 - C]^{-1}D = \begin{bmatrix}
1638 \\
2575 \\
1562
\end{bmatrix} \]

That is, \(x_1 = 1638.00\), \(x_2 = 2575.00\), and \(x_3 = 1562.00\). All direct and indirect costs can now be determined by substituting these values in Table 3, as shown in Table 4.

From Table 4 we learn that department \(P_1\) pays $1123.75 for the services it receives from \(S_1\), \(S_2\), \(S_3\), and \(P_2\) pays $1174.40 for the services it receives from these departments. The procedure we have followed charges the direct costs of the service departments to the production departments, and each production department is charged according to the services it utilizes. Furthermore, the total cost for \(P_1\) and \(P_2\) is $5898.15, and this figure approximates the sum of the direct costs of the three service departments and the two production departments, $5900. The results are consistent with conventional accounting procedure. Discrepancies that occur are due to rounding.

**TABLE 4**

<table>
<thead>
<tr>
<th>Department</th>
<th>Total Costs, Dollars</th>
<th>Direct Costs, Dollars</th>
<th>Indirect Costs for Services from Departments, Dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(S_1)  (S_2)  (S_3)</td>
</tr>
<tr>
<td>(S_1)</td>
<td>1638</td>
<td>600</td>
<td>409.50  386.25  234.30</td>
</tr>
<tr>
<td>(S_2)</td>
<td>2575</td>
<td>1100</td>
<td>573.30  515.00  390.50</td>
</tr>
<tr>
<td>(S_3)</td>
<td>1562</td>
<td>600</td>
<td>163.80  257.50  546.70</td>
</tr>
<tr>
<td>(P_1)</td>
<td>3223.75</td>
<td>2100</td>
<td>245.70  643.75  234.30</td>
</tr>
<tr>
<td>(P_2)</td>
<td>2674.40</td>
<td>1500</td>
<td>245.70  772.50  156.20</td>
</tr>
</tbody>
</table>
Finally, a comment should be made about the allocation of charges for services as shown in Table 3. How is it determined that 25% of the total cost of $S_1$ should be charged to $S_1$, 35% to $S_2$, 10% to $S_3$, 15% to $P_1$, and 15% to $P_2$? The services of each department can be measured in some suitable unit, and each department can be charged according to the number of these units of service it receives. If 20% of the accounting items concern a given department, that department is charged 20% of the total cost of the accounting department. When services are not readily measurable, the allocation basis is subjectively determined.

**EXERCISE 3.4 APPLICATION IN ACCOUNTING**

**Answers Begin on Page AN–17.**

1. **Accounting** Consider a firm with 2 service departments and 3 production departments as described by the data in the table:

<table>
<thead>
<tr>
<th>Department</th>
<th>Total Costs</th>
<th>Direct Costs from Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$x_1$</td>
<td>2000 $\frac{1}{9}x_1$ $\frac{3}{9}x_2$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$x_2$</td>
<td>1000 $\frac{3}{9}x_1$ $\frac{1}{9}x_2$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>$x_3$</td>
<td>2500 $\frac{1}{9}x_1$ $\frac{2}{9}x_2$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$x_4$</td>
<td>1500 $\frac{3}{9}x_1$ $\frac{1}{9}x_2$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$x_5$</td>
<td>3000 $\frac{1}{9}x_1$ $\frac{2}{9}x_2$</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>10,000</strong></td>
<td>$x_1$ $x_2$</td>
</tr>
</tbody>
</table>

Determine whether this accounting problem has a solution. If it does, find the total costs. Prepare a table similar to Table 4. Show that the total of the service charges allocated to $S_1$ and $S_2$ is equal to the sum of the direct costs of the service departments $S_1$ and $S_2$.

2. **Accounting** Follow the directions of Problem 1 for the accounting problem described by the following data:

<table>
<thead>
<tr>
<th>Department</th>
<th>Total Costs</th>
<th>Direct Costs from Departments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$x_1$</td>
<td>500 0.20$x_1$ 0.10$x_2$ 0.10$x_3$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$x_2$</td>
<td>1000 0.40$x_1$ 0.15$x_2$ 0.30$x_3$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$x_3$</td>
<td>500 0.10$x_1$ 0.05$x_2$ 0.30$x_3$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>$x_4$</td>
<td>2000 0.20$x_1$ 0.35$x_2$ 0.20$x_3$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$x_5$</td>
<td>1500 0.10$x_1$ 0.35$x_2$ 0.10$x_3$</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>5500</strong></td>
<td>$x_1$ $x_2$ $x_3$</td>
</tr>
</tbody>
</table>

3. **Accounting** A local firm has two service departments, purchasing and legal, and three production departments, $P_1$, $P_2$, and $P_3$. The total costs of each department are unknown and are denoted $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$. The direct costs of manufacture and the indirect costs of service are listed in the table below.

<table>
<thead>
<tr>
<th>Department</th>
<th>Total Costs</th>
<th>Direct Cost in Dollars</th>
<th>Indirect Costs for Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$x_1$</td>
<td>800 0.20$x_1$ 0.10$x_2$</td>
<td></td>
</tr>
<tr>
<td>$S_2$</td>
<td>$x_2$</td>
<td>4000 0.10$x_1$ 0.30$x_2$</td>
<td></td>
</tr>
<tr>
<td>$P_1$</td>
<td>$x_3$</td>
<td>1500 0.20$x_1$ 0.10$x_2$</td>
<td></td>
</tr>
<tr>
<td>$P_2$</td>
<td>$x_4$</td>
<td>500 0.30$x_1$ 0.20$x_2$</td>
<td></td>
</tr>
<tr>
<td>$P_3$</td>
<td>$x_5$</td>
<td>1200 0.20$x_1$ 0.30$x_2$</td>
<td></td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>8000</strong></td>
<td>$x_1$ $x_2$</td>
<td></td>
</tr>
</tbody>
</table>

Find the total costs. Prepare a table similar to Table 4. Show that the total of the service charges allocated to $P_1$, $P_2$, and $P_3$ is equal to the sum of the direct costs of the service departments $S_1$ and $S_2$.

**Application in Statistics: The Method of Least Squares**

In Section 1.4 we were given a set of data points $(x, y)$ and we drew the scatter diagram representing the relation between $x$ and $y$. We analyzed the scatter diagram to determine if the relation between the variables was linear. When it was linear, we selected two
points and found an equation of the line containing the points. We then compared our line to the line of best fit, found using a graphing utility. In this application we define the line of best fit and learn how matrices are used to find the line of best fit.

The method of least squares refers to a technique that is often used in data analysis to find the "best" linear equation that fits a given collection of experimental data. (We shall see a little later just what we mean by the word best.) It is a technique employed by statisticians, economists, business forecasters, and those who try to interpret and analyze data. Before we begin our discussion, we will need to define the transpose of a matrix.

**Definition**

Let \( A \) be a matrix of dimension \( m \times n \). The transpose of \( A \), written \( A^T \), is the \( n \times m \) matrix obtained from \( A \) by interchanging the rows and columns of \( A \).

The first row of \( A^T \) is the first column of \( A \); the second row of \( A^T \) is the second column of \( A \); and so on.

**EXAMPLE 1** Finding the Transpose of a Matrix

If

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad C = [1 \ 0 \ -1]
\]

then

\[
A^T = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\]

**NOW WORK PROBLEM 1.**

Note how dimensions are reversed in finding \( A^T \): \( A \) has the dimension \( 2 \times 3 \), while \( A^T \) has the dimension \( 3 \times 2 \), and so on.

We begin our discussion of the method of least squares with an example.

Suppose a product has been sold over time at various prices and that we have some data that show the demand for the product (in thousands of units) in terms of its price (in dollars). If we use \( x \) to represent price and \( y \) to represent demand, the data might look like the information in Table 5.

Suppose also that we have reason to assume that \( y \) and \( x \) are linearly related. That is, we assume we can write

\[
y = mx + b
\]

for some, as yet unknown, \( m \) and \( b \). In other words, we are assuming that when demand is graphed against price, the resulting graph will be a line. Our belief that \( y \) and \( x \) are linearly related might be based on past experience or economic theory.

If we plot the data from Table 5 in a scatter diagram (Figure 7), it seems pretty clear that no single line passes through the plotted points. Though no line will pass...
through all the points, we might still ask: “Is there a line that ‘best fits’ the points?” That is, we seek a line of the sort in Figure 8.

**Statement of the Problem**

Given a set of noncollinear points, find the line that “best fits” these points.

We still need to clarify our use of the phrase “best fits.” Suppose there are four data points, labeled from left to right as \((x_1, y_1), (x_2, y_2),\) and so on. See Figure 9. Now suppose the equation of the line \(L\) in Figure 9 is

\[
  y = mx + b
\]

where we do not know \(m\) and \(b\). If \((x_1, y_1)\) is actually on the line, it would be true that

\[
  y_1 = mx_1 + b
\]

But, since we can’t be sure \((x_1, y_1)\) lies on \(L\), the above equation may not be true. That is, there may be a nonzero difference between \(y_1\) and \(mx_1 + b\). We designate this difference by \(e_1\), where \(e_1\) represents the “error.” Then

\[
  e_1 = y_1 - (mx_1 + b)
\]

so, solving for \(y_1\), we have

\[
  y_1 = mx_1 + b + e_1
\]

What we have just said about \((x_1, y_1)\) we can repeat for the remaining data points \((x_i, y_i), i = 2, 3, 4\), obtaining

\[
  e_i = y_i - (mx_i + b)
\]

or, equivalently,

\[
  y_i = mx_i + b + e_i \quad i = 1, 2, 3, 4
\]

Geometrically, \(|e_i|\) measures the vertical distance between the line \(L\) and the data point \((x_i, y_i)\).

Recall that our problem is to find a line \(L\) that “best fits” the data points \((x_i, y_i)\). Intuitively, we would seek a line \(L\) for which the \(e_i\)’s are all small. Since some \(e_i\)’s may be positive and some may be negative, it will not do to just add them up. To eliminate the signs, we square each \(e_i\). The method of least squares consists of finding a line \(L\) for which the sum of the squares of the \(e_i\)’s is as small as possible. Since finding the line \(y = mx + b\) is the same as finding \(m\) and \(b\), we restate the least squares problem for this example as follows:
The Least Squares Problem

Given the data points \((x_i, y_i), i = 1, 2, 3, 4\), find \(m\) and \(b\) satisfying

\[
y_i = mx_i + b + e_i \quad i = 1, 2, 3, 4
\]

so that \(e_1^2 + e_2^2 + e_3^2 + e_4^2\) is minimized.

A solution to the problem can be compactly expressed using matrix language. Since a derivation requires either calculus or more advanced linear algebra, we omit the proof and simply present the solution:

Theorem

Solution to the Least Squares Problem

The line of best fit

\[
y = mx + b
\]

to a set of four points \((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\) is obtained by solving the system of two equations in two unknowns

\[
A^TAX = A^TY
\]

for \(m\) and \(b\), where

\[
A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \end{bmatrix} \quad X = \begin{bmatrix} m \\ b \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}
\]

**Example 2** Finding the Line of Best Fit

Use least squares to find the line of best fit for the points

\[(4, 9) \quad (5, 8) \quad (9, 6) \quad (12, 3)\]

**Solution** We use Equations (5) to set up the matrices \(A\) and \(Y\):

\[
A = \begin{bmatrix} 4 & 1 \\ 5 & 1 \\ 9 & 1 \\ 12 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 9 \\ 8 \\ 6 \\ 3 \end{bmatrix}
\]
Equation (4), $A^TAX = A^TY$, becomes

$$
\begin{bmatrix}
4 & 5 & 9 & 12 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
m \\
b
\end{bmatrix}
= 
\begin{bmatrix}
4 & 5 & 9 & 12 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
9 \\
8 \\
6 \\
3
\end{bmatrix}
$$

This reduces to a system of two equations in two unknowns:

$$
\begin{align*}
26m + 30b &= 166 \\
30m + 4b &= 26
\end{align*}
$$

The solution, which you can verify, is given by

$$
m = -\frac{29}{41} \quad \text{and} \quad b = \frac{484}{41}
$$

The least squares solution to the problem is given by the line

$$
y = -\frac{29}{41}x + \frac{484}{41}
$$

Note, for example, that corresponding to the value $x = 5$, the above line of “best fit” has $y = -\frac{29}{41} \cdot 5 + \frac{484}{41} = 8.27$, while the experimentally observed demand had value 8. Were we to use our computed line to approximate the connection between price and demand, we would predict that a selling price of $x = 6$ would yield a demand of $y = -\frac{29}{41} \cdot 6 + \frac{484}{41} = 7.56$.

**COMMENT** The line of best fit obtained in Example 2 is

$$
y = -0.7073170732x + 11.80487805
$$

In Figure 10, we compare the line of best fit obtained above to that generated by a graphing utility and show the graph of the line of best fit on the scatter diagram.

**FIGURE 10**

**NOW WORK PROBLEM 9.**

**The General Least Squares Problem**

We presented the method of least squares using an example with only four data points. The general least squares problem is given next.
Chapter 3 Matrices

Theorem

The general least squares problem consists of finding the line of best fit to a set of \( n \) data points:

\[
(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)
\]

If we let

\[
A = \begin{bmatrix}
    x_1 & 1 \\
    x_2 & 1 \\
    \vdots & \vdots \\
    x_n & 1
\end{bmatrix}, \quad
X = \begin{bmatrix}
    m \\
    b
\end{bmatrix}, \quad
Y = \begin{bmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n
\end{bmatrix}
\]

then the line of best fit to the data points in Equation (6) is

\[
y = mx + b
\]

where \( X = \begin{bmatrix}
    m \\
    b
\end{bmatrix} \) is found by solving the system of two equations in two unknowns,

\[
A^TAX = A^TY
\]

EXERCISE 3.4 APPLICATION IN STATISTICS

Skill Building

In Problems 1–6, compute \( A^T \).

1. \( A = \begin{bmatrix}
    4 & 1 & 2 \\
    3 & 1 & 0
\end{bmatrix} \)

2. \( A = \begin{bmatrix}
    5 & 2 & -1 \\
    1 & 3 & 6 \\
    1 & -1 & 2
\end{bmatrix} \)

3. \( A = \begin{bmatrix}
    8 \\
    6 \\
    3
\end{bmatrix} \)

4. \( A = \begin{bmatrix}
    -1 & 6 & 4
\end{bmatrix} \)

5. \( A = 5 \begin{bmatrix}
    2 \\
    3
\end{bmatrix} \)

6. \( A = 3 \begin{bmatrix}
    3 & 1 \\
    7
\end{bmatrix} \)

7. A matrix is symmetric if \( A^T = A \). Which of the following matrices are symmetric?

(a) \( \begin{bmatrix}
    1 & 1 \\
    1 & 0 \\
    3 & 2
\end{bmatrix} \)

(b) \( \begin{bmatrix}
    0 & 1 \\
    1 & 4 \\
    3 & 7
\end{bmatrix} \)

(c) \( \begin{bmatrix}
    1 & 2 & 3 \\
    2 & 4 & 5 \\
    3 & 5 & 1
\end{bmatrix} \)

Need a symmetric matrix be square?

8. Show that the matrix \( A^TA \) is always symmetric.

Applications

9. Supply Equation The table shows the supply (in thousands of units) of a product at various prices (in dollars):

| Price, \( x \) | 3 | 5 | 6 | 7 |
| Supply, \( y \) | 10 | 13 | 15 | 16 |

(a) Find the least squares line of best fit to the above data.

(b) Use the equation of this line to estimate the supply of the product at a price of $8.
10. **Study Time vs. Performance** Data giving the number of hours a person studied compared to his or her performance on an exam are given in the table:

<table>
<thead>
<tr>
<th>Hours Studied (x)</th>
<th>Exam Score (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>74</td>
</tr>
<tr>
<td>4</td>
<td>85</td>
</tr>
<tr>
<td>6</td>
<td>90</td>
</tr>
<tr>
<td>8</td>
<td>92</td>
</tr>
</tbody>
</table>

(a) Find a least squares line of best fit to the above data.
(b) What prediction would this line make for a student who studied 9 hours?

11. **Advertising vs. Sales** A business would like to determine the relationship between the amount of money spent on advertising and its total weekly sales. Over a period of 5 weeks it gathers the following data:

<table>
<thead>
<tr>
<th>Amount Spent on Advertising (in thousands) (x)</th>
<th>Weekly Sales Volume (in thousands) (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>17</td>
<td>61</td>
</tr>
<tr>
<td>11</td>
<td>55</td>
</tr>
<tr>
<td>18</td>
<td>60</td>
</tr>
<tr>
<td>21</td>
<td>70</td>
</tr>
</tbody>
</table>

Find a least squares line of best fit to the above data.

12. **Drug Concentration vs. Time** The following data show the connection between the number of hours a drug has been in a person’s body and its concentration in the body.

<table>
<thead>
<tr>
<th>Number of Hours</th>
<th>Drug Concentration (parts per million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.1</td>
</tr>
<tr>
<td>4</td>
<td>1.6</td>
</tr>
<tr>
<td>6</td>
<td>1.4</td>
</tr>
<tr>
<td>8</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(a) Fit a least squares line to the above data.
(b) Use the equation of the line to estimate the drug concentration after 5 hours.

13. **Medicine** The number of persons in the United States with diagnosed diabetes for the years 1994–2008 is given below.

<table>
<thead>
<tr>
<th>Year</th>
<th>Number (in millions) of persons with Diagnosed Diabetes in the United States</th>
</tr>
</thead>
<tbody>
<tr>
<td>1994</td>
<td>8.1</td>
</tr>
<tr>
<td>1995</td>
<td>8.0</td>
</tr>
<tr>
<td>1996</td>
<td>8.8</td>
</tr>
<tr>
<td>1997</td>
<td>9.4</td>
</tr>
<tr>
<td>1998</td>
<td>10.5</td>
</tr>
<tr>
<td>1999</td>
<td>11.1</td>
</tr>
<tr>
<td>2000</td>
<td>12.0</td>
</tr>
<tr>
<td>2001</td>
<td>12.9</td>
</tr>
<tr>
<td>2002</td>
<td>13.6</td>
</tr>
<tr>
<td>2003</td>
<td>14.3</td>
</tr>
<tr>
<td>2004</td>
<td>15.2</td>
</tr>
<tr>
<td>2005</td>
<td>16.3</td>
</tr>
<tr>
<td>2006</td>
<td>16.8</td>
</tr>
<tr>
<td>2007</td>
<td>17.9</td>
</tr>
<tr>
<td>2008</td>
<td>24.0</td>
</tr>
</tbody>
</table>

Let $N$ be the dependent variable representing the number (in millions) of persons with diagnosed diabetes in the United States and let $t$ be the independent variable representing time in years. Let $t = 0$ correspond to the year 1994. Use matrix methods to find an equation of the line of best fit for this data and then use this equation to predict the number of persons that will have diagnosed diabetes in the United States in the year 2011.

**Source:** Centers for Disease Control and Prevention, United States Department of Health and Human Services, and American Diabetes Association.

14. **Baseball Salaries** The average annual salary for major league baseball players is given below for the indicated years.

<table>
<thead>
<tr>
<th>Year</th>
<th>Average Annual Salary for Major League Baseball Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>1975</td>
<td>$44,676</td>
</tr>
<tr>
<td>1980</td>
<td>$143,756</td>
</tr>
<tr>
<td>1985</td>
<td>$371,571</td>
</tr>
<tr>
<td>1990</td>
<td>$578,930</td>
</tr>
<tr>
<td>1995</td>
<td>$1,071,029</td>
</tr>
<tr>
<td>2000</td>
<td>$1,998,034</td>
</tr>
<tr>
<td>2005</td>
<td>$2,652,655</td>
</tr>
<tr>
<td>2009</td>
<td>$3,240,000</td>
</tr>
</tbody>
</table>
Let \( S \) be the dependent variable representing the average annual salary for major league baseball players and let \( t \) be the independent variable representing time in years. Let \( t = 0 \) correspond to the year 1975. Use matrix methods to find an equation of the line of best fit for this data and then use this equation to predict the average annual salary for major league baseball players in the year 2011.

**Source:** Baseball Almanac and CBS Sports MLB

### CHAPTER 3 REVIEW OBJECTIVES

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<th>You should be able to</th>
<th>Review Exercises</th>
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<td>1–16</td>
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<tr>
<td></td>
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<td>1, 9</td>
</tr>
<tr>
<td></td>
<td>8, 9, 10</td>
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<td></td>
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<td></td>
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<tr>
<td></td>
<td>13, 17</td>
<td>6 Solve applied problems involving matrices (p. 116)</td>
<td>26, 27</td>
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<td>3, 4, 6</td>
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</tr>
<tr>
<td></td>
<td>7, 8, 9</td>
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<td>9, 15, 16</td>
</tr>
<tr>
<td></td>
<td>5, 10, 11</td>
<td>3 Solve applied problems involving matrices (p. 129)</td>
<td>27</td>
</tr>
<tr>
<td>3.3</td>
<td>2, 3, 4, 5, 6</td>
<td>1 Find the inverse of a matrix (p. 136)</td>
<td>17–22</td>
</tr>
<tr>
<td></td>
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<td>28</td>
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<tr>
<td></td>
<td>9</td>
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<td>28</td>
</tr>
<tr>
<td></td>
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<td>25</td>
</tr>
</tbody>
</table>

### THINGS TO KNOW

- **Matrix** (p. 105)
  - Dimension of a matrix (p. 105)
  - Square matrix (p. 105)
  - Zero matrix (p. 109)
  - Identity matrix (p. 128)
  - Inverse of a matrix (p. 135)

### REVIEW EXERCISES

**Answers Begin on Page AN–18**

*In Problems 1–16, use the following matrices to compute each expression. State the dimension of the resulting matrix.*

\[
A = \begin{bmatrix}
1 & 0 \\
2 & 4 \\
-1 & 2 
\end{bmatrix} \quad B = \begin{bmatrix}
4 & -3 & 0 \\
1 & 1 & -2 
\end{bmatrix} \quad C = \begin{bmatrix}
3 & -4 \\
1 & 5 \\
5 & -2 
\end{bmatrix}
\]

1. \( A + C \)
2. \( A - C \)
3. \( 6A \)
4. \( -4B \)
5. \( AB \)
6. \( BA \)
7. \( CB \)
8. \( BC \)
9. \( (A + C)B \)
10. \( B(2C - A) \)
11. \( 3A + 2C \)
12. \( 5C - 2A \)
13. \( C + 0 \)
14. \( A + (-A) \)
15. \( AB + CB \)
16. \( AI_2 \)
In Problems 17–22, find the inverse, if it exists, of each matrix.

17. \[
\begin{bmatrix}
3 & 0 \\
-2 & 1
\end{bmatrix}
\]
18. \[
\begin{bmatrix}
4 & 1 \\
3 & 1
\end{bmatrix}
\]
19. \[
\begin{bmatrix}
4 & 2 \\
6 & 3
\end{bmatrix}
\]
20. \[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 6
\end{bmatrix}
\]
21. \[
\begin{bmatrix}
4 & 3 & -1 \\
0 & 2 & 2 \\
3 & -1 & 0
\end{bmatrix}
\]
22. \[
\begin{bmatrix}
1 & 2 & -3 \\
4 & 6 & 2 \\
-1 & -6 & 9
\end{bmatrix}
\]

23. What must be true about \(x, y, z, w\), if we require \(AB = BA\) for the matrices
\[
A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

24. Let \(t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}\), with \(t_1 + t_2 = 1\), and let \(A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \) and \(tA = t\).

Find \(t\) such that \(tA = t\).

25. **Cryptography** Use the correspondence
\[
\begin{array}{cccccccccccccccc}
A & B & C & D & E & F & G & H & I & J & K & L & M & N & O & P \\
Q & R & S & T & U & V & W & X & Y & Z & & & & & & \\
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & \\
\end{array}
\]
and the matrices \(A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}\) to encode the following messages:
(a) HEY DUDE WHATS UP
(b) CALL ME ON MY CELL
(c) Use matrix \(A\) to decode the message: 75 78 45 42 72 64 35 42 17 26 67 62.

26. **Ordering Coffee** Bill is in charge of ordering coffee for three coffee houses. Based on historical sales, each week he orders 100 pounds of Arabica coffee, 150 pounds of French Roast, 200 pounds of a Hazelnut/Columbian blend, and 75 pounds of Breakfast Blend for the downtown location; 50 pounds of Arabica coffee, 100 pounds of French Roast, 150 pounds of a Hazelnut/Columbian blend, and 150 pounds of Breakfast Blend for the train-station location; and 200 pounds of Arabica coffee, 250 pounds of French Roast, 300 pounds of a Hazelnut/Columbian blend, and 100 pounds of Breakfast Blend for the mall location.

(a) Express this information using a \(4 \times 3\) matrix \(A\). Be sure to label the rows and columns.

(b) For a week in which the downtown store is closed due to a holiday, he reduces his order for the downtown location by 20%, reduces his order for the train station location by 10%, and increases his order for the mall location by 25%. Express this information using a \(4 \times 3\) matrix \(B\).

(c) Find \(A + B\) and interpret the entries.

(d) Find \(3A + B\) and interpret the entries.

27. **Stock Purchases** On February 3, 2010, Microsoft stock sold for $28.50 per share, Intel for $19.50 per share, and Dell for $13.30 per share. At these prices, Juanita, Debbie, and Dawn each purchased shares in these three companies as shown in the table.

<table>
<thead>
<tr>
<th></th>
<th>Microsoft</th>
<th>Intel</th>
<th>Dell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Juanita</td>
<td>20</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>Debbie</td>
<td>15</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>Dawn</td>
<td>10</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

(a) Express the price per share of each stock on February 3, 2010, as a \(1 \times 3\) matrix \(A\). Be sure to label the row and columns.

(b) Express the shares purchased of the three stocks by the three investors as a \(3 \times 3\) matrix \(B\). Be sure to label the rows and columns so you can answer part (c).

(c) Find \(AB\) and interpret the entries.
Chapter 3 Project

THE OPEN LEONTIEF MODEL AND THE AMERICAN ECONOMY

The open Leontief model, introduced in Section 3.4, is used to model the economies of many countries throughout the world, as well as the global economy itself. The first example Leontief himself used was the American economy in 1947. He divided the economy into 42 industries (or sectors) for his original model. For purposes of this example, the data from the 42 sectors used by Leontief has been collected into just 3: agriculture, manufacturing, and services (shown in Table 1). The open sector or demand vector (which consumes only goods and services) is also present.

<table>
<thead>
<tr>
<th>TABLE 1 EXCHANGE OF GOODS AND SERVICES IN THE U.S. FOR 1947 (IN BILLIONS OF 1947 DOLLARS)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Agriculture</td>
</tr>
<tr>
<td>Manufacturing</td>
</tr>
<tr>
<td>Services</td>
</tr>
</tbody>
</table>

1. Create the matrix $A$ and the demand vector $D_0$ for the open Leontief model from Table 1.
2. Calculate the production vector $X_0$ for the open Leontief model from Table 1.
3. Now suppose that the demand vector is changed to

$$D_1 = \begin{bmatrix} 40.24 \\ 60.02 \\ 130.65 \end{bmatrix}$$

Notice that the only change from the original demand vector $D_0$ to the new demand vector $D_1$ is the addition of 1 unit of consumer demand in the agricultural sector. Calculate a new production vector $X_1$ for this new demand vector $D_1$.
4. Compute $X_1 - X_0$.
   You should find that the difference in the production vectors $X_0$ and $X_1$ is the first column of the matrix $(I - A)^{-1}$. This is not a coincidence. It can be shown that the difference in the old and new production vectors equals the first column of $(I - A)^{-1}$. This observation leads to an important interpretation of the entries of $(I - A)^{-1}$:

**Interpretation:** The $(i,j)$ entry in the matrix $(I - A)^{-1}$ is the amount by which industry $i$ must change its production level to satisfy an increase of one unit in the demand from industry $j$.

5. How much would the service production level need to increase if demand from the agriculture sector is increased by 1 unit?
6. How much would the manufacturing production level need to increase if demand from the agricultural sector is increased by 1 unit?

The most up-to-date available input–output table for the American economy is the 2007 table. The entire table (which is officially called an I-O Use Table) may be found in the publication “Annual Input–Output Accounts of the United States Economy, 2007,” which may be downloaded from the Bureau of Economic Analysis web site: www.bea.gov.
**TABLE 2  EXCHANGE OF GOODS AND SERVICES IN THE U.S. FOR 2007 (IN MILLIONS OF 2007 DOLLARS)**

<table>
<thead>
<tr>
<th>Industry</th>
<th>Agric.</th>
<th>Mining</th>
<th>Const.</th>
<th>Manuf.</th>
<th>T.C.U.</th>
<th>Trade</th>
<th>F.I.R.E</th>
<th>Services</th>
<th>Other</th>
<th>Demand</th>
<th>Total Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture</td>
<td>79,195.5</td>
<td>0.7</td>
<td>1,737.8</td>
<td>196,474.3</td>
<td>56.8</td>
<td>3,463.2</td>
<td>2,643.7</td>
<td>20,268.3</td>
<td>2,252.3</td>
<td>64,207.5</td>
<td>370,299.9</td>
</tr>
<tr>
<td>Mining</td>
<td>476.8</td>
<td>61,827.8</td>
<td>8,541.6</td>
<td>390,478.5</td>
<td>116,780.9</td>
<td>112.2</td>
<td>1,966.1</td>
<td>552.5</td>
<td>-146,059</td>
<td>451,012.5</td>
<td></td>
</tr>
<tr>
<td>Construction</td>
<td>1,136.1</td>
<td>123.2</td>
<td>1,362.4</td>
<td>8,822.7</td>
<td>11,458.4</td>
<td>9,259.3</td>
<td>44,325.7</td>
<td>37,531.7</td>
<td>70,555.3</td>
<td>1,185,993</td>
<td>1,370,567.5</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>59,159.8</td>
<td>48,144.4</td>
<td>308,310.1</td>
<td>1,609,532.2</td>
<td>226,949.1</td>
<td>143,737.7</td>
<td>103,338.9</td>
<td>455,521.6</td>
<td>317,079</td>
<td>1,740,597</td>
<td>5,012,369.7</td>
</tr>
<tr>
<td>Transportation, Communication, and Utilities</td>
<td>14,071.1</td>
<td>13,892.9</td>
<td>37,376.4</td>
<td>245,562.7</td>
<td>443,308.6</td>
<td>117,348.5</td>
<td>159,041.1</td>
<td>336,302.9</td>
<td>181,636</td>
<td>2,552,503.9</td>
<td></td>
</tr>
<tr>
<td>Trade</td>
<td>13,539.3</td>
<td>7,128.3</td>
<td>114,366.6</td>
<td>316,133</td>
<td>44,835.3</td>
<td>61,454.1</td>
<td>29,883</td>
<td>99,774</td>
<td>37,668.7</td>
<td>1,759,363</td>
<td>2,484,145</td>
</tr>
<tr>
<td>Finance, Insurance, and Real Estate</td>
<td>24,156.9</td>
<td>30,653.9</td>
<td>42,733.9</td>
<td>121,341.5</td>
<td>145,708</td>
<td>178,698.3</td>
<td>901,967.6</td>
<td>515,522.9</td>
<td>100,578</td>
<td>4,515,756.2</td>
<td></td>
</tr>
<tr>
<td>Services</td>
<td>7,822.5</td>
<td>40,436.6</td>
<td>118,336.3</td>
<td>452,113.1</td>
<td>318,883</td>
<td>350,236.7</td>
<td>434,053.2</td>
<td>893,276.3</td>
<td>401,008</td>
<td>3,719,337</td>
<td>6,735,503.5</td>
</tr>
<tr>
<td>Other</td>
<td>85.3</td>
<td>72.3</td>
<td>1,421.4</td>
<td>3,170.3</td>
<td>7,631.6</td>
<td>10,257</td>
<td>10,906.8</td>
<td>41,005.2</td>
<td>9,904.3</td>
<td>2,278,087</td>
<td>2,362,541.2</td>
</tr>
</tbody>
</table>

7. Find the consumption matrix $A$ and demand vector $D_0$ for this table.
8. Find the production vector $X$ for the consumption matrix $A$ and demand vector $D_0$.
9. If the demand for construction increases by 1 million, how much will the production of the transportation, communication, and utilities sector have to increase to compensate?
10. Which three sectors are most affected by an increase of $1$ million of demand for construction?
11. If the entire $j$th column of the matrix $(I - A)^{-1}$ is added, the result is the amount of new production in the **entire economy** caused by an increase of $1$ million in demand for the $j$th sector. How much new production is produced in the entire economy by an increase of $1$ million in demand for construction? This amount is referred to in economic literature as a **backward linkage**.
12. What would it mean for the $(i, j)$ entry in $(I - A)^{-1}$ to be zero?
In Chapter 1, we discussed linear equations and applications that involve linear equations. In Chapter 2, we studied systems of linear equations. In this chapter, we discuss systems of linear inequalities and an important application involving linear equations and systems of linear inequalities: linear programming.

Whenever the analysis of a problem leads to minimizing or maximizing a linear expression in which the variables obey a collection of linear inequalities, a solution may be obtained using a linear programming model.

Historically, linear programming evolved out of the need to solve problems involving resource allocation by the U.S. Army during World War II. Among those who worked on such problems was George Dantzig, who later gave a general formulation of the linear programming problem and offered a method for solving it, called the simplex method. This method is discussed in Chapter 5.

In this chapter we study ways to solve linear programming problems that involve only two variables. As a result, we can use a geometric approach utilizing the graph of a system of linear inequalities to solve the problem.
Many applications contain language such as
“I only have at most $50 to spend.”
“No more than 40 hours of labor is available.”
“Invest at least as much in stocks as in bonds.”

To express statements such as these in mathematical terms requires an inequality. For example, if \( x \) is the amount to be invested in stocks and \( y \) is the amount to be invested in bonds, the statement “invest more in stocks than in bonds” can be expressed by the inequality \( x > y \). This is an example of a linear inequality in two variables \( x \) and \( y \).

Recall that a linear equation (linear equality) in two variables \( x \) and \( y \) is an equation of the form
\[
Ax + By = C
\]
where \( A \), \( B \), and \( C \) are real numbers and \( A \) and \( B \) are not both zero. If in Equation (1) we replace the equal sign by an inequality symbol, namely, one of the symbols \( <, >, \leq \), or \( \geq \), we obtain a linear inequality in two variables \( x \) and \( y \).

For example, the expressions
\[
3x + 2y \geq 4 \quad 3x + 5y < -8 \quad x - y > 0
\]
are each linear inequalities in two variables. The first of these is called a nonstrict inequality since the expression is satisfied when \( 3x + 2y = 4 \), as well as when \( 3x + 2y > 4 \). The remaining two linear inequalities are strict.

**Graph Linear Inequalities**

The graph of a linear inequality in two variables \( x \) and \( y \) is the set of all points \((x, y)\) for which the inequality is satisfied.

**EXAMPLE 1** Graphing a Linear Inequality

Graph the linear inequality: \( 2x + 3y \geq 6 \)

**SOLUTION** The linear inequality \( 2x + 3y \geq 6 \) is equivalent to \( 2x + 3y > 6 \) or \( 2x + 3y = 6 \). So we begin by graphing the line \( 2x + 3y = 6 \), noting that any point on this line will satisfy the inequality \( 2x + 3y \geq 6 \). See Figure 1(a).
Now we test a few points, such as \((-1, -1), (5, 5), (4, 0), (-4, 0)\), to see if they satisfy the inequality. We do this by substituting the coordinates of each point into the expression \(2x + 3y\) and determining whether the result is \(\geq 6\) or \(< 6\). Remember, we want \(2x + 3y \geq 6\).

\[
\begin{align*}
2x + 3y \\
(-1, -1): & \quad 2(-1) + 3(-1) = -2 - 3 = -5 < 6 \quad \text{Not part of the graph} \\
(5, 5): & \quad 2(5) + 3(5) = 25 > 6 \quad \text{Part of the graph} \\
(4, 0): & \quad 2(4) + 3(0) = 8 > 6 \quad \text{Part of the graph} \\
(-4, 0): & \quad 2(-4) + 3(0) = -8 < 6 \quad \text{Not part of the graph}
\end{align*}
\]

Notice that the two points \((4, 0)\) and \((5, 5)\) that are part of the graph both lie on one side of the line \(2x + 3y = 6\), while the points \((-4, 0)\) and \((-1, -1)\) that are not part of the graph lie on the other side. This is not an accident. The graph of the inequality consists of all points on the same side of the line as \((4, 0)\) and \((5, 5)\). The shaded region of Figure 1(b) illustrates the graph of the inequality.

The inequality in Example 1 is nonstrict, so the corresponding line is part of the graph of the inequality. If the inequality is strict, the corresponding line is not part of the graph of the inequality. We will indicate a strict inequality by using dashes to graph the line.

**Steps for Graphing a Linear Inequality**

**STEP 1** Graph the corresponding linear equation, a line \(L\). If the inequality is nonstrict, graph \(L\) using a solid line; if the inequality is strict, graph \(L\) using dashes.

**STEP 2** Select a test point \(P\) not on the line \(L\).

**STEP 3** Substitute the coordinates of the test point \(P\) into the given inequality. If the coordinates of this point \(P\) satisfy the linear inequality, then all points on the same side of \(L\) as the point \(P\) satisfy the inequality. If the coordinates of the point \(P\) do not satisfy the linear inequality, then all points on the opposite side of \(L\) from \(P\) satisfy the inequality.

**STEP 4** Shade (or strike) the region to be included.
Graph the linear inequality: $2x - y < -4$

**SOLUTION**

**STEP 1** The corresponding linear equation is the line

$L: \quad 2x - y = -4$

Since the inequality is strict, points on $L$ are not part of the graph of the linear inequality. Graph $L$ using a dashed line to indicate this fact. See Figure 2(a).

**STEP 2** Select a point not on the line $L$ to be tested, for example, $(0, 0)$.

**STEP 3** Test $(0, 0)$:

$2x - y = 2(0) - 0 = 0 \quad 2x - y < -4$

Since $0$ is not less than $-4$, the point $(0, 0)$ does not satisfy the inequality. As a result, all points on the opposite side of $L$ from $(0, 0)$ satisfy the inequality.

**STEP 4** The graph of $2x - y < -4$ is the shaded region of Figure 2(b).

\[ \text{NOW WORK PROBLEM 15.} \]

**EXAMPLE 3** Graphing Linear Inequalities

Graph:

(a) $x \leq 3$       (b) $2x \leq y$

**SOLUTION**

(a) The corresponding linear equation is $x = 3$, a vertical line. If we choose $(0, 0)$ as the test point, we find that it satisfies the inequality $[0 \leq 3]$, so all points to the left of, and on, the vertical line also satisfy the inequality. See Figure 3(a).

(b) The corresponding linear equation is $2x = y$. We choose the point $(0, 2)$ as the test point. The point $(0, 2)$ satisfies the inequality $[2(0) \leq 2]$, so all points on the same side of the line as $(0, 2)$ also satisfy the inequality. See Figure 3(b).
The set of points belonging to the graph of a linear inequality [for example, the shaded region in Figure 3(b)] is called a **half-plane**.

**NOW WORK PROBLEM 13.**

**COMMENT** A graphing utility can be used to obtain the graph of a linear inequality. See “Using a Graphing Utility to Graph Inequalities” in Appendix C, Section C.4, for a discussion.

### 2 Graph Systems of Linear Inequalities

Most applications that involve a linear inequality involve more than one linear inequality. For example, if \( x \) is the amount invested in stocks and \( y \) the amount invested in bonds, then both \( x \) and \( y \) must be nonnegative. If we also are told
- “invest at least as much in stocks as in bonds”
- “invest no more than \$5000 in bonds”

then we are led to the inequalities

\[
\begin{align*}
  x & \geq y & y & \leq 5000 \\
  x & \geq 0 & y & \geq 0 
\end{align*}
\]

This is an example of a **system of linear inequalities**.

A **system of linear inequalities** is a collection of two or more linear inequalities. To **graph** a system of inequalities containing two variables \( x \) and \( y \) we locate all the points \((x, y)\) whose coordinates satisfy **each** of the linear inequalities of the system.

### EXAMPLE 4 Determining Whether a Point Belongs to the Graph of a System of Two Linear Inequalities

Determine which of the following points are part of the graph of the system of linear inequalities:

\[
\begin{align*}
  2x + y & \leq 6 & (1) \\
  x - y & \geq 3 & (2)
\end{align*}
\]

(a) \( P_1 = (6, 0) \)  
(b) \( P_2 = (3, 5) \)  
(c) \( P_3 = (0, 0) \)  
(d) \( P_4 = (3, -2) \)

**SOLUTION** We check to see if the given point satisfies each of the inequalities of the system.

(a) \( P_1 = (6, 0) \)

\[
\begin{align*}
  2x + y &= 2(6) + 0 = 12 & x - y &= 6 - 0 = 6 \\
  2x + y &\leq 6 & x - y &\geq 3
\end{align*}
\]

\( P_1 \) satisfies inequality (2) but not inequality (1), so \( P_1 \) is not part of the graph of the system.

(b) \( P_2 = (3, 5) \)

\[
\begin{align*}
  2x + y &= 2(3) + 5 = 11 & x - y &= 3 - 5 = -2 \\
  2x + y &\leq 6 & x - y &\geq 3
\end{align*}
\]

\( P_2 \) satisfies neither inequality (1) nor inequality (2), so \( P_2 \) is not part of the graph of the system.
EXAMPLE 5 Graphing a System of Two Linear Inequalities

Graph the system: \[ \begin{align*}
2x - y &\leq -4 \\
x + y &\geq -1
\end{align*} \]

**SOLUTION** First graph each inequality separately. See Figures 5(a) and 5(b).

The solution of the system consists of all points common to these two half-planes. The region shaded gray in Figure 5(c) represents the solution of the system.
The lines in the system of linear inequalities given in Example 5 intersect. If the two lines of a system of two linear inequalities are parallel, the system of linear inequalities may or may not have a solution. Examples of such situations follow.

**EXAMPLE 6**

**Graphing a System of Two Linear Inequalities**

Graph the system: \[
\begin{align*}
2x - y &\leq -4 \\
2x - y &\leq -2
\end{align*}
\]

**SOLUTION**
First graph each inequality separately. See Figures 6(a) and 6(b). The region shaded gray in Figure 6(c) represents the solution of the system.

Notice that the solution of this system is the same as that of the single linear inequality \(2x - y \leq -4\).
EXAMPLE 7  Graphing a System of Two Linear Inequalities

The solution of the system

\[
\begin{align*}
2x - y & \geq -4 \\
2x - y & \leq -2
\end{align*}
\]

is the region shaded gray in Figure 7.

FIGURE 7

EXAMPLE 8  Graphing a System of Two Linear Inequalities

The system

\[
\begin{align*}
2x - y & \leq -4 \\
2x - y & \geq -2
\end{align*}
\]

has no solution, as Figure 8 indicates (no gray region).

FIGURE 8

NOW WORK PROBLEM 37.

Until now, we have considered systems of only two linear inequalities. The next example is of a system of four linear inequalities. As we shall see, the technique for graphing such systems is the same as that used for graphing systems of two linear inequalities in two variables.
EXAMPLE 9  Graphing a System of Four Linear Inequalities

Graph the system:
\[
\begin{align*}
L_1: & \quad x + y \geq 2 \\
L_2: & \quad 2x + y \geq 3 \\
L_3: & \quad x \geq 0 \\
L_4: & \quad y \geq 0
\end{align*}
\]

The lines \( L_1 \) and \( L_2 \) intersect at the point \((1, 1)\), which you should verify. The inequalities \( x \geq 0 \) and \( y \geq 0 \) indicate that the graph of the system lies in quadrant I. The graph of the system consists of that part of the graphs of the inequalities \( x + y \geq 2 \) and \( 2x + y \geq 3 \) that lie in quadrant I. See the dark blue region in Figure 9.

EXAMPLE 10  Graphing a System of Four Linear Inequalities

Graph the system:
\[
\begin{align*}
L_1: & \quad x + y \leq 2 \\
L_2: & \quad 2x + y \leq 3 \\
L_3: & \quad x \geq 0 \\
L_4: & \quad y \geq 0
\end{align*}
\]

The lines associated with these linear inequalities are the same as those of the previous example. Again the inequalities \( x \geq 0 \), \( y \geq 0 \) indicate that the graph of the system lies in quadrant I. The graph of the system consists of the overlapping part of the graphs of the inequalities \( x + y \leq 2 \) and \( 2x + y \leq 3 \) that lie in quadrant I. See Figure 10.

Now work Problem 39(a).
Some Terminology

Compare the graphs of the systems of linear inequalities given in Figures 9 and 10. The graph in Figure 9 is said to be unbounded in the sense that it extends infinitely far in some direction. The graph in Figure 10 is bounded in the sense that it can be enclosed by some rectangle of sufficiently large dimension. See Figure 11.

The boundary of each of the graphs in Figures 9 and 10 consists of line segments. In fact, the graph of any system of linear inequalities will have line segments as boundaries. The point of intersection of two line segments that form the boundary is called a corner point* of the graph. For example, the graph of the system given in Example 9 has the corner points (0, 3), (1, 1), and (2, 0). See Figure 9. The graph of the system given in Example 10 has the corner points (0, 2), (0, 0), (1, 1). See Figure 10.

We shall soon see that the corner points of the graph of a system of linear inequalities play a major role in the procedure for solving linear programming problems.

NOW WORK PROBLEM 39(b).

3 Solve Applied Problems Involving Systems of Linear Inequalities

EXAMPLE 11 Analyzing a Mixture Problem

Nutt’s Nuts has 75 pounds of cashews and 120 pounds of peanuts. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of cashews and 12 ounces of peanuts and a high-grade mixture that contains 8 ounces of cashews and 8 ounces of peanuts.

(a) Use $x$ to denote the number of packages of the low-grade mixture and use $y$ to denote the number of packages of the high-grade mixture to be made. Write a system of linear inequalities that describes the possible number of each kind of package.

(b) Graph the system and list the corner points.

SOLUTION (a) We begin by naming the variables:

$$x = \text{Number of packages of low-grade mixture}$$

$$y = \text{Number of packages of high-grade mixture}$$

* Some books use the term vertex.
The only meaningful values for $x$ and $y$ are nonnegative values, so

$$x \geq 0 \quad \text{and} \quad y \geq 0$$

Next, note that there is a limit to the number of pounds of cashews and peanuts available. That is, the total number of pounds of cashews cannot exceed 75 pounds (1200 ounces), and the number of pounds of peanuts cannot exceed 120 pounds (1920 ounces). This means that

$$\begin{align*}
\text{Ounces of cashews required for low-grade mixture} & \quad \text{Number of packages of low-grade mixture} \\
\text{Ounces of cashews required for high-grade mixture} & \quad \text{Number of packages of high-grade mixture}
\end{align*}$$

cannot exceed 1200

$$\begin{align*}
\text{Ounces of peanuts required for low-grade mixture} & \quad \text{Number of packages of low-grade mixture} \\
\text{Ounces of peanuts required for high-grade mixture} & \quad \text{Number of packages of high-grade mixture}
\end{align*}$$

cannot exceed 1920

In terms of the data given and the variables introduced, these statements can be written compactly as

$$4x + 8y \leq 1200$$
$$12x + 8y \leq 1920$$

The system of linear inequalities that gives the possible values of $x$ and $y$ is

$$\begin{align*}
4x + 8y & \leq 1200 \\
12x + 8y & \leq 1920 \\
x & \geq 0 \\
y & \geq 0
\end{align*}$$

(b) The system of linear inequalities given above can be simplified to the equivalent form

$$\begin{align*}
x + 2y & \leq 300 \quad \text{Divide both sides by 4.} \\
3x + 2y & \leq 480 \quad \text{Divide both sides by 4.}
\end{align*}$$

$$\begin{align*}
x & \geq 0 \\
y & \geq 0
\end{align*}$$

The graph of the system is given in Figure 12. Notice that the corner points of the graph are labeled. Three are easy to identify by inspection: (0, 0), (0, 150), and (160, 0).

The remaining one (90, 105) is found by solving the system of equations

$$\begin{align*}
x + 2y & = 300 \\
3x + 2y & = 480
\end{align*}$$

By subtracting the first equation from the second, we find $2x = 180$ or $x = 90$. Back-substituting in the first equation, we find $y = 105$. 
EXAMPLE 12  Analyzing Investments

The Text and Academic Authors Association (TAA) plans to invest up to $50,000 in the following two funds based on their average annual return for the past 5 years as of December 31, 2009: ICON Bond I at 4.35% and Frost Dividend Value Equity A at 5.25%. A minimum of $10,000 is to go into the ICON Bond fund and a maximum of $25,000 into the Frost Dividend fund. The governing board of TAA requires that the total average annual yield from these deposits be at least $800.

(a) Let \(x\) represent the amount to be invested in the ICON Bond fund and let \(y\) represent the amount to be invested in the Frost Dividend fund. Write a system of inequalities representing possible amounts to be invested by TAA in these two funds.

(b) Graph the solution set for this system of inequalities and list its corner points.

(c) Interpret the meaning of each corner point.

Source: www.moneycentral.msn.com

SOLUTION  

(a) The variables \(x\) and \(y\) represent, respectively, the amount, in dollars, invested in the ICON Bond fund and the Frost Dividend fund. As a result, we must have

\[x \geq 0 \quad y \geq 0\]

The statement “plans to deposit up to $50,000” requires that

\[x + y \leq 50,000\]

The statement “a minimum of $10,000 in the ICON Bond fund and a maximum of $25,000 in the Frost Dividend fund” requires that

\[x \geq 10,000 \quad y \leq 25,000\]
Finally, the statement “requires a total annual yield of at least $800” means that the average yearly earnings from the ICON Bond fund (0.0435x) plus the average yearly earnings from the Frost Dividend fund (0.0525y) be at least $800. That is,

\[0.0435x + 0.0525y \geq 800\]

Putting all this together, the possible amounts for \(x\) and \(y\) must obey the system of linear inequalities

\[
\begin{align*}
x & \geq 0 \\
y & \geq 0 \\
x + y & \leq 50,000 \\
x & \geq 10,000 \\
y & \leq 25,000 \\
0.0435x + 0.0525y & \geq 800
\end{align*}
\]

(b) The graph of the system is given in Figure 13. Every point \((x, y)\) in the dark gray region will satisfy the conditions given.

(c) Each corner point represents an amount \(x\) and an amount \(y\) that satisfy the conditions given. For example, the corner point \((10000, 25000)\) indicates that $10,000 is invested in the ICON Bond fund and $25,000 is invested in the Frost Dividend fund. The average yearly earnings for this scenario are $0.0435 \times 10000 + 0.0525 \times 25000 = $1747.50. See Table 1 for a listing of each corner point and the average yearly earnings.

\[
\begin{array}{|c|c|}
\hline
\text{Corner Point} & \text{Average Yearly Earnings} \\
\hline
(18391, 0) & $800 \\
(50000, 0) & $2175 \\
(10000, 6952) & $800 \\
(10000, 25000) & $1748 \\
(25000, 25000) & $2400 \\
\hline
\end{array}
\]

All values are rounded to the nearest dollar.

---

**EXERCISE 4.1** Answers Begin on Page AN–19.

**‘Are You Prepared?’ Problems** Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. **True or False**
   - (a) \(-3 < -5\) (b) \(-2 > -1\) (pp. A–16 to A–18)
   - The solution of the inequality \(2x + 3 < 9\) is _________. (pp. A–24 to A–25)
   - Graph the line \(2x + 3y = 6\). (pp. 2–14)

2. **True or False** The lines
   \[2x + y = 6\] and \[y = 2x + 2\]
   are parallel. (pp. 22–26)

3. Find the point of intersection of the lines \(3x + 2y = 480\) and \(x + 2y = 300\). (pp. 22–26)

**Concepts and Vocabulary**

6. The graph of a linear inequality is called a(n) _________.

7. **True or False** Sometimes the graph of a system of linear inequalities will have no points.

8. **True or False** The graph of a system of linear inequalities is sometimes unbounded.
Skill Building

In Problems 9–20, graph each inequality:

9. \( x \geq 0 \)
10. \( y \geq 0 \)
11. \( x < 4 \)
12. \( y \leq 6 \)
13. \( y \geq 1 \)
14. \( x > 2 \)
15. \( 2x + y \leq 4 \)
16. \( 3x + 2y \geq 6 \)
17. \( 5x + y \geq 10 \)
18. \( x + 2y > 4 \)
19. \( x + 5y < 5 \)
20. \( 3x + y \leq 3 \)

21. Without graphing, determine which of the points
\( P_1 = (3, 8), P_2 = (12, 9), P_3 = (5, 1) \) are part of the graph of the following system:
\[
\begin{align*}
  x + 3y &\geq 0 \\
-3x + 2y &\geq 0
\end{align*}
\]

22. Without graphing, determine which of the points
\( P_1 = (9, -5), P_2 = (12, -4), P_3 = (4, 1) \) are part of the graph of the following system:
\[
\begin{align*}
  x + 4y &\leq 0 \\
5x + 2y &\geq 0
\end{align*}
\]

23. Without graphing, determine which of the points
\( P_1 = (2, 3), P_2 = (10, 10), P_3 = (5, 1) \) are part of the graph of the following system:
\[
\begin{align*}
3x + 2y &\geq 0 \\
x + y &\leq 15
\end{align*}
\]

24. Without graphing, determine which of the points
\( P_1 = (2, 6), P_2 = (12, 4), P_3 = (4, 2) \) are part of the graph of the following system:
\[
\begin{align*}
2x - 5y &\leq 0 \\
x + 3y &\leq 15
\end{align*}
\]

In Problems 25–32, determine which region \( a, b, c, \) or \( d \) represents the graph of the given system of linear inequalities. The regions \( a, b, c, \) and \( d \) are nonoverlapping regions bounded by the indicated lines.

25. \( 5x - 4y \leq 8 \)
\( 2x + 5y \leq 23 \)

26. \( 4x - 5y \leq 0 \)
\( 4x + 2y \leq 28 \)

27. \( 2x - 3y \geq -3 \)
\( 2x + 3y \leq 16 \)

28. \( 6x - 5y \leq 5 \)
\( 2x + 4y \geq 30 \)

29. \( 5x - 3y \geq 3 \)
\( 2x + 6y \geq 30 \)

30. \( 5x - 5y \geq 10 \)
\( 6x + 4y \geq 48 \)
In Problems 31–38, graph each system of inequalities.

31. \[
\begin{align*}
2x + 3y & \leq 6 \\
x - y & \geq 2
\end{align*}
\]
32. \[
\begin{align*}
3x - 2y & \geq 6 \\
x + y & \leq 2
\end{align*}
\]
33. \[
\begin{align*}
4x - 3y & \geq 12 \\
x + 3y & \geq 12
\end{align*}
\]
34. \[
\begin{align*}
6x + 3y & \leq 6 \\
x - 3y & \geq 1
\end{align*}
\]
35. \[
\begin{align*}
3x + 2y & \leq 6 \\
x + 2y & \geq 0
\end{align*}
\]
36. \[
\begin{align*}
x - 2y & \leq -4 \\
2x - 4y & \geq 0
\end{align*}
\]
37. \[
\begin{align*}
5x - 2y & \leq -10 \\
10x - 4y & \geq 0
\end{align*}
\]
38. \[
\begin{align*}
x + y & \leq 1 \\
x + y & \leq 2
\end{align*}
\]

In Problems 39–50, (a) graph each system of linear inequalities and (b) determine whether the graph is bounded or unbounded. List each corner point of the graph.

39. \[
\begin{align*}
x + y & \geq 2 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
40. \[
\begin{align*}
2x + 3y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
41. \[
\begin{align*}
x + y & \geq 2 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
42. \[
\begin{align*}
x + y & \leq 2 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
43. \[
\begin{align*}
x + y & \geq 2 \\
x & \leq 8 \\
2x + y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
44. \[
\begin{align*}
x + y & \leq 2 \\
x & \leq 8 \\
2x + y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
45. \[
\begin{align*}
x + y & \leq 2 \\
x & \leq 8 \\
2x + y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
46. \[
\begin{align*}
x + y & \leq 2 \\
x & \leq 8 \\
2x + y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
47. \[
\begin{align*}
x + 2y & \geq 1 \\
y & \leq 4 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
48. \[
\begin{align*}
x + 2y & \geq 1 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
49. \[
\begin{align*}
x + 2y & \geq 1 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
50. \[
\begin{align*}
x + 2y & \geq 1 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

Applications

51. Mixture Problem Nutt’s Nuts has 60 pounds of almonds and 90 pounds of peanuts available. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of almonds and 12 ounces of peanuts and a high-grade mixture that contains 8 ounces of almonds and 8 ounces of peanuts.

(a) Use $x$ to denote the number of packages of the low-grade mixture and use $y$ to denote the number of packages of the high-grade mixture. Write a system of linear inequalities that describes the possible number of each kind of package.

(b) Graph the system and list the corner points.

52. Mixture Problem Nutt’s Nuts has 60 pounds of cashews and 90 pounds of peanuts available. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of cashews and 12 ounces of peanuts and a high-grade mixture that contains 10 ounces of cashews and 6 ounces of peanuts.

(a) Use $x$ to denote the number of packages of the low-grade mixture and use $y$ to denote the number of packages of the high-grade mixture. Write a system of linear inequalities that describes the possible number of each kind of package.

(b) Graph the system and list the corner points.

53. Manufacturing Mike’s Famous Toy Trucks company manufactures two kinds of toy trucks—a dumpster and a tanker. In the manufacturing process, each dumpster requires 3 hours of grinding and 4 hours of finishing, while each tanker requires 2 hours of grinding and 3 hours of finishing. The company has two grinders and three finishers, each of whom works at most 40 hours per week.

(a) Using $x$ to denote the number of dumpsters and $y$ to denote the number of tankers, write a system of linear inequalities that describes the possible numbers of each truck that can be manufactured per week.

(b) Graph the system and list its corner points.

54. Manufacturing Repeat Problem 53 if the company only has one grinder and two finishers available, each of whom works at most 40 hours per week.

55. Financial Planning The Harpers have up to $25,000 to invest. As their financial advisor, you recommend they make the following investments in two funds based on their average yearly performance for the past 5 years ending December 31, 2009: at least $15,000 in a Dividend Equity fund with an average yearly return of 6% and at most $10,000 in a Global Bond fund with an average yearly return of 9%.

(a) Using $x$ to denote the amount to be invested in the Dividend Equity fund and $y$ to denote the amount to be invested in the Global Bond fund, write a system of inequalities that describes this situation.
56. **Financial Planning** Use the information supplied in Problem 55, along with the fact that the couple will invest at least $20,000, to answer parts (a), (b), and (c).

57. **Nutrition** A farmer prepares feed for livestock by combining two types of grain. Each unit of the first grain contains 1 unit of protein and 5 units of iron while each unit of the second grain contains 2 units of protein and 1 unit of iron. Each animal must receive at least 5 units of protein and 16 units of iron each day.

(a) Write a system of linear inequalities that describes the possible amounts of each grain the farmer needs to prepare. Be sure to name the variables.

(b) Graph the system and list the corner points.

58. **Investment Strategy** Kathleen wishes to invest up to a total of $40,000 in class AA bonds and stocks. Furthermore, she believes that the amount invested in class AA bonds should be at most one-third of the amount invested in stocks.

(a) Write a system of linear inequalities that describes the possible amount of investments in each security. Be sure to name the variables.

(b) Graph the system and list the corner points.

59. **Nutrition** To maintain an adequate daily diet, nutritionists recommend the following: at least 85 g of carbohydrate, 70 g of fat, and 50 g of protein. An ounce of food A contains 5 g of carbohydrate, 3 g of fat, and 2 g of protein, while an ounce of food B contains 4 g of carbohydrate, 3 g of fat, and 3 g of protein.

(a) Write a system of linear inequalities that describes the possible quantities of each food. Be sure to name the variables.

(b) Graph the system and list the corner points.

60. **Transportation** A microwave company has two plants, one on the East Coast and one in the Midwest. It takes 25 hours (packing, transportation, and so on) to transport an order of microwaves from the eastern plant to its central warehouse and 20 hours from the Midwest plant to its central warehouse. It costs $80 to transport an order from the eastern plant to the central warehouse and $40 from the midwestern plant to its central warehouse. There are 1000 work-hours available for packing, transportation, and so on, and $3000 for transportation cost.

(a) Write a system of linear inequalities that describes the transportation system. Be sure to name the variables.

(b) Graph the system and list the corner points.

61. **Financial Planning** In March 2010, a couple plans to invest $30,000 in two funds based on their average yearly performance for the past 5 years ending December 31, 2009: ICON Bond fund with an average return of 4.35% and ING Mid Cap Opportunities fund with an average return of 5.05%. They wish to invest a minimum of $3000 in the ICON Bond fund and a minimum of $5000 in the ING Mid Cap Opportunities fund. They also want the total average annual yield from these funds to be at least $500.

(a) Let \( x \) represent the amount to be invested in the ICON Bond fund and let \( y \) represent the amount to be invested in the ING Mid Cap Opportunities fund. Write a system of inequalities representing the possible amounts to be invested in these two funds.

(b) Graph the solution set for this system of inequalities and list its corner points.

(c) Interpret the meaning of each corner point.

62. **Financial Planning** The members of an investment club decide to deposit at least $30,000 and up to $100,000 in two funds based on their average annual returns for the past 5 years: the American Independent Stock I fund at 5% and the Multi Sector Bond Fixed Income fund at 4.75%. Also they decide that the amount invested in the American Independent Stock I fund should be at least $8000 more than the amount invested in the Multi Sector Bond fund I.

(a) Let \( x \) represent the amount to be invested in the American Independent Stock I fund and let \( y \) represent the amount to be invested in the Multi Sector Bond Fixed Income fund. Write a system of inequalities representing the possible amounts to be invested in these two funds.

(b) Graph the solution set for this system of inequalities and list its corner points.

(c) Calculate the total annual yield from both accounts at each corner point of the solution set.

Source: morningstar.com

63. **Mutual Funds** The table lists two mutual funds: the John Hancock Large Cap Equity I fund and the T Rowe Price Latin America fund.

<table>
<thead>
<tr>
<th>Fund</th>
<th>Average Annual Rate of Return for the Five-Year Period Ending 12-31-09</th>
</tr>
</thead>
<tbody>
<tr>
<td>J Hancock Large Cap Equity</td>
<td>9.97%</td>
</tr>
<tr>
<td>T Rowe Price Latin America</td>
<td>23.65%</td>
</tr>
</tbody>
</table>

The manager of a pension fund is considering investing up to $200,000 in these two mutual funds. She concludes that the amount invested in the T Rowe Price Latin America fund should be at least three times as much as the amount invested in the John Hancock Large Cap Equity fund and that a minimum of $20,000 should be invested in the John Hancock Large Cap Equity fund.

(a) Let \( x \) represent the amount to be invested in the John Hancock Large Cap Equity fund and let \( y \) represent the amount to be invested in the T Rowe Price Latin America fund. Write a system of inequalities representing the possible amounts to be invested by this manager in these two funds.
4.2 A Geometric Approach to Linear Programming Problems with Two Variables

(b) Graph the solution set for this system of inequalities and list its corner points.

(c) Calculate a projected annual return from both investments at each corner point of the solution set. Assume that the annual rate of return for each mutual fund will be equal to the average annual rate of return for the five-year period ending on 12-31-09.

Source: biz.yahoo.com

64. Home Mortgages On March 1, 2010, Fremont Bank offered no-closing-cost home mortgage at a 5.125% annual rate for 30-year fixed-rate loans and a 4.375% annual rate for 15-year fixed-rate loans. Suppose Fremont Bank is planning to allocate up to $30 million for these two kinds of loans, with a minimum of $5 million for each type of loan. Fremont Bank has also decided that the amount allocated to the 15-year loans should be no more than half the amount allocated to the 30-year loans.

(a) Let $x$ represent the amount to be allocated to the 30-year fixed-rate loans, and let $y$ represent the amount to be allocated to the 15-year fixed-rate loans. Write a system of inequalities representing possible amounts to be allocated to these two types of loans.

(b) Graph the solution set for this system of inequalities and list its corner points.

(c) For each corner point, calculate the total amount of interest received by Fremont Bank in the first month.

Source: Fremont Bank

Discussion and Writing

65. Make up a system of linear inequalities that has no solution.

66. Make up a system of linear inequalities that has a single point as a solution.

‘Are You Prepared?’ Answers

1. (a) False (b) False 2. $x \leq 3$ 3. $y \geq 0$ 4. (90, 105) 5. False

4.2 A Geometric Approach to Linear Programming Problems with Two Variables*

OBJECTIVES

1. Identify a linear programming problem with two variables (p. 192)
2. Solve a linear programming problem with two variables (p. 193)

* Because the word programming has come to be associated with the writing of computer code, many people have opted to change from the terminology linear programming to linear optimization. For now, we shall continue to use the historical and more familiar term linear programming in this chapter and the next chapter.
1 Identify a Linear Programming Problem with Two Variables

To help see the characteristics of a linear programming problem, look again at Example 11 of the previous section.

Nutt’s Nuts has 75 pounds of cashews and 120 pounds of peanuts. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of cashews and 12 ounces of peanuts and a high-grade mixture that contains 8 ounces of cashews and 8 ounces of peanuts.

Suppose that in addition to the information given above, we also know what the profit will be on each type of mixture. For example, suppose the profit is $0.25 on each package of the low-grade mixture and is $0.45 on each package of the high-grade mixture. The question of importance to the manager is “How many packages of each type of mixture should be prepared to maximize the profit?”

If \( P \) symbolizes the profit, \( x \) the number of packages of low-grade mixture, and \( y \) the number of high-grade packages, then the question can be restated as “What are the values of \( x \) and \( y \) so that the expression

\[
P = 0.25x + 0.45y
\]

is a maximum?”

This problem is typical of a linear programming problem. It requires that a certain linear expression, in this case the profit, be maximized. This linear expression is called the objective function. Furthermore, the problem requires that the maximum profit be achieved under certain restrictions or constraints, each of which are linear inequalities involving the variables. The linear programming problem may be restated as

Maximize

\[
P = 0.25x + 0.45y
\]

subject to the conditions that

\[
\begin{align*}
x + 2y &\leq 300 \quad \text{Cashew constraint} \\
3x + 2y &\leq 480 \quad \text{Peanut constraint} \\
x &\geq 0 \quad \text{Nonnegativity constraint} \\
y &\geq 0 \quad \text{Nonnegativity constraint}
\end{align*}
\]

In general, every linear programming problem has two components:

1. A linear objective function to be maximized or minimized.
2. A collection of linear inequalities that must be satisfied simultaneously.

### Definition

**Linear Programming Problem**

A linear programming problem in two variables, \( x \) and \( y \), consists of maximizing or minimizing an objective function

\[
z = Ax + By
\]

where \( A \) and \( B \) are given real numbers, not both zero, subject to certain conditions or constraints expressible as a system of linear inequalities in \( x \) and \( y \). The points that satisfy all the constraints are called feasible points.
Let’s look at this definition more closely. To maximize (or minimize) the quantity 
\[ z = Ax + By \]
means to locate the points \((x, y)\) that result in the largest (or smallest) 
value of \(z\). But not all points \((x, y)\) are eligible. Only the points that obey all the 
constraints are potential solutions. That is, only feasible points are eligible to be solutions.

2 Solve a Linear Programming Problem with Two Variables

In a linear programming problem we want to find the feasible point that maximizes (or 
minimizes) the objective function.  

By a solution to a linear programming problem we mean a feasible point \((x, y)\), 
together with the value of the objective function at that point, which maximizes (or 
minimizes) the objective function. If none of the feasible points maximizes (or minimizes) 
the objective function, or if there are no feasible points, then the linear programming 
problem has no solution.

**EXAMPLE 1 Solving a Linear Programming Problem**

Minimize 
\[ z = x + 2y \]

subject to the constraints
\[
\begin{align*}
    x + y &\geq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

**SOLUTION** The objective function to be minimized is \(z = x + 2y\). The constraints are the linear inequalities
\[
\begin{align*}
    x + y &\geq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

Begin by graphing the constraints. See Figure 14. The shaded portion illustrates the set of 
feasible points.

To see if there is a smallest \(z\), graph \(z = x + 2y\) for some choice of \(z\), say, \(z = 3\). See 
Figure 15. By moving the line \(x + 2y = 3\) parallel to itself, we can observe what happens 
for different values of \(z\). Since we want a minimum value for \(z\), we try to move \(z = x + 2y\) 
down as far as possible while keeping some part of the line within the set of feasible points. 
The “best” solution is obtained when the line just touches a corner point of the set of 
feasible points. If you refer to Figure 15, you will see that the best solution is \(x = 1, y = 0\), 
which yields \(z = 1\). There is no other feasible point for which \(z\) is smaller.

The next example illustrates a linear programming problem that has no solution.
EXAMPLE 2  A Linear Programming Without a Solution

Maximize

\[ z = x + 2y \]

subject to the constraints

\[
\begin{align*}
    x + y &\geq 1 \\
    x &\geq 0 \\
    y &\geq 0
\end{align*}
\]

First, graph the constraints. See Figure 16. The shaded portion illustrates the set of feasible points. Notice that the graph is unbounded.

The graphs of the objective function \( z = x + 2y \) for \( z = 2, z = 8, \) and \( z = 12 \) are also shown in Figure 16. Observe that we continue to get larger values for \( z \) by moving the graph of the objective function upward. But there is no feasible point that will make \( z \) largest. No matter how large a value is assigned to \( z \), there is a feasible point that will give a larger value. Since there is no feasible point that makes \( z \) largest, we conclude that this linear programming problem has no solution.

Notice that the corresponding minimum problem—Minimize \( z = x + 2y \) subject to the constraints given in Example 2—has a solution. The minimum value of \( z \) is \( z = 1 \), obtained at the feasible point \((1,0)\).

Examples 1 and 2 demonstrate that sometimes a linear programming problem has a solution and sometimes it does not.

Theorem  Some Conditions Relating to the Solution of a Linear Programming Problem

Consider a linear programming problem with the set \( R \) of feasible points and objective function \( z = Ax + By \).

1. If \( R \) is the empty set, then the linear programming problem has no solution and \( z \) has neither a maximum nor a minimum value.
2. If \( R \) is bounded, then \( z \) has both a maximum and a minimum value on \( R \).
3. If \( R \) is unbounded, \( A > 0, B > 0 \), and the constraints include \( x \geq 0 \) and \( y \geq 0 \), then \( z \) has a minimum value on \( R \) but not a maximum value (see Example 2).

The conditions listed above are not exhaustive. When the set of feasible solutions is unbounded, it is best to graph the set of feasible solutions and some typical values of the objective function to determine whether a solution exists or not.

In Example 1 we found that the feasible point that minimizes \( z \) occurs at a corner point. In fact, if there are feasible points minimizing (or maximizing) the objective function, at least one will be at a corner point of the set of feasible points.
Theorem

**Fundamental Theorem of Linear Programming with Two Variables**

Consider a linear programming problem with the set $R$ of feasible points and objective function $z = Ax + By$, where $x \geq 0, y \geq 0$. If a linear programming problem has a solution, it is located at a corner point of the set $R$ of feasible points; if a linear programming problem has multiple solutions, at least one of them is located at a corner point of the set $R$ of feasible points. In either case, the corresponding value of the objective function is unique.

Since the objective function attains its maximum or minimum value at the corner points of the set of feasible points, we can outline a procedure for solving a linear programming problem provided that it has a solution.

**Steps for Solving a Linear Programming Problem**

If a linear programming problem has a solution, follow these steps to find it:

**STEP 1** Write an expression for the quantity that is to be maximized or minimized (the objective function).

**STEP 2** Determine all the constraints and graph the set of feasible points.

**STEP 3** List the corner points of the set of feasible points.

**STEP 4** Determine the value of the objective function at each corner point.

**STEP 5** Select the maximum or minimum value of the objective function.

**EXAMPLE 3 Solving a Linear Programming Problem**

Maximize and minimize the objective function

$$z = x + 5y$$

subject to the constraints

$$\begin{align*}
\begin{cases}
& x + 4y \leq 12 \\
& x \leq 8 \\
& x + y \geq 2 \\
& x \geq 0 \\
& y \geq 0
\end{cases}
\end{align*}$$

**SOLUTION**

**STEP 1** The objective function is $z = x + 5y$.

**STEP 2** The constraints consist of a system of five linear inequalities. Graph the system. See Figure 17. The shaded region illustrates the set of feasible points. Notice in Figure 17 that we have labeled each line from the system of linear inequalities. We have also
labeled the corner points. Since the set of feasible points is bounded, we know the objective function has both a maximum and a minimum value in the region.

**Figure 17**

<table>
<thead>
<tr>
<th>Corner Point (x, y)</th>
<th>Value of Objective Function z = x + 5y</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 3)</td>
<td>z = 0 + 5(3) = 15</td>
</tr>
<tr>
<td>(8, 1)</td>
<td>z = 8 + 5(1) = 13</td>
</tr>
<tr>
<td>(8, 0)</td>
<td>z = 8 + 5(0) = 8</td>
</tr>
<tr>
<td>(2, 0)</td>
<td>z = 2 + 5(0) = 2</td>
</tr>
<tr>
<td>(0, 2)</td>
<td>z = 0 + 5(2) = 10</td>
</tr>
</tbody>
</table>

**STEP 3** The corner points of the set of feasible points are

(0, 3) (8, 1) (8, 0) (2, 0) (0, 2)

**STEP 4** Evaluate the objective function z = x + 5y at each corner point. See Table 2.

**STEP 5** The maximum value of z is 15, and it occurs at the point (0, 3). The minimum value of z is 2, and it occurs at the point (2, 0).

**NOW WORK PROBLEMS 5 AND 21.**

The Fundamental Theorem of Linear Programming indicates that it is possible for a feasible point that is not a corner point to minimize (or maximize) the objective function. This occurs if the slope of the objective function is the same as the slope of one of the boundaries of the set of feasible points and if the two adjacent corner points are solutions. Then all the points on the line segment joining the two corner points are also solutions. The following example illustrates this situation.

**EXAMPLE 4 Solving a Linear Programming Problem with Multiple Solutions**

Minimize

$$z = x + 2y$$

subject to the constraints

$$\begin{align*}
x + y & \geq 1 \\
2x + 4y & \geq 3 \\
x & \geq 0 \\
y & \geq 0
\end{align*}$$
4.2 A Geometric Approach to Linear Programming Problems with Two Variables

SOLUTION

STEP 1 The objective function is \( z = x + 2y \).

STEP 2 The constraints consist of a system of four linear inequalities. The graph is given in Figure 18. Although the set of feasible points is unbounded, the linear programming problem will have a solution since condition 2 of the Criteria for the Existence of a Solution is satisfied. Check this for yourself.

FIGURE 18

STEP 3 The corner points are \((0, 1)\), \(\left(\frac{1}{2}, \frac{1}{2}\right)\), and \(\left(\frac{3}{2}, 0\right)\).

STEP 4 At each corner point, the value of the objective function is:

\[
(0, 1): \quad z = x + 2y = 0 + 2(1) = 2
\]

\[
\left(\frac{1}{2}, \frac{1}{2}\right): \quad z = x + 2y = \frac{1}{2} + 2\left(\frac{1}{2}\right) = 1.5
\]

\[
\left(\frac{3}{2}, 0\right): \quad z = x + 2y = \frac{3}{2} + 2(0) = 1.5
\]

STEP 5 The minimum value is 1.5, obtained at the corner points \(\left(\frac{1}{2}, \frac{1}{2}\right)\) and \(\left(\frac{3}{2}, 0\right)\).

This linear program problem has multiple solutions. To see why, graph the objective equation \( z = x + 2y \) for some choice of \( z \) and move it down. See Figure 19. We find that a minimum is reached when \( z = \frac{3}{2} \). In fact, any point on the line \( 2x + 4y = 3 \) between the adjacent corner points \(\left(\frac{1}{2}, \frac{1}{2}\right)\) and \(\left(\frac{3}{2}, 0\right)\) and including these corner points will minimize the objective function. Of course, the reason any feasible point on \( 2x + 4y = 3 \) minimizes the objective equation \( z = x + 2y \) is that these two lines each have slope \(-\frac{1}{2}\).

FIGURE 19

NOW WORK PROBLEM 33.
In Problems 29–36, minimize (if possible) the quantity subject to the given constraints.

In Problems 21–28, maximize (if possible) the quantity subject to the given constraints.

In Problems 15–20, list the corner points for each collection of constraints of a linear programming problem.

In Problems 5–14, the figure on the right illustrates the graph of the set of feasible points of a linear programming problem. Find the maximum and minimum values of each objective function.

### Concepts and Vocabulary

1. In a linear programming problem, the function to be maximized (or minimized) is called the _______ _______.
2. In a linear programming problem, any point that satisfies the system of constraints is called a(n) _______ _______.
3. True or False Every linear programming problem has a solution.
4. True or False Some linear programming problems have multiple solutions.

### Skill Building

In Problems 5–14, the figure on the right illustrates the graph of the set of feasible points of a linear programming problem. Find the maximum and minimum values of each objective function.

5. \( z = 2x + 3y \)
6. \( z = 3x + 2y \)
7. \( z = x + y \)
8. \( z = 3x + 3y \)
9. \( z = x + 6y \)
10. \( z = 6x + y \)
11. \( z = 3x + 4y \)
12. \( z = 4x + 3y \)
13. \( z = 10x + y \)
14. \( z = x + 10y \)

In Problems 15–20, list the corner points for each collection of constraints of a linear programming problem.

15. \( \begin{align*}
x & \leq 13 \\
4x + 3y & \geq 12 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
16. \( \begin{align*}
x & \leq 8 \\
2x + 3y & \geq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
17. \( \begin{align*}
y & \leq 10 \\
x + y & \leq 12 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
18. \( \begin{align*}
y & \leq 8 \\
2x + y & \geq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
19. \( \begin{align*}
x & \leq 10 \\
4x + 3y & \geq 12 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
20. \( \begin{align*}
x & \leq 9 \\
2x + 3y & \geq 24 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)

In Problems 21–28, maximize (if possible) the quantity \( z = 5x + 7y \) subject to the given constraints.

21. \( \begin{align*}
x + y & \leq 2 \\
y & \geq 1 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
22. \( \begin{align*}
2x + 3y & \leq 6 \\
x & \leq 2 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
23. \( \begin{align*}
x + y & \geq 2 \\
2x + 3y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
24. \( \begin{align*}
x + y & \geq 2 \\
2x + 3y & \leq 12 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
25. \( \begin{align*}
x + y & \leq 2 \\
x + y & \leq 8 \\
2x + y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
26. \( \begin{align*}
x + y & \leq 2 \\
x + y & \leq 8 \\
x + 2y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
27. \( \begin{align*}
x + y & \leq 10 \\
x & \geq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
28. \( \begin{align*}
x + y & \leq 8 \\
x & \geq 2 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)

In Problems 29–36, minimize (if possible) the quantity \( z = 2x + 3y \) subject to the given constraints.

29. \( \begin{align*}
x + y & \leq 2 \\
y & \leq x \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
30. \( \begin{align*}
3x + y & \leq 3 \\
x & \geq x \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
31. \( \begin{align*}
x + y & \geq 2 \\
x + 3y & \leq 12 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
32. \( \begin{align*}
x + y & \leq 8 \\
x + 3y & \geq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \)
4.3 Models Utilizing Linear Programming with Two Variables

**OBJECTIVES**

1. Solve applied problems using linear programming with two variables (p. 199)

**Discussion and Writing**

53. Find conditions for a linear programming problem to have a solution (maximum, minimum, or both) if the set of feasible point is unbounded. Assume the constraints \( x \geq 0, y \geq 0 \) are not present. Further assume there are only two constraints whose corresponding equations intersect. Hint: Compare the slopes of the objection with the equations corresponding to the constraints.

In Problems 37–44, find the maximum and minimum values (if possible) of the given objective function subject to the constraints.

37. \( z = x + y \)

38. \( z = 2x + 3y \)

39. \( z = 5x + 2y \)

40. \( z = x + 2y \)

41. \( z = 3x + 4y \)

42. \( z = 3x + 6y \)

43. \( z = 10x + y \)

44. \( z = x + 10y \)

45. Find the maximum and minimum values of \( z = 18x + 30y \) subject to the constraints \( 3x + 3y \geq 9, -x + 4y \leq 12 \), and \( 4x - y \leq 12 \), where \( x \geq 0 \) and \( y \geq 0 \).

46. Find the maximum and minimum values of \( z = 20x + 16y \) subject to the constraints \( 4x + 3y \geq 12, -2x + 4y \leq 16 \), and \( 6x - y \leq 18 \), where \( x \geq 0 \) and \( y \geq 0 \).

47. Find the maximum and minimum values of \( z = 7x + 6y \) subject to the constraints \( 2x + 3y \geq 6, -3x + 4y \leq 8 \), and \( 5x - y \leq 15 \), where \( x \geq 0 \) and \( y \geq 0 \).

48. Find the maximum and minimum values of \( z = 6x + 3y \) subject to the constraints \( 2x + 2y \geq 4, -x + 5y \leq 10 \), and \( 3x - 3y \leq 6 \), where \( x \geq 0 \) and \( y \geq 0 \).

49. Maximize \( z = -20x + 30y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 5x + 3y \geq 15 \), and \( -3x + 3y \leq 21 \).

50. Maximize \( z = -10x + 10y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 6x + y \geq 6 \), and \( -3x + y \leq 7 \).

51. Maximize \( z = -12x + 24y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 3x + 3y \geq 9 \), and \( -3x + 2y \leq 14 \).

52. Maximize \( z = -20x + 10y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 4x + 3y \geq 12 \), and \( -3x + y \leq 7 \).

4.3 Models Utilizing Linear Programming with Two Variables

Now let’s solve the problem of the cashews and peanuts discussed at the start of the previous section.
EXAMPLE 1 Maximizing Profit

Nutt’s Nuts has 75 pounds of cashews and 120 pounds of peanuts available. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of cashews and 12 ounces of peanuts and a high-grade mixture that contains 8 ounces of cashews and 8 ounces of peanuts. The profit is $0.25 on each package of the low-grade mixture and $0.45 on each package of the high-grade mixture. How many packages of each type of mixture should be prepared to maximize the profit?

SOLUTION Let \( x \) denote the number of packages of the low-grade mixture and \( y \) the number of packages of the high-grade mixture. If \( P \) denotes the profit, then

\[
P = 0.25x + 0.45y
\]

STEP 1 We want to maximize \( P \) so \( P = 0.25x + 0.45y \) is the objective function.

STEP 2 Look back at Example 11 of Section 4.1. The constraints consist of the system of linear inequalities

\[
\begin{align*}
x + 2y &\leq 300 & \text{The cashew constraint} \\
3x + 2y &\leq 480 & \text{The peanut constraint} \\
x &\geq 0 & \text{The nonnegativity constraint} \\
y &\geq 0 & \text{The nonnegativity constraint}
\end{align*}
\]

See Figure 20 for the graph of the set of feasible points. Notice that this set is bounded, so the linear programming problem has a solution.

STEP 3 The corner points are (0, 0), (0, 150), (160, 0), and (90, 105).

STEP 4 In Table 3 we evaluate the objective function \( P \) at each corner point.

TABLE 3

<table>
<thead>
<tr>
<th>Corner Point ((x, y))</th>
<th>Value of Objective Function ( P = (0.25)x + (0.45)y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>( P = (0.25)(0) + (0.45)(0) = 0 )</td>
</tr>
<tr>
<td>(0, 150)</td>
<td>( P = (0.25)(0) + (0.45)(150) = 67.50 )</td>
</tr>
<tr>
<td>(160, 0)</td>
<td>( P = (0.25)(160) + (0.45)(0) = 40.00 )</td>
</tr>
<tr>
<td>(90, 105)</td>
<td>( P = (0.25)(90) + (0.45)(105) = 69.75 )</td>
</tr>
</tbody>
</table>
A maximum profit is obtained if 90 packages of low-grade mixture and 105 packages of high-grade mixture are made. The maximum profit obtainable under the conditions described is $69.75.

NOW WORK PROBLEM 1.

EXAMPLE 2 Maximizing Profit

Mike’s Famous Toy Trucks manufactures two kinds of toy trucks—a standard model and a deluxe model. In the manufacturing process each standard model requires 2 hours of grinding and 2 hours of finishing, and each deluxe model needs 2 hours of grinding and 4 hours of finishing. The company has two grinders and three finishers, each of whom works at most 40 hours per week. Each standard model toy truck brings a profit of $3 and each deluxe model a profit of $4. Assuming that every truck made will be sold, how many of each should be made to maximize profit?

SOLUTION First, name the variables:

- \( x \) = Number of standard models made
- \( y \) = Number of deluxe models made

The quantity to be maximized is the profit, which we denote by \( P \). Since each standard models brings a profit of $3 and each deluxe model a profit of $4 we have

\[
P = 3x + 4y
\]

This is the objective function.

To manufacture one standard model requires 2 grinding hours and to make one deluxe model requires 2 grinding hours. The number of grinding hours needed to manufacture \( x \) standard and \( y \) deluxe models is

\[
2x + 2y
\]

But the total amount of grinding time available is only 80 hours per week. This means we have the constraint

\[
2x + 2y \leq 80 \quad \text{Grinding time constraint}
\]

Similarly, for the finishing time we have the constraint

\[
2x + 4y \leq 120 \quad \text{Finishing time constraint}
\]

By simplifying each of these constraints and adding the nonnegativity constraints \( x \geq 0 \) and \( y \geq 0 \), we can list all the constraints for this problem:

\[
\begin{align*}
\text{(1)} & : x + y \leq 40 \\
\text{(2)} & : x + 2y \leq 60 \\
\text{(3)} & : x \geq 0 \\
\text{(4)} & : y \geq 0
\end{align*}
\]

Figure 21 illustrates the set of feasible points, which is bounded.
The corner points of the set of feasible points are 
(0, 0)  (0, 30)  (40, 0)  (20, 20) 
Table 4 lists each corner point and the corresponding value of the objective equation:

**TABLE 4**

<table>
<thead>
<tr>
<th>Corner Point (x, y)</th>
<th>Value of Objective Function P = 3x + 4y</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>P = 0</td>
</tr>
<tr>
<td>(0, 30)</td>
<td>P = $120</td>
</tr>
<tr>
<td>(40, 0)</td>
<td>P = $120</td>
</tr>
<tr>
<td>(20, 20)</td>
<td>P = 3(20) + 4(20) = $140</td>
</tr>
</tbody>
</table>

A maximum profit is obtained if 20 standard trucks and 20 deluxe trucks are manufactured. The maximum profit is $140.

**EXAMPLE 3  Financial Planning**

A couple has up to $30,000 to invest in mutual funds. Their broker recommends investing in two funds based on their average annual return for the 5 years ending on December 31, 2009: the Templeton Global Bond fund yielding 8% and the Franklin International Small Cap Growth fund yielding 12%. After some consideration the couple decides to invest at most $12,000 in the Franklin International Small Cap Growth fund and at least $6000 in the Templeton Global Bond fund. They also want the amount invested in the Templeton Global Bond fund to exceed or equal the amount invested in the Franklin International Small Cap Growth fund. What should the broker recommend if the couple (quite naturally) wants to maximize the return on their investment?

**Source:** Franklin Templeton

**SOLUTION**

First, name the variables:

- x = Amount invested in the Global Bond fund
- y = Amount invested in the Franklin International Small Cap Growth fund

The quantity to be maximized, the couple's return on investment, which we denote by P, is

\[ P = 0.08x + 0.12y \]

This is the objective function.

The conditions specified by the problem are

**Up to $30,000 available to invest**

\[ x + y \leq 30,000 \]  

**Invest at most $12,000 in the Franklin International Small Cap Growth fund**

\[ y \leq 12,000 \]

**Invest at least $6000 in the Global Bond fund**

\[ x \geq 6000 \]

**Amount in the Global Bond fund must exceed or equal the amount in the Franklin International Small Cap growth fund**

\[ x \geq y \]

In addition, we must have the conditions \( x \geq 0 \) and \( y \geq 0 \). The total list of constraints is

\[
\begin{align*}
  x + y &\leq 30,000 \quad (1) \\
  y &\leq 12,000 \quad (2) \\
  x &\geq 6000 \quad (3) \\
  x &\geq y \quad (4) \\
  x &\geq 0 \quad (5) \\
  y &\geq 0 \quad (6)
\end{align*}
\]
Figure 22 illustrates the set of feasible points, which is bounded. The corner points of the set of feasible points are

$(6000, 0) \quad (6000, 6000) \quad (12000, 12000) \quad (18000, 12000) \quad (30000, 0)$

The corresponding return on investment at each corner point is

$(6000, 0): \quad P = 0.08(6000) + 0.12(0) = $480$

$(6000, 6000): \quad P = 0.08(6000) + 0.12(6000) = 480 + 720 = $1200$

$(12000, 12000): \quad P = 0.08(12,000) + 0.12(12,000) = 960 + 1440 = $2400$

$(18000, 12000): \quad P = 0.08(18,000) + 0.12(12,000) = 1440 + 1440 = $2880$

$(30000, 0): \quad P = 0.08(30,000) + 0.12(0) = $2400$

The maximum return on investment is $2880, obtained by placing $18,000 in the global Bond fund and $12,000 in the Franklin International Small Cap growth fund.

**EXAMPLE 4 Manufacturing Vitamin Pills—Maximizing Profit**

A pharmaceutical company makes two types of vitamins at its New Jersey plant—a high-potency, antioxidant vitamin and a vitamin enriched with added calcium. Each high-potency vitamin contains, among other things, 500 mg of vitamin C and 40 mg of calcium and generates a profit of $0.10 per tablet. A calcium-enriched vitamin tablet contains 100 mg of vitamin C and 400 mg of calcium and generates a profit of $0.05 per tablet. Each day the company has available 235 kg of vitamin C and 156 kg of calcium for use. Assuming all vitamins made are sold, how many of each type of vitamin should be manufactured to maximize profit?

**Source:** Centrum Vitamin Supplements.

**SOLUTION** First name the variables:

$x = \text{Number (in thousands) of high-potency vitamins to be produced}$

$y = \text{Number (in thousands) of calcium-enriched vitamins to be produced}$
We want to maximize the profit, $P$, which is given by:

$$ P = 0.10x + 0.05y \quad x \text{ and } y \text{ in thousands} $$

Since 1 kg = 1,000,000 mg, the constraints, in mg, take the form

$$\begin{align*}
500x + 100y &\leq 235,000 & \text{vitamin C constraint (in thousands of mg)} \\
40x + 400y &\leq 156,000 & \text{calcium constraint (in thousands of mg)} \\
x &\geq 0 & \text{nonnegativity constraints (in thousands)} \\
y &\geq 0 & 
\end{align*}$$

Figure 23 illustrates the set of feasible points, which is bounded. The corner points of the set of feasible points are

$$ (0, 0) \quad (0, 390) \quad (400, 350) \quad (470, 0) $$

Since $x$ and $y$ are in thousands, the profit corresponding to each corner point is

$$\begin{align*}
(0, 0): & \quad P = 0.10(0) + 0.05(0) = 0 = \$0 \\
(0, 390): & \quad P = 0.10(0) + 0.05(390) = 19.5 \text{ thousand} = \$19,500 \\
(400, 350): & \quad P = 0.10(400) + 0.05(350) = 57.5 \text{ thousand} = \$57,500 \\
(470, 0): & \quad P = 0.10(470) + 0.05(0) = 47 \text{ thousand} = \$47,000
\end{align*}$$

The maximum profit is $57,500, obtained when 400,000 high-potency vitamins are produced ($x = 400$ thousand units) and 350,000 calcium-enriched vitamins are produced ($y = 350$ thousand units).

NOW WORK PROBLEM 5.

**EXAMPLE 5 Minimizing Production Costs**

A factory produces gasoline engines and diesel engines. Each week the factory is obligated to deliver at least 20 gasoline engines and at least 15 diesel engines. Due to physical limitations, however, the factory cannot make more than 60 gasoline engines nor more than 40 diesel engines in any given week. Finally, to prevent layoffs, a total of at least 50 engines must be produced.

(a) If gasoline engines cost $450 each to produce and diesel engines cost $550 each to produce, how many of each should be produced per week to minimize the cost?
What is the excess capacity of the factory? That is, how many of each kind of engine is being produced in excess of the number that the factory is obligated to deliver?

**SOLUTION (a)** First name the variables:

- $x =$ the number of gasoline engines produced
- $y =$ the number of diesel engines produced

If $C$ is the weekly cost of production in dollars, then

$$C = 450x + 550y$$

This is the objective function to be minimized.

The constraints are

- $x \geq 20$ At least 20 gasoline engines produced per week
- $y \geq 15$ At least 15 diesel engines produced per week
- $x \leq 60$ No more than 60 gasoline engines produced
- $y \leq 40$ No more than 40 diesel engines produced
- $x + y \geq 50$ At least 50 engines produced
- $x \geq 0, y \geq 0$ Nonnegativity constraints

Figure 24 shows the graph of the set of feasible points.

The corner points are $(20, 30), (20, 40), (60, 40), (60, 15)$, and $(35, 15)$. The cost $C$ for each of these levels of production is

- $(20, 30): C = 450x + 550y = 450(20) + 550(30) = \$25,500$
- $(20, 40): C = 450(20) + 550(40) = \$31,000$
- $(60, 40): C = 450(60) + 550(40) = \$49,000$
- $(60, 15): C = 450(60) + 550(15) = \$35,250$
- $(35, 15): C = 450(35) + 550(15) = \$24,000$

The minimum cost is $\$24,000$, obtained when 35 gasoline engines and 15 diesel engines are produced.

**b)** The weekly obligation of the factory is to produce at least 20 gasoline engines and at least 15 diesel engines. The minimum cost of production produces an excess of 15 gasoline engines.

**Now work Problem 7.**
Applications

1. **Farm Management** A farmer has 70 acres of land available for planting either soybeans or wheat. The cost of preparing the soil, the workdays required, and the expected profit per acre planted for each type of crop are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Soybeans</th>
<th>Wheat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preparation cost per acre</td>
<td>$60</td>
<td>$30</td>
</tr>
<tr>
<td>Workdays required per acre</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Profit per acre</td>
<td>$180</td>
<td>$100</td>
</tr>
</tbody>
</table>

The farmer cannot spend more than $1800 in preparation costs nor use more than a total of 120 workdays.

(a) How many acres of each crop should be planted to maximize the profit?

(b) What is the maximum profit?

(c) What is the maximum profit if the farmer is willing to spend no more than $2400 on preparation?

2. **Farm Management** A small farm in Illinois has 100 acres of land available on which to grow corn and soybeans. The following table shows the cultivation cost per acre, the labor cost per acre, and the expected profit per acre. The column on the right shows the amount of money available for each of these expenses. Find the number of acres of each crop that should be planted to maximize profit.

<table>
<thead>
<tr>
<th></th>
<th>Soybeans</th>
<th>Corn</th>
<th>Money Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cultivation cost per acre</td>
<td>$40</td>
<td>$60</td>
<td>$1800</td>
</tr>
<tr>
<td>Labor cost per acre</td>
<td>$60</td>
<td>$60</td>
<td>$2400</td>
</tr>
<tr>
<td>Profit per acre</td>
<td>$200</td>
<td>$250</td>
<td></td>
</tr>
</tbody>
</table>

3. **Investment Strategy** An investment broker wants to invest up to $20,000. She can invest in two mutual funds based on their yearly average return for the 5 years ending December 31, 2009: the Franklin Natural Resources fund yielding 10% and the Oppenheimer Developing Markets A fund yielding 15%. She wants to invest at least $5000 in the Franklin Natural Resources fund and no more than $8000 in the Oppenheimer Developing Markets A fund. How much should she invest in each type of fund to maximize her return?

4. **Investment Strategy** A financial consultant wishes to invest up to a total of $30,000 in two types of securities, one that yields 10% per year and another that yields 8% per year. Furthermore, she believes that the amount invested in the first security should be at most one-third of the amount invested in the second security. What investment program should the consultant pursue in order to maximize income?

5. **Manufacturing Vitamin Pills** A pharmaceutical company has 300 kg of vitamin C and 220 kg of calcium available each day for the manufacture of high-potency, antioxidant vitamins and vitamins enriched with added calcium. If each high-potency vitamin contains, among other things, 500 mg of vitamin C and 40 mg of calcium and generates a profit of $0.10 per tablet, and each calcium-enriched vitamin tablet contains 100 mg of vitamin C and 400 mg of calcium and generates a profit of $0.05 per tablet, how many of each type of vitamin should be manufactured to maximize profit?

Source: Centrum Vitamin Supplements

6. **Inventory** Blink Appliances has a sale on microwaves and stoves. Each microwave requires 2 hours to unpack and set up, and each stove requires 1 hour. The storeroom space is limited to 50 items. The budget of the store allows only 80 hours of employee time for unpacking and setup. Microwaves sell for $300 each, and stoves sell for $200 each. How many of each should the store order to maximize revenue?

7. **Banquet Seating** A banquet hall offers two types of tables for rent: 6-person rectangular tables at a cost of $28 each and 10-person round tables at a cost of $52 each. Kathleen would like to rent the hall for a wedding banquet and needs tables for 250 people. The room can have a maximum of 35 tables and the hall only has 15 rectangular tables available. How many of each type of table should be rented to minimize cost and what is the minimum cost?

Source: facilities.princeton.edu

8. **Spring Break** The student activities department of a community college plans to rent buses and vans for a spring-break trip. Each bus has 40 regular seats and 1 handicapped seat; each van has 8 regular seats and 3 handicapped seats. The rental cost is $350 for each van and $975 for each bus. If 320 regular and 36 handicapped seats are required for the trip, how many vehicles of each type should be rented to minimize cost?

Source: www.busrates.com

9. **Nutrition** As part of his weight-loss routine, Eric plans to only eat 6-inch Turkey Breast and 6-inch Veggie Delite®
12. **Dietary Requirements**  A diet is to contain at least 400 units of vitamins, 500 units of minerals, and 1400 calories. Two foods are available: $F_1$, which costs $0.05 per unit, and $F_2$, which costs $0.03 per unit. A unit of food $F_1$ contains 2 units of vitamins, 1 unit of minerals, and 4 calories; a unit of food $F_2$ contains 1 unit of vitamins, 2 units of minerals, and 4 calories. Find the minimum cost for a diet that consists of a mixture of these two foods and also meets the minimal nutrition requirements.

13. **Dietary Requirements**  A certain diet requires at least 60 units of carbohydrates, 45 units of protein, and 30 units of fat each day. Each ounce of Supplement A provides 5 units of carbohydrates, 3 units of protein, and 4 units of fat. Each ounce of Supplement B provides 2 units of carbohydrates, 2 units of protein, and 1 unit of fat. If Supplement A costs $1.50 per ounce and Supplement B costs $1.00 per ounce, how many ounces of each supplement should be taken daily to minimize the cost of the diet?

14. **Production Scheduling**  In a factory, machine 1 produces 8-inch pliers at the rate of 60 units per hour and 6-inch pliers at the rate of 70 units per hour. Machine 2 produces 8-inch pliers at the rate of 40 units per hour and 6-inch pliers at the rate of 20 units per hour. It costs $50 per hour to operate machine 1, and machine 2 costs $30 per hour to operate. The production schedule requires that at least 240 units of 8-inch pliers and at least 140 units of 6-inch pliers be produced during each 10-hour day. Which combination of machines will cost the least money to operate?

15. **Home Mortgages**  Fremont Bank offered no-closing-cost home mortgages at a 5.125% annual rate for 30-year fixed-rate loans and at a 4.375% annual rate for 15-year fixed-rate loans. Suppose that Fremont Bank is planning to allocate up to $72 million for these two kinds of loans, with a minimum of $15 million for each kind of loan. Also, Fremont Bank has decided that the amount allocated to the 30-year loans should be at most twice the amount allocated to the 15-year loans. How much should Fremont Bank allocate to each type of loan in order to maximize the total amount of interest that would be received in the first month of these loans? What is the total amount of interest that would be received in the first month, if the maximum solution is implemented?

16. **Return on Investment**  An investment broker is instructed by her client to invest $20,000 in two funds based on their average annual returns for 5 years ending December 31, 2009: a Global Bond fund yielding 9% and the Yachtman fund yielding 7%. The client wants to invest at least $8000 in the Yachtman fund and no more than $12,000 in the Global Bond fund.

(a) How much should the broker recommend that the client place in each investment to maximize income if the client insists that the amount invested in the Yachtman fund must equal or exceed the amount placed in the Global Bond fund?

(b) How much should the broker recommend that the client place in each investment to maximize income if the client insists that the amount invested in the Yachtman fund must not exceed the amount placed in the Global Bond fund?

17. **Maximizing Profit on Ice Skates**  A factory manufactures two kinds of ice skates: racing skates and figure skates. The racing skates require 6 work-hours in the fabrication department, whereas the figure skates require 4 work-hours there. The racing skates require 1 work-hour in the finishing department, whereas the figure skates require 2 work-hours there. The fabricating department has available at most 120 work-hours per day, and the finishing department has no more than 40 work-hours per day available. If the profit on each racing skate is $10 and the profit on each figure skate is $12, how many of each should be manufactured each day to maximize profit? (Assume that all skates made are sold.)
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18. **Maximizing Profit on Figurines** A factory manufactures two kinds of ceramic figurines: a dancing girl and a mermaid. Each requires three processes: molding, painting, and glazing. The daily labor available for molding is no more than 90 work-hours, labor available for painting does not exceed 120 work-hours, and labor available for glazing is no more than 60 work-hours. The dancing girl requires 3 work-hours for molding, 6 work-hours for painting, and 2 work-hours for glazing. The mermaid requires 3 work-hours for molding, 4 work-hours for painting, and 3 work-hours for glazing. If the profit on each figurine is $25 for dancing girls and $30 for mermaids, how many of each should be produced each day to maximize profit? If management decides to produce the number of each figurine that maximizes profit, determine which of these processes has work-hours assigned to it that are not used.

19. **Pollution Control** A chemical plant produces two compounds A and B. For each compound A produced, 2 cubic feet of carbon monoxide and 6 cubic feet of sulfur dioxide are emitted into the atmosphere; to produce compound B, 4 cubic feet of carbon monoxide and 3 cubic feet of sulfur dioxide are emitted into the atmosphere. Government pollution standards permit the manufacturer to emit a maximum of 3000 cubic feet of carbon monoxide and 5400 cubic feet of sulfur dioxide per week. The manufacturer can sell all of the compounds that it produces and makes a profit of $1.50 per unit for compound A and $1.00 per unit for compound B. Determine the number of units of each compound to be produced each week to maximize profit without exceeding government standards.

20. **Baby Food Servings** Gerber Banana Plum Granola costs $0.89 per 5.5-oz serving; each serving contains 140 calories, 31 g of carbohydrates, and 0% of the recommended daily allowance of vitamin C. Gerber Mixed Fruit Carrot Juice costs $0.79 per 4-oz serving; each serving contains 60 calories, 13 g of carbohydrates, and 100% of the recommended daily allowance of vitamin C. Determine how many servings of each of the above foods would be needed to provide a child at least 160 calories, 40 g of carbohydrates, and 70% of the recommended daily allowance of vitamin C at minimum cost. Fractions of servings are permitted.

Source: Gerber Web site and Safeway Stores, Inc.

21. **Production Scheduling** A company produces two types of steel. Type 1 requires 2 hours of melting, 4 hours of cutting, and 10 hours of rolling per ton. Type 2 requires 3 hours of melting, 1 hour of cutting, and 5 hours of rolling per ton. Forty hours are available for melting, 20 for cutting, and 60 for rolling. Each ton of Type 1 produces $240 profit, and each ton of Type 2 yields $80 profit. Find the maximum profit and the production schedule that will produce this profit.

22. **TV Advertising** Nielsen TV ratings indicated that during the week of June 15–21, 2009, the news program Dateline NBC had an audience of 5.6 million viewers, while the news program 60 Minutes on CBS had an audience of 8.3 million viewers. An advertiser has a budget of up to $1,000,000 to purchase 30-second commercial timeslots on these two news programs during the next three months. Suppose that each 30-second commercial timeslot on Dateline NBC costs $30,000 and each 30-second commercial timeslot on 60 Minutes costs $40,000. The advertiser wants to purchase at least eight 30-second timeslots on each of the programs, Dateline NBC and 60 Minutes, but has stipulated that at least two-thirds of the 30-second timeslots that are purchased should be on Dateline NBC. How many 30-second timeslots should be purchased on each of these news programs to maximize the total number of viewers? Assume that audience size will remain the same during the subsequent three months as it was during the week of June 15–21, 2009. How many viewers were there?

Source: for by the numbers.com

23. **Diet** Danny’s Chicken Farm is a producer of frying chickens. In order to produce the best fryers possible, the regular chicken feed is supplemented by four vitamins. The minimum amount of each vitamin required per 100 ounces of feed is: vitamin 1, 50 units; vitamin 2, 100 units; vitamin 3, 60 units; vitamin 4, 180 units. Two supplements are available: supplement I costs $0.03 per ounce and contains 5 units of vitamin 1 per ounce, 25 units of vitamin 2 per ounce, 2 units of vitamin 3 per ounce, and 35 units of vitamin 4 per ounce. Supplement II costs $0.04 per ounce and contains 25 units of vitamin 1 per ounce, 10 units of vitamin 3 per ounce, and 20 units of vitamin 4 per ounce. How much of each supplement should Danny buy to add to each 100 ounces of feed in order to minimize his cost but still have the desired vitamin amounts present?

24. **Risk Management** The table below shows the price per share at the close of trading on March 1, 2010, the price to earnings ratio, and the dividend paid in the last quarter for Exxon Mobil and Wells Fargo.

<table>
<thead>
<tr>
<th></th>
<th>Price per Share 3-1-10</th>
<th>Price to Earnings Ratio (P/E Ratio)</th>
<th>Last Quarterly Dividend per Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exxon Mobil Corporation</td>
<td>$65.40</td>
<td>16.51</td>
<td>$0.42</td>
</tr>
<tr>
<td>General Dynamics</td>
<td>$73.50</td>
<td>11.97</td>
<td>$0.38</td>
</tr>
</tbody>
</table>

A pension fund has decided to invest up to $800,000 to purchase shares of stock in the above two companies. It wants the total dividends next quarter from these stocks to be at least $4500, but will not invest more than 60% of the $800,000 in any one of these companies. When purchasing stock, this pension fund minimizes risk, which it calculates as follows:

\[
\text{Risk} = (\text{Amount invested in Stock A})(\text{P/E Ratio for Stock A}) + (\text{Amount invested in Stock B})(\text{P/E Ratio for Stock B})
\]

How many shares of stock in each company should be purchased to minimize risk? Assume dividends paid next quarter are the same as in the previous quarters.

Source: moneycentral.msn.com
25. **Financial Planning** In March 2010, a couple plans to invest $45,000 in two funds based on their average yearly performance for the past 5 years ending December 31, 2009: ICON Bond fund with an average return of 4.35% and ING Mid Cap Opportunities with an average return of 5.05%. They wish to invest a minimum of $10,000 in each of the accounts with a maximum investment of $15,000 in the ICON Bond fund. They also decide that the amount invested in the ING Mid Cap Opportunities fund cannot exceed twice the amount deposited in the ICON Bond fund. How much should this couple invest in each fund in order to maximize their total annual yield?

26. **Maximizing Rates of Return** The table lists two mutual funds: the John Hancock Large Cap Equity I fund and the T Rowe Price Latin America fund.

<table>
<thead>
<tr>
<th>Fund</th>
<th>Rate of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>J Hancock Large Cap Equity I</td>
<td>9.97%</td>
</tr>
<tr>
<td>T Rowe Price Latin America</td>
<td>23.65%</td>
</tr>
</tbody>
</table>

The manager of a pension fund plans to invest up to $1,000,000 in these two mutual funds. He concludes that the amount invested in the John Hancock Large Cap Equity fund should be at least one fourth of the amount invested in the T Rowe Price Latin America fund and will invest a minimum of $175,000 in the John Hancock Large Cap Equity fund. Also, not more than 70% of the total amount invested should be invested in any one of these funds. How much should be invested in each mutual fund to maximize the total projected annual return? Assume that the projected annual return for each mutual fund equals the average annual total rate of return during the 5-year period ending on December 31, 2009. What is the maximum total projected annual return?

27. **Minimizing Travel Costs** A national sales company is planning to conduct a weeklong sales meeting in San Francisco on August 14. On July 14, one month from that date, the lowest round trip airfare from New York City (NYC) to San Francisco is $343 and the lowest round-trip airfare from Chicago to San Francisco is $329. There are 28 sales representatives based in NYC and 22 sales representatives based in Chicago who could travel to San Francisco for this meeting. A total of at least 40 sales representatives from Chicago and NYC must attend this meeting with at least 12 from Chicago and at least 16 from NYC. How many sales representatives based in NYC and how many sales representatives based in Chicago should be sent to the San Francisco sales meeting, so as to minimize the total airfare? What is the minimum total airfare?

**Source:** Orbitz.com

28. **Maximizing Income** J. B. Rug Manufacturers has available 1200 square yards of wool and 1000 square yards of nylon for the manufacture of two grades of carpeting: high-grade, which sells for $500 per roll, and low-grade, which sells for $300 per roll. Twenty square yards of wool and 40 square yards of nylon are used in a roll of high-grade carpet, and 40 square yards of nylon are used in a roll of low-grade carpet. Forty work-hours are required to manufacture each roll of the high-grade carpet, and 20 work-hours are required for each roll of the low-grade carpet, at an average cost of $6 per work-hour. A maximum of 800 work-hours are available. The cost of wool is $5 per square yard and the cost of nylon is $2 per square yard. How many rolls of each type of carpet should be manufactured to maximize income? \[\text{Income} = \text{Revenue from sale} - (\text{Production cost for material } + \text{ labor})\]

29. The rug manufacturer in Problem 28 finds that maximum income occurs when no high-grade carpet is produced. If the price of the low-grade carpet is kept at $300 per roll, in what price range should the high-grade carpet be sold so that income is maximized by selling some rolls of each type of carpet? Assume all other data remain the same.

**Discussion and Writing**

30. Refer to Example 5. What would you do to eliminate the excess capacity?

**CHAPTER 4 REVIEW**

**OBJECTIVES**

<table>
<thead>
<tr>
<th>Section</th>
<th>Examples</th>
<th>You should be able to</th>
<th>Review Exercises</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>1, 2, 3</td>
<td>1 Graph linear inequalities (p. 176)</td>
<td>1–14</td>
</tr>
<tr>
<td></td>
<td>4, 5, 6, 7, 8, 9, 10</td>
<td>2 Graph systems of linear inequalities (p. 179)</td>
<td>9–14</td>
</tr>
<tr>
<td></td>
<td>11, 12</td>
<td>3 Solve applied problems involving systems of linear inequalities (p. 184)</td>
<td>34</td>
</tr>
<tr>
<td>4.2</td>
<td>1, 2</td>
<td>1 Identify a linear programming problem with two variables (p. 192)</td>
<td>35–41</td>
</tr>
<tr>
<td></td>
<td>1, 2, 3, 4</td>
<td>2 Solve a linear programming problem with two variables (p. 193)</td>
<td>15–32</td>
</tr>
<tr>
<td>4.3</td>
<td>1, 2, 3, 4, 5</td>
<td>1 Solve applied problems using linear programming with two variables (p. 199)</td>
<td>33, 35–41</td>
</tr>
</tbody>
</table>
THINGS TO KNOW

Graphs of Inequalities (p. 176) The graph of a strict inequality is represented by a dashed line and the half-plane satisfying the inequality.
The graph of a nonstrict inequality is represented by a solid line and the half-plane satisfying the inequality.

Graphs of Systems of Linear Inequalities (pp. 179–183) The graph of a system of linear inequalities is the set of all points that satisfy each inequality in the system.
The graph is called bounded if some rectangle can be drawn around it.
The graph is called unbounded if it extends infinitely far in at least one direction.

Corner Point (p. 184) A corner point is the intersection of two line segments that form the boundary of the graph of a system of linear inequalities.

Linear Programming (p. 192) Maximize (or minimize) a linear objective function, \( z = Ax + By \), subject to certain conditions, or constraints, expressible as linear inequalities in \( x \) and \( y \). A feasible point \( (x, y) \) is a point that satisfies the constraints of a linear programming problem.

Solution to a Linear Programming Problem (p. 193) A solution to a linear programming problem is a feasible point that maximizes (or minimizes) the objective function together with the value of the objective function at that point.

Location of Solution (p. 195) If a linear programming problem has a solution, it is located at a corner point of the graph of the feasible points.
If a linear programming problem has multiple solutions, at least one of them is located at a corner point of the graph of the feasible points.
In either case, the corresponding value of the objective function is unique.

REVIEW EXERCISES Answers begin on page AN–22.
Blue problem numbers indicate the author’s suggestions for use in a practice test.

In Problems 1–4, graph each linear inequality:
1. \( x + 3y \leq 0 \)  2. \( 4x + y \geq 0 \)
3. \( 5x + y > 10 \)  4. \( 2x + 3y < 6 \)

5. Without graphing, determine which of the points \( P_1 = (4, -3), P_2 = (2, -6), P_3 = (8, -3) \) are part of the graph of the following system:
   \[
   \begin{align*}
   x + 2y & \leq 8 \\
   2x - y & \geq 4 
   \end{align*}
   \]

6. Without graphing, determine which of the points \( P_1 = (8, 6), P_2 = (2, -5), P_3 = (4, 1) \) are part of the graph of the following system:
   \[
   \begin{align*}
   5x - y & \geq 2 \\
   x - 4y & \leq -2 
   \end{align*}
   \]

In Problems 7–8, determine which region—\( a, b, c, \) or \( d \)—represents the graph of the given system of linear inequalities.

7. \[
   \begin{align*}
   6x - 4y & \leq 12 \\
   3x + 2y & \leq 18 
   \end{align*}
   \]

8. \[
   \begin{align*}
   6x - 5y & \geq 5 \\
   6x + 6y & \leq 60 
   \end{align*}
   \]
In Problems 27–32, solve each linear programming problem.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Constraints</th>
<th>Objective Function</th>
</tr>
</thead>
</table>
| 27. | \[
\begin{align*}
3x + 2y & \leq 12 \\
x + y & \geq 4 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\] | Maximize \[z = 2x + 3y\] |
| 28. | \[
\begin{align*}
x + 2y & \leq 4 \\
2x + y & \leq 4 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\] | Maximize \[z = 4x + y\] |
| 29. | \[
\begin{align*}
x & \leq 9 \\
y & \leq 8 \\
x + y & \geq 3 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\] | Maximize \[z = x + 2y\] |
| 30. | \[
\begin{align*}
x + 2y & \geq 2 \\
3x + 2y & \leq 12 \\
y & \leq 5 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\] | Minimize \[z = 3x + 4y\] |
| 31. | \[
\begin{align*}
x + 2y & \geq 8 \\
3x + y & \geq 6 \\
x & \leq 8 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\] | Minimize \[z = 3x + 2y\] |
| 32. | \[
\begin{align*}
x & \leq 10 \\
y & \leq 12 \\
2x + y & \leq 10 \\
x - y & \geq 0 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\] | Minimize \[z = 2x + 5y\] |
33. **Blending Coffee** Bill’s Coffee House, a store that specializes in coffee, has available 75 pounds of pure Columbian coffee and 120 pounds of a house-brand coffee. These will be blended into 1 pound packages as follows: An economy blend that contains 4 ounces of pure Columbian coffee and 12 ounces of the house-brand coffee and a superior blend that contains 8 ounces of pure Columbian coffee and 8 ounces of the house-brand coffee.

(a) Using \( x \) to denote the number of packages of the economy blend and \( y \) to denote the number of packages of the superior blend, write a system of linear inequalities that describes the possible number of packages of each kind of blend.

(b) Graph the system and label the corner points.

(c) A profit of \$0.30 per package is made on the economy blend, whereas a profit of \$0.40 per package is made on the superior blend. How many packages of each mixture should be prepared to achieve a maximum profit? Assume that all packages prepared can be sold.

34. **Manufacturing Trucks** Mike’s Toy Truck Company manufactures two models of toy trucks, a standard model and a deluxe model. Each standard model requires 2 hours for painting and 3 hours for detail work; each deluxe model requires 3 hours for painting and 4 hours for detail work. Two painters and three detail workers are employed by the company, and each works 40 hours per week.

(a) Using \( x \) to denote the number of standard model trucks and \( y \) to denote the number of deluxe model trucks, write a system of linear inequalities that describes the possible number of each model of truck that can be manufactured in a week.

(b) Graph the system and label the corner points.

35. **Maximizing Profit** A ski manufacturer makes two types of skis: downhill and cross-country. Using the information given in the table below, how many of each type of ski should be made for a maximum profit to be achieved? What is the maximum profit?

<table>
<thead>
<tr>
<th></th>
<th>Downhill</th>
<th>Cross-Country</th>
<th>Maximum Time Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturing Time per Ski</td>
<td>2 hours</td>
<td>1 hour</td>
<td>40 hours</td>
</tr>
<tr>
<td>Finishing Time per Ski</td>
<td>1 hour</td>
<td>1 hour</td>
<td>32 hours</td>
</tr>
<tr>
<td>Profit per Ski</td>
<td>$70</td>
<td>$50</td>
<td></td>
</tr>
</tbody>
</table>

36. **Production Scheduling** A company sells two types of shoes. The first uses 2 units of leather and 2 units of synthetic material and yields a profit of \$8 per pair. The second type requires 5 units of leather and 1 unit of synthetic material and gives a profit of \$10 per pair. If there are 40 units of leather and 16 units of synthetic material available, how many pairs of each type of shoe should be manufactured to maximize profit? What is the maximum profit?

37. **Baby Food Servings** Gerber Banana Oatmeal and Peach costs \$0.79 per 4-oz serving; each serving contains 90 calories, 19 g of carbohydrates, and 45% of the recommended daily allowance of vitamin C. Gerber Mixed Fruit Juice costs \$0.65 per 4-oz serving; each serving contains 60 calories, 15 g of carbohydrates, and 100% of the recommended daily allowance of vitamin C. Determine how many servings of each of the above foods would be needed to provide a child with at least 130 calories, 30 g of carbohydrates, and 60% of the recommended daily allowance of vitamin C at minimum cost. Fractions of servings are permitted.

Source: Gerber Web site and Safeway Stores, Inc.

38. **Maximizing Profit** A company makes two explosives: low density and high density. Due to storage problems, a maximum of 100 pounds of low density and 150 pounds of high density explosives can be mixed and packaged each week. One pound of low density takes 60 hours to mix and 70 hours to package; 1 pound of high density takes 40 hours to mix and 40 hours to package. The mixing department has at most 7200 work-hours available each week, and packaging has at most 7800 work-hours available. If the profit for 1 pound of low density is \$60 and for 1 pound of high density is \$40, what is the maximum profit possible each week?

39. **Manufacturing Vitamin Pills** A pharmaceutical company makes two types of vitamins at its New Jersey plant—a high-potency, antioxidant vitamin and a vitamin enriched with added calcium. Each high-potency vitamin contains, among other things, 500 mg of vitamin C, 40 mg of calcium, and 100 mg of magnesium and generates a profit of \$0.10 per tablet. A calcium-enriched vitamin tablet contains 100 mg of vitamin C, 400 mg of calcium, and 40 mg of magnesium and generates a profit of \$0.05 per tablet. Each day the company has available 300 kg of vitamin C, 122 kg of calcium, and 65 kg of magnesium for use in the production of the vitamins. How many of each type of vitamin should be manufactured to maximize profit?

Source: Centrum Vitamin Supplements.

40. **Mixture** A company makes two kinds of animal food, A and B, which contain two food supplements. It takes 2 pounds of the first supplement and one pound of the second to make a dozen cans of food A, and 4 pounds of the first supplement and 5 pounds of the second to make a dozen cans of food B. On a certain day 80 pounds of the first supplement and 70 pounds of the second are available. How many
cans of food A and food B should be made to maximize company profits, if the profit on a dozen cans of food A is $3.00 and the profit on a dozen cans of food B is $10.00?

41. Maximizing Yield

The purchase price paid on March 1, 2010, and the last quarterly dividend paid per share for stock in Boeing and Honeywell were as follows:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Price per Share</th>
<th>Last Quarterly Dividend per Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boeing</td>
<td>$64</td>
<td>0.42</td>
</tr>
<tr>
<td>Honeywell</td>
<td>$40</td>
<td>0.30</td>
</tr>
</tbody>
</table>

A mutual fund has decided to invest up to $500,000 to purchase shares of stock in these two companies. It will purchase a minimum of 2000 shares of stock in each of the above companies but will not invest more than 60% of the total amount in any one of these companies. How many shares of stock in each company should be purchased to maximize the total projected dividend for the next quarter? Use the dividend for the last quarter as the projected dividend for the next quarter. What is the maximum total projected dividend?

Source: moneycentral.msn.com

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### BALANCING NUTRIENTS

In preparing a recipe, you must decide what ingredients and how much of each ingredient you will use. In these health-conscious days, you may also want to consider the amount of certain nutrients in your recipe. You may even be interested in minimizing some quantities (like calories or fat) or maximizing others (like carbohydrates or protein). Linear programming techniques can help to do this.

For example, consider making a very simple trail mix from dry-roasted, unsalted peanuts and seedless raisins. Table 1 lists the amounts of various dietary quantities for these ingredients. The amounts are given per serving of the ingredient.

**TABLE 1 NUTRIENTS IN PEANUTS AND RAISINS**

<table>
<thead>
<tr>
<th>Nutrient</th>
<th>Peanuts Serving Size = 1 Cup</th>
<th>Raisins Serving Size = 1 Cup</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calories (kcal)</td>
<td>850</td>
<td>440</td>
</tr>
<tr>
<td>Protein (g)</td>
<td>34.57</td>
<td>4.67</td>
</tr>
<tr>
<td>Fat (g)</td>
<td>72.50</td>
<td>0.67</td>
</tr>
<tr>
<td>Carbohydrates (g)</td>
<td>31.40</td>
<td>114.74</td>
</tr>
</tbody>
</table>


Suppose that you want to make at most 6 cups of trail mix for a day hike. You don’t want either ingredient to dominate the mixture, so you want the amount of raisins to be at least \( \frac{1}{2} \) of the amount of peanuts and the amount of peanuts to be at least \( \frac{1}{2} \) of the amount of raisins. You want the entire amount of trail mix you make to have fewer than 4000 calories, and you want to maximize the amount of carbohydrates in the mix.

1. Let \( x \) be the number of cups of peanuts you will use, let \( y \) be the number of cups of raisins you will use, and let \( c \) be the amount of carbohydrates in the mix. Find the objective function.
2. What constraints must be placed on the objective function?
3. Graph the set of feasible points for this problem.
4. Find the number of cups of peanuts and raisins that maximize the amount of carbohydrates in the mix.
5. How many grams of carbohydrates are in a cup of the final mix? How many calories?
6. Under all of the constraints given above, what recipe for trail mix will maximize the amount of protein in the mix? How many grams of protein are in a cup of this mix? How many calories?
7. Suppose you decide to eat at least 3 cups of the trail mix. Keeping the constraints given above, what recipe for trail mix will minimize the amount of fat in the mix?
8. How many grams of carbohydrates are in this mix?
9. How many grams of protein are in this mix?
10. Which of the three trail mixes would you use? Why?
Use the following information to do Problems 1–3:
The Random Company manufactures two products, Zeta and Beta. Each product must pass through two processing operations. All materials are introduced at the start of process 1. There are no work-in-process inventories. Random may produce either one product exclusively or various combinations of both products subject to the following constraints:

<table>
<thead>
<tr>
<th></th>
<th>Process No. 1</th>
<th>Process No. 2</th>
<th>Contribution Margin per Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zeta</td>
<td>1 hour</td>
<td>1 hour</td>
<td>$4.00</td>
</tr>
<tr>
<td>Beta</td>
<td>2 hours</td>
<td>3 hours</td>
<td>$5.25</td>
</tr>
<tr>
<td>Total capacity in hours per day</td>
<td>1000 hours</td>
<td>1275 hours</td>
<td></td>
</tr>
</tbody>
</table>

A shortage of technical labor has limited Beta production to 400 units per day. There are no constraints on the production of Zeta other than the hour constraints in the above schedule. Assume that all relationships between capacity and production are linear, and that all of the above data and relationships are deterministic rather than probabilistic.

1. **CPA Exam** Given the objective to maximize total contribution margin, what is the production constraint for process 1?
   - (a) Zeta + Beta ≤ 1000
   - (b) Zeta + 2 Beta ≤ 1000
   - (c) Zeta + Beta ≥ 1000
   - (d) Zeta + 2 Beta ≥ 1000

2. **CPA Exam** Given the objective to maximize total contribution margin, what is the labor constraint for production of Beta?
   - (a) Beta ≤ 400
   - (b) Beta ≥ 400
   - (c) Beta ≤ 425
   - (d) Beta ≥ 425

3. **CPA Exam** What is the objective function of the data presented?
   - (a) $Zeta + 2\text{Beta} = 9.25$
   - (b) $(4.00)Zeta + (3(5.25))\text{Beta} = \text{Total contribution margin}$
   - (c) $(4.00)Zeta + (5.25)\text{Beta} = \text{Total contribution margin}$
   - (d) $2(4.00)Zeta + 3(5.25)\text{Beta} = \text{Total contribution margin}$

4. **CPA Exam** Williamson Manufacturing intends to produce two products, X and Y. Product X requires 6 hours of time on machine 1 and 12 hours of time on machine 2. Product Y requires 4 hours of time on machine 1 and no time on machine 2. Both machines are available for 24 hours. Assuming that the objective function of the total contribution margin is $2X + 1Y$, what product mix will produce the maximum profit?
   - (a) No units of product X and 6 units of product Y.
   - (b) 1 unit of product X and 4 units of product Y.
   - (c) 2 units of product X and 3 units of product Y.
   - (d) 4 units of product X and no units of product Y.

5. **CPA Exam** Quepea Company manufactures two products, Q and P, in a small building with limited capacity. The selling price, cost data, and production time are given below:

<table>
<thead>
<tr>
<th>Product Q</th>
<th>Product P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selling price per unit</td>
<td>$20</td>
</tr>
<tr>
<td>Variable costs of producing and selling a unit</td>
<td>$12</td>
</tr>
<tr>
<td>Hours to produce a unit</td>
<td>3</td>
</tr>
</tbody>
</table>

Based on this information, the profit maximization objective function for a linear programming solution may be stated as
   - (a) Maximize $20Q + 17P$.
   - (b) Maximize $12Q + 13P$.
   - (c) Maximize $3Q + 1P$.
   - (d) Maximize $8Q + 4P$. 

* Copyright © 1991–2010 by the American Institute of Certified Public Accountants, Inc. All rights reserved. Reprinted with permission.
6. CPA Exam Patsy, Inc., manufactures two products, $X$ and $Y$. Each product must be processed in each of three departments: machining, assembling, and finishing. The hours needed to produce one unit of product per department and the maximum possible hours per department follow:

<table>
<thead>
<tr>
<th>Department</th>
<th>Production Hours per Unit $X$</th>
<th>Production Hours per Unit $Y$</th>
<th>Maximum Capacity in Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machining</td>
<td>2</td>
<td>1</td>
<td>420</td>
</tr>
<tr>
<td>Assembling</td>
<td>2</td>
<td>2</td>
<td>500</td>
</tr>
<tr>
<td>Finishing</td>
<td>2</td>
<td>3</td>
<td>600</td>
</tr>
</tbody>
</table>

Other restrictions follow:

$$X \geq 50 \quad Y \geq 50$$

The objective function is to maximize profits where $\text{profit} = 4X + 2Y$. Given the objective and constraints, what is the most profitable number of units of $X$ and $Y$, respectively, to manufacture?

(a) 150 and 100  
(b) 165 and 90  
(c) 170 and 80  
(d) 200 and 50

7. CPA Exam Milford Company manufactures two models, medium and large. The contribution margin expected is $12 for the medium model and $20 for the large model. The medium model is processed 2 hours in the machining department and 4 hours in the polishing department. The large model is processed 3 hours in the machining department and 6 hours in the polishing department. How would the formula for determining the maximization of total contribution margin be expressed?

(a) $5X + 10Y$  
(b) $6X + 9Y$  
(c) $12X + 20Y$  
(d) $12X(2 + 4) + 20Y(3 + 6)$

8. CPA Exam Hale Company manufactures products $A$ and $B$, each of which requires two processes, polishing and grinding. The contribution margin is $3 for product $A$ and $4 for product $B$. The illustration shows the maximum number of units of each product that may be processed in the two departments.

Considering the constraints (restrictions) on processing, which combination of products $A$ and $B$ maximizes the total contribution margin?

(a) 0 units of $A$ and 20 units of $B$.  
(b) 20 units of $A$ and 10 units of $B$.  
(c) 30 units of $A$ and 0 units of $B$.  
(d) 40 units of $A$ and 0 units of $B$.

9. CPA Exam Johnson, Inc., manufactures product $X$ and product $Y$, which are processed as follows:

<table>
<thead>
<tr>
<th>Machine A</th>
<th>Machine B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product $X$</td>
<td>6 hours</td>
</tr>
<tr>
<td>Product $Y$</td>
<td>9 hours</td>
</tr>
</tbody>
</table>

The contribution margin is $12 for product $X$ and $7 for product $Y$. The available time daily for processing the two products is 120 hours for machine $A$ and 80 hours for machine $B$. How would the restriction (constraint) for machine $B$ be expressed?

(a) $4X + 5Y$  
(b) $4X + 5Y \leq 80$  
(c) $6X + 9Y \leq 120$  
(d) $12X + 7Y$

10. CPA Exam A small company makes only two products, with the following two production constraints representing two machines and their maximum availability:

$$2X + 3Y \leq 18$$
$$2X + Y \leq 10$$

Where

$X = \text{Units of the first product}$

$Y = \text{Units of the second product}$

If the profit equation is $Z = 4X + 2Y$, the maximum possible profit is

(a) $20$  
(b) $21$  
(c) $18$  
(d) $24$  
(e) Some profit other than those given above
Questions 11–13 are based on the Jarten Company, which manufactures and sells two products. Demand for the two products has grown to such a level that Jarten can no longer meet the demand with its facilities. The company can work a total of 600,000 direct labor-hours annually using three shifts. A total of 200,000 hours of machine time is available annually. The company plans to use linear programming to determine a production schedule that will maximize its net return.

The company spends $2,000,000 in advertising and promotion and incurs $1,000,000 for general and administrative costs. The unit sale price for model A is $27.50; model B sells for $75.00 each. The unit manufacturing requirements and unit cost data are as shown below. Overhead is assigned on a machine-hour (MH) basis.

<table>
<thead>
<tr>
<th></th>
<th>Model A</th>
<th>Model B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw material</td>
<td>$ 3</td>
<td>$ 7</td>
</tr>
<tr>
<td>Direct labor</td>
<td>1 DLH @ $8</td>
<td>1.5 DLH @ $8</td>
</tr>
<tr>
<td>Variable overhead</td>
<td>0.5 MH @ $12</td>
<td>2.0 MH @ $12</td>
</tr>
<tr>
<td>Fixed overhead</td>
<td>0.5 MH @ $4</td>
<td>2.0 MH @ $4</td>
</tr>
<tr>
<td>Cost</td>
<td>$19</td>
<td>$51</td>
</tr>
</tbody>
</table>

11. **CMA Exam** The objective function that would maximize Jarten’s net income is
   (a) $10.50A + 32.00B$
   (b) $8.50A + 24.00B$
   (c) $27.50A + 75.00B$
   (d) $19.00A + 51.00B$
   (e) $17.00A + 43.00B$

12. **CMA Exam** The constraint function for the direct labor is
   (a) $1A + 1.5B \leq 200,000$
   (b) $8A + 12B \leq 600,000$
   (c) $8A + 12B \leq 200,000$
   (d) $1A + 1.5B \leq 4,800,000$
   (e) $1A + 1.5B \leq 600,000$

13. **CMA Exam** The constraint function for the machine capacity is
   (a) $6A + 24B \leq 200,000$
   (b) $(1/0.5)A + (1.5/2.0)B \leq 800,000$
   (c) $0.5A + 2B \leq 200,000$
   (d) $(0.5 + 0.5)A + (2 + 2)B \leq 200,000$
   (e) $(0.5 \times 1) + (1.5 \times 2.00) \leq (200,000 \times 600,000)$

14. **CPA Exam** Boaz Company manufactures two models, medium (X) and large (Y). The contribution margin expected is $24 for the medium model and $40 for the large model. The medium model is processed 2 hours in the machining department and 4 hours in the polishing department. The large model is processed 3 hours in the machining department and 6 hours in the polishing department. If total contribution margin is to be maximized, using linear programming, how would the objective function be expressed?
   (a) $24X(2 + 4) + 40Y(3 + 6)$
   (b) $24X + 40Y$
   (c) $6X + 9Y$
   (d) $5X + 10Y$
In Chapter 4 we described a geometric method (using graphs) for solving linear programming problems. Unfortunately, this method is useful only when there are no more than two variables and the number of constraints is small.

Since most practical linear programming problems involve systems of several thousand linear inequalities containing several thousand variables, more sophisticated techniques need to be used. One of these methods is the *simplex method*, the subject of this chapter.

That hiking trip to Yosemite is still going to happen. But when you told the others who are going about your trail mix of unsalted peanuts and raisins, they all said “borrrring.” It was decided to use the unsalted peanuts and raisins mixed with M&Ms and salted mini-pretzels. Now it’s more complicated to determine in what proportions these four ingredients should be mixed, and there remain the issues of carbohydrates, protein, calories, and fat. Fortunately, you also finished the simplex method before midterm, so you can handle the problem just fine. The Chapter Project at the end of the chapter will guide you.

**OUTLINE**

5.1 The Simplex Tableau; Pivoting

5.2 The Simplex Method: Solving Maximum Problems in Standard Form

5.3 Solving Minimum Problems Using the Duality Principle

5.4 The Simplex Method for Problems Not in Standard Form

- Chapter Review
- Chapter Project
- Mathematical Questions from Professional Exams

**A Look Back, A Look Forward**

In Chapter 4 we described a geometric method (using graphs) for solving linear programming problems. Unfortunately, this method is useful only when there are no more than two variables and the number of constraints is small.

Since most practical linear programming problems involve systems of several thousand linear inequalities containing several thousand variables, more sophisticated techniques need to be used. One of these methods is the *simplex method*, the subject of this chapter.

The simplex method is a way to solve linear programming problems involving many inequalities and variables. Developed by George Dantzig in 1946, it is particularly well suited for computerization. In 1984, Narendra Karmarkar of Bell Laboratories discovered a way of solving large linear programming problems that led to methods that compete with and often outperform the simplex method.

A discussion of LINDO, a software package that performs the simplex method, is provided in Appendix B.
Introduction

If we have a large number of variables or constraints, it is still true that if an optimal solution exists, it will be found at a corner point of the set of feasible points. In fact, we could find these corner points by writing all the equations corresponding to the inequalities of the problem and then proceeding to solve all possible combinations of these equations. We would, of course, have to discard any solutions that are not feasible (because they do not satisfy one or more of the constraints). Then we could evaluate the objective function at the remaining corner points.

Just how difficult is this procedure? Well, if there were just 4 variables and 7 constraints, we would have to solve all possible combinations of 4 equations chosen from a set of 7 equations—that would be 35 solutions in all. Each of these solutions would then have to be tested for feasibility. So even for this relatively small number of variables and constraints, the work would be quite tedious. In the real world of applications, it is fairly common to encounter problems with several thousands of variables and constraints. Of course, such problems must be solved by computer. Even so, choosing a more efficient problem-solving strategy might reduce the computer’s running time from hours to seconds or, for very large problems, from years to hours.

A more systematic approach involves choosing a solution at one corner point of the feasible set, then moving from there to another corner point at which the objective function has a better value, and continuing in this way until the best possible value is found. The simplex method is a very efficient and popular way of doing this.

The simplex method is not a single algorithm but rather is a strategy for solving complex linear programming problems. There are many implementations, for example, of rules for pivoting (discussed shortly) that go under the simplex “umbrella.” We present here one of these implementations. Our intention is to provide an introduction to the simplex method, which is a very complex and dynamic theory.
Determine a Maximum Problem Is in Standard Form

**Definition**

**Standard Form of a Maximum Problem**

A linear programming problem in which the objective function is to be maximized is referred to as a maximum problem. A maximum problem is said to be in standard form* provided two conditions are met:

**Condition 1** All the variables are nonnegative.

**Condition 2** The other constraints can be written as a linear expression that is less than or equal to a positive constant.

---

**EXAMPLE 1** Determining Whether a Maximum Problem Is in Standard Form

Determine which of the following maximum problems are in standard form.†

(a) Maximize

\[ z = 5x_1 + 4x_2 \]

subject to the constraints

\[ 3x_1 + 4x_2 \leq 120 \]
\[ 4x_1 + 3x_2 \leq 20 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

(b) Maximize

\[ z = 8x_1 + 2x_2 + 3x_3 \]

subject to the constraints

\[ 4x_1 + 8x_2 \leq 120 \]
\[ 3x_2 + 4x_3 \leq 120 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

(c) Maximize

\[ z = 6x_1 - 8x_2 + x_3 \]

subject to the constraints

\[ 3x_1 + x_2 \leq 10 \]
\[ 4x_1 - x_2 \leq 5 \]
\[ x_1 - x_2 - x_3 \geq -3 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

* Other books may give a different meaning to “standard form.”
† Due to the nature of solving linear programming problems using the simplex method, we shall find it convenient to use subscripted variables throughout this chapter.
(d) Maximize

\[ z = 8x_1 + x_2 \]

subject to the constraints

\[ 3x_1 + 4x_2 \leq 2 \]
\[ x_1 + x_2 \leq 6 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

**SOLUTION**

(a) This is a maximum problem containing two variables \( x_1 \) and \( x_2 \). Since both variables are nonnegative and since the other constraints are each written as linear expressions less than or equal to a positive constant, the maximum problem is in standard form.

(b) This is a maximum problem containing three variables \( x_1, x_2, \) and \( x_3 \). Since the variable \( x_3 \) is not given as nonnegative, the maximum problem is not in standard form.

(c) This is a maximum problem containing three variables \( x_1, x_2, \) and \( x_3 \). Each variable is nonnegative. The set of constraints

\[ 3x_1 + 4x_2 \leq 120 \]
\[ 4x_1 + 3x_2 \leq 20 \]

are each written as linear expressions less than or equal to a positive constant, the maximum problem is in standard form.

(d) The maximum problem contains two variables \( x_1 \) and \( x_2 \), each of which is nonnegative. Of the other constraints, the first one, \( 3x_1 + 4x_2 \geq 2 \), does not conform. The maximum problem is not in standard form. Notice that we cannot modify this problem to place it in standard form. Even though multiplying by \(-1\) will change the \( \geq \) to \( \leq \), in so doing the \( 2 \) will change to \(-2\).

**NOW WORK PROBLEMS 7 AND 17.**

2 Set Up the Initial Simplex Tableau

In order to apply the simplex method to a maximum problem in standard form, we need to first

1. Introduce *slack variables.*
2. Construct the *initial simplex tableau.*
5.1 The Simplex Tableau: Pivoting

EXAMPLE 2 Setting Up the Initial Simplex Tableau

Return to a maximum problem solved in Chapter 4. (Refer to Example 2, page 201.)

Maximize

\[ P = 3x_1 + 4x_2 \]

subject to the constraints

\[
\begin{align*}
2x_1 + 4x_2 &\leq 120 \\
2x_1 + 2x_2 &\leq 80 \\
x_1 \geq 0 & \quad x_2 \geq 0
\end{align*}
\]

First, observe that this maximum problem is in standard form.

Next, recall that an inequality of the form \(2x_1 + 4x_2 \leq 120\) means that there is a nonnegative number, which we designate by \(s_1\), so that

\[ 2x_1 + 4x_2 + s_1 = 120 \quad \text{and} \quad s_1 \geq 0 \]

This number \(s_1\) is a variable. It must be nonnegative since it is the difference between 120 and an expression that is less than or equal to 120. We call it a slack variable since it “takes up the slack” between the left and right sides of the inequality.

Similarly, for the constraint \(2x_1 + 2x_2 \leq 80\), we introduce the slack variable \(s_2\). Then the variable \(s_2\) obeys

\[ 2x_1 + 2x_2 + s_2 = 80 \quad \text{and} \quad s_2 \geq 0 \]

Finally, we write the objective function \(P = 3x_1 + 4x_2\) as

\[ P - 3x_1 - 4x_2 = 0 \]

In effect, we have now replaced our original system of constraints and the objective function by a system of three equations containing five variables, \(P, x_1, x_2, s_1,\) and \(s_2\):

\[
\begin{align*}
2x_1 + 4x_2 + s_1 &= 120 \\
2x_1 + 2x_2 + s_2 &= 80 \\
P - 3x_1 - 4x_2 &= 0
\end{align*}
\]

where

\[ x_1 \geq 0 \quad x_2 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \]

The augmented matrix for this system is given below:

\[
\begin{bmatrix}
P & x_1 & x_2 & s_1 & s_2 \\
0 & 2 & 4 & 1 & 0 & 120 \\
0 & 2 & 2 & 0 & 1 & 80 \\
1 & -3 & -4 & 0 & 0 & 0
\end{bmatrix}
\]

If we write the augmented matrix in the form given next, we have the initial simplex tableau for the maximum problem:

\[
\begin{array}{cccccc}
\text{BV} & P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
\hline
s_1 & 0 & 2 & 4 & 1 & 0 & 120 \\
s_2 & 0 & 2 & 2 & 0 & 1 & 80 \\
P & 1 & -3 & -4 & 0 & 0 & 0
\end{array}
\]
Display (1) and Display (2) are equivalent. The bottom row of the initial simplex tableau represents the objective function \( P - 3x_1 - 4x_2 = 0 \) and is called the objective row. The rows above it represent the constraints:

\[
\begin{align*}
2x_1 + 4x_2 + s_1 & = 120 \\
2x_1 + 2x_2 + s_2 & = 80
\end{align*}
\]

We separate the objective row from the rows representing the constraints with a horizontal rule. Notice that we have written the symbol for each variable above the column in which its coefficients appear. The notation \( BV \) stands for basic variables. These are the variables that have a coefficient of 1 and 0s elsewhere in their column. The notation \( RHS \) stands for right-hand side, that is, the numbers to the right of the equal sign in each equation.

So far, we have seen this much of the simplex method:

\[\square\]

**For a Maximum Problem in Standard Form**

- The constraints are changed from inequalities to equations by the introduction of additional variables—one for each constraint and all nonnegative—called slack variables.
- These equations, together with the one that involves the objective function, are placed in the initial simplex tableau.

**EXAMPLE 3 Setting Up the Initial Simplex Tableau**

The following maximum problems are in standard form. For each one introduce slack variables and set up the initial simplex tableau.

(a) Maximize

\[
P = 3x_1 + 2x_2 + x_3
\]

subject to the constraints

\[
\begin{align*}
3x_1 + x_2 + x_3 & \leq 30 \\
5x_1 + 2x_2 + x_3 & \leq 24 \\
x_1 + x_2 + 4x_3 & \leq 20 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*}
\]

(b) Maximize

\[
P = x_1 + 4x_2 + 3x_3 + x_4
\]

subject to the constraints

\[
\begin{align*}
2x_1 + x_2 & \leq 10 \\
3x_1 + x_2 + x_3 + 2x_4 & \leq 18 \\
x_1 + x_2 + x_3 + x_4 & \leq 14 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0 \\
x_4 & \geq 0
\end{align*}
\]

**SOLUTION**

(a) Write the objective function in the form

\[
P = 3x_1 - 2x_2 - x_3 = 0
\]
5.1 The Simplex Tableau: Pivoting

For each constraint introduce a nonnegative slack variable to obtain the following system of equations:

\[
\begin{align*}
3x_1 + x_2 + x_3 + s_1 &= 30 \\
5x_1 + 2x_2 + x_3 + s_2 &= 24 \\
x_1 + x_2 + 4x_3 + s_3 &= 20
\end{align*}
\]

where \( x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \quad s_3 \geq 0 \)

These equations, together with the objective function \( P \), give the initial simplex tableau:

\[
\begin{array}{cccccccc}
BV & P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \\
s_1 & 0 & 3 & 1 & 1 & 1 & 0 & 0 & 30 \\
s_2 & 0 & 5 & 2 & 1 & 0 & 1 & 0 & 24 \\
s_3 & 0 & 1 & 1 & 4 & 0 & 0 & 1 & 20 \\
P & 1 & -3 & -2 & -1 & 0 & 0 & 0 & 0
\end{array}
\]

(b) Write the objective function in the form

\[ P - x_1 - 4x_2 - 3x_3 - x_4 = 0 \]

For each constraint introduce a nonnegative slack variable to obtain the system of equations:

\[
\begin{align*}
2x_1 + x_2 + s_1 &= 10 \\
3x_1 + x_2 + x_3 + 2x_4 + s_2 &= 18 \\
x_1 + x_2 + x_3 + x_4 + s_3 &= 14
\end{align*}
\]

where \( x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \quad s_3 \geq 0 \)

These equations, together with the objective function \( P \), give the initial simplex tableau:

\[
\begin{array}{cccccccc}
BV & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & RHS \\
s_1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 10 \\
s_2 & 0 & 3 & 1 & 1 & 2 & 0 & 1 & 0 & 18 \\
s_3 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 14 \\
P & 1 & -1 & -4 & -3 & -1 & 0 & 0 & 0 & 0
\end{array}
\]

Notice that in each initial simplex tableau, an identity matrix appears under the columns headed by \( P \) and the slack variables. Notice too that the right-hand column (RHS) always contains nonnegative constants.

**NOW WORK PROBLEM 23.**

3 Perform Pivot Operations

Before going any further in our discussion of the simplex method, we need to discuss the matrix operation known as *pivoting*. The first thing one does in a pivot operation is to choose a *pivot element*. For now the pivot element will be specified. The method for selecting a pivot element in a simplex tableau will be discussed in the next section.
Definition

Pivoting

To pivot a matrix about a given element, called the pivot element, is to apply certain row operations so that the pivot element is replaced by a 1 and all other entries in the same column, called the pivot column, become 0s. The row containing the pivot element is called the pivot row.

Steps for Pivoting

STEP 1 In the pivot row where the pivot element appears, divide each entry by the pivot element, which we assume is not 0. This row operation causes the pivot element to become 1.

STEP 2 Obtain 0s elsewhere in the pivot column by performing row operations using the revised pivot row.

The row operations used in Steps 1 and 2 are the same as those used in Chapter 3.

To illustrate the pivot operation, we continue with the initial simplex tableau obtained in Example 2. See Display (2) on page 221.

EXAMPLE 4 Performing a Pivot Operation

Perform a pivot operation on the initial simplex tableau given in Display (2) on page 221 and repeated below in Display (3). The pivot element is circled and the pivot row and pivot column are indicated.

\[
\begin{array}{cccccc}
\text{BV} & P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
 s_1 & 0 & 2 & 4 & 1 & 0 & 120 \\
 s_2 & 0 & \bigcirc & 2 & 0 & 1 & 80 \\
 P & 1 & -3 & -4 & 0 & 0 & 0 \\
\end{array}
\]

(3)

SOLUTION

STEP 1 In this tableau the pivot column is column \(x_1\) and the pivot row is row \(s_2\). Step 1 of the pivoting procedure tells us to divide the pivot row by the pivot element 2, so we use the row operation

\[
R_2 = \frac{1}{2} r_2
\]

which gives

\[
\begin{array}{cccccc}
P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
 0 & 2 & 4 & 1 & 0 & 120 \\
 0 & 1 & 1 & 0 & \frac{1}{2} & 40 \\
 1 & -3 & -4 & 0 & 0 & 0 \\
\end{array}
\]

Revised pivot row \(R_2\)
STEP 2

The revised pivot row is row $R_2$. To obtain 0s elsewhere in the pivot column, multiply row $R_2$ by $-2$ and add it to row 1; then multiply row $R_2$ by 3 and add it to row 3. The row operations specified are

$$
R_1 = -2R_2 + r_1 \quad \text{and} \quad R_3 = 3R_2 + r_3
$$

The new tableau looks like this:

$$
\begin{array}{cccccc}
P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
0 & 0 & 2 & 1 & -1 & 40 \\
0 & 1 & 1 & 0 & \frac{1}{2} & 40 \\
1 & 0 & -1 & 0 & \frac{3}{2} & 120 \\
\end{array}
$$

(4)

This completes the pivot operation since the pivot column (column $x_1$) has a 1 in the pivot row and has 0s everywhere else.

\[\text{NOW WORK PROBLEM 35(a).}\]

4

Analyze a Tableau

Just what has the pivot operation done? To see, look again at the initial simplex tableau in Display (3). Observe that the entries in columns $P$, $s_1$, and $s_2$ can be reordered to form an identity matrix ($I_3$). This makes it easy to solve for $s_1$, $s_2$, and $P$ using the other variables as parameters:

$$
\begin{align*}
2x_1 + 4x_2 + s_1 &= 120 \quad \text{or} \quad s_1 = -2x_1 - 4x_2 + 120 \\
2x_1 + 2x_2 + s_2 &= 80 \quad \text{or} \quad s_2 = -2x_1 - 2x_2 + 80 \\
P - 3x_1 - 4x_2 &= 0 \quad \text{or} \quad P = 3x_1 + 4x_2
\end{align*}
$$

The variables $s_1$, $s_2$, and $P$ are the original basic variables (BV) listed in the tableau.

After pivoting, we obtain the tableau given in Display (4). Notice that in this form, it is easy to solve for $s_1$, $x_1$, and $P$ in terms of $x_2$ and $s_2$.

$$
\begin{align*}
2x_2 + s_1 - s_2 &= 40 \quad \text{or} \quad s_1 = -2x_2 + s_2 + 40 \\
x_1 + x_2 + \frac{1}{2}s_2 &= 40 \quad \text{or} \quad x_1 = -x_2 - \frac{1}{2}s_2 + 40 \\
P - x_2 + \frac{3}{2}s_2 &= 120 \quad \text{or} \quad P = x_2 - \frac{3}{2}s_2 + 120
\end{align*}
$$

(5)

The variables $s_1$, $x_1$, and $P$ are the new basic variables of the tableau. The variables $x_2$ and $s_2$ are the **nonbasic variables**. The result of pivoting is that $x_1$ becomes a basic variable, while $s_2$ becomes a nonbasic variable.

Because the pivot process changes the basic variables from $s_1$, $s_2$, $P$ to $s_1$, $x_1$, $P$, we write Display (4) as

$$
\begin{array}{cccccc}
\text{BV} & P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
s_1 & 0 & 0 & 2 & 1 & -1 & 40 \\
x_1 & 0 & 1 & 1 & 0 & \frac{1}{2} & 40 \\
P & 1 & 0 & -1 & 0 & \frac{3}{2} & 120 \\
\end{array}
$$

(6)
Notice in Display (5) that if we let the value of the nonbasic variables $x_2$ and $s_2$ equal 0, then the basic variables $s_1, x_1,$ and $P$ are

\[
\begin{align*}
    s_1 &= -2x_2 + s_2 + 40 = -2(0) + 0 + 40 = 40 \\
    x_1 &= -x_2 - \frac{1}{2}s_2 + 40 = 0 - \frac{1}{2}(0) + 40 = 40 \\
    P &= x_2 - \frac{3}{2}s_2 + 120 = 0 - 0 + 120 = 120
\end{align*}
\]

That is, the current value of $P$ is 120. Notice also that the values of $s_1, x_1,$ and $P$ are the ones found across from them in the far right column under RHS of the tableau in Display (6).

Because $P = x_2 - \frac{3}{2}s_2 + 120$ and $x_2 \geq 0, s_2 \geq 0,$ the value of $P$ can be increased beyond 120 when $x_2 > 0$ and $s_2 = 0.$ In other words, $P$ is not yet a maximum.

**Analyzing a Tableau: Finding Current Values**

The current values of the objective function and the basic variables in a tableau are found as follows:

- From the tableau, write the equation corresponding to each row.
- Solve the bottom equation for $P$ and the remaining equations for the basic variables.
- Set each nonbasic variable equal to zero to obtain the current values of $P$ and the basic variables.

**EXAMPLE 5** Performing a Pivot Operation; Analyzing a Tableau

Perform another pivot operation on the tableau given in Display (6), using the circled pivot element in the tableau below. Then analyze the new tableau.

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>40</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>40</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>120</td>
</tr>
</tbody>
</table>

**Current values**

\[
\begin{align*}
    s_1 &= 40 \\
    x_1 &= 40 \\
    P &= 120
\end{align*}
\]

**SOLUTION**

**STEP 1** The pivot element, 2, is in row 1. Use the row operation

\[ R_1 = \frac{1}{2} r_1 \]
5.1 The Simplex Tableau: Pivoting

To obtain

\[
\begin{array}{cccc|c}
P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 20 \\
0 & 1 & 1 & 0 & \frac{1}{2} & 40 \\
1 & 0 & -1 & 0 & \frac{3}{2} & 120 \\
\end{array}
\]

**STEP 2** To get 0’s in the pivot column \( x_2 \), use the row operations

\[
R_2 = -R_1 + r_2 \quad R_3 = R_1 + r_3
\]

Then

\[
\begin{array}{cccc|c}
\text{BV} & P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
x_2 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 20 \\
x_1 & 0 & 1 & 0 & -\frac{1}{2} & 1 & 20 \\
P & 1 & 0 & 0 & \frac{1}{2} & 1 & 140 \\
\end{array}
\]

In the tableau given in Display (7), the basic variables are \( x_2, x_1, \) and \( P \). The variables \( s_1 \) and \( s_2 \) are the nonbasic variables. The result of pivoting caused \( x_2 \) to become a basic variable and \( s_1 \) to become a nonbasic variable. Finally, the equations represented by Display (7) can be written as

\[
x_2 = -\frac{1}{2} s_1 + \frac{1}{2} s_2 + 20
\]

\[
x_1 = \frac{1}{2} s_1 - s_2 + 20
\]

\[
P = -\frac{1}{2} s_1 - s_2 + 140
\]

Let the nonbasic variables \( s_1 \) and \( s_2 \) equal 0. Then the current values of the basic variables are

\[
P = 140 \quad x_2 = 20 \quad x_1 = 20
\]

The second pivot has improved the value of \( P \) from 120 to 140.

Because \( P = -\frac{1}{2} s_1 - s_2 + 140 \) and \( s_1 \geq 0, s_2 \geq 0 \), the value of \( P \) cannot increase beyond 140 because any values of \( s_1 \) and \( s_2 \), other than 0, reduce the value of \( P \). This means we have maximized \( P \).

\[\blacksquare\]
Look at the sequence of tableaus from Examples 4 and 5 and interpret each tableau using a graph.

In the initial tableau, the three rows represent:

\[
\begin{align*}
2x_1 + 4x_2 + s_1 &= 120 & s_1 &\geq 0 & 2x_1 + 4x_2 &\leq 120 \\
2x_1 + 2x_2 + s_2 &= 80 & s_2 &\geq 0 & 2x_1 + 2x_2 &\leq 80 \\
P - 3x_1 - 4x_2 &= 0 & & & P &= 3x_1 + 4x_2
\end{align*}
\]

Set up graphs using \(x_1\) as the horizontal axis and \(x_2\) as the vertical axis to illustrate each of these tableaus. See Figure 1.

As Figure 1 shows the pivoting process takes us from one corner point \((0, 0)\), to another corner point \((40, 0)\), ending at the corner point \((20, 20)\), where \(P\) is maximized.
EXAMPLE 6 Performing a Pivot Operation; Analyzing a Tableau

The following maximum problem is in standard form:
Maximize
\[ P = 2x_1 + 3x_2 \]
subject to the constraints
\[ 3x_1 + 2x_2 \leq 200 \]
\[ x_1 + 3x_2 \leq 150 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

(a) Set up the initial simplex tableau.

(b) Using the indicated pivot element, perform a pivot operation.

(c) Analyze the resulting simplex tableau.

SOLUTION

(a) Rewrite the objective function in the form
\[ P - 2x_1 - 3x_2 = 0 \]

Introduce two slack variables, \( s_1 \) and \( s_2 \), and obtain the system of equations
\[
\begin{align*}
3x_1 + 2x_2 + s_1 &= 200 \\
x_1 + 3x_2 + s_2 &= 150 \\
P - 2x_1 - 3x_2 &= 0
\end{align*}
\]

where
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \]

The initial simplex tableau is

<table>
<thead>
<tr>
<th>BV</th>
<th>( P )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>150</td>
</tr>
<tr>
<td>( P )</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) In the tableau below, the pivot element is circled and the pivot row and pivot column are marked.

\[
\begin{array}{ccccccc}
\text{Pivot column} & & & & & & \\
\downarrow & & & & & & \\
\text{Pivot row} & & & & & & \\
<table>
<thead>
<tr>
<th>BV</th>
<th>( P )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>150</td>
</tr>
<tr>
<td>( P )</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
\end{array}
\]
230 Chapter 5 Linear Programming: Simplex Method

**STEP 1** To pivot, first use the row operation $R_1 = \frac{1}{3}r_1$ to obtain a 1 in the pivot row and pivot column.

```
Pivot column
\[
\begin{bmatrix}
P & x_1 & x_2 & s_1 & s_2 & RHS \\
0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 & \frac{200}{3} \\
0 & 1 & 3 & 0 & 1 & 150 \\
1 & -2 & -3 & 0 & 0 & 0
\end{bmatrix}
\]
```

**STEP 2** To obtain 0s elsewhere in the pivot column, use the row operations:

- $R_2 = -R_1 + r_2$
- $R_3 = 2R_1 + r_3$

The result is the tableau:

```
BV [P x_1 x_2 s_1 s_2 RHS]
\[
\begin{bmatrix}
x_1 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & \frac{200}{3} \\
s_2 & 0 & 0 & \frac{7}{3} & -\frac{1}{3} & 1 & \frac{250}{3} \\
P & 1 & 0 & -\frac{5}{3} & \frac{2}{3} & 0 & \frac{400}{3}
\end{bmatrix}
\]
```

(c) In the tableau above, the basic variables are $x_1, s_2$, and $P$. The nonbasic variables are $x_2$ and $s_1$. The equations represented by the tableau are

- $x_1 = \frac{200}{3} - \frac{2}{3}x_2 - \frac{1}{3}s_1$
- $s_2 = \frac{250}{3} - \frac{7}{3}x_2 + \frac{1}{3}s_1$
- $P = \frac{400}{3} + \frac{5}{3}x_2 - \frac{2}{3}s_1$

Let the nonbasic variables $x_2$ and $s_1$ equal 0. Then the current values of the basic variables are

- $x_1 = \frac{200}{3}$
- $s_2 = \frac{250}{3}$
- $P = \frac{400}{3}$

Because $P = \frac{400}{3} + \frac{5}{3}x_2 - \frac{2}{3}s_1$, and $x_2 \geq 0$ and $s_1 \geq 0$, we see that $P$ can be further increased by choosing a positive value for the nonbasic variable $x_2$ and leaving $s_1 = 0$.

- **NOW WORK PROBLEM 35(b) AND 35(c).**

**COMMENT** The pivot operation can be done using a graphing utility such as the TI-84 Plus. The augmented matrix contained in the initial tableau is entered into the calculator. Once the pivot element has been identified, the elementary row operations built into the graphing utility are used to obtain a new tableau with a pivot element of one and all other elements in the pivot column replaced with zeros. This is best seen using an example.
5.1 The Simplex Tableau: Pivoting 231

**USING TECHNOLOGY**

**EXAMPLE 7** Performing a Pivot Operation Using a Graphing Utility

In the initial tableau below, the pivot element is circled and the pivot row and pivot column are marked. Use a graphing utility to perform a pivot operation.

\[
\begin{array}{cccccc}
\text{BV} & P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
\text{Pivot row} & s_1 & 0 & 3 & 2 & 1 & 0 \\
p & 1 & -2 & -3 & 0 & 0 & 0 \\
\end{array}
\]

**SOLUTION**

First enter the initial tableau into the graphing utility. See Figure 2. The matrix has 3 rows and 6 columns and has been named matrix \([A]\). Using the pivot element 3 as indicated above, follow the steps below using the elementary row operations in the calculator to obtain the new tableau.

**FIGURE 2**
The initial tableau

**FIGURE 3**
The simplex tableau after one pivot

**STEP 1** Make the pivot element one by dividing the pivot row by 3. On a TI-84 Plus, the operation is

\[
\text{matrix math E: } *\text{row}(1/3, [A], 1) 
\]

**STEP 2** Obtain 0’s elsewhere in the pivot column by performing row operations using the revised pivot row from Step 1. To obtain 0s elsewhere in the pivot column, multiply row 1 by \(-1\) and add it to row 2, then multiply row 1 by 2 and add it to row 3. On a TI-84 Plus, the operations are

\[
\text{matrix math F: } *\text{row} + (\text{hit enter}) \\
*\text{row} + (-1, \text{ANS}, 1, 2) \\
\text{matrix math F: } *\text{row} + (\text{hit enter}) \\
*\text{row} + (2, \text{ANS}, 1, 3)
\]

Figure 3 shows the result.
In Problems 7–16, determine which maximum problems are in standard form. Do not attempt to solve them!

**Skill Building**

In Problems 7–16, determine which maximum problems are in standard form. Do not attempt to solve them!

7. Maximize

\[ P = 2x_1 + x_2 \]

subject to the constraints

\[ x_1 + x_2 \leq 5 \]
\[ 2x_1 + 3x_2 \leq 2 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]

8. Maximize

\[ P = 3x_1 + 4x_2 \]

subject to the constraints

\[ 3x_1 + x_2 \leq 6 \]
\[ x_1 + 4x_2 \leq 74 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]

9. Maximize

\[ P = 3x_1 + x_2 + x_3 \]

subject to the constraints

\[ x_1 + x_2 + x_3 \leq 6 \]
\[ 2x_1 + 3x_2 + 4x_3 \leq 10 \]
\[ x_1 \geq 0 \]

10. Maximize

\[ P = 2x_1 + x_2 + 4x_3 \]

subject to the constraints

\[ x_1 + x_2 + x_3 \leq 10 \]
\[ x_2 \geq 0 \]

11. Maximize

\[ P = 3x_1 + x_2 + x_3 \]

subject to the constraints

\[ x_1 + x_2 + x_3 \leq 8 \]
\[ 2x_1 + x_2 + 4x_3 \geq 6 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]

12. Maximize

\[ P = 2x_1 + x_2 + 4x_3 \]

subject to the constraints

\[ 2x_1 + x_2 + x_3 \leq -1 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]

13. Maximize

\[ P = 2x_1 + x_2 \]

subject to the constraints

\[ x_1 + x_2 \geq -6 \]
\[ 2x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]

14. Maximize

\[ P = 3x_1 + x_2 \]

subject to the constraints

\[ x_1 + 3x_2 \leq 4 \]
\[ 2x_1 - x_2 \geq 1 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]

15. Maximize

\[ P = 2x_1 + x_2 + 3x_3 \]

subject to the constraints

\[ x_1 + x_2 - x_3 \leq 10 \]
\[ x_2 + x_3 \leq 4 \]
\[ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0 \]

16. Maximize

\[ P = 2x_1 + 2x_2 + 3x_3 \]

subject to the constraints

\[ x_1 - x_2 + x_3 \leq 6 \]
\[ x_1 \leq 4 \]
\[ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0 \]
In Problems 17–22, each maximum problem is not in standard form. Determine if the problem can be modified so as to be in standard form. If it can, write the modified version.

29. Maximize

\[ P = x_1 + x_2 \]

subject to the constraints

\[ 3x_1 - 4x_2 \leq -6 \]
\[ x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

In Problems 23–30, each maximum problem is in standard form. For each one introduce slack variables and set up the initial simplex tableau.

23. Maximize

\[ P = 2x_1 + x_2 + 3x_3 \]

subject to the constraints

\[ 5x_1 + 2x_2 + x_3 \leq 20 \]
\[ 6x_1 - x_2 + 4x_3 \leq 24 \]
\[ x_1 + x_2 + 4x_3 \leq 16 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

24. Maximize

\[ P = 3x_1 + 2x_2 + x_3 \]

subject to the constraints

\[ 3x_1 + 2x_2 - x_3 \leq 10 \]
\[ x_1 - x_2 + 3x_3 \leq 12 \]
\[ 2x_1 + x_2 + x_3 \leq 6 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

26. Maximize

\[ P = 2x_1 + 3x_2 \]

subject to the constraints

\[ 1.2x_1 - 2.1x_2 \leq 0.5 \]
\[ 0.3x_1 + 0.4x_2 \leq 1.5 \]
\[ x_1 + x_2 \leq 0.7 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

29. Maximize

\[ P = 3x_1 + 4x_2 + 2x_3 \]

subject to the constraints

\[ 3x_1 + x_2 + 4x_3 \leq 5 \]
\[ x_1 - x_2 \leq 5 \]
\[ 2x_1 - x_2 + x_3 \leq 6 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]
In Problems 31–34, each maximum problem can be modified so as to be in standard form. Write the modified version and, for each one, introduce slack variables and set up the initial simplex tableau.

31. Maximize

\[ P = x_1 + 2x_2 + 5x_3 \]

subject to the constraints

\[-x_1 + 2x_2 + 3x_3 \geq -10 \]
\[3x_1 + x_2 - x_3 \geq -12 \]
\[x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

32. Maximize

\[ P = 2x_1 + 4x_2 + x_3 \]

subject to the constraints

\[-2x_1 - 3x_2 + x_3 \geq -8 \]
\[-3x_1 + x_2 - 2x_3 \geq -12 \]
\[2x_1 + x_2 + x_3 \leq 10 \]
\[x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

33. Maximize

\[ P = 2x_1 + 3x_2 + x_3 + 6x_4 \]

subject to the constraints

\[-x_1 + x_2 + 2x_3 + x_4 \leq 10 \]
\[x_1 - x_2 + x_3 - x_4 \geq -8 \]
\[x_1 + x_2 + x_3 + x_4 \leq 9 \]
\[x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \]

34. Maximize

\[ P = x_1 + 5x_2 + 3x_3 + 6x_4 \]

subject to the constraints

\[x_1 - x_2 + 2x_3 - 2x_4 \geq -8 \]
\[-x_1 + x_2 + x_3 - x_4 \geq -10 \]
\[x_1 + x_2 + x_3 + x_4 \leq 12 \]
\[x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \]

In Problems 35–40:

(a) perform a pivot operation on each tableau. The pivot element is circled.

(b) Using the new tableau obtained, write the corresponding system of equations.

(c) Analyze the new tableau.

In Section 5.2, you will be asked to solve Problems 41–46.

41. **A Toy Story, Part I** A toy store has shelf space to hold at most 60 baby dolls of three different types. Sleeping baby dolls cost $6 each, talking baby dolls cost $7.50 each, and walking baby dolls cost $9 each. The store can afford to spend up to $405 for the dolls. The sleeping, talking, and walking baby dolls resell for $12, $13.50, and $17 each, respectively. The store wishes to maximize its profit by ordering the appropriate number of each type of doll. (Remember: Profit = Revenue – Cost.)

(a) Define the variables.

(b) Write the maximum problem in standard form.

(c) Introduce slack variables.

(d) Setup the initial tableau.
42. A Toy Story, Part II
Concerned about having an appropriate variety of baby dolls for customers, the manager of the toy store from Problem 41 decides to change the ordering requirements as follows: The number of sleeping baby dolls ordered should be at most 6 more than the number of talking dolls ordered, and the number of talking dolls ordered should be at most 9 more than the number of walking dolls. Rework Problem 41 with these additional constraints.

43. Clothing
Steven owns a company that produces men’s shirts, jackets, and pants, each of which require time for cutting and sewing. One shirt requires 10 minutes of cutting and 20 minutes of sewing, one jacket requires 20 minutes of cutting and 40 minutes of sewing, and one pair of pants requires 20 minutes of cutting and 30 minutes of sewing. Steven has 500 minutes of cutting time available each day and 900 minutes of sewing time. His company makes a profit of $19 for each shirt, $34 for each jacket, and $15 on each pair of pants. Steven wants to maximize his profit, but can produce at most 30 items in one day.
(a) Define the variables.
(b) Write the maximum problem in standard form.
(c) Introduce slack variables.
(d) Set up the initial tableau.

44. Fruit Trees
Suzanne owns an orchard and wants to plant some additional fruit trees to obtain maximum revenue. She will use at most 43,560 square feet of land and can plant pear, peach, and plum trees. She wants the combined total of peach trees and plum trees to be no more than 150, but also wants to make sure that the number of pear trees does not exceed the number of peach trees by more than 10. The table provides additional information.

(a) Define the variables.
(b) Write the maximum problem in standard form.
(c) Set up the initial tableau.

45. Investments
Kami has up to $25,000 to invest and can divide it among three investments: (1) a money market account that pays 1.05% annually, (2) a mutual fund that pays 5% annually, and (3) a certificate of deposit (CD) that pays 1.66% annually. She wants to maximize her return, but does not want to invest more than $10,000 between the mutual fund and the CD. In addition, the amount invested in the CD cannot exceed the amount invested in the money market by more than $1500.
(a) Define the variables.
(b) Write the maximum problem in standard form.
(c) Introduce slack variables.
(d) Set up the initial tableau.

46. Travel Arrangements
A textbook publishing company has 10 sales representatives stationed in Chicago and 15 sales representatives stationed in Boston. The company wants to send representatives to two different conventions being held simultaneously in Atlanta and Seattle. The company would like to fly up to 12 reps to the Atlanta convention and up to 8 reps to the Seattle convention. A round-trip ticket from Chicago to Atlanta costs approximately $400, from Chicago to Seattle costs approximately $450, from Boston to Atlanta costs approximately $400, and from Boston to Seattle costs approximately $475. (Source: priceline.com) The total flight costs for the trips cannot exceed $7700. The company needs to determine how many sales representatives should fly from Chicago to Atlanta, from Chicago to Seattle, from Boston to Atlanta, and from Boston to Seattle in order to maximize the number of representatives at the conferences.
(a) Define the variables.
(b) Write the maximum problem in standard form.
(c) Introduce slack variables.
(d) Set up the initial tableau.
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

OBJECTIVES

1. Solve maximum problems in standard form using the simplex method (p. 236)
2. Determine if a tableau is final, requires additional pivoting, or indicates no solution (p. 240)
3. Solve applied problems involving maximum problems in standard form (p. 243)

1. Solve Maximum Problems in Standard Form Using the Simplex Method

The steps given on page 237 are used to solve a maximum problem in standard form. We return to Examples 4 and 5 of the previous section to help see the rationale for each step.

Look back at Example 4 on page 224. The initial simplex tableau given there is repeated in the far left column. The tableau obtained after one pivot is given next, and then the tableau obtained after another pivot, this one being the final tableau.

The pivot element is circled. Why did we select this particular entry? Remember, we want to maximize \( P \) and the objective row tells us we can improve the value of \( P \) by increasing either \( x_1 \) or \( x_2 \). We’ll choose \( x_1 \). This corresponds to choosing the first negative entry in the objective row and identifies the pivot column. But we cannot increase \( x_1 \) without bound, as we will eventually leave the set of feasible solutions. Look at the constraints:

\[
2x_1 + 4x_2 \leq 120 \quad \text{and} \quad 2x_1 + 2x_2 \leq 80
\]

If \( x_2 = 0 \), then

\[
x_1 \leq \frac{120}{2} = 60 \quad \text{and} \quad x_1 \leq \frac{80}{2} = 40
\]

so \( x_1 \leq 40 \). This number \( \left( \frac{80}{2} \right) \) is the smallest quotient obtained when the RHS (80) is divided by the corresponding entry in the pivot column (2) and identifies the pivot row.

Why do we pivot again? Now

\[
P = x_2 - \frac{3}{2}s_2 + 120.
\]

Since \( x_2 \geq 0 \) and \( s_2 \geq 0 \), we can improve \( P \) by increasing \( x_2 \) and letting \( s_2 = 0 \). This corresponds to choosing the first negative entry in the objective row (\( -1 \)) establishing the pivot column \( x_2 \). Do you see why the pivot row is \( s_2 \)?

Why is this a final tableau? Here

\[
P = -\frac{1}{2}s_1 - s_2 + 140.
\]

Since \( s_1 \geq 0 \) and \( s_2 \geq 0 \), and their coefficients are negative, \( P \) is maximized when \( s_1 = 0 \) and \( s_2 = 0 \). The negative coefficients of \( P \) correspond to no negative entries in the objective row.
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

The Simplex Method for Solving a Maximum Problem in Standard Form

**STEP 1**
Set up the initial simplex tableau.

**STEP 2**
Look at the entries in the objective row, excluding the RHS entry.

- If there are negative entries, choose the first negative entry going from left to right.* This entry identifies the pivot column.
- If all the entries are positive or zero, STOP. This is a final tableau. A solution has been found.

**STEP 3**
Look at the entries in the pivot column.

- For each positive entry above the objective row, divide the corresponding RHS entry by the entry in the pivot column. Pick the smallest nonnegative quotient. This identifies the pivot row. (In case of ties, use either row.)
- If all the entries are zero or negative, STOP. The problem is unbounded and has no solution.

**STEP 4**
Identify the pivot element and pivot.

**STEP 5**
Repeat STEPS 2, 3, and 4 until a final tableau is obtained or no solution is found.

---

**EXAMPLE 1 Using the Simplex Method**

Maximize

\[ P = 3x_1 + 4x_2 \]

subject to the constraints

\[ 2x_1 + 4x_2 \leq 120 \\
2x_1 + 2x_2 \leq 80 \\
x_1 \geq 0 \quad x_2 \geq 0 \]

* By choosing the first negative entry going from left to right, we invoke part of Bland’s rule, which prevents cycling, going back and forth between the same collection of tableaus. In reality, the choice of a pivot element is usually driven by other issues. As it is beyond the scope of this book, we shall not deal with such issues.
Chapter 5  Linear Programming: Simplex Method

**SOLUTION**

**STEP 1** This is a maximum problem in standard form. To obtain the initial simplex tableau, proceed as follows: The objective function is rewritten in the form

\[ P = 3x_1 - 4x_2 = 0 \]

After introducing slack variables \( s_1 \) and \( s_2 \), the constraints take the form

\[
\begin{align*}
2x_1 + 4x_2 + s_1 &= 120 \\
2x_1 + 2x_2 + s_2 &= 80
\end{align*}
\]

where \( x_1 \geq 0 \), \( x_2 \geq 0 \), \( s_1 \geq 0 \), and \( s_2 \geq 0 \).

The initial simplex tableau is

\[
\begin{array}{c|cccccc}
BV & P & x_1 & x_2 & s_1 & s_2 & RHS \\
\hline
s_1 & 0 & 2 & 4 & 1 & 0 & 120 \\
s_2 & 0 & 2 & 2 & 0 & 1 & 80 \\
\hline
P & 1 & -3 & -4 & 0 & 0 & 0 \\
\end{array}
\]

**STEP 2** The first negative entry in the objective row is \(-3\). The pivot column is column \( x_1 \).

**STEP 3** For each positive entry above the objective row in the pivot column, form the quotient of the corresponding RHS entry divided by the positive entry.

<table>
<thead>
<tr>
<th>BV</th>
<th>Positive entry, ( x_1 )</th>
<th>RHS</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>2</td>
<td>120</td>
<td>( 120 \div 2 = 60 )</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>2</td>
<td>80</td>
<td>( 80 \div 2 = 40 )</td>
</tr>
</tbody>
</table>

**STEP 4** The smallest nonnegative value is 40, so the pivot row is row \( s_2 \). The tableau below shows the pivot element (circled) and the current values.

The pivot element is 2. Next, we pivot by using the row operations:

1. \( R_2 = \frac{1}{2}r_2 \)
2. \( R_1 = -2R_2 + r_1 \quad R_3 = 3R_2 + r_3 \)

After pivoting, we have the tableau:

\[
\begin{array}{c|cccccc}
BV & P & x_1 & x_2 & s_1 & s_2 & RHS \\
\hline
s_1 & 0 & 0 & 2 & 1 & -1 & 40 \\
x_1 & 0 & 1 & 1 & 0 & \frac{1}{2} & 40 \\
\hline
P & 1 & 0 & -1 & 0 & \frac{3}{2} & 120 \\
\end{array}
\]

Notice that \( x_1 \) is now a basic variable, replacing \( s_2 \).
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

**STEP 5** Repeat Step 2.

**STEP 2** The first negative entry in the objective row is $-1$. The pivot column is column $x_2$.

**STEP 3** For each positive entry above the objective row in the pivot column, form the quotient of the corresponding RHS entry by the positive entry.

<table>
<thead>
<tr>
<th>BV</th>
<th>Positive entry, $x_2$</th>
<th>RHS</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>2</td>
<td>40</td>
<td>$40 \div 2 = 20$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>40</td>
<td>$40 \div 1 = 40$</td>
</tr>
</tbody>
</table>

The smallest nonnegative value is 20, so the pivot row is row $s_1$. The tableau below shows the pivot element circled.

**STEP 4** The pivot element is 2. Next, pivot by using the row operations:

1. $R_1 = \frac{1}{2} r_1$
2. $R_2 = -R_1 + r_2 \quad R_3 = R_1 + r_3$

The result is the tableau:

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>40</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>40</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>120</td>
</tr>
</tbody>
</table>

**Current values**

- $s_1 = 40$
- $x_1 = 40$
- $P = 120$

**STEP 5** Repeat STEP 2.

**STEP 2** There are no negative entries in the objective row so we STOP.

The maximum value of $P$ is

$$P = 140$$

and it occurs at

$$x_1 = 20 \quad x_2 = 20 \quad s_1 = 0 \quad s_2 = 0$$

The tableau in Display (3) is called the **final tableau** because with this tableau the maximum value is found.
Determine If a Tableau Is Final, Requires Additional Pivoting, or Indicates No Solution

**EXAMPLE 2 Analyzing a Tableau**

Determine whether each tableau below (1) is a final tableau. [If it is, give the solution.]

(2) requires additional pivoting. [If so, identify the pivot element.]

(3) indicates no solution.

(a) BV $P x_1 x_2 s_1 s_2$ RHS

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>

(b) BV $P x_1 x_2 s_1 s_2$ RHS

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>2</th>
<th>−1</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>−1</td>
<td>40</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>

(c) BV $P x_1 x_2 s_1 s_2$ RHS

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>0.5</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0.25</td>
<td>125</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>−4</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

**SOLUTION**

(a) The first negative entry in the objective row is $−2$. The pivot column is column $x_2$. Since all the entries in the pivot column are zero or negative, the problem is unbounded and has no solution.

(b) The objective row contains no negative entries, so this is a final tableau. The solution is

$P = 220$ when $x_1 = 40 \quad x_2 = 20 \quad s_1 = 0 \quad s_2 = 0$

(c) The first negative entry in the objective row is $−4$. The pivot column is column $x_1$. Since both entries in the pivot column are positive, compute the quotients of the RHS divided by the corresponding entries in the pivot column

$60 \div 1 = 60 \quad 125 \div 3 = 41 \frac{2}{3}$

Since $41 \frac{2}{3} < 60$, $x_2$ is the pivot row and 3 is the pivot element.

**NOW WORK PROBLEM 5.**

**EXAMPLE 3 Solving a Maximum Problem Using the Simplex Method**

Maximize

$$P = 6x_1 + 8x_2 + x_3$$
subject to the constraints

\[
\begin{align*}
3x_1 + 5x_2 + 3x_3 & \leq 20 \\
x_1 + 3x_2 + 2x_3 & \leq 9 \\
6x_1 + 2x_2 + 5x_3 & \leq 30 \\
x_1 \geq 0 & \quad x_2 \geq 0 & \quad x_3 \geq 0
\end{align*}
\]

**SOLUTION** Notice that the maximum problem is in standard form. By introducing slack variables \( s_1, s_2, \) and \( s_3 \), the constraints take the form

\[
\begin{align*}
3x_1 + 5x_2 + 3x_3 + s_1 & = 20 \\
x_1 + 3x_2 + 2x_3 + s_2 & = 9 \\
6x_1 + 2x_2 + 5x_3 + s_3 & = 30 \\
\end{align*}
\]

where \( x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \)
\( s_1 \geq 0 \quad s_2 \geq 0 \quad s_3 \geq 0 \)

Rewriting the objective function as

\[ P - 6x_1 - 8x_2 - x_3 = 0 \]

the initial simplex tableau takes the form

\[
\begin{array}{ccccccc}
\text{Pivot column} \\
\text{BV} & P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \text{RHS} & \text{Current values} & \text{RHS} \div x_1 \\
\hline
s_1 & 0 & 3 & 5 & 3 & 1 & 0 & 0 & 20 & s_1 = 20 & 20 \div 3 = 6 \frac{2}{3} \\
s_2 & 0 & 1 & 3 & 2 & 0 & 1 & 0 & 9 & s_2 = 9 & 9 \div 1 = 9 \\
s_3 & 0 & 6 & 2 & 5 & 0 & 0 & 1 & 30 & s_3 = 30 & 30 \div 6 = 5 \\
P & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P = 0 \\
\end{array}
\]

The pivot column is the column containing the first negative entry in the objective row (\(-6\) in column \( x_1 \)). The pivot row is obtained by dividing each entry in the RHS column by the corresponding entry in the pivot column and selecting the smallest nonnegative quotient. The pivot row is row \( s_3 \). The pivot element is 6, which is circled.

Pivot: 1. \( R_3 = \frac{1}{6}r_3 \) 2. \( R_1 = -3R_3 + r_1 \) \( R_2 = -R_3 + r_2 \) \( R_4 = 6R_3 + r_4 \)

\[
\begin{array}{ccccccc}
\text{Pivot column} \\
\text{BV} & P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \text{RHS} & \text{Current values} & \text{RHS} \div x_2 \\
\hline
s_1 & 0 & 0 & \frac{4}{3} & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 5 & s_1 = 5 & 5 \div 4 = 1.25 \\
s_2 & 0 & 0 & \frac{8}{3} & \frac{7}{6} & 0 & 1 & -\frac{1}{6} & 4 & s_2 = 4 & 4 \div \frac{8}{3} = 1.5 \\
x_1 & 0 & 1 & \frac{5}{3} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 5 & x_1 = 5 & 5 \div \frac{1}{3} = 15 \\
P & 1 & 0 & -6 & 4 & 0 & 0 & 1 & 30 & P = 30 \\
\end{array}
\]
Chapter 5  Linear Programming: Simplex Method

The value of \( P \) has improved to 30, but the negative entry, \(-6\), in the objective row indicates further improvement is possible. We determine the next pivot element to be 4, which is circled.

Pivot: 1. \( R_1 = \frac{1}{4} r_1 \)  \( R_2 = -\frac{8}{3} R_1 + r_2 \)  \( R_3 = -\frac{1}{3} R_1 + r_3 \)  \( R_4 = 6R_1 + r_4 \)

The solution is found by looking at the current values of the final tableau. The maximum value of \( P \) is obtained when \( x_1 = \frac{55}{12} \), \( x_2 = \frac{5}{4} \), and \( x_3 = 0 \).

\[ \text{NOW WORK PROBLEM 13.} \]

**EXAMPLE 4  Solving a Maximum Problem Using the Simplex Method**

Maximize

\[ P = 4x_1 + 4x_2 + 3x_3 \]

subject to the constraints

\[ 2x_1 + 3x_2 + x_3 \leq 6 \]
\[ x_1 + 2x_2 + 3x_3 \leq 6 \]
\[ x_1 + x_2 + x_3 \leq 5 \]
\[ x_1 \geq 0 \hspace{1cm} x_2 \geq 0 \hspace{1cm} x_3 \geq 0 \]

**SOLUTION** The maximum problem is in standard form. Introduce slack variables \( s_1, s_2, s_3 \), and write the constraints as

\[ 2x_1 + 3x_2 + x_3 + s_1 = 6 \]
\[ x_1 + 2x_2 + 3x_3 + s_2 = 6 \]
\[ x_1 + x_2 + x_3 + s_3 = 5 \]

where

\[ x_1 \geq 0 \hspace{1cm} x_2 \geq 0 \hspace{1cm} x_3 \geq 0 \]
\[ s_1 \geq 0 \hspace{1cm} s_2 \geq 0 \hspace{1cm} s_3 \geq 0 \]

Since \( P - 4x_1 - 4x_2 - 3x_3 = 0 \), the initial simplex tableau is

\[ \text{RHS} \]

\[ \begin{array}{cccccccc}
0 & 1 & -4 & -3 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 1 & 0 & 0 & 6 & \text{RHS} = x_1 \times 6 \\
1 & 2 & 3 & 0 & 1 & 0 & 6 & 6 \div 1 = 6 \\
0 & 1 & 1 & 0 & 0 & 1 & 5 & 5 \div 1 = 5 \\
\end{array} \]

\[ \begin{array}{cccccccc}
0 & 1 & -4 & -3 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 1 & 0 & 0 & 6 & \text{RHS} = x_1 \times 6 \\
1 & 2 & 3 & 0 & 1 & 0 & 6 & 6 \div 1 = 6 \\
0 & 1 & 1 & 0 & 0 & 1 & 5 & 5 \div 1 = 5 \\
\end{array} \]
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

Since −4 is the first negative entry in the objective row, the pivot column is \( x_1 \). The pivot row is row 1. (Do you see why?) The pivot element is 2, which is circled.

Pivot 1. \( R_1 = \frac{1}{2} r_1 \)  
2. \( R_2 = -R_1 + r_2 \)  
3. \( R_3 = -R_1 + r_3 \)  
4. \( R_4 = 4R_1 + r_4 \)

The pivot element is \( \frac{5}{2} \), which is circled.

Pivot 1. \( R_2 = \frac{2}{5} r_2 \)  
2. \( R_1 = -\frac{1}{2} R_2 + r_1 \)  
3. \( R_3 = -\frac{1}{2} R_2 + r_3 \)  
4. \( R_4 = R_2 + r_4 \)

This is a final tableau. The solution is \( P = \frac{66}{5} \), obtained when \( x_1 = \frac{12}{5} \), \( x_2 = 0 \), \( x_3 = \frac{6}{5} \).

3 Solve Applied Problems Involving Maximum Problems in Standard Form

**EXAMPLE 5 Maximizing Profit**

Mike's Famous Toy Trucks specializes in making four kinds of toy trucks: a delivery truck, a dump truck, a garbage truck, and a gasoline truck. Three machines—a metal casting machine, a paint spray machine, and a packaging machine—are used in the production of these trucks. The maximum time available per week for each machine is given as metal casting, 4000 hours; paint spray, 1800 hours; and packaging, 1000 hours. The time, in hours, each machine works to make each type of truck and the profit for each truck are given in Table 1.

**(a)** How many of each type truck should be produced weekly to maximize profit? Assume that every truck made is sold.

**(b)** Analyze the solution.

*Many applied problems require integer solutions and, as such, are integer programming problems. Such problems require special methods and, strictly speaking, are not linear programming problems. In this book we shall not distinguish between the two.*
SOLUTION  (a) Let $x_1$, $x_2$, $x_3$, and $x_4$ denote the number of delivery trucks, dump trucks, garbage trucks, and gasoline trucks, respectively, to be made. If $P$ denotes the profit to be maximized, we have this problem:

Maximize

$$P = 0.5x_1 + x_2 + 1.5x_3 + 2x_4$$

subject to the constraints

$$2x_1 + 2.5x_2 + 2x_3 + 2x_4 \leq 4000$$
$$x_1 + 1.5x_2 + x_3 + 2x_4 \leq 1800$$
$$0.5x_1 + 0.5x_2 + x_3 + x_4 \leq 1000$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0$$

This maximum problem is in standard form so we introduce slack variables $s_1$, $s_2$, and $s_3$, and write the initial simplex tableau. To help in doing the required arithmetic, we’ll use fractions.

The pivot element 1 is circled.

Pivot: 1. $R_2 = r_2$  2. $R_1 = -2R_2 + r_1$  3. $R_3 = -\frac{1}{2}R_2 + r_3$  4. $R_4 = \frac{1}{2}R_2 + r_4$
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

The pivot element \( \frac{3}{2} \) is circled.

Pivot: 1. \( R_2 = \frac{2}{3} r_2 \) 2. \( R_1 = \frac{1}{2} R_2 + r_1 \) \( R_3 = \frac{1}{4} R_2 + r_3 \) \( R_4 = \frac{1}{4} R_2 + r_4 \)

<table>
<thead>
<tr>
<th>BV</th>
<th>( P )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>RHS</th>
<th>Current values</th>
<th>RHS + ( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>( -\frac{4}{3} )</td>
<td>1</td>
<td>( -\frac{5}{3} )</td>
<td>0</td>
<td>1000</td>
<td>( s_1 = 1000 )</td>
<td>3000</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>( \frac{2}{3} )</td>
<td>1</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{4}{3} )</td>
<td>0</td>
<td>( \frac{2}{3} )</td>
<td>0</td>
<td>1200</td>
<td>( x_2 = 1200 )</td>
<td>1800</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>( \frac{1}{6} )</td>
<td>0</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>0</td>
<td>( -\frac{1}{3} )</td>
<td>1</td>
<td>400</td>
<td>( s_3 = 400 )</td>
<td>600</td>
</tr>
<tr>
<td>( P )</td>
<td>1</td>
<td>( \frac{1}{6} )</td>
<td>0</td>
<td>( -\frac{5}{6} )</td>
<td>( -\frac{2}{3} )</td>
<td>0</td>
<td>( \frac{2}{3} )</td>
<td>0</td>
<td>1200</td>
<td>( P = 1200 )</td>
<td></td>
</tr>
</tbody>
</table>

The pivot element \( \frac{2}{3} \) is circled.

Pivot: 1. \( R_3 = \frac{3}{2} r_3 \) 2. \( R_1 = \frac{1}{3} R_3 + r_1 \) \( R_2 = \frac{2}{3} R_3 + r_2 \) \( R_4 = \frac{5}{6} R_3 + r_4 \)

<table>
<thead>
<tr>
<th>BV</th>
<th>( P )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>RHS</th>
<th>Current values</th>
<th>RHS + ( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td>0</td>
<td>( -\frac{3}{2} )</td>
<td>1</td>
<td>( -\frac{3}{2} )</td>
<td>( -\frac{1}{2} )</td>
<td>800</td>
<td>( s_1 = 800 )</td>
<td></td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>1</td>
<td>( -1 )</td>
<td>800</td>
<td>( x_2 = 800 )</td>
<td>800</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( -\frac{1}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>600</td>
<td>( x_3 = 600 )</td>
<td>1200</td>
</tr>
<tr>
<td>( P )</td>
<td>1</td>
<td>( \frac{3}{8} )</td>
<td>0</td>
<td>0</td>
<td>( -\frac{1}{4} )</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{5}{4} )</td>
<td>1700</td>
<td>( P = 1700 )</td>
<td></td>
</tr>
</tbody>
</table>

The pivot element 1 is circled.

Pivot: 1. \( R_2 = r_2 \) 2. \( R_1 = \frac{3}{2} R_2 + r_1 \) \( R_3 = \frac{1}{2} R_2 + r_3 \) \( R_4 = \frac{1}{4} R_2 + r_4 \)

<table>
<thead>
<tr>
<th>BV</th>
<th>( P )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>RHS</th>
<th>Current values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>1</td>
<td>( \frac{3}{2} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( -2 )</td>
<td>2000</td>
<td>( s_1 = 2000 )</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( -1 )</td>
<td>800</td>
<td>( x_4 = 800 )</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>0</td>
<td>( -\frac{1}{2} )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( -1 )</td>
<td>2</td>
<td>200</td>
<td>( x_3 = 200 )</td>
</tr>
<tr>
<td>( P )</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>1900</td>
<td>( P = 1900 )</td>
</tr>
</tbody>
</table>

This is a final tableau. The maximum profit is \( P = 1900 \), obtained when

\[
\begin{align*}
    x_1 &= 0 \\
    x_2 &= 0 \\
    x_3 &= 200 \\
    x_4 &= 800
\end{align*}
\]
(b) Observe that for maximum profit, no delivery trucks and no dump trucks are produced. The practical considerations of this are that delivery trucks and dump trucks are either too costly to produce or too little profit is being gained from their sale. Since the slack variable $s_1$ has a value of 2000 for maximum $P$ and, since $s_1$ represents the number of hours the metal casting machine is idle, it may be possible to release this machine for other duties. Also note that both the paint spray and packaging machines are operating at full capacity. This means that to increase productivity, more paint spray and packaging capacity is required.

Now Work Problem 29.

Analyzing the Simplex Method

To justify some of the steps in the simplex method, we analyze more carefully Example 5. In the initial tableau, the objective function is

$$P = 0.5x_1 + x_2 + 1.5x_3 + 2x_4$$

If we set the nonbasic variables equal to zero, $x_1 = x_2 = x_3 = x_4 = 0$, we get the profit $P = 0$. Of course, this profit can easily be improved by manufacturing some trucks. We decide to increase $x_1$, the number of delivery trucks. But by how much can $x_1$ be increased? That is determined by the constraints.

Referring to Table 1, we see that each delivery truck takes 2 hours to cast, 1 hour to paint, and $\frac{1}{2}$ hour to package. The limited time available for each task restricts the number of trucks that can be made. So we look at the constraints for producing delivery trucks:

- metal casting: 4000 hours available at 2 hours per truck means 2000 delivery trucks can be manufactured.
- paint spraying: 1800 hours available at 1 hour per truck means 1800 delivery trucks can be manufactured.
- packaging: 1000 hours available at $\frac{1}{2}$ hour per truck means 2000 delivery trucks can be manufactured.

The paint spraying constraint indicates that we can make at most 1800 delivery trucks. So we choose row $s_2$ to be the pivot row. This will increase $x_1$ to as large an amount as possible, namely 1800.

After pivoting, the tableau indicates that if we make $x_1 = 1800$ delivery trucks, but no other trucks ($x_2 = x_3 = x_4 = 0$), we will make a profit $P$ of $900$. However, the negative entries in the objective row indicate that the profit can be improved.

Each iteration of the simplex method consists of choosing an “entering” variable from the nonbasic variables and a “leaving” variable from the basic variables, using a selective criterion so that the value of the objective function is not decreased. The iterative procedure stops when no variable can enter from the nonbasic variables that improves profit. This is illustrated in the final tableau, which shows the maximum profit is obtained by producing $x_3 = 200$ garbage trucks and $x_4 = 800$ gasoline trucks.

The reasoning behind the simplex method for standard maximum problems is not too complicated, and the process of “moving to a better solution” is made easy simply by following the rules. Briefly, the pivoting strategy works like this:

The choice of the pivot column forces us to pivot a variable that improves the value of the objective function.

The choice of the pivot row prevents us from making this variable too large to be feasible.
Geometry of the Simplex Method

The maximum value of the objective function, if it exists, will occur at one of the corner points of the set of feasible points. The simplex method is designed to move from corner point to corner point of the set of feasible points, at each stage improving the value of the objective function until a solution is found or no solution is indicated. More precisely, the geometry behind the simplex method is outlined below.

1. A given tableau corresponds to a corner point of the set of feasible points.
2. The operation of pivoting moves us to an adjacent corner point, where the objective function has a value at least as large as it did at the previous corner point.
3. The process continues until the final tableau is reached—which produces a corner point that maximizes the objective function—or no solution is indicated.

The Unbounded Case

So far in our discussion, it has always been possible to continue to choose pivot elements until the problem has been solved. But it may turn out that all the entries in a pivot column of a tableau are 0 or negative at some stage. In this case, the problem is unbounded and a maximum solution does not exist.

For example, consider the initial simplex tableau

\[
\begin{bmatrix}
\text{BV} & P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
0 & 0 & -1 & 1 & 1 & 0 & 2 \\
3 & 5 & 0 & 1 & -1 & 0 & 1 & 2 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The pivot element is 1 in column \(x_1\), row \(s_2\). After pivoting, the tableau becomes

\[
\begin{bmatrix}
\text{BV} & P & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
0 & 0 & 0 & 0 & 1 & 1 & 4 \\
0 & 1 & -1 & 0 & 1 & 2 \\
1 & 0 & -3 & 0 & 2 & 4 \\
\end{bmatrix}
\]

Now the only negative entry in the objective row is \(-3\) in column \(x_2\). Notice that it is impossible to choose a pivot element in that column. To see why, we go back to the initial tableau. The objective function is \(P = 2x_1 + x_2\) and the constraints are: \(-x_1 + x_2 \leq 2, x_1 - x_2 \leq 2, x_1 \geq 0, \text{ and } x_2 \geq 0\). The set of feasible points, shown in Figure 4, is unbounded and the objective function is heading in the direction of the unboundedness.

Summary of the Simplex Method

The general procedure for solving a maximum problem in standard form using the simplex method can be outlined as follows:

1. The maximum problem is stated in standard form as

Maximize

\[P = c_1x_1 + c_2x_2 + \cdots + c_nx_n\]

subject to the constraints

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m
\]

where

\[x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0\]

\[b_1 > 0, b_2 > 0, \ldots, b_m > 0\]
2. Introduce slack variables $s_1, s_2, \ldots, s_m$ so that the constraints take the form of equalities:
\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 &= b_2 \\
    \vdots & \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m &= b_m
\end{align*}
\]
$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$
$s_1 \geq 0, s_2 \geq 0, \ldots, s_m \geq 0$

3. Write the objective function in the form
\[
P - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0
\]

4. Set up the initial simplex tableau
\[
\begin{array}{cccccc}
    P & x_1 & x_2 & \cdots & x_n & s_1 & s_2 & \cdots & s_m & \text{RHS} \\
    \hline
    s_1 & 0 & a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 & b_1 \\
    s_2 & 0 & a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 & b_2 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    s_m & 0 & a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 & b_m \\
    \hline
    P & 1 & -c_1 & -c_2 & \cdots & -c_n & 0 & 0 & \cdots & 0 & 0
\end{array}
\]

5. Pivot until

(a) All the entries in the objective row are nonnegative. This is a final tableau from which a solution can be read.

Or until

(b) The pivot column is a column whose entries are negative or zero. In this case the problem is unbounded and there is no solution.

The flowchart in Figure 5 illustrates the steps to be used in solving standard maximum linear programming problems.
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

**USING TECHNOLOGY**

**EXAMPLE 6 Using Excel**

Use Excel to solve Example 5:

Maximize

\[ P = 0.5x_1 + x_2 + 1.5x_3 + 2x_4 \]

subject to the constraints

\[
\begin{align*}
2x_1 + 2.5x_2 + 2x_3 + 2x_4 & \leq 4000 \\
 x_1 + 1.5x_2 + x_3 + 2x_4 & = 1800 \\
 0.5x_1 + 0.5x_2 + x_3 + x_4 & = 1000 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0 \\
x_4 & \geq 0
\end{align*}
\]

**SOLUTION**

**STEP 1** Set up an Excel spreadsheet containing the variables, the objective function, and the constraints as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Variables</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Delivery Truck, ( x_1 )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Dump Truck, ( x_2 )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>Garbage Truck, ( x_3 )</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>Gasoline Truck, ( x_4 )</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>Objective</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Maximize Profit</td>
<td>( = 0.5B2 + 1<em>B3 + 1.5</em>B4 + 2*B5 )</td>
</tr>
<tr>
<td>11</td>
<td>Constraints</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Amount Used</td>
<td>Maximum</td>
</tr>
<tr>
<td>13</td>
<td>Metal Casting</td>
<td>( = 2<em>B2 + 2.5</em>B3 + 2<em>B4 + 2</em>B5 )</td>
</tr>
<tr>
<td>14</td>
<td>Paint Spray</td>
<td>( = B2 + 1.5<em>B3 + B4 + 2</em>B5 )</td>
</tr>
<tr>
<td>15</td>
<td>Packaging</td>
<td>( = 0.5<em>B2 + 0.5</em>B3 + B4 + B5 )</td>
</tr>
</tbody>
</table>

You should obtain:
STEP 2 Click on Tools and then Solver to get the screen below.

- **Set Target Cell**: Is the cell of the optimization function.
- **Equal To**: Allows you to maximize or minimize the objective function.
- **By Changing Cells**: The variable cells.
- **Subject to the Constraints**: The system of inequalities.

STEP 3 Input: Set Target Cell; B9; Equal To: Max; By Changing Cells: B2 to B5. Just click on the appropriate cells.

STEP 4 Add the constraints.
- Click cursor into Subject to the Constraints entry box.
- Press Add button. You should see:

  - Cell Reference is the cell containing the formula.
  - Constraint is the cell containing the maximum.
  - You must also enter the constraint that the variables must be greater than or equal to zero.
  - To enter the Metal Casting constraint, put the cursor in the Cell Reference box and click on cell B13; then put the cursor in the Constraint box and click on C13.

  - Click on Add button and repeat this for Paint Spray and Packaging constraints.
  - To add the constraint that the variables are greater than or equal to zero:
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

- Click on the variable cells.
- Select $\geq$ in the middle box.
- Type in zero in the constraint box.

**STEP 5** Your Solver box should look like this:

**STEP 6** Click on Options box and make sure Assume Linear Model is checked.

**STEP 7** Find the solution: Click on Solve and highlight Answer.
The solution is in the Excel spreadsheet below.

<table>
<thead>
<tr>
<th>Variables</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Delivery Truck, x₁</td>
<td>0</td>
</tr>
<tr>
<td>Dump Truck, x₂</td>
<td>0</td>
</tr>
<tr>
<td>Garbage Truck, x₃</td>
<td>200</td>
</tr>
<tr>
<td>Gasoline Truck, x₄</td>
<td>800</td>
</tr>
</tbody>
</table>

Objective

Maximize Profit 1900

Constraints

<table>
<thead>
<tr>
<th>Amount Used</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metal Casting</td>
<td>2000</td>
</tr>
<tr>
<td>Paint Spray</td>
<td>1800</td>
</tr>
<tr>
<td>Packaging</td>
<td>1000</td>
</tr>
</tbody>
</table>

To get a maximum profit of $1900, you should produce 0 delivery trucks, 0 dump trucks, 200 garbage trucks, and 800 gasoline trucks. Only 2000 hours of metal casting are used. The other constraints are used at full capacity.

Now work Problem 29 using Excel.

**EXERCISE 5.2 Answers Begin on Page AN–25.**

**Concepts and Vocabulary**

1. **True or False** The pivot element is sometimes located in the objective row.
2. **True or False** When a tableau is obtained that has only nonnegative entries in the objective row, it is a final tableau.
3. **True or False** If a tableau is obtained and the pivot column has entries that are positive or zero, the maximum problem will have no solution.
4. **True or False** For a maximum problem in standard form, the pivot _________ is located by selecting the first negative entry in the objective row.

**Skill Building**

In Problems 5–12, determine which of the following statements is true about each tableau:

(a) It is the final tableau.
(b) It requires additional pivoting.
(c) It indicates no solution to the problem.

If the answer is (a), write down the solution; if the answer is (b), indicate the pivot element.

5. BV

<table>
<thead>
<tr>
<th>P</th>
<th>x₁</th>
<th>x₂</th>
<th>s₁</th>
<th>s₂</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>s₁</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>x₂</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
<td>0</td>
<td>1/4</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

6. BV

<table>
<thead>
<tr>
<th>P</th>
<th>x₁</th>
<th>x₂</th>
<th>s₁</th>
<th>s₂</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₁</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>x₂</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>−1</td>
<td>1/2</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
</tr>
</tbody>
</table>
5.2 The Simplex Method: Solving Maximum Problems in Standard Form

<table>
<thead>
<tr>
<th>Problem</th>
<th>Maximize</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>RHS</th>
</tr>
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<tbody>
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<td>7.</td>
<td>$BV$</td>
<td>$P$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>RHS</td>
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<td>$x_2$</td>
<td>$s_1$</td>
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<td>$x_2$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>RHS</td>
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<td>10.</td>
<td>$BV$</td>
<td>$P$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>RHS</td>
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<tr>
<td>11.</td>
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<td>$P$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>RHS</td>
</tr>
<tr>
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<td>12.</td>
<td>$BV$</td>
<td>$P$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>RHS</td>
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In Problems 13–28, use the simplex method to solve each maximum problem.

13. Maximize

$$P = 5x_1 + 7x_2$$

subject to the constraints

$$2x_1 + 3x_2 \leq 12$$
$$3x_1 + x_2 \leq 12$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

14. Maximize

$$P = x_1 + 5x_2$$

subject to the constraints

$$2x_1 + x_2 \leq 10$$
$$x_1 + 2x_2 \leq 10$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

15. Maximize

$$P = 5x_1 + 7x_2$$

subject to the constraints

$$x_1 + 2x_2 \leq 2$$
$$2x_1 + x_2 \leq 2$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

16. Maximize

$$P = 5x_1 + 4x_2$$

subject to the constraints

$$x_1 + x_2 \leq 2$$
$$2x_1 + 3x_2 \leq 6$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

17. Maximize

$$P = 3x_1 + x_2$$

subject to the constraints

$$x_1 + x_2 \leq 2$$
$$2x_1 + 3x_2 \leq 12$$
$$3x_1 + x_2 \leq 12$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

18. Maximize

$$P = 3x_1 + 5x_2$$

subject to the constraints

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 6$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

19. Maximize

$$P = 2x_1 + x_2 + x_3$$

subject to the constraints

$$-2x_1 + x_2 - 3x_3 \leq 4$$
$$x_1 - 2x_2 + x_3 \leq 2$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

20. Maximize

$$P = 4x_1 + 2x_2 + 5x_3$$

subject to the constraints

$$x_1 + 3x_2 + 2x_3 \leq 30$$
$$2x_1 + x_2 + 3x_3 \leq 12$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

21. Maximize

$$P = 2x_1 + x_2 + 3x_3$$

subject to the constraints

$$x_1 + 2x_2 + x_3 \leq 25$$
$$3x_1 + 2x_2 + 3x_3 \leq 30$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

22. Maximize

$$P = 6x_1 + 3x_2 + 2x_3$$

subject to the constraints

$$2x_1 + 2x_2 + x_3 \leq 30$$
$$2x_1 + 2x_2 + x_3 \leq 12$$
$$x_1 \geq 0 \quad x \geq 0 \quad x_3 \geq 0$$

23. Maximize

$$P = 2x_1 + 4x_2 + x_3 + x_4$$

subject to the constraints

$$2x_1 + x_2 + 2x_3 + 3x_4 \leq 12$$
$$2x_2 + x_3 + 2x_4 \leq 20$$
$$2x_1 + x_2 + 4x_3 \leq 16$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0$$

24. Maximize

$$P = 2x_1 + 4x_2 + x_3$$

subject to the constraints

$$-x_1 + 2x_2 + 3x_3 \leq 6$$
$$-x_1 + 4x_2 + 5x_3 \leq 5$$
$$x_1 + 5x_2 + 7x_3 \leq 7$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$
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25. Maximize

\[ P = 2x_1 + x_2 + x_3 \]

subject to the constraints

\[ \begin{align*}
x_1 + 2x_2 + 4x_3 & \leq 20 \\
2x_1 + 4x_2 + 4x_3 & \leq 60 \\
3x_1 + 4x_2 + x_3 & \leq 90 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

26. Maximize

\[ P = x_1 + 2x_2 + 4x_3 \]

subject to the constraints

\[ \begin{align*}
8x_1 + 5x_2 - 4x_3 & \leq 30 \\
-2x_1 + 6x_2 + x_3 & \leq 5 \\
-2x_1 + 2x_2 + x_3 & \leq 15 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

27. Maximize

\[ P = x_1 + 2x_2 + 4x_3 - x_4 \]

subject to the constraints

\[ \begin{align*}
5x_1 + 4x_3 + 6x_4 & \leq 20 \\
4x_1 + 2x_2 + 2x_3 + 8x_4 & \leq 40 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0 \\
x_4 & \geq 0
\end{align*} \]

28. Maximize

\[ P = x_1 + 2x_2 - x_3 + 3x_4 \]

subject to the constraints

\[ \begin{align*}
2x_1 + 4x_2 + 5x_3 + 6x_4 & \leq 24 \\
4x_1 + 4x_2 + 2x_3 + 2x_4 & \leq 4 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0 \\
x_4 & \geq 0
\end{align*} \]

Applications

29. Process Utilization  A jean manufacturer makes three types of jeans, each of which goes through three manufacturing phases—cutting, sewing, and finishing. The number of minutes each type of product requires in each of the three phases is given below:

<table>
<thead>
<tr>
<th>Jean</th>
<th>Cutting</th>
<th>Sewing</th>
<th>Finishing</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>8</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>II</td>
<td>12</td>
<td>18</td>
<td>8</td>
</tr>
<tr>
<td>III</td>
<td>18</td>
<td>24</td>
<td>12</td>
</tr>
</tbody>
</table>

There are 5200 minutes of cutting time, 6000 minutes of sewing time, and 2200 minutes of finishing time each day. The company can sell all the jeans it makes and makes a profit of $4 on each Jean I, $4.50 on each Jean II, and $6 on each Jean III. What number of jeans in each category should be made each day to maximize profits?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

30. Process Utilization  A company manufactures three types of toys A, B, and C. Each requires rubber, plastic, and aluminum as listed below:

<table>
<thead>
<tr>
<th>Toy</th>
<th>Rubber</th>
<th>Plastic</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

The company has available 600 units of rubber, 800 units of plastic, and 1400 units of aluminum. The company makes a profit of $4, $3, and $2 on toys A, B, and C, respectively. Assuming all toys manufactured can be sold, determine a production order so that profit is a maximum.

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

31. Maximizing Profit  Three kinds of wrapping paper are sold door-to-door by the local lacrosse team to raise funds for equipment. Birthday wrapping paper costs $3 a roll, takes (on average) 10 minutes to sell, and costs $0.50 to deliver to the customer. The holiday wrapping paper costs $5 a roll, takes 15 minutes to sell, and is left with the customer at the time of sale. The wedding wrapping paper costs $4 a roll, takes 12 minutes to sell, and costs $1 to deliver to the customer. During the fund-raiser, the team is allowed to sell up to $500 worth of the three types of wrapping paper and is allowed delivery expenses not to exceed $75. The team members’ times can be at most a total of 30 hours selling paper during the weeklong fund-raiser. Their profits are $1 for each roll of birthday paper and holiday paper they sell and $2 for each roll of wedding paper. What combination of sales of each type of paper will lead to a maximum profit for the team, and what is that profit?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.
32. **Resource Allocations** A large hospital classifies its surgical operations into three categories according to their length and charges a fee of $1200, $1800, and $2400, respectively, for each of the categories. The average time of the operations in the three categories is 30 minutes, 1 hour, and 2 hours, respectively. The hospital has four operating rooms, each of which can be used for 10 hours per day. If the total number of operations cannot exceed 60, how many of each type should the hospital schedule to maximize its revenues?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

33. **Mixture Problem** Nutt’s Nut Company has 500 pounds of peanuts, 100 pounds of pecans, and 50 pounds of cashews on hand. They package three types of 5-pound cans of nuts: Can I contains 3 pounds of peanuts, 1 pound of pecans, and 1 pound of cashews; Can II contains 4 pounds of peanuts, \(\frac{1}{2}\) pound of pecans, and \(\frac{1}{2}\) pound of cashews; and Can III contains 5 pounds of peanuts. The selling price is $28 for Can I, $24 for Can II, and $21 for Can III. How many cans of each kind should be made to maximize revenue?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

34. **Maximizing Profit** One of the methods used by the Alexander Company to separate copper, lead, and zinc from ores is the flotation separation process. This process consists of three steps: oiling, mixing, and separation. These steps must be applied for 2, 2, and 1 hour, respectively, to produce 1 unit of copper; 2, 3, and 1 hour, respectively, to produce 1 unit of lead; and 1, 1, and 3 hours, respectively, to produce 1 unit of zinc. The oiling and separation phases of the process can be in operation for a maximum of 10 hours a day, while the mixing phase can be in operation for a maximum of 11 hours a day. The Alexander Company makes a profit of $45 per unit of copper, $30 per unit of lead, and $35 per unit of zinc. The demand for these metals is unlimited. How many units of each metal should be produced daily by use of the flotation process to achieve the highest profit?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

35. **A Toy Story, Part I** (Refer to Problem 41 in Section 5.1.) A toy store has shelf space to hold at most 60 baby dolls of three different types. Sleeping baby dolls cost $6 each, talking baby dolls cost $7.50 each, and walking baby dolls cost $9 each. The store can afford to spend up to $405 for the dolls. The sleeping, talking, and walking baby dolls resell for $12, $13.50, and $17 each, respectively. The store wishes to maximize its profit by ordering the appropriate number of each type of doll. (Remember: Profit = Revenue − Cost.)

(a) How many of each type of doll should be ordered to maximize profit?
(b) What is the maximum profit?

36. **A Toy Story, Part II** Solve Problem 42 in Section 5.1.

37. **Clothing** (Refer to Problem 43 in Section 5.1.) Steven owns a company that produces men’s shirts, jackets, and pants, each of which require time for cutting and sewing. One shirt requires 10 minutes of cutting and 20 minutes of sewing, one jacket requires 20 minutes of cutting and 40 minutes of sewing, and one pair of pants requires 20 minutes of cutting and 30 minutes of sewing. Steven has 500 minutes of cutting time available each day and 900 minutes of sewing time. His company makes a profit of $19 for each shirt, $34 for each jacket, and $15 on each pair of pants. Steven wants to maximize his profit, but can produce at most 30 items in one day.

(a) Determine how many of each clothing item should be made.
(b) What is the maximum daily profit?

38. **Maximizing Profit** A wrought iron manufacturer produces chandeliers, stair railings, and gate latches, each of which must have its parts blacksmithed, assembled, and crated for shipping. Each chandelier requires 3 hours to blacksmith, 5 hours to assemble, and 0.1 hour to crate and returns a profit of $10. Each set of stair railings requires 10 hours to blacksmith, 8 hours to assemble, and 0.6 hour to crate and returns a profit of $25. Each gate latch requires 1 hour to blacksmith, 1 hour to assemble, and 0.1 hour to crate and returns a profit of $3. The manufacturer has 30,000, 40,000, and 120 hours available weekly for blacksmithing, assembling, and crating, respectively. How many units of each product should be manufactured to maximize profit?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

39. **Mixture** The Marathon oil refinery blends high and low octane gasoline into three intermediate grades: regular, premium, and super premium. The regular grade consists of 60% high octane and 40% low octane, the premium consists of 70% high octane and 30% low octane, and the super premium consists of 80% high octane and 20% low octane. The company has available 140,000 gallons of high octane and 120,000 gallons of low octane but can mix only 225,000 gallons. Regular gas sells for $2.80 per gallon, premium sells for $2.97 per gallon, and super premium sells for $3.08 per gallon. How
many gallons of each grade should the company mix in order to maximize revenues?
(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

40. Mixture Rework Problem 39 under the additional assumption that the combined total number of gallons produced by the refinery cannot exceed 200,000 gallons.

41. Investment A financial consultant has at most $90,000 to invest in stocks, corporate bonds, and municipal bonds. The average yields for stocks, corporate bonds, and municipal bonds are 8%, 5%, and 3%, respectively. Determine how much she should invest in each security to maximize the return on her investments. She has decided that her investment in stocks should not exceed half her funds and that the difference between her investment in corporate bonds and municipal bonds should not be more than 20% of her investment.
(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

42. Investment Rework Problem 41 under the assumption that no more than $25,000 can be invested in stocks and that the difference between her investment in corporate bonds and municipal bonds should not be more than 40% of her investment.

43. Crop Planning A farmer has at most 200 acres of farmland suitable for cultivating crops of soybeans, corn, and wheat. The costs for cultivating soybeans, corn, and wheat are $40, $50, and $30 per acre, respectively. The farmer has a maximum of $18,000 available for land cultivation. Soybeans, corn, and wheat require 20, 30, and 15 hours per acre of labor, respectively, and there is a maximum of 4200 hours of labor available. If the farmer expects to make a profit of $70, $90, and $50 per acre on soybeans, corn, and wheat, respectively, how many acres of each crop should he plant in order to maximize his profit?
(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

44. Crop Planning Repeat Problem 43 with the following data:

<table>
<thead>
<tr>
<th></th>
<th>Soybeans</th>
<th>Corn</th>
<th>Wheat</th>
<th>Maximum Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost</td>
<td>$30</td>
<td>$40</td>
<td>$20</td>
<td>$12,000</td>
</tr>
<tr>
<td>Hours</td>
<td>10</td>
<td>20</td>
<td>18</td>
<td>3,600</td>
</tr>
<tr>
<td>Profit</td>
<td>$50</td>
<td>$60</td>
<td>$40</td>
<td></td>
</tr>
</tbody>
</table>

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.

45. Maximizing Profit A large TV manufacturer has warehouse facilities for storing its 52-inch color TVs in Chicago, New York, and Denver. Each month the city of Atlanta is shipped at most four hundred 52-inch TVs. The cost of transporting each TV to Atlanta from Chicago, New York, and Denver averages $50, $40, and $80, respectively, while the cost of labor required for packing averages $6, $8, and $4, respectively. Suppose $20,000 is allocated each month for transportation costs and $3,000 is allocated for labor costs. If the profit on each TV stored in Chicago is $70, in New York is $80, and in Denver is $40, how should monthly shipping arrangements be scheduled to maximize profit?
(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

46. Fruit Trees (Refer to Problem 44 in Section 5.1.) Suzanne owns an orchard and wants to plant some additional fruit trees to obtain maximum revenue. She will use at most 43,560 square feet of land and can plant pear, peach, and plum trees. She wants the combined total of peach trees and plum trees to be no more than 150 but also wants to make sure that the number of pear trees does not exceed the number of peach trees by more than 10.
(a) Determine how many of each type of tree Suzanne should plant and the maximum revenue.
(b) What is the maximum revenue?

<table>
<thead>
<tr>
<th>Tree</th>
<th>Area needed (ft²)</th>
<th>Yield (bushels)</th>
<th>Revenue (dollars/bushel)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pear</td>
<td>120</td>
<td>1/2</td>
<td>8.94</td>
</tr>
<tr>
<td>Peach</td>
<td>450</td>
<td>4</td>
<td>14.48</td>
</tr>
<tr>
<td>Plum</td>
<td>320</td>
<td>2</td>
<td>10.28</td>
</tr>
</tbody>
</table>

Source: Virginia Cooperative Extension.

47. Investments (Refer to Problem 45 in Section 5.1.) Kami has up to $25,000 to invest and can divide it among three investments to maximize her return: (1) a money market account that pays 1.05% annually, (2) a mutual fund that pays 5% annually, and (3) a certificate of deposit (CD) that pays 1.66% annually. She does not want to invest more than $10,000 between the mutual fund and the CD. In addition, the amount invested in the CD cannot exceed the amount invested in the money market by more than $1500.
(a) Determine how much Kami should put into each investment.
(b) What is the maximum return?
Source: moneyrates.com
5.3 Solving Minimum Problems Using the Duality Principle

48. Travel Arrangements  (Refer to Problem 46 in Section 5.1.) A textbook publishing company has 10 sales representatives stationed in Chicago and 15 sales representatives stationed in Boston. The company wants to send representatives to two different conventions being held simultaneously in Atlanta and Seattle. The company would like to fly up to 12 reps to the Atlanta convention and up to 8 reps to the Seattle convention. A round-trip ticket from Chicago to Atlanta costs approximately $400, from Chicago to Seattle costs approximately $450, from Boston to Atlanta costs approximately $400, and from Boston to Seattle costs approximately $475. (Source: priceline.com) The total flight costs for the trips cannot exceed $7700.

(a) Determine how many sales representative should fly from Chicago to Atlanta, from Chicago to Seattle, from Boston to Atlanta, and from Boston to Seattle in order to maximize the number of representatives at the conferences.

(b) What is the maximum number of reps that will attend the conferences?

5.3 Solving Minimum Problems Using the Duality Principle

A linear programming problem in which the objective function is to be minimized is referred to as a minimum problem. There are two ways to solve minimum problems:

1. Change the minimum problem to a maximum problem by using the fact that minimizing $z$ is the same as maximizing $P = -z$.
2. Apply the duality principle, which states that the solution of the minimum problem can be obtained by finding the solution of the dual program, a maximum problem.

Notice that both of these methodologies require solving a certain maximum problem. The choice of which to use usually is determined by the size of the initial tableau of the maximum problem—the smaller size being preferable.

We discuss the duality principle in this section and the second methodology in the next section.

Theorem

Duality Principle

If a minimum problem has a solution, the minimum value of the objective function of the minimum problem equals the maximum value of the objective function of the dual problem, a maximum problem.
As our purpose in this chapter is to introduce the simplex method, we shall only apply the duality principle to a particular class of minimum problems, those that are in standard form.

**Definition**

**Standard Form of a Minimum Problem**

A minimum problem is said to be in standard form provided the following three conditions are met:

- **CONDITION 1** All the variables are nonnegative.
- **CONDITION 2** All other constraints are written as linear expressions that are greater than or equal to a constant.
- **CONDITION 3** The objective function is expressed as a linear expression with nonnegative coefficients.

**EXAMPLE 1 Determining a Minimum Problem Is in Standard Form**

Determine which of the following minimum problems are in standard form.

**PROBLEM** (a) Minimize

\[ C = 2x_1 + 3x_2 \]

subject to the constraints

\[ x_1 + 3x_2 = 24 \]
\[ 2x_1 + x_2 = 18 \]
\[ x_1 \geq 0 \quad x_2 = 0 \]

**SOLUTION** (a) Since Conditions 1, 2, and 3 are met, this minimum problem is in standard form.

**PROBLEM** (b) Minimize

\[ C = 3x_1 - x_2 + 4x_3 \]

subject to the constraints

\[ 3x_1 + x_2 + x_3 \geq 12 \]
\[ x_1 + x_2 + x_3 \geq 8 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

**SOLUTION** (b) Conditions 1 and 2 are met, but Condition 3 is not met, since the coefficient of \( x_2 \) in the objective function is negative. This minimum problem is not in standard form.
PROBLEM (c) Minimize

\[ C = 2x_1 + x_2 + x_3 \]

subject to the constraints

\[ x_1 - 3x_2 + x_3 \leq 12 \]
\[ x_1 + x_2 + x_3 \geq 1 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

SOLUTION (c) Conditions 1 and 3 are met, but Condition 2 is not met, since the first constraint

\[-x_1 + 3x_2 - x_3 \geq -12\]

is not written with a \( \geq \) sign. The minimum problem as stated is not in standard form. Notice, however, that by multiplying by \(-1\), we can write this constraint as

\[-x_1 + 3x_2 - x_3 \geq -12\]

Written in this way, the minimum problem is in standard form.

PROBLEM (d) Minimize

\[ C = 2x_1 + x_2 + 3x_3 \]

subject to the constraints

\[-x_1 + 2x_2 + x_3 \geq -2 \]
\[ x_1 + x_2 + x_3 \geq 6 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

SOLUTION (d) Conditions 1, 2, and 3 are each met, so this minimum problem is in standard form.

Now work Problem 9.

2 Obtain the Dual Problem of a Minimum Problem in Standard Form

To apply the Duality Principle, we need to find the dual problem of the minimum problem.

EXAMPLE 2 Obtaining the Dual Problem of a Minimum Problem in Standard Form

Obtain the dual problem of the following minimum problem:

Minimize

\[ C = 300x_1 + 480x_2 \]

subject to the constraints

\[ x_1 + 3x_2 \geq 0.25 \]
\[ 2x_1 + 2x_2 \geq 0.45 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

SOLUTION First notice that the minimum problem is in standard form. Begin by writing a matrix that represents the constraints and the objective function:

\[
\begin{bmatrix}
  1 & 3 & 0.25 \\
  2 & 2 & 0.45 \\
  300 & 480 & 0
\end{bmatrix}
\]

Constraint: \( x_1 + 3x_2 \geq 0.25 \)
Constraint: \( 2x_1 + 2x_2 \geq 0.45 \)
Objective function: \( C = 300x_1 + 480x_2 \)
Now form the matrix that has as columns the rows of the above matrix by taking column 1 above and writing it as row 1 below, taking column 2 above and writing it as row 2 below, and taking column 3 above and writing it as row 3 below.

\[
\begin{bmatrix}
1 & 2 & 300 \\
3 & 2 & 480 \\
0.25 & 0.45 & 0
\end{bmatrix}
\]

This matrix is called the **transpose** of the first matrix.

From this transpose matrix, create the following maximum problem:

Maximize

\[ P = 0.25y_1 + 0.45y_2 \]

subject to the conditions

\[
\begin{align*}
y_1 + 2y_2 & \leq 300 \\
3y_1 + 2y_2 & \leq 480 \\
y_1 & \geq 0 \\
y_2 & \geq 0
\end{align*}
\]

This maximum problem is the **dual** of the given minimum problem.

The steps to use for obtaining the dual problem are listed below.

### Steps for Obtaining the Dual Problem

**STEP 1** Write the minimum problem in standard form.

**STEP 2** Construct a matrix that represents the constraints and the objective function, placing the objective function in the bottom row.

**STEP 3** Interchange the rows and columns to form the matrix of the dual problem.

**STEP 4** Translate this matrix into a maximum problem in standard form.

---

## EXAMPLE 3 Obtaining the Dual Problem of a Minimum Problem in Standard Form

Find the dual of the following minimum problem:

Minimize

\[ C = 2x_1 + 3x_2 \]

subject to the constraints

\[
\begin{align*}
2x_1 + x_2 & \geq 6 \\
x_1 + 2x_2 & \geq 4 \\
x_1 + x_2 & \geq 5 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\]

### SOLUTION

**STEP 1** The minimum problem is in standard form.

**STEP 2** The matrix that represents the constraints and the objective function is

\[
\begin{bmatrix}
2 & 1 & 6 \\
2 & 2 & 4 \\
1 & 1 & 5 \\
2 & 3 & 0
\end{bmatrix}
\]

**Constraint:** \( 2x_1 + x_2 \geq 6 \)

**Constraint:** \( x_1 + 2x_2 \geq 4 \)

**Constraint:** \( x_1 + x_2 \geq 5 \)

**Objective function:** \( C = 2x_1 + 3x_2 \)
Interchanging rows and columns, we obtain the matrix

\[
\begin{bmatrix}
2 & 1 & 1 & 2 \\
1 & 2 & 1 & 3 \\
6 & 4 & 5 & 0
\end{bmatrix}
\]

Constraint: \(2y_1 + y_2 + y_3 \leq 2\)
Constraint: \(y_1 + 2y_2 + y_3 \leq 3\)
Objective function: \(P = 6y_1 + 4y_2 + 5y_3\)

This matrix represents the following maximum problem:
Maximize

\[P = 6y_1 + 4y_2 + 5y_3\]

subject to the constraints

\[2y_1 + y_2 + y_3 \leq 2\]
\[y_1 + 2y_2 + y_3 \leq 3\]
\[y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0\]

This maximum problem is in standard form and is the dual problem of the minimum problem.

Some observations about Example 3:

1. The variables \((x_1, x_2)\) of the minimum problem are different from the variables of its dual problem \((y_1, y_2, y_3)\).
2. The minimum problem has three constraints and two variables, while the dual problem has two constraints and three variables. (In general, if a minimum problem has \(m\) constraints and \(n\) variables, its dual problem will have \(n\) constraints and \(m\) variables.)
3. The inequalities defining the constraints are \(\geq\) for the minimum problem and \(\leq\) for the maximum problem.
4. Since the coefficients in the objective function to be minimized are positive, the dual problem has nonnegative numbers to the right of the \(\leq\) signs.
5. We follow the custom of denoting an objective function by \(C\) (for Cost), if it is to be minimized; and \(P\) (for Profit), if it is to be maximized.

Now work Problem 15.

### Solve a Minimum Problem in Standard Form Using the Duality Principle

In Examples 2 and 3, notice that the dual of a minimum problem in standard form is a maximum problem in standard form so it can be solved by using techniques discussed in the previous section.

### Example 4: Solving a Minimum Problem in Standard Form by Using the Duality Principle

Continue with Example 3 and solve the maximum problem using the simplex method. Obtain the solution of the minimum problem.
SOLUTION  The maximum problem obtained in Example 3 (STEP 4) is repeated below.
Maximize
\[ P = 6y_1 + 4y_2 + 5y_3 \]
subject to the constraints
\[ \begin{align*}
2y_1 + y_2 + y_3 & \leq 2 \\
y_1 + 2y_2 + y_3 & \leq 3 \\
y_1 & \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0
\end{align*} \]
Introduce slack variables \( s_1 \) and \( s_2 \) to obtain
\[ \begin{align*}
2y_1 + y_2 + y_3 + s_1 &= 2, \quad s_1 \geq 0 \\
y_1 + 2y_2 + y_3 + s_2 &= 3, \quad s_2 \geq 0
\end{align*} \]
The objective function \( P \) is written as \( P = 6y_1 - 4y_2 - 5y_3 = 0 \).
The initial simplex tableau is

\[
\begin{array}{ccccccc|c}
BV & P & y_1 & y_2 & y_3 & s_1 & s_2 & RHS \\
\hline
s_1 & 0 & 2 & 1 & 1 & 1 & 0 & 2 \\
s_2 & 0 & 1 & 2 & 1 & 0 & 1 & 3 \\
P & 1 & -6 & -4 & -5 & 0 & 0 & 0 \\
\end{array}
\]

The pivot element, 2, is circled.
Pivot: 1. \( R_1 = \frac{1}{2}r_1 \)  2. \( R_2 = -R_1 + r_2 \)  \( R_3 = 6R_1 + r_3 \)

\[
\begin{array}{ccccccc|c}
BV & P & y_1 & y_2 & y_3 & s_1 & s_2 & RHS \\
\hline
y_1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \\
s_2 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 2 \\
P & 1 & 0 & -1 & -2 & 3 & 0 & 6 \\
\end{array}
\]

The pivot element \( \frac{3}{2} \) is circled.
Pivot: 1. \( R_2 = \frac{2}{3}r_2 \)  2. \( R_1 = -\frac{1}{2}R_2 + r_1 \)  \( R_3 = R_2 + r_3 \)

\[
\begin{array}{ccccccc|c}
BV & P & y_1 & y_2 & y_3 & s_1 & s_2 & RHS \\
\hline
y_1 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\
y_2 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\
P & 1 & 0 & 0 & \frac{5}{3} & \frac{8}{3} & \frac{2}{3} & \frac{22}{3} \\
\end{array}
\]

The pivot \( \frac{1}{3} \) is circled.
Pivot: 1. \( R_2 = \frac{4}{3}r_2 \)  2. \( R_1 = -\frac{1}{2}R_2 + r_1 \)  \( R_3 = R_2 + r_3 \)

\[
\begin{array}{ccccccc|c}
BV & P & y_1 & y_2 & y_3 & s_1 & s_2 & RHS \\
\hline
y_1 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\
y_2 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\
P & 1 & 0 & 0 & \frac{5}{3} & \frac{8}{3} & \frac{2}{3} & \frac{22}{3} \\
\end{array}
\]
5.3 Solving Minimum Problems Using the Duality Principle

The pivot element 1/3 is circled.

Pivot: 1. $R_1 = 3r_1$ 2. $R_2 = \frac{1}{3}R_1 + r_2$  $R_3 = \frac{5}{3}R_1 + r_3$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_3$</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>-1</td>
<td>9</td>
</tr>
</tbody>
</table>

The pivot element 1 is circled.

Pivot: 1. $R_2 = r_2$ 2. $R_1 = R_2 + r_1$  $R_3 = R_2 + r_3$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_3$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

This is a final tableau, so an optimal solution has been found. We read from it that the solution to the maximum problem is

$P = 10$  $y_1 = 0$  $y_2 = 0$  $y_3 = 2$

The duality principle states that the minimum value of the objective function in the original problem is the same as the maximum value in the dual; that is,

$C = 10$

But which values of $x_1$ and $x_2$ will yield this minimum value? As it turns out, the value of $x_1$ is found in the objective row in column $s_1$($x_1 = 5$) and $x_2$ is found in the objective row in column $s_2$($x_3 = 0$). As a consequence, the solution to the minimum problem can be read from the right end of the objective row of the final tableau of the maximum problem:

$x_1 = 5$  $x_2 = 0$  $C = 10$

NOW WORK PROBLEM 21.

Solving a Minimum Problem in Standard Form Using the Duality Principle

**STEP 1** Write the dual problem, a maximum problem.

**STEP 2** Solve this maximum problem by the simplex method.

**STEP 3** The minimum value of the objective function $C$ will appear in the lower right corner of the final tableau; it is equal to the maximum value of the dual objective function $P$. The values of the variables that give rise to the minimum value are located in the objective row in the slack variable columns.
EXAMPLE 5 Solving a Minimum Problem in Standard Form Using the Duality Principle

Minimize

\[ C = 6x_1 + 8x_2 + x_3 \]

subject to the constraints

\[ \begin{align*}
3x_1 + 5x_2 + 3x_3 & \geq 20 \\
x_1 + 3x_2 + 2x_3 & \geq 9 \\
6x_1 + 2x_2 + 5x_3 & \geq 30 \\
x_1 & \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*} \]

SOLUTION

**STEP 1** This minimum problem is in standard form. The matrix representing this problem is

\[
\begin{bmatrix}
3 & 5 & 3 & 20 \\
1 & 3 & 2 & 9 \\
6 & 2 & 5 & 30 \\
6 & 8 & 1 & 0
\end{bmatrix}
\]

Constraint: \(3x_1 + 5x_2 + 3x_3 \geq 20\)

Constraint: \(x_1 + 3x_2 + 2x_3 \geq 9\)

Constraint: \(6x_1 + 2x_2 + 5x_3 \geq 30\)

Objective function: \(C = 6x_1 + 8x_2 + x_3\)

Interchange rows and columns to get

\[
\begin{bmatrix}
3 & 1 & 6 & 6 \\
5 & 3 & 2 & 8 \\
3 & 2 & 5 & 1 \\
20 & 9 & 30 & 0
\end{bmatrix}
\]

Constraint: \(3y_1 + y_2 + 6y_3 \leq 6\)

Constraint: \(5y_1 + 3y_2 + 2y_3 \leq 8\)

Constraint: \(3y_1 + 2y_2 + 5y_3 \leq 1\)

Objective function: \(P = 20y_1 + 9y_2 + 30y_3\)

The dual problem is:

Maximize

\[ P = 20y_1 + 9y_2 + 30y_3 \]

subject to the constraints

\[ \begin{align*}
3y_1 + y_2 + 6y_3 & \leq 6 \\
5y_1 + 3y_2 + 2y_3 & \leq 8 \\
3y_1 + 2y_2 + 5y_3 & \leq 1 \\
y_1 & \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0
\end{align*} \]

**STEP 2** Introduce nonnegative slack variables \(s_1, s_2,\) and \(s_3\). The initial tableau for this problem is

<table>
<thead>
<tr>
<th>BV</th>
<th>(P)</th>
<th>(y_1)</th>
<th>(y_2)</th>
<th>(y_3)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s_1)</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>(s_2)</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>(s_3)</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(P)</td>
<td>1</td>
<td>-20</td>
<td>-9</td>
<td>-30</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
5.3 Solving Minimum Problems Using the Duality Principle

The final tableau (as you should verify) is

\[
\begin{array}{cccccccccc}
BV & P & y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & RHS & \text{Current value} \\
\hline
s_1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 5 & s_1 = 5 \\
s_2 & 0 & 0 & -\frac{1}{3} & -\frac{19}{3} & 0 & 1 & -\frac{5}{3} & \frac{19}{3} & s_2 = \frac{19}{3} \\
y_1 & 0 & 1 & \frac{2}{3} & \frac{5}{3} & 0 & 0 & 1 & \frac{1}{3} & y_1 = \frac{1}{3} \\
\hline
P & 1 & 0 & \frac{13}{3} & \frac{10}{3} & 0 & 0 & \frac{20}{3} & \frac{20}{3} & P = \frac{20}{3} \\
\end{array}
\]

The solution to the maximum problem is
\[
P = \frac{20}{3} \quad y_1 = \frac{1}{3} \quad y_2 = 0 \quad y_3 = 0
\]

**STEP 3** For the minimum problem, the values of \(x_1, x_2, \) and \(x_3\) are read as the entries in the objective row in the columns under \(s_1, s_2,\) and \(s_3,\) respectively. The solution to the minimum problem is
\[
C = \frac{20}{3} \quad x_1 = 0 \quad x_2 = 0 \quad x_3 = \frac{20}{3}
\]

**USING TECHNOLOGY**

**EXAMPLE 6 Using Excel**

Solve Example 5 using Excel.

**SOLUTION**

**STEP 1** Set up the Excel spreadsheet. The spreadsheet below is presented with the formulas revealed.
STEP 2  Set up Solver. Solver can be found under the toolbar command Tools.

The only differences between maximize and minimize problems are
(a) Equal To: must be min.
(b) Subject to the Constraints: must be \( \geq \).

STEP 3  Click on Solve. The solution is given below.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Variables</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( X_1 )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( X_2 )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>( X_3 )</td>
<td>6.6667</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Objective</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Minimize</td>
<td>6.6667</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Constraints</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Amount Used</td>
<td>Minimum</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>13.33333</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>33.33333</td>
</tr>
</tbody>
</table>

So \( x_1 = 0, x_2 = 0, x_3 = 6.6667 \) with a minimum value of \( C = 6.6667 \).

EXERCISE 5.3  Answers Begin on Page AN–27.

‘Are You Prepared?’  Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. What is the dimension of the matrix \[
\begin{pmatrix}
2 & -1 & 0 & 10 \\
-5 & 6 & 4 & 20
\end{pmatrix}
\]?

2. The matrix \( A = \begin{pmatrix} 1 & 3 & -5 \\ 8 & 0 & 6 \end{pmatrix} \) has 2 rows and 3 columns. Form the \( 3 \times 2 \) matrix that has for rows the columns of \( A \) and has for columns the rows of \( A \) (pp. 104–116).
5.3 Solving Minimum Problems Using the Duality Principle

Concepts and Vocabulary

3. **True or False** The objective function of a minimum problem in standard form is a linear expression with nonnegative coefficients.

5. **True or False** If a minimum problem in standard form has a solution, it can be found by solving the dual problem, which is a maximum problem in standard form.

7. The __________ __________ states that the solution of a minimum problem, if it exists, is the same as the solution of the maximum problem, which is its dual.

8. **True or False** In using the duality principle, the solution of the minimum problem equals the negative of the solution of the maximum problem.

Skill Building

In Problems 9–14, determine which of the following minimum problems are in standard form.

9. Minimize
   \[ C = 2x_1 + 3x_2 \]
   subject to the constraints
   \[ 4x_1 - x_2 \geq 2 \]
   \[ x_1 + x_2 \geq 1 \]
   \[ x_1 \geq 0 \quad x_2 \geq 0 \]

10. Minimize
    \[ C = 3x_1 + 5x_2 \]
    subject to the constraints
    \[ 3x_1 - x_2 \geq 4 \]
    \[ x_1 - 2x_2 \geq 3 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]

11. Minimize
    \[ C = 2x_1 - x_2 \]
    subject to the constraints
    \[ 2x_1 - x_2 \geq 1 \]
    \[ -2x_1 \geq -3 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]

12. Minimize
    \[ C = 2x_1 + 3x_2 \]
    subject to the constraints
    \[ x_1 - x_2 \leq 3 \]
    \[ 2x_1 + 3x_2 \geq 4 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]

13. Minimize
    \[ C = 3x_1 + 7x_2 + x_3 \]
    subject to the constraints
    \[ x_1 + x_3 \leq 6 \]
    \[ 2x_1 + x_2 \geq 4 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

14. Minimize
    \[ C = x_1 - x_2 + x_3 \]
    subject to the constraints
    \[ x_1 + x_2 \geq 6 \]
    \[ 2x_1 - x_3 \geq 4 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

In Problems 15–20, write the dual problem of each minimum problem.

15. Minimize
    \[ C = 2x_1 + 3x_2 \]
    subject to the constraints
    \[ x_1 + x_2 \geq 2 \]
    \[ 2x_1 + 3x_2 \geq 6 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]

16. Minimize
    \[ C = 3x_1 + 4x_2 \]
    subject to the constraints
    \[ 2x_1 + x_2 \geq 2 \]
    \[ 2x_1 + x_2 \geq 6 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]

17. Minimize
    \[ C = 3x_1 + 2x_2 + x_3 \]
    subject to the constraints
    \[ x_1 + x_2 + x_3 \geq 5 \]
    \[ 2x_1 + x_2 \geq 4 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

18. Minimize
    \[ C = 2x_1 + x_2 + x_3 \]
    subject to the constraints
    \[ 2x_1 + x_2 + x_3 \geq 4 \]
    \[ x_1 + 2x_2 + x_3 \geq 6 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

19. Minimize
    \[ C = 3x_1 + 4x_2 + x_4 \]
    subject to the constraints
    \[ x_1 + x_2 + x_3 + 2x_4 \geq 60 \]
    \[ 3x_1 + 2x_2 + x_3 + 2x_4 \geq 90 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \]

In Problems 21–28, solve each minimum problem using the duality principle.

21. Minimize
    \[ C = 6x_1 + 3x_2 \]
    subject to the constraints
    \[ x_1 + x_2 \geq 2 \]
    \[ 2x_1 + 6x_2 \geq 6 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]

22. Minimize
    \[ C = 3x_1 + 4x_2 \]
    subject to the constraints
    \[ x_1 + x_2 \geq 3 \]
    \[ 2x_1 + x_2 \geq 4 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]

23. Minimize
    \[ C = 6x_1 + 3x_2 \]
    subject to the constraints
    \[ x_1 + x_2 \geq 4 \]
    \[ 3x_1 + 4x_2 \geq 12 \]
    \[ x_1 \geq 0 \quad x_2 \geq 0 \]
Chapter 5  Linear Programming: Simplex Method

24. Minimize
\[ C = 2x_1 + 3x_2 + 4x_3 \]
subject to the constraints
\[
\begin{align*}
  x_1 - 2x_2 - 3x_3 & \geq -2 \\
  x_1 + x_2 + x_3 & \geq 2 \\
  2x_1 & + x_3 \geq 3 \\
  x_1 \geq 0 & \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*}
\]

25. Minimize
\[ C = x_1 + 2x_2 + x_3 \]
subject to the constraints
\[
\begin{align*}
  x_1 - 3x_2 + 4x_3 & \geq 12 \\
  3x_1 + x_2 + 2x_3 & \geq 10 \\
  x_1 - x_2 - x_3 & \geq -8 \\
  x_1 \geq 0 & \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*}
\]

26. Minimize
\[ C = x_1 + 2x_2 + 4x_3 \]
subject to the constraints
\[
\begin{align*}
  x_1 - x_2 + 3x_3 & \geq 4 \\
  2x_1 + 2x_2 - 3x_3 & \geq 6 \\
  -x_1 + 2x_2 + 3x_3 & \geq 2 \\
  x_1 \geq 0 & \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*}
\]

Applications

29. Number of Employees  A "weekends only" furniture store is open Friday through Sunday and needs to ensure the store has enough sales employees working each day. Employees work exactly two days per week and are paid a flat rate each day. The employee requirement and pay for each day are given in the table. Determine the number of sales employees the store must schedule each day to meet its requirements while minimizing its payroll. What is the minimum payroll each week? How many employees are scheduled to work a Friday–Sunday schedule?

<table>
<thead>
<tr>
<th>Day</th>
<th>Employees needed</th>
<th>Pay per day</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friday</td>
<td>at least 6</td>
<td>$100</td>
</tr>
<tr>
<td>Saturday</td>
<td>at least 15</td>
<td>$150</td>
</tr>
<tr>
<td>Sunday</td>
<td>at least 8</td>
<td>$150</td>
</tr>
</tbody>
</table>

Mrs. Mintz and her friends would like to order at least 4 bowls of soup, 9 salads, 6 sandwiches, and 5 orders of pasta and keep the cost as low as possible.

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

30. Production Schedule  A company owns two mines. Mine A produces 1 ton of high-grade ore, 3 tons of medium-grade ore, and 5 tons of low-grade ore each day. Mine B produces 2 tons of each grade of ore per day. The company needs at least 80 tons of high-grade ore, at least 160 tons of medium-grade ore, and at least 200 tons of low-grade ore each day. How many days should each mine be operated to minimize costs if it costs $2000 per day to operate each mine?

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.
(b) Solve the linear programming problem.
(c) Analyze the solution.

31. Menu Planning  Fresh Starts Catering offers the following lunch menu:

<table>
<thead>
<tr>
<th>Lunch #1</th>
<th>Soup, salad, sandwich</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lunch #2</td>
<td>Salad, pasta</td>
</tr>
<tr>
<td>Lunch #3</td>
<td>Salad, sandwich, pasta</td>
</tr>
<tr>
<td></td>
<td>$6.20</td>
</tr>
<tr>
<td></td>
<td>$7.40</td>
</tr>
<tr>
<td></td>
<td>$9.10</td>
</tr>
</tbody>
</table>

32. Inventory Control  A toy store stocks three sizes of dolls: Small, Medium, and Large. Each Small doll occupies 1 square foot of shelf space, each Medium doll occupies 2 square feet, and each Large doll occupies 3 square feet. The store has 120 square feet available for storage. Surveys show that the store should have on hand at least 12 Small dolls and at least 30 Small and Medium dolls combined. Each Small doll costs
the store $8, each Medium doll $6, and each Large doll $10. Minimize the cost to the store.

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

33. Diet Preparation Mr. Jones needs to supplement his diet with at least 50 mg of calcium and 8 mg of iron daily. The minerals are available in two types of vitamin pills, P and Q. Pill P contains 3 mg of calcium and 2 mg of iron, while pill Q contains 10 mg of calcium and 1 mg of iron. If each pill P costs 3 cents and each pill Q costs 4 cents, how could Mr. Jones minimize the cost of adding the minerals to his diet?

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

34. Diet Planning A health clinic dietician is planning a meal consisting of three foods whose ingredients are summarized as follows:

<table>
<thead>
<tr>
<th>Units of Protein</th>
<th>Units of Carbohydrates</th>
<th>Units of Iron</th>
<th>Calories</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food I</td>
<td>10</td>
<td>4</td>
<td>80</td>
</tr>
<tr>
<td>Food II</td>
<td>15</td>
<td>8</td>
<td>120</td>
</tr>
<tr>
<td>Food III</td>
<td>20</td>
<td>1</td>
<td>100</td>
</tr>
</tbody>
</table>

The dietician wishes to determine the number of units of each food to use to create a balanced meal containing at least 40 units of protein, 6 units of carbohydrates, and 12 units of iron, with as few calories as possible. How many units of each food should be used in order to minimize calories?

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

35. Dried Fruit, Part I Katy wants to mix at least 2 kilograms of some dried fruit that will meet her dietary requirements with the least amount of sodium. She wants the mix to provide at least 400 mg of calcium and at least 250 mg of vitamin C. She has the following fruits available for the mix and wants at least 500 grams of apricots and pears combined. How much of each type of fruit should be included in the mix and what is the minimum amount of sodium?

<table>
<thead>
<tr>
<th>Fruit</th>
<th>Calcium (mg)</th>
<th>Vitamin C (mg)</th>
<th>Sodium (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apricot</td>
<td>67</td>
<td>12</td>
<td>26</td>
</tr>
<tr>
<td>Peaches</td>
<td>48</td>
<td>18</td>
<td>16</td>
</tr>
<tr>
<td>Pears</td>
<td>35</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

Source: National Dried Fruit Association, UK

Note: Table entries are mg per 100 g of fruit.

36. Dried Fruit, Part II Refer to Problem 35. Katy is concerned about the variety of her dried fruit mix and wants to make sure the mix also includes some apricots. Rework Problem 35 with the added constraint that there needs to be at least 300 g of apricots included.

5.4 The Simplex Method for Problems Not in Standard Form

So far, we have developed the simplex method only for solving linear programming problems in standard form. In this section we develop the simplex method for linear programming problems that cannot be written in standard form.
Solve a Maximum Problem Not in Standard Form

Recall that for a maximum problem to be in standard form each constraint must be of the form
\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b_1 \quad b_1 > 0 \]
That is, each constraint is a linear expression less than or equal to a positive constant. When the constraints are of any other form (greater than or equal to, or equal to, or with a zero or negative constraint on the right), we have a maximum problem that is not in standard form.

**EXAMPLE 1 Solving a Maximum Problem Not in Standard Form**

Maximize
\[ P = 20x_1 + 15x_2 \]
subject to the constraints
\[
\begin{align*}
  x_1 + x_2 & \geq 7 & (1) \\
  9x_1 + 5x_2 & \leq 45 & (2) \\
  2x_1 + x_2 & \geq 8 & (3) \\
  x_1 & \geq 0 \quad x_2 & \geq 0
\end{align*}
\]

**SOLUTION**

Observe this is a maximum problem that is not in standard form. Further, it cannot be modified so as to be in standard form.

**STEP 1**

Write each constraint, except the nonnegative constraints, as an inequality with the variables on the left side of a \( \leq \) sign.

Multiply the first and third inequalities by \(-1\). The result is that the constraints become
\[
\begin{align*}
  -x_1 - x_2 & \leq -7 & (1) \\
  9x_1 + 5x_2 & \leq 45 & (2) \\
  -2x_1 - x_2 & \leq -8 & (3) \\
  x_1 & \geq 0 \quad x_2 & \geq 0
\end{align*}
\]

**STEP 2**

Introduce nonnegative slack variables on the left side of each inequality to form an equality.

Introduce the slack variables \( s_1, s_2, s_3 \) to obtain
\[
\begin{align*}
  -x_1 - x_2 + s_1 & = -7 \\
  9x_1 + 5x_2 + s_2 & = 45 \\
  -2x_1 - x_2 + s_3 & = -8
\end{align*}
\]
where
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \quad s_3 \geq 0 \]
STEP 3 Set up the initial simplex tableau.

**STEP 3**

<table>
<thead>
<tr>
<th>BV</th>
<th>P</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-7</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>9</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-8</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
<td>-20</td>
<td>-15</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

This initial tableau represents the solution $x_1 = 0, x_2 = 0, s_1 = -7, s_2 = 45, s_3 = -8$. This is not a feasible point. The two negative entries in the right-hand column violate the nonnegativity requirement. Whenever this occurs, the simplex algorithm requires an Alternative Pivoting Strategy.

**Alternative Pivoting Strategy**

**STEP 4** Whenever negative entries occur in the right-hand column RHS of the constraint equations, the pivot element is selected as follows:

- **Pivot row:** Identify the most negative entry in the RHS column and its corresponding basic variable (BV). (Ignore the objective row.) The basic variable BV chosen identifies the pivot row. Because the objective row is ignored, it can never be the pivot row.
- **Pivot column:** Go from left to right along the pivot row until a negative entry is found. (Ignore the RHS.) This entry identifies the pivot column and is the pivot element. If there are no negative entries in the pivot row except the one in the RHS column, then the problem has no solution.

**STEP 5** Pivot.

1. If, in the new tableau, negative entries appear on the RHS of the constraint equations, repeat **STEP 4**.
2. If, in the new tableau, only nonnegative entries appear on the RHS of the constraint equations, then the tableau represents a maximum problem in standard form and the steps on page 237 of Section 5.2 are followed.

To continue with the example, notice there are negative entries in the RHS column of Display (1). Follow **STEP 4** of the Alternative Pivoting Strategy.

**STEP 4** The most negative entry in the RHS column is $-8$, which corresponds to the basic variable $s_3$. Going across from left to right along row $s_3$, the first negative entry we find is $-2$ in column $x_1$. The pivot column is $x_1$ and $-2$ is the pivot element.

* The alternative pivoting strategy will lead to a tableau that represents a maximum problem in standard form or it will result in a contradictory equation in some row. We shall only deal with problems of the first type.
STEP 5 Pivot: 1. $R_3 = -\frac{1}{2}r_3$  
2. $R_1 = R_3 + r_1$  $R_2 = -9R_3 + r_2$  $R_4 = 20R_3 + r_4$

The new tableau has a negative entry, $-3$, in the RHS column so repeat Step 4.

STEP 4 The pivot row corresponds to slack variable $s_1$. Going across, the first negative entry is $-\frac{1}{2}$, so the pivot column is column $x_2$. The pivot element is $-\frac{1}{2}$, which is circled.

STEP 5 Pivot: 1. $R_1 = -2r_1$  
2. $R_2 = -\frac{1}{2}R_1 + r_2$  $R_3 = -\frac{1}{2}R_1 + r_3$  $R_4 = 5R_1 + r_4$

The smaller of these is 1, so the pivot row is row $x_1$. The pivot element is 1, which is circled.

Since the objective row still contains a negative entry, pivot again. The pivot column is $s_3$; the only eligible row is $s_2$, so the pivot element is 5, which is circled.

Pivot: 1. $R_2 = \frac{1}{5}r_2$  
2. $R_1 = R_2 + r_1$  $R_3 = R_2 + r_3$  $R_4 = 15R_2 + r_4$
2 Solve a Minimum Problem Not in Standard Form

In Section 5.3, we used the duality principle to solve minimum problems that were in standard form. While the duality principle can also be used to solve minimum problems that are not in standard form, in this section we will solve such minimum problems using the methodology below:

If the objective function \( z \) is to be minimized, the minimum value of \( z \) is the same as the maximum value of \( P = -z \), subject to the same set of constraints.

**Solving a Minimum Problem**

- If \( z \) is to be minimized, let \( P = -z \).
- Solve the linear programming problem: Maximize \( P \) subject to the same constraints as the minimum problem.
- Use the principle that

\[
\text{Minimum of } z = -\text{Maximum of } P
\]

**EXAMPLE 2 Solving a Minimum Problem Not in Standard Form**

Minimize
\[
z = 5x_1 + 6x_2
\]
subject to the constraints
\[
\begin{align*}
x_1 + x_2 & \leq 10 \\
x_1 + 2x_2 & \geq 12 \\
2x_1 + x_2 & \geq 12 \\
x_1 & \geq 3 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\]
SOLUTION Change the problem from
\[ \text{Minimize } z = 5x_1 + 6x_2 \]
to
\[ \text{Maximize } P = -z = -5x_1 - 6x_2 \]
and follow the steps for a maximum problem not in standard form.

**STEP 1** Write each constraint with a ≤.
\[ x_1 + x_2 \leq 10 \]
\[ -x_1 - 2x_2 \leq -12 \]
\[ -2x_1 - x_2 \leq -12 \]
\[ -x_1 \leq -3 \]

**STEP 2** Introduce nonnegative slack variables to form equalities:
\[ x_1 + x_2 + s_1 = 10 \]
\[ -x_1 - 2x_2 + s_2 = -12 \]
\[ -2x_1 - x_2 + s_3 = -12 \]
\[ -x_1 + s_4 = -3 \]
where \( x_1 \geq 0 \), \( x_2 \geq 0 \), \( s_1 \geq 0 \), \( s_2 \geq 0 \), \( s_3 \geq 0 \), \( s_4 \geq 0 \)
The objective function is \( P + 5x_1 + 6x_2 = 0 \).

**STEP 3** Set up the initial simplex tableau

<table>
<thead>
<tr>
<th>BV</th>
<th>( P )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( s_4 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-12</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-12</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>( P )</td>
<td>1</td>
<td>5</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Because of the negative entries in the RHS column, follow the Alternative Pivoting Strategy given in **STEP 4**.

**STEP 4** The pivot row is \( s_2 \). The pivot column is \( x_1 \). The pivot element is \(-1\), which is circled.

**STEP 5** Pivot:
1. \( R_2 = -r_2 \)
2. \( R_1 = -R_2 + r_1 \)
   \[ R_3 = 2R_2 + r_3 \]
   \[ R_4 = R_2 + r_4 \]
   \[ R_5 = -5R_2 + r_5 \]

<table>
<thead>
<tr>
<th>BV</th>
<th>( P )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( s_4 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>( P )</td>
<td>1</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>-60</td>
</tr>
</tbody>
</table>

The new tableau has a negative entry, \(-2\), in the RHS column. Repeat **STEP 4**.

**STEP 4** The pivot row is \( s_1 \); the pivot column is \( x_2 \); the pivot element is \(-1\), which is circled.
5.4 The Simplex Method for Problems Not in Standard Form

STEP 5

Pivot: 1. $R_1 = -r_1$
2. $R_2 = -2r_1 + r_2$  $R_3 = -3r_1 + r_3$  $R_4 = -2r_1 + r_4$  $R_5 = 4r_1 + r_5$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$s_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-52</td>
</tr>
</tbody>
</table>

The smallest of these is 2. The pivot row is row $s_2$. The pivot element is 3, which is circled.

Pivot: 1. $R_3 = \frac{1}{3}r_3$
2. $R_1 = R_3 + r_1$  $R_2 = -2R_3 + r_2$  $R_4 = -2R_3 + r_4$  $R_5 = 4R_3 + r_5$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{2}{3}$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$s_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{2}{3}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{7}{3}$</td>
<td>$\frac{4}{3}$</td>
<td>0</td>
<td>-44</td>
</tr>
</tbody>
</table>

This is a final tableau. The maximum value of $P$ is $-44$, so the minimum value of $z$ is 44. This occurs when $x_1 = 4$ and $x_2 = 4$.

NOW WORK PROBLEM 11.

3 Solve Maximum/Minimum Problems with Equality Constraints

So far we have only dealt with constraints that are inequalities. What do we do if one or more of the constraints is an equation? Remember that we introduce a slack variable in order to get an equality. When some of the constraints are given as equalities, the job of getting an equation is already accomplished. So, with constraints that are equations, we merely use the equations without any slack variables. However, before anything is done to the initial tableau, we must pivot on each row that came about because of an equation. We can choose to pivot on any nonbasic variable in that row.
Procedure to Follow When Some Constraints Are Equations

- Set up a tableau in the usual way using slack variables for those constraints that are inequalities and entering without modification any constraints that are equations.
- For each row in the tableau that represents an equation, pivot on a nonbasic variable in that row.
- Work with the resulting tableau as usual.

EXAMPLE 3 Solving a Minimum Problem with Equality Constraints

Minimize

\[ z = 7x_1 + 5x_2 + 6x_3 \]

subject to the constraints

\[ \begin{align*}
  x_1 + x_2 + x_3 &= 10 \\
  x_1 + 2x_2 + 3x_3 &\leq 19 \\
  2x_1 + 3x_2 &\geq 21 \\
  x_1 &\geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*} \]

**SOLUTION** Maximize \( P = -z = -7x_1 - 5x_2 - 6x_3 \) subject to the constraints

\[ \begin{align*}
  x_1 + x_2 + x_3 &= 10 \quad (1) \\
  x_1 + 2x_2 + 3x_3 &\leq 19 \quad (2) \\
  2x_1 + 3x_2 &\geq 21 \quad (3) \\
  x_1 &\geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*} \]

**STEP 1** Rewrite the \( \geq \) inequality constraint (3) as a \( \leq \) inequality:

\[ \begin{align*}
  x_1 + x_2 + x_3 &= 10 \quad (1) \\
  x_1 + 2x_2 + 3x_3 &\leq 19 \quad (2) \\
  -2x_1 - 3x_2 &\leq -21 \quad (3)
\end{align*} \]

**STEP 2** Introduce nonnegative slack variables for the inequalities (2) and (3):

\[ \begin{align*}
  x_1 + x_2 + x_3 &= 10 \\
  x_1 + 2x_2 + 3x_3 + s_1 &= 19 \\
  -2x_1 - 3x_2 + s_2 &= -21
\end{align*} \]

where \( x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \)

**STEP 3** Set up the tableau:

\[
\begin{array}{cccccc|c}
BV & P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 10 \\
0 & 1 & 2 & 3 & 1 & 0 & 0 & 19 \\
0 & -2 & -3 & 0 & 0 & 1 & 0 & -21 \\
1 & 7 & 5 & 6 & 0 & 0 & 0 & 0
\end{array}
\]
The first row represents a constraint that is an equation. We pivot on that row, choosing $x_1$ as the pivot column.

Pivot: 1. $R_1 = r_1$  2. $R_2 = -R_1 + r_2$  $R_3 = 2R_1 + r_3$  $R_4 = -7R_1 + r_4$

Now we have the initial simplex tableau. Because of the negative entry in the RHS column, we follow the Alternative Pivoting Strategy.

**STEP 4**
The pivot row is row $s_2$; the pivot column is column $x_2$. The pivot element is $-1$, which is circled.

**STEP 5**
Pivot: 1. $R_3 = -r_3$  2. $R_1 = -R_3 + r_1$  $R_2 = -R_3 + r_2$  $R_4 = 2R_3 + r_4$

Since the new tableau has only nonnegative entries in the RHS column (remember, the objective row is ignored), it represents a maximum problem in standard form. Since the tableau is not a final tableau (the objective row contains negative entries), we use the standard pivoting strategy of Section 5.2. The pivot column is column $x_3$. Form the nonnegative quotients:

$9 \div 3 = 3  \quad 8 \div 4 = 2$

The smaller of these is 2, so the pivot row is row $s_1$. The pivot element is 4, which is circled.

Pivot: 1. $R_2 = \frac{1}{4}r_2$  2. $R_1 = -3R_2 + r_1$  $R_3 = 2R_2 + r_3$  $R_4 = 5R_2 + r_4$

This tableau represents a maximum problem in standard form. But it is not a final tableau. The pivot column is column $s_2$. Form the quotients:

$3 \div \frac{1}{4} = 12  \quad 2 \div \frac{1}{4} = 8$
The smaller of these is 8. The pivot row is \( x_3 \); the pivot element is \( \frac{1}{4} \), which is circled.

Pivot: 1. \( R_2 = 4r_2 \)

2. \( R_1 = -\frac{1}{4}R_2 + r_1 \)

\( R_3 = \frac{1}{2}R_2 + r_3 \)

\( R_4 = \frac{3}{4}R_2 + r_4 \)

This is a final tableau. The maximum value of \( P \) is \(-52\), so the minimum value of \( z \) is 52. This occurs when \( x_1 = 1 \), \( x_2 = 9 \), and \( x_3 = 0 \).

\[ \begin{array}{ccccccc|c}
 BV & P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \\
 x_1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\
s_2 & 0 & 0 & 0 & 4 & 1 & 1 & 8 \\
x_2 & 0 & 0 & 1 & 2 & 1 & 0 & 9 \\
P & 1 & 0 & 0 & 3 & 2 & 0 & -52 \\
\end{array} \]

\( x_1 = 1 \)

\( s_2 = 8 \)

\( x_2 = 9 \)

\( P = -52 \)

\[ \checkmark \text{CHECK:} \text{ Verify the results obtained in Example 3 using Excel.} \]

\[ \text{NOW WORK PROBLEM 9.} \]

\[ \text{EXAMPLE 4 Minimizing Cost} \]

The Red Tomato Company operates two plants for canning its tomatoes and has two warehouses for storing the finished products until they are purchased by retailers. The schedule shown in the table represents the per case shipping costs from plant to warehouse.

<table>
<thead>
<tr>
<th>Warehouse</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant I</td>
<td>$0.25</td>
<td>$0.18</td>
</tr>
<tr>
<td>Plant II</td>
<td>$0.25</td>
<td>$0.14</td>
</tr>
</tbody>
</table>

Each week plant I can produce at most 450 cases and plant II can produce at most 350 cases of tomatoes. Also, each week warehouse A requires at least 300 cases and warehouse B requires at least 500 cases. If we represent the number of cases shipped from plant I to warehouse A by \( x_1 \), from plant I to warehouse B by \( x_2 \), and so on, the above data can be represented by the following table:

<table>
<thead>
<tr>
<th>Warehouse</th>
<th>A</th>
<th>B</th>
<th>Maximum Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant I</td>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>450</td>
</tr>
<tr>
<td>Plant II</td>
<td>( x_3 )</td>
<td>( x_4 )</td>
<td>350</td>
</tr>
<tr>
<td>Minimum Demand</td>
<td>300</td>
<td>500</td>
<td></td>
</tr>
</tbody>
</table>

The company wants to arrange its shipments from the plants to the warehouses so that the requirements of the warehouses are met and shipping costs are kept at a minimum. How should the company proceed?

\[ \text{SOLUTION} \quad \text{We model this situation using linear programming. The linear programming problem is stated as follows:} \]
5.4 The Simplex Method for Problems Not in Standard Form

Minimize the cost equation

\[ C = 0.25x_1 + 0.18x_2 + 0.25x_3 + 0.14x_4 \]

subject to the constraints

\[
\begin{align*}
x_1 + x_2 & \leq 450 \\
x_3 + x_4 & \leq 350 \\
x_1 + x_3 & \geq 300 \\
x_2 + x_4 & \geq 500 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0 \\
x_4 & \geq 0 
\end{align*}
\]

The maximum problem is:

Maximize

\[ P = -C = -0.25x_1 - 0.18x_2 - 0.25x_3 - 0.14x_4 \]

subject to the same constraints.

**STEP 1** Write each constraint with a \( \leq \).

\[
\begin{align*}
x_1 + x_2 & \leq 450 \\
x_3 + x_4 & \leq 350 \\
-x_1 - x_3 & \leq -300 \\
-x_2 - x_4 & \leq -500 
\end{align*}
\]

**STEP 2** Introduce nonnegative slack variables to form equalities:

\[
\begin{align*}
x_1 + x_2 + s_1 & = 450 \\
x_3 + x_4 + s_2 & = 350 \\
-x_1 - x_3 + s_3 & = -300 \\
-x_2 - x_4 + s_4 & = -500 
\end{align*}
\]

\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \quad s_3 \geq 0 \quad s_4 \geq 0 \]

**STEP 3** Set up the initial simplex tableau:

<table>
<thead>
<tr>
<th>BV</th>
<th>P</th>
<th>x_1</th>
<th>x_2</th>
<th>x_3</th>
<th>x_4</th>
<th>s_1</th>
<th>s_2</th>
<th>s_3</th>
<th>s_4</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>s_1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>450</td>
</tr>
<tr>
<td>s_2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>350</td>
</tr>
<tr>
<td>s_3</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-300</td>
</tr>
<tr>
<td>s_4</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-500</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
<td>0.25</td>
<td>0.18</td>
<td>0.25</td>
<td>0.14</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Because of the negative entries in the RHS column, we follow the Alternative Pivoting Strategy.

**STEP 4** The pivot row is row \( s_4 \); the pivot column is column \( x_2 \). The pivot element is \(-1\), which is circled.

**STEP 5** Pivot: 1. \( R_4 = -r_4 \) 2. \( R_1 = -R_4 + r_1 \) \( R_5 = -0.18R_4 + r_5 \)

<table>
<thead>
<tr>
<th>BV</th>
<th>P</th>
<th>x_1</th>
<th>x_2</th>
<th>x_3</th>
<th>x_4</th>
<th>s_1</th>
<th>s_2</th>
<th>s_3</th>
<th>s_4</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>s_1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-50</td>
</tr>
<tr>
<td>s_2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>350</td>
</tr>
<tr>
<td>s_3</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-300</td>
</tr>
<tr>
<td>x_2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>500</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
<td>-0.04</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.18</td>
<td>-90</td>
</tr>
</tbody>
</table>

The negative entries in the RHS require that we repeat STEP 4.
STEP 4 The pivot row is $s_3$; the pivot column is $x_1$. The pivot element is $-1$, which is circled.

Pivot: 1. $R_3 = -r_3$  2. $R_1 = -R_3 + r_1$  $R_5 = -0.25R_3 + r_5$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\boxed{1}$</td>
<td>$-1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$-350$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>350</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>500</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-0.04$</td>
<td>0</td>
<td>0</td>
<td>0.25</td>
<td>0.18</td>
<td>$-165$</td>
</tr>
</tbody>
</table>

Since there is still a negative value in the RHS column, we repeat STEP 4.

STEP 4 The pivot row is $s_1$; the pivot column is $x_3$. The pivot element is $-1$, which is circled.

Pivot: 1. $R_1 = -r_1$  2. $R_2 = -R_3 + r_2$  $R_3 = -R_1 + r_3$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>$-1$</td>
<td>1</td>
<td>350</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\boxed{1}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-50$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>500</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-0.04$</td>
<td>0</td>
<td>0</td>
<td>0.25</td>
<td>0.18</td>
<td>$-165$</td>
</tr>
</tbody>
</table>

The $-50$ in the RHS column requires that we repeat STEP 4.

STEP 4 The pivot row is $x_1$; the pivot column is $x_4$. The pivot element is $-1$, which is circled.

Pivot: 1. $R_3 = -r_3$  2. $R_1 = -R_3 + r_1$  $R_4 = -R_3 + s_4$  $R_5 = 0.04R_3 + r_5$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>50</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>450</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>$-0.04$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-0.04$</td>
<td>0</td>
<td>0.25</td>
<td>0.14</td>
<td>$-163$</td>
</tr>
</tbody>
</table>

Since the new tableau has only nonnegative entries in the RHS column (remember, the objective row is ignored), it represents a maximum problem in standard form. Since the tableau is not a final tableau (the objective row contains negative entries), we use the standard pivoting strategy of Section 5.2. The pivot column is column $x_1$. Form the quotients:

$$300 \div 1 = 300 \quad 450 \div 1 = 450$$

The smaller of these is $300$. The pivot row is row $x_3$. The pivot element is $1$, which is circled.

Pivot: 1. $R_1 = r_1$  2. $R_3 = R_1 + r_3$  $R_4 = -R_1 + r_4$  $R_5 = 0.04R_1 + r_5$

<table>
<thead>
<tr>
<th>BV</th>
<th>$P$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>350</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>150</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-0.04$</td>
<td>0</td>
<td>$-0.04$</td>
<td>0</td>
<td>0.21</td>
<td>0.14</td>
</tr>
</tbody>
</table>
In Problems 5–10, solve each maximum problem not in standard form.

5. Maximize

\[ P = 3x_1 + 4x_2 \]

subject to the constraints

\[ x_1 + x_2 \leq 12 \]
\[ 5x_1 + 2x_2 \geq 36 \]
\[ 7x_1 + 4x_2 \geq 14 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

6. Maximize

\[ P = 5x_1 + 2x_2 \]

subject to the constraints

\[ x_1 + x_2 \geq 11 \]
\[ 2x_1 + 3x_2 \geq 24 \]
\[ x_1 + 3x_2 \geq 18 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

7. Maximize

\[ P = 3x_1 + 2x_2 - x_3 \]

subject to the constraints

\[ x_1 + 3x_2 + x_3 \leq 9 \]
\[ 2x_1 + 3x_2 - x_3 \geq 2 \]
\[ 3x_1 - 2x_2 + x_3 \geq 5 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

8. Maximize

\[ P = 3x_1 + 2x_2 - x_3 \]

subject to the constraints

\[ 2x_1 - x_2 - x_3 \leq 2 \]
\[ x_1 + 2x_2 + x_3 \geq 2 \]
\[ x_1 - 3x_2 - 2x_3 \leq -5 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

9. Maximize

\[ P = 3x_1 + 2x_2 \]

subject to the constraints

\[ 2x_1 + x_2 \leq 4 \]
\[ x_1 + x_2 = 3 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

10. Maximize

\[ P = 45x_1 + 27x_2 + 18x_3 + 36x_4 \]

subject to the constraints

\[ 5x_1 + x_2 + x_3 + 8x_4 = 30 \]
\[ 2x_1 + 4x_2 + 3x_3 + 2x_4 = 30 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]
\[ x_4 \geq 0 \]

**Concepts and Vocabulary**

1. **True or False** In using the Alternate Pivoting Strategy, some of the slack variables can be negative.

2. A minimum problem that is not in standard form and whose objective function is \( z \) can be solved by changing it to a maximum problem whose objective function is \( P = -z \).

3. **True or False** In solving a maximum problem not in standard form, the initial tableau can have negative entries in the right-hand column RHS.

4. **True or False** If a constraint is an equality, it can be ignored.

---

**Skill Building**

In Problems 5–10, solve each maximum problem not in standard form.
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In Problems 11–12, solve each minimum problem not in standard form.

11. Minimize
   \[ z = 6x_1 + 8x_2 + x_3 \]
   subject to the constraints
   \[
   \begin{align*}
   3x_1 + 5x_2 + 3x_3 & \geq 20 \\
   x_1 + 3x_2 + 2x_3 & \geq 9 \\
   6x_1 + 2x_2 + 5x_3 & \geq 30 \\
   x_1 + x_2 + x_3 & \leq 10 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0 \\
   x_3 & \geq 0
   \end{align*}
   \]

12. Minimize
   \[ z = 2x_1 + x_2 + x_3 \]
   subject to the constraints
   \[
   \begin{align*}
   3x_1 - x_2 - 4x_3 & \leq -12 \\
   x_1 + 3x_2 + 2x_3 & \geq 10 \\
   x_1 - x_2 + x_3 & \leq 8 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0 \\
   x_3 & \geq 0
   \end{align*}
   \]

Applications

13. Shipping Private Motors, Inc., has two plants, M1 and M2, which manufacture engines; the company also has two assembly plants, A1 and A2, which assemble the cars. M1 can produce at most 600 engines per week. M2 can produce at most 400 engines per week. A1 needs at least 500 engines per week and A2 needs at least 300 engines per week. Following is a table of charges to ship engines to assembly plants.

<table>
<thead>
<tr>
<th>Plant</th>
<th>A1 Cost</th>
<th>A2 Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>$400</td>
<td>$100</td>
</tr>
<tr>
<td>M2</td>
<td>$200</td>
<td>$300</td>
</tr>
</tbody>
</table>

How many engines should be shipped each week from each engine plant to each assembly plant? [Hint: Consider four variables: \( x_1 \) = number of units shipped from M1 to A1, \( x_2 \) = number of units shipped from M1 to A2, \( x_3 \) = number of units shipped from M2 to A1, and \( x_4 \) = number of units shipped from M2 to A2.]

(a) Formulate a linear programming problem that models the problem.

(b) Solve the linear programming problem.

(c) Analyze the solution.

14. Shipping Schedule A cell phone manufacturer must fill orders from two retailers. The first retailer, R1, has ordered 55 cell phones, while the second retailer, R2, has ordered 75 cell phones. The manufacturer has the cell phones stored in two warehouses, W1 and W2. There are 100 sets in W1 and 120 sets in W2. The shipping costs per cell phone are $8 from W1 to R1; $12 from W1 to R2; $13 from W2 to R1; and $7 from W2 to R2. Find the number of cell phones to be shipped from each warehouse to each retailer if the total shipping cost is to be a minimum.

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

15. Shipping Schedule A GPS systems manufacturer must fill orders from two dealers. The first dealer, D1, has ordered 20 GPS systems, while the second dealer, D2, has ordered 30 GPS systems. The manufacturer has the GPS systems stored in two warehouses, W1 and W2. There are 40 GPS systems in W1 and 15 GPS systems in W2. The shipping costs per GPS system are $8 from W1 to D1; $12 from W1 to D2; $13 from W2 to D1; and $7 from W2 to D2. Find the number of GPS systems to be shipped from each warehouse to each dealer if the total shipping cost is to be a minimum.

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

16. Mixture Minimize the cost of preparing the following mixture, which is made up of three foods, I, II, III. Food I costs $2 per unit, food II costs $1 per unit, and food III costs $3 per unit. Each unit of food I contains 2 ounces of protein and 4 ounces of carbohydrate; each unit of food II has 3 ounces of protein and 2 ounces of carbohydrate; and each unit of food III has 4 ounces of protein and 2 ounces of carbohydrate. The mixture must contain at least 20 ounces of protein and 15 ounces of carbohydrate.

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

17. Advertising A local appliance store has decided on an advertising campaign utilizing newspaper and radio. Each dollar spent on newspaper advertising is expected to reach
50 people in the “Under $25,000” and 40 in the “Over $25,000” bracket. Each dollar spent on radio advertising is expected to reach 70 people in the “Under $25,000” and 20 people in the “Over $25,000” bracket. If the store wants to reach at least 100,000 people in the “Under $25,000” and at least 120,000 in the “Over $25,000” bracket, how should it proceed so that the cost of advertising is minimized?

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

18. **Minimizing Materials** Quality Oak Tables, Inc., has an individual who does all its finishing work, and it wishes to use him in this capacity at least 6 hours each day. The assembly area can be used at most 8 hours each day. The company has three models of oak tables, T1, T2, T3. T1 requires 1 hour for assembly, 2 hours for finishing, and 9 board feet of oak. T2 requires 1 hour for assembly, 1 hour for finishing, and 9 board feet of oak. T3 requires 2 hours for assembly, 1 hour for finishing, and 3 board feet of oak. If we wish to minimize the board feet of oak used, how many of each model should be made?

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

19. **Nutrition** The serving sizes and nutritional content per serving of certain canned vegetables are shown in the table below.

<table>
<thead>
<tr>
<th>Canned Vegetable</th>
<th>Serving Size</th>
<th>Fiber (grams)</th>
<th>Sugar (grams)</th>
<th>Calories</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole Kernel Corn</td>
<td>1/2 cup</td>
<td>2</td>
<td>4</td>
<td>80</td>
</tr>
<tr>
<td>Sweet Peas</td>
<td>1/2 cup</td>
<td>2</td>
<td>4</td>
<td>80</td>
</tr>
<tr>
<td>Cut Blue Lake Green Beans</td>
<td>1/2 cup</td>
<td>3</td>
<td>6</td>
<td>70</td>
</tr>
<tr>
<td>Sliced Carrots</td>
<td>1/2 cup</td>
<td>4</td>
<td></td>
<td>30</td>
</tr>
</tbody>
</table>

*Source: Nutri-Facts.com*

A school nutritionist will make a dish of mixed vegetables by combining these four vegetables. She will need 400 servings (200 cups) of the mixture. The mixed vegetables must have at least 2.25 grams of fiber per serving and at most 4.25 grams of sugar per serving. In addition, the dish must contain at least as much corn as green beans, and at least half as much carrots as peas. How many cups of each type vegetable should the nutritionist use in order to minimize the number of calories in the mixture? What will be the minimal number of calories per serving?

20. **Maximizing Salespeople at a Trade Show** Suppose that a computer company has 12 sales representatives based in New York City and 18 sales representatives based in San Francisco. To staff an upcoming trade show in Dallas and another trade show in Chicago, this company wants to send at least 5 of these sales representatives to Dallas and at least 5 to Chicago. According to Travelocity, for April 2010, the lowest round-trip airfares (including taxes and fees) for a one-week stay, with tickets purchased two weeks in advance, are as follows: San Francisco to Dallas: $385; San Francisco to Chicago: $211; New York (LaGuardia) to Dallas: $273; New York (LaGuardia) to Chicago: $182. For these two trade shows, this computer company has a total airfare budget of $4830. Determine how many sales representatives this company should send from New York City to each trade show and how many sales representatives should be sent from San Francisco to each trade show in order to satisfy the staffing requirements and stay within the company’s airfare budget while maximizing the total number of sales representatives that are sent.

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.

(b) Solve the linear programming problem. Round answers to the nearest integer.

(c) Analyze the solution.

*Source: Travelocity*

21. **Pension Funds** A pension fund has decided to invest $50,000 in the four high-yield stocks listed in the table below.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Price per share 4-12-10</th>
<th>Yield</th>
<th>Price/Earnings Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duke Energy Corp.</td>
<td>$16.26</td>
<td>5.90%</td>
<td>19.79</td>
</tr>
<tr>
<td>H. J. Heinz</td>
<td>$45.99</td>
<td>3.65%</td>
<td>16.33</td>
</tr>
<tr>
<td>General Electric Co.</td>
<td>$18.71</td>
<td>2.14%</td>
<td>18.23</td>
</tr>
<tr>
<td>Ferrellgas Partners LP</td>
<td>$23.14</td>
<td>8.64%</td>
<td>37.78</td>
</tr>
</tbody>
</table>

*Source: TD Ameritrade.*

The pension fund has decided to invest at least 10% of the $50,000 in the H. J. Heinz stock and at least 20% in each of the other three stocks. Further, it has decided that at most 50% of the $50,000 can be invested in the Duke Energy Corp. and the General Electric Co. combined. Determine how many shares of each stock should be purchased in order to minimize the average price/earnings ratio of the stocks purchased, while conforming to the above requirements. Note that to minimize the average price/earnings ratio of the stocks purchased, it is sufficient to minimize the sum of the four products obtained by multiplying the price/earnings ratio for each stock and the amount invested in that stock. What is the average price/earnings ratio (rounded to the nearest hundredth) for the optimal investment plan? What is the annual yield in dollars for the optimal investment plan? Assume that it is possible to purchase a fraction of a share.
In Problems 1–8, determine which maximum problems are in standard form.

1. Maximize
   \[ P = 2x_1 + 3x_2 \]
   subject to the constraints
   \[
   \begin{align*}
   x_1 + 2x_2 & \leq 4 \\
   3x_1 + x_2 & \leq 8 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0
   \end{align*}
   \]

2. Maximize
   \[ P = x_1 + 5x_2 \]
   subject to the constraints
   \[
   \begin{align*}
   x_1 + x_2 & \leq 5 \\
   2x_1 - x_2 & \leq 8 \\
   x_1 + 3x_2 & \leq 15 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0
   \end{align*}
   \]

3. Maximize
   \[ P = 5x_1 + x_2 + x_3 \]
   subject to the constraints
   \[
   \begin{align*}
   x_1 + x_2 + x_3 & \leq 30 \\
   x_1 + 4x_3 & \leq 10 \\
   2x_1 + x_2 + x_3 & \leq 15 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0 \\
   x_3 & \geq 0
   \end{align*}
   \]
4. Maximize 
\[ P = 2x_1 + 3x_2 + x_3 \]
subject to the constraints
\[ x_1 + 2x_2 + x_3 \leq 12 \]
\[ 2x_1 + x_2 + 4x_3 \leq 20 \]
\[ x_1 + x_2 + x_3 \leq 8 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

5. Maximize 
\[ P = x_1 + 2x_2 + x_3 \]
subject to the constraints
\[ x_1 + 3x_2 \geq 4 \]
\[ 2x_1 + 5x_2 + x_3 \leq 8 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

6. Maximize 
\[ P = 5x_1 + 2x_2 + x_3 \]
subject to the constraints
\[ x_1 + 3x_2 + x_3 \leq 50 \]
\[ x_1 + 2x_2 + 5x_3 = 8 \]
\[ x_1 \geq 0 \quad x_3 \geq 0 \]

7. Maximize 
\[ P = x_1 + 3x_2 \]
subject to the constraints
\[ x_1 + x_2 \leq 10 \]
\[ 3x_1 + x_2 \leq 20 \]
\[ x_1 \leq 0 \]

8. Maximize 
\[ P = 4x_1 + 5x_2 \]
subject to the constraints
\[ x_1 + 3x_2 \geq 10 \]
\[ 3x_1 + x_2 \geq 20 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

In Problems 9–14, set up the initial tableau for each maximum problem.

9. Maximize 
\[ P = 2x_1 + x_2 + 3x_3 \]
subject to the constraints
\[ 2x_1 + 5x_2 + x_3 \leq 100 \]
\[ x_1 + 3x_2 + x_3 \leq 80 \]
\[ 2x_1 + 3x_2 + 3x_3 \leq 120 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

10. Maximize 
\[ P = x_1 + 5x_2 + 3x_3 \]
subject to the constraints
\[ x_1 + x_2 + x_3 \leq 50 \]
\[ 2x_1 + 3x_2 + x_3 \leq 80 \]
\[ x_1 + 2x_2 + 5x_3 \leq 100 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

11. Maximize 
\[ P = 6x_1 + 3x_2 \]
subject to the constraints
\[ x_1 + 5x_2 \leq 200 \]
\[ 5x_1 + 3x_2 \leq 450 \]
\[ x_1 + x_2 \leq 120 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

12. Maximize 
\[ P = 2x_1 + x_2 \]
subject to the constraints
\[ x_1 + x_2 \leq 20 \]
\[ 2x_1 + 3x_2 \leq 50 \]
\[ 3x_1 + x_2 \leq 30 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

13. Maximize 
\[ P = x_1 + 2x_2 + x_3 + 4x_4 \]
subject to the constraints
\[ x_1 + 3x_2 + x_3 + 2x_4 \leq 20 \]
\[ 4x_1 + x_2 + x_3 + 6x_4 \leq 80 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \]

14. Maximize 
\[ P = x_1 + x_2 + 3x_3 + 4x_4 \]
subject to the constraints
\[ 2x_1 + 3x_2 + 5x_3 + x_4 \leq 200 \]
\[ x_1 + x_2 + 6x_3 + x_4 \leq 180 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \]
In Problems 15–22, use each tableau to:

(a) Choose the pivot element and perform a pivot.
(b) Write the system of equations resulting from pivoting in part (a).
(c) Determine if the new tableau: (1) is the final tableau and, if so, write the solution; (2) requires additional pivoting and, if so, choose the new pivot element, or (3) indicates no solution exists for the problem.

15. BV

<table>
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<th>P</th>
<th>x₁</th>
<th>x₂</th>
<th>s₁</th>
<th>s₂</th>
<th>RHS</th>
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16. BV

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17. BV

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18. BV

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19. BV

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20. BV

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21. BV

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22. BV

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<td>0</td>
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In Problems 23–26, use the simplex method to solve each maximum problem.

23. Maximize

\[ P = 100x₁ + 200x₂ + 50x₃ \]

subject to the constraints

\[ 5x₁ + 5x₂ + 10x₃ \leq 1000 \]
\[ 10x₁ + 8x₂ + 5x₃ \leq 2000 \]
\[ 10x₁ + 5x₂ \leq 500 \]
\[ x₁ \geq 0 \quad x₂ \geq 0 \quad x₃ \geq 0 \]

24. Maximize

\[ P = x₁ + 2x₂ + x₃ \]

subject to the constraints

\[ 3x₁ + x₂ + x₃ \leq 3 \]
\[ x₁ - 10x₂ - 4x₃ \leq 20 \]
\[ x₁ \geq 0 \quad x₂ \geq 0 \quad x₃ \geq 0 \]
25. Maximize
\[ P = 40x_1 + 60x_2 + 50x_3 \]
subject to the constraints
\[ \begin{align*}
2x_1 + 2x_2 + x_3 & \leq 8 \\
x_1 - 4x_2 + 3x_3 & \leq 12 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

26. Maximize
\[ P = 2x_1 + 8x_2 + 10x_3 + x_4 \]
subject to the constraints
\[ \begin{align*}
x_1 + 2x_2 + x_3 + x_4 & \leq 50 \\
3x_1 + x_2 + 2x_3 + x_4 & \leq 100 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0 \\
x_4 & \geq 0
\end{align*} \]

In Problems 27–32, determine if the minimum problem is in standard form.

27. Minimize
\[ C = 2x_1 + 3x_2 \]
subject to the constraints
\[ \begin{align*}
x_1 + 2x_2 & \geq 4 \\
3x_1 + x_2 & \geq 12 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*} \]

28. Minimize
\[ C = x_1 + 3x_2 \]
subject to the constraints
\[ \begin{align*}
3x_1 + 2x_2 & \geq 20 \\
x_1 + 5x_2 & \geq 40 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*} \]

29. Minimize
\[ C = 2x_1 + x_2 + 3x_3 \]
subject to the constraints
\[ \begin{align*}
x_1 + x_2 + x_3 & \leq 15 \\
x_1 - x_2 + 3x_3 & \leq 20 \\
2x_1 + 2x_2 + 4x_3 & \leq 25 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

30. Minimize
\[ C = x_1 + 2x_2 + x_3 \]
subject to the constraints
\[ \begin{align*}
2x_1 + 3x_2 + x_3 & \leq 400 \\
x_1 + 6x_2 + 3x_3 & \leq 250 \\
3x_1 + x_2 + x_3 & \leq 300 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

31. Minimize
\[ C = x_1 + 2x_2 + x_3 \]
subject to the constraints
\[ \begin{align*}
3x_1 + x_2 + x_3 & \leq 50 \\
x_1 + 2x_2 + x_3 & \leq 60 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

32. Minimize
\[ C = 2x_1 + 2x_2 + x_3 \]
subject to the constraints
\[ \begin{align*}
2x_1 + 3x_2 + x_3 & \leq 100 \\
x_1 + 2x_2 + 2x_3 & \leq 160 \\
x_1 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

In Problems 33–36, write the dual of the minimum problem.

33. Minimize
\[ C = 2x_1 + x_2 \]
subject to the constraints
\[ \begin{align*}
2x_1 + 2x_2 & \geq 8 \\
x_1 - x_2 & \geq 2 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*} \]

34. Minimize
\[ C = 4x_1 + 2x_2 \]
subject to the constraints
\[ \begin{align*}
x_1 + 2x_2 & \geq 4 \\
x_1 + 4x_2 & \geq 6 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*} \]

35. Minimize
\[ C = 5x_1 + 4x_2 + 2x_3 \]
subject to the constraints
\[ \begin{align*}
x_1 + x_2 + x_3 & \geq 100 \\
2x_1 + x_2 & \geq 50 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0
\end{align*} \]

36. Minimize
\[ C = 2x_1 + x_2 + 3x_3 + x_4 \]
subject to the constraints
\[ \begin{align*}
x_1 + x_2 + x_3 + x_4 & \geq 50 \\
3x_1 + x_2 + 2x_3 + x_4 & \geq 100 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_3 & \geq 0 \\
x_4 & \geq 0
\end{align*} \]

37. Solve Problem 33 using the duality principle.

38. Solve Problem 34 using the duality principle.

39. Solve Problem 35 using the duality principle.

40. Solve Problem 36 using the duality principle.
In Problems 41–46, solve each linear programming problem not in standard form.

41. Maximize

\[ P = 3x_1 + 5x_2 \]

subject to the constraints

\[ x_1 + x_2 \geq 2 \]
\[ 2x_1 + 3x_2 \leq 12 \]
\[ 3x_1 + 2x_2 \leq 12 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

42. Maximize

\[ P = 2x_1 + 4x_2 \]

subject to the constraints

\[ 2x_1 + x_2 \geq 4 \]
\[ x_1 + x_2 \leq 9 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

43. Minimize

\[ C = 2x_1 + 3x_2 \]

subject to the constraints

\[ x_1 + x_2 \geq 3 \]
\[ x_1 + x_2 \leq 9 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

44. Minimize

\[ C = 3x_1 + 6x_2 + 2x_3 \]

subject to the constraints

\[ x_2 + x_3 \leq 4 \]
\[ x_1 + 4x_2 + x_3 \leq 3 \]
\[ x_1 + x_2 + x_3 \geq 1 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

45. Maximize

\[ P = 300x_1 + 200x_2 + 450x_3 \]

subject to the constraints

\[ 4x_1 + 3x_2 + 5x_3 \leq 140 \]
\[ x_1 + x_2 + x_3 = 30 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

46. Maximize

\[ P = 2x_1 + x_2 + 3x_3 \]

subject to the constraints

\[ x_1 + x_2 + 2x_3 \leq 6 \]
\[ 3x_1 + 2x_2 + x_3 \leq 24 \]
\[ 5x_1 - 15x_2 + 5x_3 \leq 10 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

47. Maximizing Profit

The finishing process in the manufacture of cocktail tables and end tables requires sanding, staining, and varnishing. The time in minutes required for each finishing process is given below:

<table>
<thead>
<tr>
<th></th>
<th>Sanding</th>
<th>Staining</th>
<th>Varnishing</th>
</tr>
</thead>
<tbody>
<tr>
<td>End table</td>
<td>8</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>Cocktail table</td>
<td>4</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

The equipment required for each process is used on one table at a time and is available for 6 hours each day. If the profit on each cocktail table is $20 and on each end table is $15, how many of each should be manufactured each day in order to maximize profit?

(a) Formulate a linear programming problem that models the problem given above. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

48. Management

The manager of a supermarket meat department finds that there are 160 pounds of round steak, 600 pounds of chuck steak, and 300 pounds of pork in stock on Saturday morning. From experience, the manager knows that half of these quantities can be sold as straight cuts. The remaining meat will have to be ground into hamburger patties and picnic patties for which there is a large weekend demand. Each pound of hamburger patties contains 20% ground round and 60% ground chuck. Each pound of picnic patties contains 30% ground pork and 50% ground chuck. The remainder of each product consists of an inexpensive nonmeat filler that the store has in unlimited quantities. How many pounds of each product should be made if the objective is to maximize the amount of meat used to make the patties?

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.

49. Minimizing Cost

The ACE Meat Market makes up a combination package of ground beef and ground pork for meat loaf. The ground beef is 75% lean (75% beef, 25% fat) and costs the market $1.89 per pound. The ground pork is 60% lean (60% pork, 40% fat) and costs the market $1.29 per pound. If the meat loaf is to be at least 70% lean, how much ground beef and ground pork should be mixed to keep cost at a minimum?

(a) Formulate a linear programming problem that models the problem. Be sure to identify all variables used.

(b) Solve the linear programming problem.

(c) Analyze the solution.
BALANCING NUTRIENTS—CONTINUED

In the Chapter 4 Project, we discussed making a very simple trail mix from dry-roasted unsalted peanuts and seedless raisins. We used graphing techniques to find which recipes maximized carbohydrates and protein and which minimized fat. The simplex method allows us to consider and to solve more complicated recipes.

Suppose we add two more ingredients to our trail mix: M&Ms Plain Chocolate Candies and hard, salted mini-pretzels. Table 1 lists the amounts of various dietary quantities for these ingredients, including serving sizes. The amounts are given per serving of the ingredient:

<table>
<thead>
<tr>
<th>Nutrient</th>
<th>Peanuts (1 cup)</th>
<th>Raisins (1 cup)</th>
<th>M&amp;Ms (1 cup)</th>
<th>Mini-Pretzels (1 cup)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calories (kcal)</td>
<td>854.10</td>
<td>435.00</td>
<td>1023.96</td>
<td>162.02</td>
</tr>
<tr>
<td>Protein (g)</td>
<td>34.57</td>
<td>4.67</td>
<td>9.01</td>
<td>3.87</td>
</tr>
<tr>
<td>Fat (g)</td>
<td>72.50</td>
<td>0.67</td>
<td>43.95</td>
<td>1.49</td>
</tr>
<tr>
<td>Carbohydrates (g)</td>
<td>31.40</td>
<td>114.74</td>
<td>148.12</td>
<td>33.68</td>
</tr>
</tbody>
</table>


Suppose that you want to make at most 10 cups of trail mix for a day hike. You don’t want any of the ingredients to dominate the mixture, so you want each of the ingredients to contribute at least 10% of the total volume of mix made. You want the entire amount of trail mix you make to have fewer than 7000 calories, and you want to maximize the amount of carbohydrates in the mix.

1. Let $x_1$ be the number of cups of peanuts you will use in the mix, $x_2$ the number of cups of raisins, $x_3$ the number of cups of M&Ms, and $x_4$ the number of cups of mini-pretzels. Let $C$ be the amount of carbohydrates in the mix. Find the objective function.

2. What constraints must be placed on the objective function?

3. Find the number of cups of peanuts, raisins, M&Ms, and mini-pretzels that maximize the amount of carbohydrates in the mix.

4. How many grams of carbohydrates are in a cup of the final mix? How many calories?

5. Under all the constraints given above, what recipe for trail mix will maximize the amount of protein in the mix? How many grams of protein are in a cup of this mix? How many calories?

6. Consider making a batch of trail mix under the following conditions: You still want to make at most 10 cups of trail mix, and you still want each of the ingredients to contribute at least 10% of the total volume of mix made. You want the entire amount of trail mix you make to have at least 1000 grams of carbohydrates, and you want to minimize the amount of fat in the mix. What recipe for trail mix will minimize the amount of fat in the mix?

7. How much fat, protein, and carbohydrates are in a cup of this mix?

8. How many calories are in a cup of this mix?
Mathematical Questions from Professional Exams

Use the following information to answer Problems 1–4.

**CPA Exam** The Ball Company manufactures three types of lamps, which are labeled A, B, and C. Each lamp is processed in two departments—I and II. Total available work-hours per day for departments I and II are 400 and 600, respectively. No additional labor is available. Time requirements and profit per unit for each lamp type are as follows:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Work-hours required in department I</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Work-hours required in department II</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Profit per unit (sales price less all variable costs)</td>
<td>$5</td>
<td>$4</td>
<td>$3</td>
</tr>
</tbody>
</table>

1. The coefficients of the objective function would be
   (a) 4, 2, 3
   (b) 2, 3, 1
   (c) 5, 4, 3
   (d) 400, 600

2. The constraints in the model would be
   (a) 2, 3, 1
   (b) 5, 4, 3
   (c) 4, 2, 3
   (d) 400, 600

3. The constraint imposed by the available work-hours in department I could be expressed as
   (a) \(4X_1 + 2X_2 + 3X_3 \leq 400\)
   (b) \(4X_1 + 2X_2 + 3X_3 = 400\)
   (c) \(2X_1 + 3X_2 + 1X_3 \leq 400\)
   (d) \(2X_1 + 3X_2 + 1X_3 \geq 400\)

4. The most types of lamps that would be included in the optimal solution would be
   (a) 2
   (b) 1
   (c) 3
   (d) 0

5. **CPA Exam** In a system of equations for a linear programming model, what can be done to equalize an inequality such as \(3X + 2Y \leq 15\)?
   (a) Nothing.
   (b) Add a slack variable.
   (c) Add a tableau.
   (d) Multiply each element by \(-1\).

Use the following information to answer Problems 6 and 7.

**CPA Exam** The Golden Hawk Manufacturing Company wants to maximize the profits on products A, B, and C. The contribution margin for each product follows:

<table>
<thead>
<tr>
<th>Product</th>
<th>Contribution Margin</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$2</td>
</tr>
<tr>
<td>B</td>
<td>$5</td>
</tr>
<tr>
<td>C</td>
<td>$4</td>
</tr>
</tbody>
</table>
The production requirements and departmental capacities, by departments, are as follows:

<table>
<thead>
<tr>
<th>Department</th>
<th>Production Requirements by Product (Hours)</th>
<th>Department Capacity (Total Hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Assembling</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Painting</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Finishing</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

6. What is the profit maximization formula for the Golden Hawk Manufacturing Company?
   (a) $2A + 5B + 4C = X \text{ (where } X = \text{ Profit)}$
   (b) $5A + 8B + 5C \leq 96,000$
   (c) $2A + 5B + 4C \leq X \text{ (where } X = \text{ Profit)}$
   (d) $2A + 5B + 4C = 96,000$

7. What is the constraint for the painting department of the Golden Hawk Manufacturing Company?
   (a) $1A + 2B + 2C \geq 38,000$
   (b) $2A + 5B + 4C \geq 38,000$
   (c) $1A + 2B + 2C \geq 38,000$
   (d) $2A + 3B + 2C \geq 30,000$

<table>
<thead>
<tr>
<th>Product</th>
<th>Profit per Unit</th>
<th>Hours Required per Unit per Department</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Machining</td>
</tr>
<tr>
<td>A</td>
<td>$10</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>$20</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>$30</td>
<td>2</td>
</tr>
<tr>
<td><strong>Total Hours per Day per Department</strong></td>
<td></td>
<td>16</td>
</tr>
</tbody>
</table>

8. CPA Exam  Watch Corporation manufactures products A, B, and C. The daily production requirements are shown above. What is Watch’s objective function in determining the daily production of each unit?
   (a) $A + B + C \leq 60$
   (b) $3A + 6B + 7C = 60$
   (c) $A + B + C \leq \text{Profit}$
   (d) $10A + 20B + 30C = \text{Profit}$

**CPA Exam** Problems 9–11 are based on a company that uses a linear programming model to schedule the production of three products. The per-unit selling prices, variable costs, and labor time required to produce these products are presented below. Total labor time available is 200 hours.

<table>
<thead>
<tr>
<th>Product</th>
<th>Selling Price</th>
<th>Variable Cost</th>
<th>Labor (Hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$4.00</td>
<td>$1.00</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>$2.00</td>
<td>$.50</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>$3.50</td>
<td>$1.50</td>
<td>3</td>
</tr>
</tbody>
</table>
9. The objective function to maximize the company's gross profit (Z) is
   (a) $4A + 2B + 3.5C = Z$
   (b) $2A + 2B + 3C = Z$
   (c) $5A + 2.5B + 5C = Z$
   (d) $3A + 1.5B + 2C = Z$
   (e) $A + B + C = Z$

10. The constraint of labor time available is represented by
    (a) $2A + 2B + 3C \leq 200$
    (b) $2A + 2B + 3C \geq 200$
    (c) $A + B + C \geq 200$
    (d) $4A + 2B + 3.5C = 200$
    (e) $A/2 + B/2 + C/3 = 200$

11. A linear programming model produces an optimal solution by
    (a) Ignoring resource constraints.
    (b) Minimizing production costs.
    (c) Minimizing both variable production costs and labor costs.
    (d) Maximizing the objective function subject to resource constraints.
    (e) Finding the point at which various resource constraints intersect.

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Finance 6

You are ready to get a car, but there are a number of factors to consider. Should you purchase or lease? Do you want a new car or will a used car do as well? Are you looking for a hybrid vehicle or an SUV? What extra accessories do you want or need? And how will you pay for the car? How much will be required for a down payment? What will the monthly payments be? The Chapter Project at the end of this chapter investigates various financing options for purchasing or leasing a car.

A Look Back, A Look Forward

In earlier mathematics courses, you studied percents. In later courses, you studied exponential expressions and logarithmic expressions, which are reviewed in Appendix A, Section A.3. In this chapter we apply these topics to the field of finance, discussing simple interest, compound interest, simple annuities, sinking funds, and amortization.
**Interest** is money paid for the use of money. The total amount of money borrowed, whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account, is called the **principal**. The **rate of interest** is the amount charged for the use of the principal for a given length of time, usually on a yearly, or *per annum*, basis.

**1. Solve Problems Involving Percents**

Rates of interest are usually expressed as percents: 10% per annum, 14% per annum, $\frac{7}{2}\%$ per annum, and so on. The word **percent** means “per hundred.” The familiar symbol for percent means to divide by one hundred. For example,

$1\% = \frac{1}{100} = 0.01$  
$12\% = \frac{12}{100} = 0.12$  
$0.3\% = \frac{0.3}{100} = \frac{3}{1000} = 0.003$

By reversing these ideas, decimals can be written as percents.

$0.35 = \frac{35}{100} = 35\%$  
$1.25 = \frac{125}{100} = 125\%$  
$0.005 = \frac{5}{1000} = \frac{0.5}{100} = 0.5\% = \frac{1}{2}\%$

*NOW WORK PROBLEMS 5 AND 13.*

**EXAMPLE 1 Working with Percents**

(a) Find 12% of 80.  
(b) What percent of 40 is 18?  
(c) 8 is 15% of what number?

**SOLUTION** (a) The English word *of* translates to *multiply* when percents are involved. For example,

$12\%$ of $80 = 12\% \times 80 = (0.12) \cdot (80) = 9.6$

(b) Let $x$ represent the unknown percent. Then

$x\% \cdot 40 = 18$  
$x \cdot \frac{40}{100} = 18$  
$x \cdot \frac{40}{100} = 18$  
$x = \frac{1800}{40}$  
$x = 45$

So, 45% of 40 is 18.

*Optional*
(c) Let \( x \) represent the number. Then

\[
8 = 15\% \text{ of } x
\]

\[
8 = 0.15x
\]

15\% = 0.15

\[
x = \frac{8}{0.15}
\]

Divide both sides by 0.15.

\[
x = 53.33
\]

Use a calculator.

So, 8 is 15\% of 53.33.

**NOW WORK PROBLEMS 21, 27, AND 31.**

---

**EXAMPLE 2 Computing State Income Tax**

A resident of Illinois has base income, after adjustment for deductions, of $18,000. The state income tax on this base income is 3\%. What tax is due?

**SOLUTION** We need to find 3\% of $18,000. Convert 3\% to its decimal equivalent and then multiply by $18,000.

\[
3\% \text{ of } 18,000 = (0.03)(18,000) = 540
\]

The state income tax due is $540.00.

---

**2 Solve Problems Involving Simple Interest**

**Definition**

**Simple Interest**

Simple interest is interest computed on the principal for the entire period it is borrowed.

**Theorem**

**Simple Interest Formula**

If a principal \( P \) is borrowed at a simple interest rate of \( R\% \) per annum (where \( R\% \) is expressed as the decimal \( r = \frac{R}{100} \)) for a period of \( t \) years, the interest \( I \) is

\[
I = Prt
\]

(1)
Theorem

The amount \( A \) owed at the end of \( t \) years is the sum of the principal \( P \) borrowed and the interest \( I \) charged. That is,

\[
A = P + I = P + Prt = P(1 + rt) \tag{2}
\]

EXAMPLE 3 Computing Interest and the Amount Due on a Loan

A loan of $250 is made for 9 months at a simple interest rate of 10% per annum. What is the interest charge? What amount is due after 9 months?

SOLUTION

The actual period the money is borrowed for is 9 months, which is \( \frac{9}{12} = \frac{3}{4} \) of a year.

The interest charge is the product of the amount borrowed, $250; the annual rate of interest, 10%, expressed as a decimal, 0.10; and the length of time in years, \( \frac{3}{4} \). Use Formula (1) to find the interest charge \( I \).

\[
I = Prt
\]

Interest charge = \((250)(0.10)\left(\frac{3}{4}\right) = 18.75
\]

Using Formula (2), the amount \( A \) due after 9 months is

\[
A = P + I = 250 + 18.75 = 268.75
\]

NOW WORK PROBLEM 35.

EXAMPLE 4 Computing the Rate of Interest on a Loan

A person borrows $1000 for a period of 6 months. What simple interest rate is being charged if the amount \( A \) that must be repaid after 6 months is $1045?

SOLUTION

The principal \( P \) is $1000, the period is 6 months so \( t = \frac{1}{2} \) year, and the amount \( A \) owed after 6 months is $1045. We substitute the known values of \( A, P, \) and \( t \) in Formula (2), and then solve for \( r \), the interest rate.

\[
A = P + Prt \tag{2}
\]

\[
1045 = 1000 + 1000r\left(\frac{1}{2}\right)
\]

\[
45 = 500r
\]

\[
r = \frac{45}{500} = 0.09
\]

The per annum rate of interest is 9%.

NOW WORK PROBLEM 41.
EXAMPLE 5 Computing the Amount Due on a Loan

A company borrows $1,000,000 for 1 month at a simple interest rate of 9% per annum. How much must the company pay back at the end of 1 month?

SOLUTION The principal \( P \) is $1,000,000, the period \( t \) is 1 month = \( \frac{1}{12} \) year, and the interest rate \( r \) is 0.09. The amount \( A \) that must be paid back is

\[
A = P(1 + rt)
\]

Formula (2)

\[
= 1,000,000 \left[ 1 + 0.09 \left( \frac{1}{12} \right) \right]
\]

\[
= 1,000,000(1.0075)
\]

\[
= $1,007,500
\]

At the end of 1 month, the company must pay back $1,007,500.

EXAMPLE 6 Computing the Proceeds of a Discounted Loan

A borrower signs a note for a discounted loan and agrees to pay the lender $1000 in 9 months at a 10% rate of interest. How much does this borrower receive?

SOLUTION The amount of the loan is \( L = $1000 \). The rate of interest is \( r = 10\% = 0.10 \). The time is \( t = 9 \) months = \( \frac{9}{12} \) year. The discount is

\[
Lrt = $1000(0.10) \left( \frac{9}{12} \right) = $75
\]

The discount is deducted from the loan amount of $1000, so that the proceeds, the amount the borrower receives, is

\[
R = L - Lrt = 1000 - 75 = $925
\]

NOW WORK PROBLEM 47.
EXAMPLE 7  Computing the Simple Rate of Interest on a Discounted Loan

What simple rate of interest is the borrower in Example 6 paying on the $925 that was borrowed for 9 months and paid back in the amount of $1000?

SOLUTION  The principal $P$ is $925$, $t$ is $\frac{3}{4}$ of a year, and the amount $A$ is $1000$. If $r$ is the simple rate of interest, then, from Formula (2),

$$A = P + Prt$$

Simplify.

Solve for $r$.

The equivalent simple rate of interest for this loan is $10.81\%$.

EXAMPLE 8  Finding the Amount of a Discounted Loan

You wish to borrow $10,000 for 3 months. If the person you are borrowing from offers a discounted loan at 8%, how much must you repay at the end of 3 months?

SOLUTION  The amount you receive, the proceeds, is $R = 10,000$; the interest rate is $r = 0.08$; and the time $t$ is 3 months $= \frac{1}{4}$ year. From Formula (3), the amount $L$ you repay obeys the equation

$$R = L(1 - rt)$$

Simplify.

Solve for $L$.

You will repay $10,204.08$ for this loan.

NOW WORK PROBLEM 51.

4  Application: Pricing Treasury Bills

Treasury bills (T-bills) are short-term securities issued by the Federal Reserve. The bills do not specify a rate of interest. They are sold at public auction with financial institutions making competitive bids. For example, a financial institution may bid $982,400 for a 3-month $1 million treasury bill. At the end of 3 months the financial institution receives $1 million, which includes the interest earned and the cost of the T-bill. This bidding process is an example of a discounted loan.

EXAMPLE 9  Bidding on Treasury Bills

How much should a bank bid on a 6-month, $500,000 treasury bill if it wants a 0.25% discounted rate of interest? *

* Based on actual rates as of March 2010.
6.1 Interest

SOLUTION The maturity value, the amount to be repaid to the bank by the government, is \( L = \$500,000 \). The rate of interest is \( r = 0.25\% = 0.0025 \). The time is \( t = 6 \) months = \( \frac{1}{2} \) year. The proceeds \( R \) to the government are

\[
R = L(1 - rt) = \$500,000 \left[ 1 - 0.0025 \left( \frac{1}{2} \right) \right] = \$500,000(0.99875) = 499,375
\]

The bank should bid \$499,375 if it wants to earn 0.0025%.

NOW WORK PROBLEM 67.

EXERCISE 6.1 Answers Begin on Page AN–32.

Concepts and Vocabulary

1. If a principal \( P \) is borrowed at a simple interest rate \( r \) (expressed as a decimal) for a period of \( t \) years, the interest charge \( I \) is _________.

2. If a lender deducts the interest from the amount of the loan at the time the loan is made, the loan is said to be _________.

Skill Building

In Problems 5–12, write each decimal as a percent.

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.60</td>
<td>60%</td>
</tr>
<tr>
<td>0.40</td>
<td>40%</td>
</tr>
<tr>
<td>1.1</td>
<td>110%</td>
</tr>
<tr>
<td>1.2</td>
<td>120%</td>
</tr>
<tr>
<td>0.06</td>
<td>6%</td>
</tr>
<tr>
<td>0.07</td>
<td>7%</td>
</tr>
<tr>
<td>0.0025</td>
<td>0.25%</td>
</tr>
<tr>
<td>0.0015</td>
<td>0.15%</td>
</tr>
</tbody>
</table>

In Problems 13–20, write each percent as a decimal.

<table>
<thead>
<tr>
<th>Percent</th>
<th>Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>25%</td>
<td>0.25</td>
</tr>
<tr>
<td>15%</td>
<td>0.15</td>
</tr>
<tr>
<td>100%</td>
<td>1.00</td>
</tr>
<tr>
<td>300%</td>
<td>3.00</td>
</tr>
<tr>
<td>6.5%</td>
<td>0.065</td>
</tr>
<tr>
<td>4.3%</td>
<td>0.043</td>
</tr>
<tr>
<td>0.05%</td>
<td>0.0005</td>
</tr>
<tr>
<td>0.012%</td>
<td>0.00012</td>
</tr>
</tbody>
</table>

In Problems 21–34, calculate the indicated quantity.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>15% of 1000</td>
<td>( 0.15 \times 1000 )</td>
</tr>
<tr>
<td>10% of 50</td>
<td>( 0.1 \times 50 )</td>
</tr>
<tr>
<td>What percent of 80 is 4?</td>
<td>( \frac{4}{80} \times 100 )</td>
</tr>
<tr>
<td>What percent of 60 is 5?</td>
<td>( \frac{5}{60} \times 100 )</td>
</tr>
<tr>
<td>What percent of 25 is 45?</td>
<td>( \frac{45}{25} \times 100 )</td>
</tr>
<tr>
<td>20 is 8% of what number?</td>
<td>( \frac{20}{0.08} )</td>
</tr>
<tr>
<td>50 is 15% of what number?</td>
<td>( \frac{50}{0.15} )</td>
</tr>
</tbody>
</table>

In Problems 35–40, find the interest due on each loan.

<table>
<thead>
<tr>
<th>Loan</th>
<th>Interest Due</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1000 borrowed for 3 months at 4% simple interest</td>
<td>( \frac{4}{100} \times 1000 \times \frac{3}{12} )</td>
</tr>
<tr>
<td>$500 borrowed for 9 months at 12% simple interest</td>
<td>( \frac{12}{100} \times 500 \times \frac{9}{12} )</td>
</tr>
<tr>
<td>$1000 borrowed for 18 months at 10% simple interest</td>
<td>( \frac{10}{100} \times 1000 \times \frac{18}{12} )</td>
</tr>
<tr>
<td>$100 borrowed for 6 months at 8% simple interest</td>
<td>( \frac{8}{100} \times 100 \times \frac{6}{12} )</td>
</tr>
<tr>
<td>$800 borrowed for 8 months at 5% simple interest</td>
<td>( \frac{5}{100} \times 800 \times \frac{8}{12} )</td>
</tr>
<tr>
<td>$100 borrowed for 24 months at 12% simple interest</td>
<td>( \frac{12}{100} \times 100 \times \frac{24}{12} )</td>
</tr>
</tbody>
</table>

In Problems 41–46, find the simple interest rate for each loan.

<table>
<thead>
<tr>
<th>Loan</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1000 borrowed; the amount owed after 6 months is $1050</td>
<td>( \frac{1050 - 1000}{1000} \times \frac{12}{6} )</td>
</tr>
<tr>
<td>$300 borrowed; the amount owed after 12 months is $400</td>
<td>( \frac{400 - 300}{300} \times \frac{12}{12} )</td>
</tr>
<tr>
<td>$900 borrowed; the amount owed after 10 months is $1000</td>
<td>( \frac{1000 - 900}{900} \times \frac{12}{10} )</td>
</tr>
<tr>
<td>$500 borrowed; the amount owed after 8 months is $600</td>
<td>( \frac{600 - 500}{500} \times \frac{12}{8} )</td>
</tr>
<tr>
<td>$600 borrowed; the amount owed after 9 months is $660</td>
<td>( \frac{660 - 600}{600} \times \frac{12}{9} )</td>
</tr>
<tr>
<td>$800 borrowed; the amount owed after 3 months is $900</td>
<td>( \frac{900 - 800}{800} \times \frac{12}{3} )</td>
</tr>
</tbody>
</table>
In Problems 47–50, find the proceeds for each discounted loan.

47. $1200 repaid in 6 months at 10%.
48. $500 repaid in 8 months at 9%.
49. $2000 repaid in 24 months at 8%.
50. $1500 repaid in 18 months at 10%.

In Problems 51–54, find the amount you must repay for each discounted loan. What is the equivalent simple interest rate for this loan?

51. The proceeds are $1200 for 6 months at 10%.
52. The proceeds are $500 for 8 months at 9%.
53. The proceeds are $2000 for 24 months at 8%.
54. The proceeds are $1500 for 18 months at 10%.

Applications

55. Buying a Stereo Madalyn wants to buy a $500 stereo set in 9 months. How much should she invest now at 3% simple interest to have the money then?

56. Interest on a Loan Mike borrows $10,000 for a period of 3 years at a simple interest rate of 10%. Determine the interest due on the loan.

57. Discounted Sheets A set of Egyptian cotton sheets has a list price of $130.00 but is on sale for $84.50. By what percent is the list price reduced?

58. Income Tax A resident of Missouri has base income, after exemptions and adjustments, of $24,000. The state income tax on this base income is $315 plus 6% of the excess over $9000. What tax is due?

59. Home Equity Loan Sarah Jane needs to borrow $5000 to repair her roof. She can get a discounted loan from her bank at an 8.99% rate of interest for 18 months. What loan amount should be used so that she will receive the $5000 she needs?

60. Equipment Loan The owner of a restaurant needs to borrow $12,000 from a bank to buy some equipment. The bank will give the owner a discounted loan at an 11% rate of interest for 9 months. What loan amount should be used so that the owner will receive $12,000?

61. Payday Loan Jake’s car needs repairs but he is short of cash. A payday loan company charges a fee of $46.55 for a two-week loan of $250. This is equivalent to a discounted loan with proceeds of $250 and a loan amount of $296.55. What per annum simple rate of interest is being charged?

Source: mycashnow.com

62. Home Equity Loan Fred’s credit union offers a simple interest home equity loan that uses a daily rate. For such loans, interest is computed using the following formula:

\[(\text{Loan balance}) \times (\text{daily rate}) \times (\text{days between payments})\]

If Fred borrows $15,000 at a daily periodic rate of 0.000144 and he pays the loan off in one single payment of $18,153.60, how long was the term of the loan?

Source: University Credit Union

63. Comparing Loans You need to borrow $1000 right now but can repay the loan in 6 months. Since you want to pay as little interest as possible, which type of loan should you take: a discounted loan at 9% per annum or a simple interest loan at 10% per annum?
64. **Comparing Loans** You need to borrow $5000 right now but can repay the loan in 9 months. Since you want to pay as little interest as possible, which type of loan should you take: a discounted loan at 8% per annum or a simple interest loan at 8.5% per annum?

65. **Used Car Loan** In April 2010, the interest rate on 36-month used car loans was 5.74%. Suppose you have a contract to buy a used car for $8000.  
(a) If a bank will lend you $8000 at a 5.74% simple rate of interest for 36 months, how much interest will be charged?  
(b) What is the total amount of the loan?  
(c) What is the monthly payment due?  
**Source:** Nationwide Bank

Problems 66–70 involve treasury bills,

66. **Bidding on Treasury Bills** How much should a bank bid on a 6-month $3 million treasury bill to earn a 0.23% discounted rate of interest?

67. **Bidding on Treasury Bills** In May 2010, a bank bid $3,996,800 on a 3-month $4 million treasury bill. What was the discounted rate of interest?  
**Source:** United States Treasury

68. **T-Bill Auctions** As the result of a treasury bill auction, 1-month T-bills were issued in April 2010, at a price of $999.87 per $1000 face value. How much interest would be received on the maturity date for an investment of $10,000? What is the per annum simple interest rate for this investment, rounded to the nearest thousandth of a percent?  
**Source:** United States Treasury

69. **T-Bill Auction** As the result of a treasury bill auction, 3-month T-bills were issued in February 2009 at a price of $993.25 per $1000 face value. How much interest would be received on the maturity date for an investment of $100,000? What is the per annum simple interest rate for this investment, rounded to the nearest thousandth of a percent?  
**Source:** United States Treasury

70. **T-Bill Auction** As the result of a treasury bill auction, 1-month T-bills were issued in January 2008 at a price of $997.48 per $1000 face value. How much interest would be received on the maturity date for an investment of $5,000? What is the per annum simple interest rate for this investment, rounded to the nearest thousandth of a percent?  
**Source:** United States Treasury

**Discussion and Writing**

71. If a store marks up an item by 10% and later discounts it by 10%, is the new price the same as the original price? Give an example to support your answer.

72. If a stock drops in price by 10% and then improves its price by 10%, is the resulting price the same as the original price? Give an example to support your answer.

73. When a store gives a 40% discount off the selling price and then gives another 10% off, is that the same as a 50% discount? Give an example to support your answer.

74. Use the results of Problems 68–70 to discuss the trend in short-term interest rates. Compare that trend to current short-term rates and those of 6 months ago.

### 6.2 Compound Interest

**PREPARING FOR THIS SECTION** Before getting started, review the following:

- Exponents and Logarithms (Appendix A, Section A.3, pp. A–28 to A–33)

**NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 311.**

**OBJECTIVES**

1. Determine the future value of a lump sum of money (p. 302)
2. Find the effective rate of interest (p. 307)
3. Determine the present value of a lump sum of money (p. 308)
4. Determine the rate of interest required to double a lump sum of money (p. 310)
5. Determine the time required to double a lump sum of money (p. 310)
Determine the Future Value of a Lump Sum of Money

In working with problems involving interest, we use the term payment period as follows:

<table>
<thead>
<tr>
<th>Payment Period</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>Once per year</td>
</tr>
<tr>
<td>Semiannually</td>
<td>Twice per year</td>
</tr>
<tr>
<td>Quarterly</td>
<td>4 times per year</td>
</tr>
<tr>
<td>Monthly</td>
<td>12 times per year</td>
</tr>
<tr>
<td>Daily</td>
<td>365 times per year</td>
</tr>
</tbody>
</table>

If the interest due at the end of each payment period is added to the principal, so that the interest computed for the next payment period is based on this new amount of the old principal plus interest, then the interest is said to have been compounded. That is, compound interest is interest paid on the initial principal and previously earned interest.

**EXAMPLE 1 Computing Compound Interest**

A bank pays interest of 4% per annum compounded quarterly. If $200 is placed in a savings account and the quarterly interest is left in the account, how much money is in the account after 1 year?

**SOLUTION**

At the end of the first quarter (3 months), the interest earned is found using the Simple Interest formula:

\[
I = Prt = (200)(0.04)\left(\frac{1}{4}\right) = $2.00
\]

The new principal is \( P + I = 200 + 2 = 202 \). The interest on this principal at the end of the second quarter is

\[
I = (202)(0.04)\left(\frac{1}{4}\right) = $2.02
\]

The interest at the end of the third quarter on the principal of \( 202 + 2.02 = 204.02 \) is

\[
I = (204.02)(0.04)\left(\frac{1}{4}\right) = $2.04
\]

The interest at the end of the fourth quarter on the principal of \( 204.02 + 2.04 = 206.06 \) is

\[
I = (206.06)(0.04)\left(\frac{1}{4}\right) = $2.06
\]

After 1 year, the total in the savings account is \( 206.06 + 2.06 = 208.12 \). These results are shown in Figure 1.

The pattern of the calculations performed in Example 1 leads to a general formula for compound interest. To fix our ideas, let \( P \) represent the principal to be invested at a per annum interest rate \( r \) that is compounded \( n \) times per year, so the time of each

\* Some banks use 360 times per year.
compounding period is \( \frac{1}{n} \) years. (For computing purposes, \( r \) is expressed as a decimal.)

The interest earned after each compounding period is given by the Simple Interest Formula, Formula (1) on page 295.

\[
\text{Interest} = \text{principal} \times \text{rate} \times \text{time} = P \cdot r \cdot \frac{1}{n} = P \left( \frac{r}{n} \right)
\]

The amount \( A \) after one compounding period is

\[
A = P + P \left( \frac{r}{n} \right) = P \left( 1 + \frac{r}{n} \right)
\]

After two compounding periods, the amount \( A \), based on the new principal \( P \left( 1 + \frac{r}{n} \right) \), is

\[
A = P \left( 1 + \frac{r}{n} \right) + P \left( 1 + \frac{r}{n} \right) \left( \frac{r}{n} \right) = P \left( 1 + \frac{r}{n} \right) \left( 1 + \frac{r}{n} \right) = P \left( 1 + \frac{r}{n} \right)^2
\]

After three compounding periods, the amount \( A \) is

\[
A = P \left( 1 + \frac{r}{n} \right)^2 + \left[ P \left( 1 + \frac{r}{n} \right) \left( \frac{r}{n} \right) \right] = P \left( 1 + \frac{r}{n} \right)^2 \left( 1 + \frac{r}{n} \right) = P \left( 1 + \frac{r}{n} \right)^3
\]

Continuing this way, after \( n \) compounding periods (1 year), the amount \( A \) is

\[
A = P \left( 1 + \frac{r}{n} \right)^n
\]

Because \( t \) years will contain \( n \cdot t \) compounding periods, after \( t \) years the amount \( A \) is

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt}
\]

**Theorem: Compound Interest Formula**

The amount \( A \) after \( t \) years due to a principal \( P \) invested at an annual interest rate \( r \) compounded \( n \) times per year is

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt} \quad (1)
\]

For example, to rework Example 1, use \( P = 200, r = 0.04, n = 4 \) (quarterly compounding), and \( t = 1 \) year to obtain

\[
A = 200 \left( 1 + \frac{0.04}{4} \right)^4 = 208.12
\]

In the Compound Interest Formula, the amount \( A \) is typically referred to as the **future value** of the account, while \( P \) is called the **present value**.
EXAMPLE 2  Working with the Compound Interest Formula

If $1000 is borrowed at an annual rate of interest of 10% and no payments are made on this loan, what is the amount after 5 years if the compounding takes place
(a) Annually?  (b) Monthly?  (c) Daily?

How much interest is due in each case?

SOLUTION  Use the Compound Interest Formula. The principal is $P = 1000$, the annual rate of interest is $r = 0.10$, and $t = 5$ years.

(a) For annual compounding, $n = 1$. The amount $A$ is

$$A = P \left(1 + \frac{r}{n}\right)^{nt} = (1000)(1 + 0.10)^5 = (1000)(1.61051) = 1610.51$$

The interest due is

$$A - P = 1610.51 - 1000.00 = 610.51$$

(b) For monthly compounding, $n = 12$. The amount $A$ is

$$A = P \left(1 + \frac{r}{n}\right)^{nt} = (1000)\left(1 + \frac{0.10}{12}\right)^{60} = (1000)(1.64531) = 1645.31$$

The interest due is

$$A - P = 1645.31 - 1000.00 = 645.31$$

(c) For daily compounding, $n = 365$. The amount $A$ is

$$A = P \left(1 + \frac{r}{n}\right)^{nt} = (1000)\left(1 + \frac{0.10}{365}\right)^{1825} = (1000)(1.64861) = 1648.61$$

The interest due is

$$A - P = 1648.61 - 1000.00 = 648.61$$

The results of Example 2 are summarized in Table 1.

<table>
<thead>
<tr>
<th>Per Annum Rate</th>
<th>Compounding Method</th>
<th>Interest Rate per Payment Period</th>
<th>Initial Principal</th>
<th>Amount after 5 Years</th>
<th>Interest Due</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>Annual</td>
<td>0.10</td>
<td>$1000</td>
<td>$1610.51</td>
<td>$610.51</td>
</tr>
<tr>
<td>10%</td>
<td>Monthly</td>
<td>$\frac{0.10}{12} = 0.00833$</td>
<td>$1000</td>
<td>$1645.31</td>
<td>$645.31</td>
</tr>
<tr>
<td>10%</td>
<td>Daily</td>
<td>$\frac{0.10}{365} = 0.000274$</td>
<td>$1000</td>
<td>$1648.61</td>
<td>$648.61</td>
</tr>
</tbody>
</table>

NOW WORK PROBLEM 9.
EXAMPLE 3 Comparing Investments Using Different Compounding Periods

Investing $10,000 at an annual rate of 4% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual compounding \( (n = 1) \):
\[ A = P \cdot (1 + r) \]
\[ = ($10,000)(1 + 0.04) = $10,400.00 \]

Semiannual compounding \( (n = 2) \):
\[ A = P \cdot \left(1 + \frac{r}{2}\right)^2 \]
\[ = ($10,000)\left(1 + \frac{0.04}{2}\right)^2 = $10,404.00 \]

Quarterly compounding \( (n = 4) \):
\[ A = P \cdot \left(1 + \frac{r}{4}\right)^4 \]
\[ = ($10,000)\left(1 + \frac{0.04}{4}\right)^4 = $10,406.04 \]

Monthly compounding \( (n = 12) \):
\[ A = P \cdot \left(1 + \frac{r}{12}\right)^{12} \]
\[ = ($10,000)\left(1 + \frac{0.04}{12}\right)^{12} = $10,407.42 \]

Daily compounding \( (n = 365) \):
\[ A = P \cdot \left(1 + \frac{r}{365}\right)^{365} \]
\[ = ($10,000)\left(1 + \frac{0.04}{365}\right)^{365} = $10,408.08 \]

From Example 3, notice that the effect of compounding more frequently is that the amount after 1 year is higher: $10,000 compounded 2 times a year at 4% results in $10,404; $10,000 compounded 12 times a year at 4% results in $10,407.42; and $10,000 compounded 365 times a year at 4% results in $10,408.08. This leads to the following question: What would happen to the amount after 1 year if the number of times that the interest is compounded were increased without bound?

Let’s find the answer. Suppose that \( P \) is the principal, \( r \) is the per annum interest rate, and \( n \) is the number of times that the interest is compounded each year. The amount after 1 year is

\[ A = P \cdot \left(1 + \frac{r}{n}\right)^n \]

Rewrite this expression as follows:

\[ A = P \cdot \left(1 + \frac{r}{n}\right)^n = P \cdot \left(1 + \frac{1}{\frac{n}{r}}\right)^n = P \cdot \left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^r = P \cdot \left[\left(1 + \frac{1}{n}\right)^{\frac{n}{r}}\right]^r \]

\[ \frac{n}{r} = \frac{1}{n} \]
\[ n = \frac{n}{r} \cdot r \]
\[ h = \frac{n}{r} \]
Now suppose that the number \( n \) of times that the interest is compounded per year gets larger and larger. This is equivalent to \( h = \frac{n}{r} \) getting larger and larger. Table 2 illustrates what happens to the expression \( \left(1 + \frac{1}{h}\right)^h\) as \( h \) takes on increasingly larger values; notice it is getting closer to a particular number, which we designate as \( e \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \frac{1}{h} )</th>
<th>( 1 + \frac{1}{h} )</th>
<th>( \left(1 + \frac{1}{h}\right)^h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td>1.2</td>
<td>2.48832</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>1.1</td>
<td>2.59374246</td>
</tr>
<tr>
<td>100</td>
<td>0.01</td>
<td>1.01</td>
<td>2.704813829</td>
</tr>
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<td>1.001</td>
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<td>1.0001</td>
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<td>100,000</td>
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<td>1,000,000</td>
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</tr>
<tr>
<td>1,000,000,000</td>
<td>10^{-9}</td>
<td>1 + 10^{-9}</td>
<td>2.718281827</td>
</tr>
</tbody>
</table>

Definition

The number \( e \) is defined as the number that the expression

\[
\left(1 + \frac{1}{h}\right)^h
\]

approaches as \( h \) gets larger and larger.

Look again at Table 2. The bottom number in the right column is the number \( e \) correct to nine decimal places and is the same as the entry given for \( e \) on your calculator (if expressed correctly to nine decimal places).* As it turns out, the number \( e \) is an irrational number, so its decimal equivalent is a nonrepeating, nonterminating decimal.

Now return to Equation (2). As the number \( n \) of compounding periods increases without bound, then \( h = \frac{n}{r} \) will also increase without bound, and the expression in brackets on the far right of Equation (2) will equal the number \( e \). The larger \( n \) gets, the closer \( \left(1 + \frac{1}{h}\right)^h \) gets to \( e^r \). In other words, no matter how frequent the compounding, the amount after 1 year has the definite ceiling \( Pe^r \).

Definition

When interest at a per annum rate \( r \) is compounded on a principal \( P \) so that the amount after 1 year is \( Pe^r \), we say the interest is **compounded continuously**.

* Most calculators have the key \( e^x \) or \( \exp(x) \), which may be used to evaluate the expression \( e^x \) for a given value of \( x \). Consult your owner’s manual.
Theorem Continuous Compounding

The amount \( A \) after \( t \) years due to a principal \( P \) invested at an annual interest rate \( r \) compounded continuously is

\[
A = Pe^{rt}
\]  

(4)

EXAMPLE 4 Using Continuous Compounding

The amount \( A \) that results from investing a principal \( P \) of $10,000 at an annual rate \( r \) of 4\% compounded continuously for a time \( t \) of 1 year is

\[
A = Pe^{rt} = 10,000e^{0.04} = (10,000)(1.040811) = 10,408.11
\]

NOW WORK PROBLEM 17.

2 Find the Effective Rate of Interest

The effective rate of interest is the equivalent annual simple rate of interest that would yield the same amount as compounding after 1 year.

For example, based on Example 4, a principal of $10,000 will result in $10,408.11 at a rate of 4\% compounded continuously. To get this same amount using a simple rate of interest would require that interest of $408.11 be earned on the principal. Since $408.11 is 4.0811\% of $10,000, a simple rate of interest of 4.0811\% is needed to equal 4\% compounded continuously. The effective rate of interest of 4\% compounded continuously is 4.0811\%.

Based on the results of Examples 3 and 4, we find the following comparisons:

<table>
<thead>
<tr>
<th></th>
<th>Annual Rate</th>
<th>Effective Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual compounding</td>
<td>4%</td>
<td>4%</td>
</tr>
<tr>
<td>Semiannual compounding</td>
<td>4%</td>
<td>4.04%</td>
</tr>
<tr>
<td>Quarterly compounding</td>
<td>4%</td>
<td>4.0604%</td>
</tr>
<tr>
<td>Monthly compounding</td>
<td>4%</td>
<td>4.0742%</td>
</tr>
<tr>
<td>Daily compounding</td>
<td>4%</td>
<td>4.0808%</td>
</tr>
<tr>
<td>Continuous compounding</td>
<td>4%</td>
<td>4.0811%</td>
</tr>
</tbody>
</table>

EXAMPLE 5 Computing the Effective Rate of Interest

On January 1, 2010, $2000 is placed in an Individual Retirement Account (IRA) that will pay interest of 3\% per annum compounded continuously.

(a) What will the IRA be worth on January 1, 2030?

(b) What is the effective annual rate of interest?

SOLUTION (a) On January 1, 2030, the initial principal of $2000 will have earned interest of 3\% compounded continuously for 20 years. The amount \( A \) after 20 years is

\[
A = Pe^{rt} = 2000e^{(0.03)(20)} = 3644.24
\]
First, compute the interest earned on $2000 at compounded continuously after 1 year.

\[ A = 2000e^{0.03(1)} = 2060.91 \]

So the interest earned after 1 year is $2060.91 - $2000.00 = $60.91. To find the effective rate of interest \( R \), use the simple interest formula \( I = Prt \), with \( I = 60.91 \), \( P = 2000 \), and \( t = 1 \).

\[ \frac{60.91}{2000} = 0.0305 \quad \text{Solve for } R. \]

The effective annual rate of interest \( R \) is 3.05%.

**NOW WORK PROBLEM 33.**

### 3 Determine the Present Value of a Lump Sum of Money

When people engaged in finance speak of the “time value of money,” they are usually referring to the present value of money. The present value of \( A \) dollars to be received at a future date is the principal that you would need to invest now so that it will grow to \( A \) dollars in the specified time period. The present value of money to be received at a future date is always less than the amount to be received, since the amount to be received will equal the present value (money invested now) plus the interest accrued over the time period.

We use the Compound Interest Formula to get a formula for present value. If \( P \) is the present value of \( A \) dollars to be received after \( t \) years at a per annum interest rate \( r \) compounded \( n \) times per year, then

\[ A = P \left( 1 + \frac{r}{n} \right)^{nt} \quad \text{Compound Interest Formula} \]

To solve for \( P \), divide both sides by \( \left( 1 + \frac{r}{n} \right)^{nt} \). The result is

\[ \frac{A}{\left( 1 + \frac{r}{n} \right)^{nt}} = P \quad \text{or} \quad P = A \left( 1 + \frac{r}{n} \right)^{-nt} \]

**Theorem**

**Present Value Formulas**

The present value \( P \) of \( A \) dollars to be received after \( t \) years, assuming a per annum interest rate \( r \) compounded \( n \) times per year, is

\[ P = A \left( 1 + \frac{r}{n} \right)^{-nt} \quad (5) \]

If the interest is compounded continuously,

\[ P = Ae^{-rt} \quad (6) \]

To prove (6), solve Formula (4) for \( P \).
**EXAMPLE 6** Computing the Present Value of $10,000

How much money should be invested now at 4% per annum so that after 2 years the amount will be $10,000 when the interest is compounded

(a) Annually? (b) Monthly? (c) Daily? (d) Continuously?

**SOLUTION** In this problem we want to find the principal $P$ needed now to get the amount $A$ after $t$ years when the interest rate is $r$. That is, we want to find the present value $P$ of $10,000.

(a) Since compounding is once per year, use Formula (5) with $n = 1$. The present value $P$ of $10,000$ is

$$P = A \left(1 + \frac{r}{n}\right)^{-nt} = 10,000 \left(1 + 0.04\right)^{-2} = 10,000(0.924556) = $9245.56$$

(b) Since compounding is 12 times per year, use Formula (5) with $n = 12$. The present value $P$ of $10,000$ is

$$P = A \left(1 + \frac{r}{n}\right)^{-nt} = 10,000 \left(1 + \frac{0.04}{12}\right)^{-24} = 10,000(0.923239) = $9232.39$$

(c) Since compounding is 365 times per year, use Formula (5) with $n = 365$. The present value $P$ of $10,000$ is

$$P = A \left(1 + \frac{r}{n}\right)^{-nt} = 10,000 \left(1 + \frac{0.04}{365}\right)^{-730} = 10,000(0.923120) = $9231.20$$

(d) Since compounding is continuous, use Formula (6).

$$P = Ae^{-rt} = 10,000e^{-0.04(2)} = 10,000(0.924556) = $9232.39$$

**NOW WORK PROBLEM 19.**

**EXAMPLE 7** Computing the Value of a Zero-coupon Bond

A zero-coupon (noninterest-bearing) bond can be redeemed in 10 years for $1000. How much should you be willing to pay for it now if you want a return of

(a) 5% compounded monthly? (b) 4% compounded continuously?

**SOLUTION** (a) We seek the present value of $1000. Use Formula (5) with $A = 1000$, $n = 12$, $r = 0.05$, and $t = 10$.

$$P = A \left(1 + \frac{r}{n}\right)^{-nt} = 1000 \left(1 + \frac{0.05}{12}\right)^{-12(10)} = $607.16$$

For a return of 5% compounded monthly, you should pay $607.16 for the bond.

(b) Here use Formula (6) with $A = 1000$, $r = 0.04$, and $t = 10$.

$$P = Ae^{-rt} = 1000e^{-0.04(10)} = $670.32$$

For a return of 4% compounded continuously, you should pay $670.32 for the bond.

**NOW WORK PROBLEM 85.**
### Determine the Rate of Interest Required to Double a Lump Sum of Money

#### EXAMPLE 8 Rate of Interest Required to Double an Investment

What annual rate of interest compounded annually should you seek if you want to double your investment in 5 years?

**SOLUTION**

If $P$ is the principal and we want $P$ to double, the amount $A$ will be $2P$. Use the Compound Interest Formula (5) with $n = 1$ and $t = 5$ to find $r$.

$$
A = P \left(1 + \frac{r}{n}\right)^{nt}
$$

Formula (5)

$$
2P = P \cdot (1 + r)^5
$$

$2 = (1 + r)^5$

Divide both sides by $P$.

$$
\sqrt[5]{2} = 1 + r
$$

Take the fifth root of each side.

$$
r = \sqrt[5]{2} - 1 \approx 1.148698 - 1 = 0.148698
$$

Solve for $r$.

The annual rate of interest needed to double the principal in 5 years is 14.87%.

**NOW WORK PROBLEM 37.**

### Determine the Time Required to Double a Lump Sum of Money

#### EXAMPLE 9 Time Required to Double an Investment

**(a)** How long will it take for an investment to double in value if it earns 5% compounded annually?

**(b)** How long will it take for an investment to double in value if it earns 5% compounded continuously?

**SOLUTION**

**(a)** If $P$ is the initial investment and we want $P$ to double, the amount $A$ we want will be $2P$. Use the Compound Interest Formula (5) with $r = 0.05$, $n = 1$, and $A = 2P$.

$$
A = P \left(1 + \frac{r}{n}\right)^{nt}
$$

Formula (5)

$$
2P = P(1 + 0.05)^t
$$

$2 = (1.05)^t$

Divide both sides by $P$.

$$
t = \log_{1.05} 2
$$

Change to a logarithmic equation.

$$
t = \frac{\log 2}{\log 1.05} \approx 14.207
$$

Apply the Change-of-Base Formula.

It will take about 14.2 years to double the investment at 5% compounded annually.

**(b)** If $P$ is the initial investment and we want $P$ to double, the amount $A$ will be $2P$. Use Formula (6) for continuously compounded interest with $r = 0.05$ and $A = 2P$.

$$
A = Pe^{rt}
$$

Formula (6)

$$
2P = Pe^{0.05t}
$$

$2 = e^{0.05t}$

Divide both sides by $P$. 

6.2 Compound Interest

Rewrite as a logarithmic equation.*

\[0.05t = \ln 2\]

Solve for \(t\).

\[t = \frac{\ln 2}{0.05} \approx 13.86\]

It will take just under 14 years to double the investment at 5% compounded continuously.

\[\text{NOW WORK PROBLEM 41.}\]

* When the base of a logarithm is the number \(e\), that is, when \(x = \log_e N\), it is customary to write the logarithm as \(x = \ln N\). Most calculators have a key \(\ln x\) for evaluating these logarithms.

**EXERCISE 6.2** Answers Begin on Page AN–32.

**‘Are You Prepared?’ Problems** Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Evaluate: (a) \((1 + 0.05)^8\) (b) \(\left(1 + \frac{0.08}{12}\right)^{-24}\) Round answers to 3 decimal places. (pp. A–28 to A–33)

2. Change the equation \(y = \log_2 16\) to an exponential equation: \(\text{__________} \) (pp. A–28 to A–33)

3. Change the equation \(2 = (1.05)^t\) to a logarithmic equation: \(\text{__________} \) (pp. A–28 to A–33)

4. Use the Change-of-Base Formula to evaluate \(\log_{1.05} 2\): \(\text{__________} \). Round your answer to 3 decimal places. (pp. A–28 to A–33)

**Concepts and Vocabulary**

5. **True or False** The more times in a year that interest is compounded, the more money there is after one year.

6. **True or False** If a rate of 9% interest is compounded monthly, the effective rate of interest is somewhat less than 9%.

7. When interest at a rate \(r\) is compounded on a principal \(P\) so that the amount after one year is \(Pe^r\), the interest is said to be \(\text{__________} \) \(\text{__________} \).

8. The principal needed now to get $1000 at a later date is called the \(\text{__________} \) \(\text{__________} \) of $1000.

**Skill Building**

In Problems 9–18, find the amount that results from each investment.

9. $100 invested at 4% compounded quarterly after a period of 2 years

10. $50 invested at 6% compounded monthly after a period of 3 years

11. $500 invested at 8% compounded quarterly after a period of \(\frac{1}{2}\) years

12. $300 invested at 2% compounded monthly after a period of \(\frac{1}{2}\) years

13. $600 invested at 5% compounded daily after a period of 3 years

14. $700 invested at 6% compounded daily after a period of 2 years

15. $10 invested at 2% compounded continuously after a period of 2 years

16. $40 invested at 7% compounded continuously after a period of 3 years

17. $100 invested at 3% compounded continuously after a period of \(\frac{3}{4}\) years

18. $100 invested at 3% compounded continuously after a period of \(\frac{3}{4}\) years

In Problems 19–28, find the principal needed now to get each amount; that is, find the present value.

19. To get $100 after 2 years at 2% compounded monthly

20. To get $75 after 3 years at 3% compounded quarterly

21. To get $1000 after 2 \(\frac{1}{2}\) years at 6% compounded daily

22. To get $800 after 3 \(\frac{1}{2}\) years at 2% compounded monthly

23. To get $600 after 2 years at 4% compounded quarterly

24. To get $300 after 4 years at 3% compounded daily
25. To get $80 after $\frac{3}{4}$ years at 2% compounded continuously
27. To get $400 after 1 year at 4% compounded continuously

In Problems 29–32, which of the two rates would yield the larger amount in 1 year?

[Hint: Start with a principal of $10,000 in each instance.]

29. 6% compounded quarterly or $6\frac{1}{4}$% compounded annually
30. 9% compounded quarterly or 9.2% compounded annually
31. 9% compounded monthly or 8.8% compounded daily
32. 8% compounded semiannually or 7.9% compounded daily

In Problems 33–36, find the effective rate of interest.

33. For 5% compounded quarterly
34. For 6% compounded monthly
35. For 5% compounded continuously
36. For 6% compounded continuously
37. What rate of interest compounded annually is required to double an investment in 3 years?
38. What rate of interest compounded annually is required to double an investment in 6 years?
39. What rate of interest compounded annually is required to triple an investment in 5 years?
40. What rate of interest compounded annually is required to triple an investment in 10 years?
41. (a) How long does it take for an investment to double in value if it is invested at 8% compounded monthly?
   (b) How long does it take if the interest is compounded continuously?
42. (a) How long does it take for an investment to triple in value if it is invested at 6% compounded monthly?
   (b) How long does it take if the interest is compounded continuously?
43. What rate of interest compounded quarterly will yield an effective interest rate of 7%?
44. What rate of interest compounded continuously will yield an effective interest rate of 6%?

Applications and Extensions

53. Time Required to Reach a Goal If Tanisha has $100 to invest at 3% per annum compounded monthly, how long will it be before she has $150? If the compounding is continuous, how long will it be?
54. Time Required to Reach a Goal If Angela has $100 to invest at 4% per annum compounded monthly, how long will it be before she has $175? If the compounding is continuous, how long will it be?
55. Time Required to Reach a Goal How many years will it take for an initial investment of $10,000 to grow to $25,000? Assume a rate of interest of 6% compounded continuously.
56. Time Required to Reach a Goal How many years will it take for an initial investment of $25,000 to grow to $80,000? Assume a rate of interest of 5% compounded continuously.

57. Price Appreciation of Homes What will a $90,000 house cost 5 years from now if the price appreciation for homes over that period averages 3% compounded annually?
58. Credit Card Interest A department store charges 1.25% per month on the unpaid balance for customers with charge accounts (interest is compounded monthly). A customer charges $200 and does not pay her bill for 6 months. What is the bill at that time?
59. Saving for a Car Jerome will be buying a used car for $15,000 in 3 years. How much money should he ask his parents for now so that, if he invests it at 5% compounded continuously, he will have enough to buy the car?
60. Paying off a Loan John will require $3000 in 6 months to pay off a loan that has no prepayment privileges. If he has the
$3000 now, how much of it should he save in an account paying 3% compounded monthly so that in 6 months he will have exactly $3000 in the account?

61. Return on a Stock George is contemplating the purchase of 100 shares of a stock selling for $15 per share. The stock pays no dividends. The history of the stock indicates that it should grow at an annual rate of 15% per year. How much will the 100 shares of stock be worth in 5 years?

62. Return on an Investment A business purchased for $650,000 in 2008 is sold in 2011 for $850,000. What is the annual rate of return for this investment?

63. Comparing Savings Plans Jim places $1000 in a bank account that pays 5.6% compounded continuously. After 1 year, will he have enough money to buy a computer system that costs $1060? If another bank will pay Jim 5.9% compounded monthly, is this a better deal?

64. Savings Plans On January 1, Kim places $1000 in a certificate of deposit that pays 2.8% compounded continuously and matures in 3 months. Then Kim places the $1000 and the interest in a passbook account that pays 1.75% compounded monthly. How much does Kim have in the passbook account on May 1?

65. Comparing IRA Investments Will invests $2000 in his IRA in a bond trust that pays 5% interest compounded semiannually. His friend Henry invests $2000 in his IRA in a certificate of deposit that pays 4.1% compounded continuously. Who will have more money after 20 years, Will or Henry?

66. Comparing Two Alternatives Suppose that April has access to an investment that will pay 5% interest compounded continuously. Which is better: to be given $1000 now so that she can take advantage of this investment opportunity or to be given $1200 three years from now?

67. Down Payment on a House Tami and Todd will need $40,000 for a down payment on a house in 4 years. How much should they deposit in a savings account now so that they will be able to do this? The bank pays 3% compounded quarterly.

68. Saving for College A newborn child receives a $3000 gift toward a college education. How much will the $3000 be worth in 17 years if it is invested at 5% compounded quarterly?

69. Gifting A child’s grandparents have opened a $6000 savings account for the child on the day of her birth. The account pays 3% compounded semiannually. The child will be allowed to withdraw the money when she reaches the age of 25. What will the account be worth at that time?

70. Effective Rates of Interest A bank advertises that it pays interest on saving accounts at the rate of 2.25% compounded daily.

(a) Find the effective rate if the bank uses 360 days in determining the daily rate.

(b) What is the effective rate if 365 days are used?

71. Length of Investment How many years will it take for an initial investment of $10,000 to grow to $25,000? Assume a rate of interest of 3% compounded daily.

72. Length of Investment How many years will it take for an initial investment of $25,000 to grow to $80,000? Assume a rate of interest of 4% compounded daily.

73. Roth IRA Jason and his wife put $4000 into a Roth IRA that pays 4% compounded annually. How much will the investment be worth after 30 years?

74. Roth IRA What annual rate of interest is required for an initial investment of $2000 in a Roth IRA to be worth $4500 in 20 years?

75. College Savings Tom and Anita have a new baby and want to start saving for college. The average tuition, fees, and room and board charges at a public four-year institution totaled $15,213 in 2008–2009. Assuming a 6% annual increase, this cost will rise to $43,423 when their child begins college in 2026. How much do they need to invest in 2010 at 8% per annum compounded continuously to afford 1 year at a public 4-year institution in 2026?

Source: Trends in College Pricing 2009, College Board

76. College Savings The average tuition, fees, and room and board charges at a private four-year institution totaled $35,636 in 2008–2009. Assuming a 5% annual increase, this cost will rise to $85,762 for the academic year 2026–2027. For their newborn baby, how much will Tom and Anita need to invest in 2010 at 7% per annum compounded continuously to afford 1 year at a private 4-year institution in 2026 (to the nearest dollar)?

Source: Trends in College Pricing 2009, College Board

77. National Debt The national debt was approximately $5.77 trillion on January 1, 2000, and approximately $12.31 trillion on January 1, 2010.

(a) Find the annual rate of growth of the national debt in this 10-year period, assuming a constant annual rate of growth.

(b) If this rate of growth continues, what would be the projected national debt on January 1, 2020?

Source: U.S. Treasury
78. **Cost of Energy** The average U.S. household will spend $2173 on energy use in the year 2010. This includes $849 for heating, $553 for appliances, $302 for water heating, $271 for cooling, and $198 for lighting.

(a) Assuming a constant annual inflation rate of 2.3%, determine the projected amounts that the average U.S. household will spend on each of the forms of energy listed above in the year 2014.

(b) If energy costs increase at an annual rate of 4%, what will the average household spend on each form of energy in 2014?

![Home Energy Cost](image)

**Home Energy Cost**
The average U.S. household will spend $2173 on energy use in 2010. How it will be spent:

- **25.4% Appliances**
- **39.1% Heating**
- **9.1% Lighting**
- **13.9% Water heating**
- **12.5% Cooling**

Source: American Council for an Energy-Efficient Economy

79. **Analyzing Interest Rates on a Mortgage** Colleen and Bill have just purchased a house for $650,000, with the seller holding a second mortgage of $100,000. They promise to pay the seller $100,000 plus all accrued interest 5 years from now. The seller offers them three interest options on the second mortgage:

(a) Simple interest at 6% per annum

(b) $\frac{3}{4}$% interest compounded monthly

(c) $\frac{5}{2}$% interest compounded continuously

Which option is best; that is, which results in the least interest on the loan?

---

**Inflation** Problems 80–84 require the following discussion. **Inflation** is a term used to describe the erosion of the purchasing power of money. For example, suppose the annual inflation rate is 3%. Then $1000 worth of purchasing power now will have only $970 worth of purchasing power in one year because 3% of the original $1000 (0.03 × 1000 = 30) has been eroded due to inflation. In general, if the inflation averages a rate of r over n years, the amount A that $P will purchase after n years is

$$A = P \cdot (1 - r)^n$$

where r is expressed as a decimal.

80. **Inflation** If the inflation rate averages 2%, how much will $1000 purchase in 3 years?

81. **Inflation** If the amount that $1000 will purchase is only $950 after 2 years, what was the average inflation rate?

82. **Inflation** If the amount that $1000 will purchase is only $930 after 2 years, what was the average inflation rate?

83. **Inflation** If the average inflation rate is 2%, how long is it until purchasing power is cut in half?

84. **Inflation** If the average inflation rate is 4%, how long is it until purchasing power is cut in half?

---

**Problems 85–88 involve zero-coupon bonds. A zero-coupon bond is a bond that is sold now at a discount and will pay its face value at the time it matures; no interest payments are made.**

85. **Zero-Coupon Bonds** A child’s grandparents are considering buying a $40,000 face value zero-coupon bond at birth so that she will have enough money for her college education 17 years later. If they want a rate of return of 4% compounded annually, what should they pay for the bond?
Problems 91–94 require the following discussion. The Consumer Price Index (CPI) indicates the relative change in price over time for a fixed basket of goods and services. It is a cost of living index that helps measure the effect of inflation on the cost of goods and services. The CPI uses the base period 1982–1984 for comparison (the CPI for this period is 100). The CPI for January 2010 was 217.6. This means that $100 in the period 1982–1984 had the same purchasing power as $217.60 in January 2010. In general, if the rate of inflation averages a rate of \( r \) over \( n \) years, then the CPI index after \( n \) years is
\[
CPI = CPI_0 \left(1 + \frac{r}{100}\right)^n
\]
where \( CPI_0 \) is the CPI index at the beginning of the \( n \)-year period.

Source: U.S. Bureau of Labor Statistics

91. Consumer Price Index
   (a) The CPI was 169.3 in January 2000 and 217.6 in January 2010. Assuming that annual inflation remained constant for this time period, determine the average annual inflation rate.
   (b) Using the inflation rate from part (a), in what year will the CPI reach 300?
92. Consumer Price Index
   If the January 2010 CPI was 217.6 and the average annual inflation rate is 2.3%, what will be the CPI in 5 years?
93. Consumer Price Index
   If the average annual inflation rate is 2.3%, how long will it take for the CPI index to double?
94. Consumer Price Index
   The base period for the CPI last changed in 1998. Under the previous weight and item structure, the CPI for 1995 was 456.5. If the average annual inflation rate was 5.57%, what year was used as the base period for the CPI?

6.2 Compound Interest

87. Zero-Coupon Bonds How much should a $10,000 face value zero-coupon bond, maturing in 10 years, be sold for now if its rate of return is to be 4% compounded annually?
88. Zero-Coupon Bonds If Pat pays $12,485 for a $25,000 face value zero-coupon bond that matures in 12 years, what is his annual rate of return?
89. Time to Double or Triple an Investment The formula
\[
t = \frac{\ln m}{n \ln \left(1 + \frac{r}{n}\right)}
\]
can be used to find the number of years \( t \) required to multiply an investment \( m \) times when \( r \) is the per annum interest rate compounded \( n \) times a year.
   (a) How many years will it take to double the value of an IRA that compounds annually at the rate of 4%?
   (b) How many years will it take to triple the value of a savings account that compounds quarterly at an annual rate of 3%?
   (c) Give a derivation of this formula.
90. Time to Reach an Investment Goal
   The formula
\[
t = \frac{\ln A - \ln P}{r}
\]
can be used to find the number of years \( t \) required for an investment \( P \) to grow to a value \( A \) when compounded continuously at an annual rate \( r \).
   (a) How long will it take to increase an initial investment of $1000 to $8000 at an annual rate of 5%?
   (b) What annual rate is required to increase the value of a $2000 IRA to $25,000 in 35 years?
   (c) Give a derivation of this formula.

Discussion and Writing

95. Explain in your own words what the term compound interest means. What does continuous compounding mean?
96. Explain in your own words the meaning of present value.
97. Critical Thinking You have just contracted to buy a house and will seek financing in the amount of $100,000. You go to several banks. Bank 1 will lend you $100,000 at the rate of 5.078% amortized over 30 years with a loan origination fee of 1.75%. Bank 2 will lend you $100,000 at the rate of 4.310% amortized over 15 years with a loan origination fee of 1.5%. Bank 3 will lend you $100,000 at the rate of 5.199% amortized over 30 years with no loan origination fee. Bank 4 will lend
you $100,000 at the rate of 4.840% amortized over 15 years with no loan origination fee. Which loan would you take? Why? Be sure to have sound reasons for your choice.

Use the information in the table to assist you. If the amount of the monthly payment does not matter to you, which loan would you take? Again, have sound reasons for your choice. Compare your final decision with others in the class. Discuss.

### ‘Are You Prepared?’ Answers

1. (a) 1.477  
   (b) 0.853  
2. $2^y = 16$  
3. $t = \log_{1.05} 2$  
4. 14.207

### 6.3 Annuities; Sinking Funds

**PREPARING FOR THIS SECTION**  
Before getting started, review the following:

- Exponents and Logarithms (Appendix A, Section A.3, pp. A–28 to A–33)

**NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 326**

**OBJECTIVES**

1. Solve problems involving annuities (p. 316)
2. Solve problems involving sinking funds (p. 320)

### 1 Solve Problems Involving Annuities

In the previous section we saw how to compute the future value of an investment when a fixed amount of money is deposited in an account that pays interest compounded periodically. Often, however, people and financial institutions do not deposit money and then sit back and watch it grow. Rather, money is invested in small amounts at periodic intervals. Examples of such investments are annual life insurance premiums, monthly deposits in a bank, installment loan payments, and dollar averaging in the stock market with 401(k) or 403(b) accounts.

An annuity is a sequence of equal periodic deposits. The periodic deposits can be annual, semiannual, quarterly, monthly, or any other fixed length of time. When the deposits are made at the same time the interest is credited, the annuity is termed ordinary. We shall concern ourselves only with ordinary annuities in this book.

The amount of an annuity is the sum of all deposits made plus all interest accumulated.

**EXAMPLE 1 Finding the Amount of an Annuity**

Find the amount of an annuity after 5 deposits if each deposit is equal to $100 and is made on an annual basis at an interest rate of 4% per annum compounded annually.

**SOLUTION**  
After 5 deposits the first $100 deposit will have accumulated interest compounded annually at 4% for 4 years. Using the Compound Interest Formula, the value $A_1$ of the first deposit of $100 after 4 years is

$$A_1 = 100(1 + 0.04)^4 = 100(1.16986) = 116.99$$
The second deposit of $100, made 1 year after the first deposit, will accumulate interest compounded at 4% for 3 years. Its value after 3 years is

\[ A_2 = \$100(1 + 0.04)^3 = \$100(1.12486) = \$112.49 \]

Similarly, the third, fourth, and fifth deposits will have the values

\[ A_3 = \$100(1 + 0.04)^2 = \$100(1.0816) = \$108.16 \]
\[ A_4 = \$100(1 + 0.04)^1 = \$100(1.04) = \$104.00 \]
\[ A_5 = \$100 \]

The amount of the annuity after 5 deposits is

\[ A_1 + A_2 + A_3 + A_4 + A_5 = \$116.99 + \$112.49 + \$108.16 + \$104.00 + \$100.00 = \$541.64 \]

Figure 2 illustrates the growth of the annuity described above.

Suppose the interest rate that an annuity earns is \( i \) per payment period (expressed as a decimal). For example, if an account pays 12% compounded monthly (12 times a year), then \( i = \frac{0.12}{12} = 0.01 \). If an account pays 8% compounded quarterly (4 times a year), then \( i = \frac{0.08}{4} = 0.02 \).

To develop a formula for the amount of an annuity, suppose that \( P \) is deposited each payment period for \( n \) payment periods in an account that earns a rate of \( i \) per payment period. When the last deposit is made at the \( n \)th payment period, the first deposit of \( \$P \) has earned interest compounded for \( n-1 \) periods, the second deposit of \( \$P \) has earned interest compounded for \( n-2 \) periods, and so on. Table 3 shows the value of each deposit after \( n \) deposits have been made.

<table>
<thead>
<tr>
<th>Deposit</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\ldots</th>
<th>( n-1 )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>( P \cdot (1 + i)^n-1 )</td>
<td>( P \cdot (1 + i)^{n-2} )</td>
<td>( P \cdot (1 + i)^{n-3} )</td>
<td>\ldots</td>
<td>( P \cdot (1 + i) )</td>
<td>( P )</td>
</tr>
</tbody>
</table>

The amount \( A \) of the annuity is the sum of the amounts shown in Table 3; that is,

\[ A = P \cdot (1 + i)^{n-1} + P \cdot (1 + i)^{n-2} + \cdots + P \cdot (1 + i) + P \]
\[ = P[1 + (1 + i) + \cdots + (1 + i)^{n-1}] \]

The expression in brackets is the sum of a geometric sequence* with \( n \) terms and a common ratio of \( (1 + i) \). As a result,

\[ A = P \left[ \frac{1 - (1 + i)^n}{1 - (1 + i)} \right] = P \frac{1 - (1 + i)^n}{-i} = P \frac{(1 + i)^n - 1}{i} \]

* The sum of the first \( n \) terms of the geometric sequence with common ratio \( r \) is

\[ 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r} \]

See Appendix A, Section A.4, for a more detailed discussion.
We have established the following result:

**Theorem**

**Amount of an Annuity**

Suppose $P$ is the deposit made at the end of each payment period for an annuity paying an interest rate of $i$ per payment period. The amount $A$ of the annuity after $n$ deposits is

$$A = P \cdot \left( \frac{(1 + i)^n - 1}{i} \right)$$  \hspace{1cm} (1)

**EXAMPLE 2** Finding the Amount of an Annuity

Find the amount of an annuity after 5 deposits if a deposit of $100 is made each year, at 4% compounded annually. How much interest is earned?

**SOLUTION**

The deposit is $P = \$100$. The number of deposits is $n = 5$ and the interest per payment period is $i = 0.04$. Using Formula (1), the amount $A$ after 5 deposits is

$$A = P \cdot \left( \frac{(1 + i)^n - 1}{i} \right) = 100 \cdot \left( \frac{(1 + 0.04)^5 - 1}{0.04} \right) = 100(5.416323) = 541.63$$

The interest earned is the amount after 5 deposits less the 5 annual payments of $100 each:

$$\text{Interest earned} = A - 500 = 541.63 - 500 = 41.63$$

**EXAMPLE 3** Finding the Amount of an Annuity

Mary decides to put aside $100 every month in a credit union that pays 5% compounded monthly. After making 8 deposits, how much money does Mary have?

**SOLUTION**

This is an annuity with $P = \$100$, $n = 8$ deposits, and interest $i = \frac{0.05}{12}$ per payment period. Using Formula (1), the amount $A$ after 8 deposits is

$$A = P \cdot \left( \frac{(1 + i)^n - 1}{i} \right) = 100 \cdot \left( \frac{\left(1 + \frac{0.05}{12}\right)^8 - 1}{\frac{0.05}{12}} \right) = 100(8.117644) = 811.76$$

Mary has $811.76 after making 8 deposits.
EXAMPLE 4 Saving for College

To save for her daughter’s college education, Ms. Miranda decides to put $50 aside every month in a bank guaranteed-interest account paying 4% interest compounded monthly. She begins this savings program when her daughter is 3 years old. How much will she have saved by the time she makes the 180th deposit? How old is her daughter at this time?

SOLUTION

This is an annuity with \( P = 50 \), \( n = 180 \) deposits, and \( i = \frac{0.04}{12} \). The amount \( A \) saved is

\[
A = P \left[ \frac{(1 + i)^n - 1}{i} \right] = 50 \left[ \frac{\left(1 + \frac{0.04}{12}\right)^{180} - 1}{\frac{0.04}{12}} \right] = 50 \left( 246.090488 \right) = 12,304.52
\]

Since there are 12 deposits per year, when the 180th deposit is made \( \frac{180}{12} = 15 \) years have passed and Ms. Miranda's daughter is 18 years old.

\[ \text{NOW WORK PROBLEM 25.} \]

EXAMPLE 5 Funding an IRA

To save for retirement, Joe, at age 35, decides to place $2000 into an Individual Retirement Account (IRA) each year for the next 30 years. What will the value of the IRA be when Joe makes his 30th deposit? Assume that the rate of return of the IRA is 4% per annum compounded annually.

SOLUTION

This is an ordinary annuity with \( n = 30 \) annual deposits of \( P = 2000 \). The rate of interest per payment period is \( i = \frac{0.04}{1} = 0.04 \). The amount \( A \) of the annuity after 30 deposits is

\[
A = P \left[ \frac{(1 + i)^n - 1}{i} \right] = 2000 \left[ \frac{(1 + 0.04)^{30} - 1}{0.04} \right] = 2000 \left( 56.084938 \right) = 112,169.88
\]

\[ \text{NOW WORK PROBLEM 25.} \]

EXAMPLE 6 Funding an IRA

If Joe had begun his IRA at age 25, instead of 35, and made 40 deposits of $2000 each, what would his IRA be worth at age 65? Assume a rate of return of 4% per annum compounded annually.

SOLUTION

Now there are \( n = 40 \) deposits of $2000 per year. The amount \( A \) of this annuity is

\[
A = P \left[ \frac{(1 + i)^n - 1}{i} \right] = 2000 \left( \frac{(1 + 0.04)^{40} - 1}{0.04} \right) = 2000 \left( 95.025516 \right) = 190,051.03
\]

Joe would have had $190,051.03 in his IRA when he is 65 if he had begun the IRA at age 25.

\[ \text{NOW WORK PROBLEM 25.} \]
EXAMPLE 7 Finding the Time It Takes to Reach a Certain Savings Goal

How long does it take to save $500,000 if you place $500 per month in an account paying 6% compounded monthly?

SOLUTION This is an annuity in which the amount $A$ is $500,000, each deposit is $P = \$500$, and the interest is $i = \frac{0.06}{12}$ per payment period. We seek the number $n$ of deposits needed to reach $500,000.

$$A = P \left[ \frac{(1 + i)^n - 1}{i} \right]$$

Formula (1)

$$500,000 = 500 \left[ \frac{\left(1 + \frac{0.06}{12}\right)^n - 1}{\frac{0.06}{12}} \right]$$

$A = 500,000; P = 500; i = \frac{0.06}{12}$

$$5 = \left(1 + \frac{0.06}{12}\right)^n - 1$$

Simplify.

$$6 = (1.005)^n$$

Simplify.

$$n = \log_{1.005} 6 = \frac{\log 6}{\log 1.005} = 359.25 \text{ months}$$

Apply the definition of a logarithm.

It takes $359 \frac{1}{4}$ months (almost 30 years) to save the $500,000.

NOW WORK PROBLEM 39.

2 Solve Problems Involving Sinking Funds

A person with a debt may decide to accumulate sufficient funds to pay off the debt by agreeing to set aside enough money each month (or quarter, or year) so that when the debt becomes payable, the money set aside each month plus the interest earned will equal the debt. The fund created by such a plan is called a **sinking fund**.

In working with sinking funds, we generally seek the deposit or payment required to reach a certain goal. In other words, we seek the payment $P$ required to obtain the amount $A$ in Formula (1).

EXAMPLE 8 Funding a Bond Obligation

The Board of Education received permission to issue $4,000,000 in bonds to build a new high school. The board is required to make payments every 6 months into a sinking fund paying 4% compounded semiannually. At the end of 12 years the bond obligation will be retired. What should each payment be?
This is an example of a sinking fund. The payment \( P \) required twice a year to accumulate $4,000,000 in 12 years (24 payments at a rate of interest of \( i = \frac{0.04}{2} = 0.02 \) per payment period) obeys Formula (1).

\[
A = P \left( \frac{(1 + i)^n - 1}{i} \right)
\]

Simplify.

Solve for \( P \).

\[
P = \frac{A}{c(1 + i)^n - 1}
\]

The board will need to make a payment of $131,484.39 every 6 months to redeem the bonds in 12 years.

\[\text{NOW WORK PROBLEM 15.}\]

### EXAMPLE 9 Depletion Investment

A gold mine is expected to yield an annual net income of $200,000 for the next 10 years, after which it will be worthless. An investor wants an annual return on his investment of 18%. If he can establish a sinking fund earning 5% annually, how much should he be willing to pay for the mine? Round the answer to the nearest dollar.

\[
P = SFC(12.57789254)
\]

Solve for \( P \).

\[
P = SFC = 0.079504575x
\]

Let \( x \) denote the price to be paid for the mine. Then 0.18\( x \) represents an 18% annual Return On Investment (ROI).

The annual Sinking Fund Contribution (SFC) needed to recover the purchase price \( x \) in 10 years obeys Formula (1), where \( A = x, n = 10, i = 0.05, \) and \( P = \text{SFC} \). Then

\[
A = P \left( \frac{(1 + i)^n - 1}{i} \right)
\]

Solve for \( P \).
The required annual Return On Investment (ROI) plus the annual Sinking Fund Contribution equals the annual net income of $200,000.

\[
\text{ROI} + \text{SFC} = \text{Annual income}
\]

\[
0.18x + 0.079504575x = 200,000
\]

\[
x = 770,699
\]

A purchase price of $770,699 will achieve the investor’s goals.

■

NOW WORK PROBLEM 33.

**EXAMPLE 10** Funding Capital Improvements

A town needs $1,000,000 for capital improvements. The town administration decides to finance the improvements with a 30-year bond that will pay 5% per annum interest twice a year. A sinking fund is to be set up 5 years in advance of the due date for the bonds to be paid off. The administration projects the sinking fund to have monthly payments and to earn 4% interest compounded monthly.

(a) What are the semiannual payments on the bond?

(b) What is the monthly payment required for the sinking fund?

(c) Construct a table that shows how the sinking fund grows over time.

**SOLUTION**

(a) The amount borrowed is $1,000,000 at 5% interest. The annual interest is \((0.05)(1,000,000) = 50,000\), so the semiannual payments are $25,000.

(b) The sinking fund payment is the value of \(P\) in Formula (1):

\[
A = P \left[ \frac{(1 + i)^n - 1}{i} \right]
\]

Here \(A = 1,000,000\), the amount to be accumulated, \(n = 60\) (5 years of monthly payments), and \(i = \frac{0.04}{12}\). The sinking fund payment \(P\) obeys

\[
1,000,000 = P \left[ \frac{\left(1 + \frac{0.04}{12}\right)^{60} - 1}{\frac{0.04}{12}} \right]
\]

\[
1,000,000 = P(66.298978)
\]

\[
P = $15,083.19
\]

The monthly sinking fund payment is $15,083.19.
(c) Table 4 shows the growth of the sinking fund over time. For example, the total in the account after payment number 12 is obtained by using Formula (1). Since the monthly payment of $15,083.19 has been made for 12 months at 4% compounded monthly, the total amount in the account at this point in time is

$$\text{Total} = \frac{15,083.19 \left( 1 + \frac{0.04}{12} \right)^{12} - 1}{\frac{0.04}{12}} = 15,083.19 \times (12.222463) = 184,353.73$$

**TABLE 4**

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Sinking Fund Deposit, $</th>
<th>Cumulative Deposits</th>
<th>Accumulated Interest, $</th>
<th>Total, $</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15,083.19</td>
<td>15,083.19</td>
<td>0</td>
<td>15,083.19</td>
</tr>
<tr>
<td>12</td>
<td>15,083.19</td>
<td>180,998.28</td>
<td>3355.45</td>
<td>184,353.73</td>
</tr>
<tr>
<td>24</td>
<td>15,083.19</td>
<td>361,996.56</td>
<td>14,221.76</td>
<td>376,218.32</td>
</tr>
<tr>
<td>36</td>
<td>15,083.19</td>
<td>542,994.84</td>
<td>32,904.92</td>
<td>575,899.76</td>
</tr>
<tr>
<td>48</td>
<td>15,083.19</td>
<td>723,993.12</td>
<td>59,723.41</td>
<td>783,716.55</td>
</tr>
<tr>
<td>60</td>
<td>15,083.10</td>
<td>904,991.31</td>
<td>95,008.69</td>
<td>1,000,000.00</td>
</tr>
</tbody>
</table>

**NOTE** The deposit for payment number 60, the final payment, is $15,083.10 because a deposit of $15,083.19 results in a total of $1,000,000.09.

**NOW WORK PROBLEM 31.**

**USING TECHNOLOGY**

**EXAMPLE 11 Using Excel**

Use Excel to solve Example 10, parts (b) and (c).

**SOLUTION** (a) Use the Payment Value function in Excel.

The syntax of the Payment Value function is

$$\text{PMT}(rate, nper, pv, fv, type)$$

*rate* is the interest rate per period.

*nper* is the total number of payments.

*pv* is the present value.

*fv* is the balance after the last payment is made.

*type* is the number 0 or 1 and indicates when payments are due. Use 0 for an ordinary annuity.
STEP 1  Set up the Excel spreadsheet as shown below:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Principal</td>
<td>Annual interest rate</td>
<td>Compounding per Year</td>
<td>Period interest rate</td>
<td>Number of periods</td>
<td>Future Value</td>
</tr>
<tr>
<td>1</td>
<td>$0.00</td>
<td>4.00%</td>
<td>12</td>
<td>0.333%</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Payment</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

STEP 2  Insert the Payment function in B4 as shown below in the fx bar.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Principal</td>
</tr>
<tr>
<td>2</td>
<td>$0.00</td>
</tr>
<tr>
<td>3</td>
<td>4.00%</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>0.333%</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
</tr>
<tr>
<td>7</td>
<td>1000000</td>
</tr>
<tr>
<td>4</td>
<td>Payment</td>
</tr>
</tbody>
</table>

The administration has to deposit $15,083.19 each month into the sinking fund.

(b) Construct a table that shows how the sinking fund grows over time.

STEP 1  Set up the column headings in row 7.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Principal</td>
<td>Annual interest rate</td>
<td>Compounding per Year</td>
<td>Period interest rate</td>
<td>Number of periods</td>
<td>Future Value</td>
</tr>
<tr>
<td>1</td>
<td>$0.00</td>
<td>4.00%</td>
<td>12</td>
<td>0.333%</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Payment</td>
<td>($15,083.19)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Payment Number</td>
<td>Sinking Fund Deposit</td>
<td>Cumulative Deposits</td>
<td>Accumulated Interest</td>
<td>Total</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
STEP 2  Set up row 8.

- Payment number will be 1.
- Sinking fund deposit is $15,083.19. This is stored in cell B4. $15,083.19 is the deposit for each month; it must be positive and not change when the rest of the table is completed. Enter the formula = –1*B4 in B8.
- Cumulative deposits is number of payments times $15,083.19. Enter the formula = A8*B8 in C8.
- The total is the future value of the sinking fund. The inputs are fixed except for the payment number, A8. Enter the formula = FV($D$2,A8,$B$4,$A$2,0).
- Accumulated interest is found by subtracting the cumulative deposits from the total. Enter the formula = E8 – C8 into D8.

The formulas are given in the table below.

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Sinking Fund Deposit</th>
<th>Cumulative Deposits</th>
<th>Accumulated Interest</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
<td>–1*$B$4</td>
<td>=A8*B8</td>
<td>=E8-C8</td>
</tr>
</tbody>
</table>

STEP 3  To complete the table, capture row 8 and drag down two rows. Change the payment number in A9 to 6 and A10 to 12. Highlight rows 9 and 10 and click and drag through row 18.

The completed table is given below.

<table>
<thead>
<tr>
<th>Principal</th>
<th>Annual Interest Rate</th>
<th>Compounding per year</th>
<th>Period Interest Rate</th>
<th>Number of periods</th>
<th>Future value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.00</td>
<td>4.00%</td>
<td>12</td>
<td>0.333%</td>
<td>60</td>
<td>100000</td>
</tr>
</tbody>
</table>

4 Payment  ($15,083.19)

NOW WORK PROBLEM 31 USING EXCEL.
EXERCISE 6.3  Answers Begin on Page AN–32.

‘Are You Prepared?’ Problems  Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Evaluate: \((1 + 0.04)^4\). Round your answer to 5 decimal places. (pp. A–28 to A–33)

2. Evaluate: \(\frac{(1 + 0.04)^{30} - 1}{0.04}\). Round your answer to 6 decimal places. (pp. A–28 to A–33)

Concepts and Vocabulary

3. An _______ is a sequence of equal periodic deposits.

4. The sum of all the equal periodic deposits and the interest earned is called the _______ of the _______.

Skill Building

In Problems 5–14, find the amount of each annuity.

5. After 10 annual deposits of $100 at 10% compounded annually

6. After 12 monthly deposits of $200 at 5% compounded monthly

7. After 12 monthly deposits of $400 at 2% compounded monthly

8. After 5 annual deposits of $1000 at 3% compounded annually

9. After 36 monthly deposits of $200 at 3% compounded monthly

10. After 40 semiannual deposits of $2000 at 2% compounded semiannually

11. After 60 monthly deposits of $100 at 4% compounded monthly

12. After 8 quarterly deposits of $1000 at 4% compounded quarterly

13. After 10 annual deposits of $9000 at 5% compounded annually

14. After 20 annual deposits of $5000 at 4% compounded annually

In Problems 15–24, find the payment required for each sinking fund.

15. The amount required is $10,000 after 5 years at 3% compounded monthly. What is the monthly payment?

16. The amount required is $5000 after 180 days at 2% compounded daily. What is the daily payment?

17. The amount required is $20,000 after 2 \(\frac{1}{2}\) years at 2% compounded quarterly. What is the quarterly payment?

18. The amount required is $50,000 after 10 years at 3% compounded semiannually. What is the semiannual payment?

19. The amount required is $25,000 after 6 months at 5 \(\frac{1}{2}\)% compounded monthly. What is the monthly payment?

20. The amount required is $100,000 after 25 years at 4 \(\frac{1}{4}\)% compounded annually. What is the annual payment?

21. The amount required is $5000 after 2 years at 4% compounded monthly. What is the monthly payment?

22. The amount required is $5000 after 2 years at 4% compounded semiannually. What is the semiannual payment?

23. The amount required is $9000 after 4 years at 5% compounded annually. What is the annual payment?

24. The amount required is $9000 after 2 years at 5% compounded quarterly. What is the quarterly payment?

Applications

25. Market Value of a Mutual Fund  Al invests $2500 a year in a mutual fund for 15 years. If the market value of the fund increases 5% per year, what will be the value of the fund after 15 deposits?

26. Value of an Annuity  Todd and Tami pay $300 every 3 months for 6 years into an ordinary annuity paying 4% compounded quarterly. What is the value of the annuity after 24 deposits?

27. Saving for a Car  Sheila wants to invest an amount every 3 months so that she will have $12,000 in 3 years to buy a new car. The account pays 2% compounded quarterly. How much should she deposit each quarter to have $12,000 after 12 deposits?

28. Saving for a House  In 4 years Colleen and Bill would like to have $30,000 for a down payment on a house. How much should they deposit each month into an account paying 4% compounded monthly to have $30,000 after 48 deposits?

29. Funding a Pension  Dan wishes to have $350,000 in a pension fund 20 years from now. How much should he deposit each month into an account paying 5% compounded monthly to have $350,000 after 240 deposits?

30. Funding a Keogh Plan  Pat has a Keogh retirement plan (this type of plan is tax-deferred until money is withdrawn). If deposits of $7500 are made each year into an account paying 6% compounded annually, how much will be in the account after 25 years?
31. **Sinking Fund Payment** A company establishes a sinking fund to provide for the payment of a $100,000 debt maturing in 4 years. Contributions to the fund are to be made each year. Find the amount of each annual deposit if interest is 3% per annum. Prepare a table showing the annual growth of the sinking fund.

32. **Paying Off Bonds** A state has $5,000,000 worth of bonds that are due in 20 years. A sinking fund is established to pay off the debt. If the state can earn 6% annually on its money, what is the annual sinking fund deposit needed? Prepare a table showing the growth of the sinking fund every 5 years.

33. **Depletion Investment** An investor wants to know the amount she should pay for an oil well expected to yield an annual return of $30,000 for the next 30 years, after which the well will be dry. Find the amount she should pay to yield a 14% annual return if a sinking fund earns 5% annually. Round the answer to the nearest dollar.

34. **Time Needed for a Million Dollars** If you deposit $10,000 every year into an account paying 4% compounded annually, how long will it take to accumulate $1,000,000?

35. **Bond Payments** A city has issued bonds to finance a new library. The bonds have a total face value of $1,000,000 and are payable in 10 years. After 7 years, a sinking fund will be started to retire the bonds. If the interest rate on the fund is expected to be 4% compounded quarterly, what are the quarterly payments?

36. **Value of an IRA**
   
   (a) Tanya invested $2000 in an IRA each year for 10 years earning 4% compounded annually. At the end of 10 years she ceased the IRA payments, but continued to invest her accumulated amount at 5% compounded annually, for the next 30 years. What was the value of her IRA investment at the end of 10 years? What was the value of her investment at the end of the next 30 years?
   
   (b) Carol waited 10 years and started her IRA investment in the 11th year and invested $2000 per year for the next 30 years at 6% compounded annually. What was the value of her investment at the end of 30 years?
   
   (c) Who had more money at the end of the 40 year period?

37. **Managing a Condo** The Crown Colony Condo Association is required by law to set aside funds to replace its roof. The current cost to replace the roof is $100,000 and it will need to be replaced in 20 years. The cost of a roof is expected to increase at the rate of 3% per year. The Condo can invest in Treasuries yielding 4% paid semiannually.

   (a) What will the roof cost in 20 years?
   
   (b) If the Condo invests in the Treasuries, what semiannual payment is required to have the funds to replace the roof in 20 years?

38. **Managing a Condo** The Crown Colony Condo Association is required by law to set aside funds to replace its common-area carpet. The current cost to replace the carpet is $20,000 and it will need to be replaced in 6 years. The cost of carpet is expected to increase at the rate of 2% per year. The Condo can invest in Treasuries yielding 5% paid semiannually.

39. **Time to Save a Million Dollars** How many years will it take to save $1,000,000 if you place $600 per month in an account that earns 7% compounded monthly?

40. **Time to Save a Million Dollars** How many years will it take to save $1,000,000 if you place $1000 per month in an account that earns 3% compounded monthly?

41. **Saving for a Trip** Angie wants to plan a trip to Hawaii with her husband on their 20th wedding anniversary in two years. She anticipates that the all-inclusive trip will cost $9500 for both of them and wants to start saving. How much should she put into an account each month paying 2.75% compounded monthly to pay for the trip in two years?

   **Source:** Hawaii-aloha.com

42. **Saving for a Trip** Shane wants to plan a trip to Australia with his wife on her 30th birthday in three years. He anticipates that the all-inclusive trip will cost $7500 for both of them and wants to start saving. How much should he put into an account each month paying 3.25% compounded monthly to pay for the trip in three years?

   **Source:** www.seeanz.com

43. **Saving for Repairs** In May 2010, Bath Community Schools asked voters to approve the renewal of a building and site capital projects sinking fund. If the sinking fund is to generate $1 million over 5 years in an account that pays 5% compounded quarterly, how much should the school district deposit into the account each quarter?

   **Source:** Bath Community Schools Board of Education

44. **Planning for Growth** Based on trends in census data, a K–12 school district passed a 5-year tax levy that increases property tax by $0.20 per $1000 valuation. The district has approximately 40,000 homes with an average value of $145,000. The monies collected are placed in a building fund account paying 4.5% compounded annually. The tax revenue is deposited at the same time that interest is credited. What will be the value of the building fund at the end of the levy period?

45. **529 College Savings Plan** Pam and Tim decide to start saving money for their daughter’s college education. They open a college savings plan (529) with a $400 initial investment and next month start to make monthly deposits of $100. If the account pays 6.00% compounded monthly, how much will the account be worth after 180 deposits of $100 each? Be sure to include the initial investment in the computation.

46. **529 College Savings Plan** Christine and Adam decide to start saving money for their son’s college education. They open a college savings plan (529) with a $1000 initial investment and in 3 months start to make quarterly deposits of $700. If the account pays 4.75% compounded quarterly, how much will the account be worth after 64 deposits of
$700 each? Be sure to include the initial investment in the computation.

47. **Lottery without Taxes** Dan won $2.6 million in a state lottery and must decide between the lump sum payment of $1,326,000 or 26 annual payments of $100,000. Assuming he can invest all proceeds in a retirement plan that pays 8.5% compounded annually, how much will his winnings be worth after 25 years? Which option should he select?

48. **Lottery with Taxes** Refer to Problem 47. Dan must pay federal taxes on his winnings. According to the 2010 federal tax schedule, he would pay a tax of $94,601 plus 35% of the amount over $349,700 if he takes the lump sum or a tax of $8,772.50 plus 25% of the amount over $63,700 if he takes the annual payments. Assuming the tax rates remain constant, determine how much his winnings will be worth after 25 years and which option Dan should select.

Source: www.illinoislottery.com; U.S. Department of the Treasury

49. **Saving for a Car** In January 2010, a new Honda Accord EX with manual transmission was listed at $23,830.

(a) Assuming a constant annual inflation rate of 3.23%, which was the annual percent increase in the U.S. Consumer Price Index for 2006, calculate a projection of what a new Honda Accord EX with manual transmission would cost in January 2014.

(b) If the sales tax where you plan to purchase this vehicle in January 2014 is 9.25%, what is the final cost of the Honda?

(c) Beginning in January 2010, suppose that you make a monthly payment into a sinking fund that earns interest at a 2.75% annual rate, compounded monthly, in order to accumulate funds to purchase the new Honda in 4 years. What monthly payment will produce a balance in this sinking fund in January 2014 that equals the projected final cost of the 2014 Honda Accord EX?


50. **Saving for a Down Payment on a Home** The median price of an existing single-family detached home in Florida was $210,200 in March 2009 and increased to $218,200 in March 2010.

(a) What is the annual rate at which homes appreciated?

(b) If the median price of an existing single-family detached home in Florida continues to increase at the annual rate found in part (a), determine the projected median price of an existing single-family detached home in Florida in March 2014.

(c) Suppose you make monthly payments into a sinking fund between March 2010 and March 2014 to generate a down payment of 10% of this projected median price. Find the monthly payment required for this sinking fund, assuming that this sinking fund earns 2.9%, compounded monthly.

Source: Florida Association of Realtors

51. **Saving for College** The average annual undergraduate college tuition and fees nationally for 2009–2010 for public 4-year institutions totaled $7020. The College Board indicated that the inflation rate for tuition in 2009 was 6.5%.

(a) Assuming that this inflation rate is constant for the next 17 years, determine the projected annual undergraduate college tuition and fees for public 4-year institutions for 2023–2024, 2024–2025, 2025–2026, and 2026–2027.

(b) Determine a quarterly sinking fund payment, beginning with the fourth quarter of 2012, so that at the end of 2023, when the 45th payment is made, the sinking fund has a value equal to the sum of the projected costs of each of the 4 years. Assume the sinking fund earns interest at a 4.2% annual rate, compounded quarterly.

Source: Trends in College Pricing 2009, College Board

52. **Saving for College** The average annual undergraduate college tuition and fees nationally for 2009–2010 for private 4-year institutions totaled $26,273. The College Board indicated that the inflation rate for tuition in 2009 was 4.4%.

(a) Assuming that this inflation rate is constant for the next 17 years, determine the projected annual undergraduate college tuition and fees for private 4-year institutions for 2023–2024, 2024–2025, 2025–2026, and 2026–2027.

(b) Determine a quarterly sinking fund payment, beginning with the fourth quarter of 2012, so that at the end of 2023, when the 45th payment is made, the sinking fund has a value equal to the sum of the projected costs of each of the 4 years. Assume the sinking fund earns interest at a 4.2% annual rate, compounded quarterly.

Source: Trends in College Pricing 2009, College Board

‘Are You Prepared?’ Answers

1. 1.16986  2. 56.084938
OBJECTIVES

1. Solve problems involving the present value of an annuity (p. 329)
2. Solve problems involving amortization (p. 332)
3. Application: Pricing corporate bonds* (p. 338)

1 Solve Problems Involving the Present Value of an Annuity

In Section 6.2 we defined present value (as it relates to the Compound Interest Formula) as the amount of money needed now to obtain an amount $A$ in the future. A similar idea is used for periodic withdrawals.

Suppose you want to withdraw $10,000 per year each year for the next 5 years from a retirement account that earns 5% compounded annually. How much money is required initially in this account for this to happen? In fact, the amount is the sum of the present values of each of the $10,000 withdrawals. This leads to the following definition.

The present value of an annuity is the sum of the present values of the withdrawals. In other words, the present value of an annuity is the amount of money needed now so that if it is invested at a rate of $i$ per payment period, $n$ equal dollar amounts can be withdrawn without any money left over.

EXAMPLE 1 Finding the Present Value of an Annuity

Compute the amount of money required to pay out $10,000 per year for 5 years at 5% compounded annually.

SOLUTION For the first $10,000 withdrawal, the present value $V_1$ (the dollars needed now to withdraw $10,000 in one year) is

\[ V_1 = 10,000(1 + 0.05)^{-1} = 10,000(0.95238095) = 9523.81 \]

For the second $10,000 withdrawal, the present value $V_2$ (the dollars needed now to withdraw $10,000 in two years) is

\[ V_2 = 10,000(1 + 0.05)^{-2} = 10,000(0.90702948) = 9070.29 \]

Similarly,

\[ V_3 = 10,000(1 + 0.05)^{-3} = 10,000(0.8638376) = 8638.38 \]

\[ V_4 = 10,000(1 + 0.05)^{-4} = 10,000(0.822702) = 8227.02 \]

\[ V_5 = 10,000(1 + 0.05)^{-5} = 10,000(0.783526) = 7835.26 \]

The present value $V$ for 5 withdrawals of $10,000 each is

\[ V = V_1 + V_2 + V_3 + V_4 + V_5 \]

\[ = 9523.81 + 9070.29 + 8638.38 + 8227.02 + 7835.26 \]

\[ = 43,294.76 \]

*Optional.
A person would need $43,294.76 now, invested at 5% per annum, in order to withdraw $10,000 per year for the next 5 years.

Table 5 summarizes the results obtained in Example 1. Table 6 lists the amount at the beginning of each year.

### Table 5

<table>
<thead>
<tr>
<th>Withdrawal</th>
<th>Present Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$10,000(1.05)^{-1} = $9523.81</td>
</tr>
<tr>
<td>2nd</td>
<td>$10,000(1.05)^{-2} = $9070.29</td>
</tr>
<tr>
<td>3rd</td>
<td>$10,000(1.05)^{-3} = $8638.38</td>
</tr>
<tr>
<td>4th</td>
<td>$10,000(1.05)^{-4} = $8227.02</td>
</tr>
<tr>
<td>5th</td>
<td>$10,000(1.05)^{-5} = $7835.26</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>$43,294.76</strong></td>
</tr>
</tbody>
</table>

### Table 6

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount at the Beginning of the Year</th>
<th>Add Interest</th>
<th>Subtract Withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43,294.76</td>
<td>2164.74</td>
<td>10,000.00</td>
</tr>
<tr>
<td>2</td>
<td>35,459.50</td>
<td>1772.97</td>
<td>10,000.00</td>
</tr>
<tr>
<td>3</td>
<td>27,232.47</td>
<td>1361.62</td>
<td>10,000.00</td>
</tr>
<tr>
<td>4</td>
<td>18,594.09</td>
<td>929.70</td>
<td>10,000.00</td>
</tr>
<tr>
<td>5</td>
<td>9,523.79</td>
<td>476.21</td>
<td>10,000.00</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We seek a formula for the present value of an annuity. Suppose an annuity earns an interest rate of \( i \) per payment period and suppose we wish to make \( n \) withdrawals of \( P \) at each payment period. The amount \( V_1 \) required for the first withdrawal (the present value of \( P \)) is

\[ V_1 = P(1 + i)^{-1} \]

The amount \( V_2 \) required for the second withdrawal is

\[ V_2 = P(1 + i)^{-2} \]

The amount \( V_n \) required for the \( n \)th withdrawal is

\[ V_n = P(1 + i)^{-n} \]

The present value \( V \) of the annuity is the sum of the amounts \( V_1, V_2, \ldots, V_n \)

\[ V = V_1 + \cdots + V_n = P(1 + i)^{-1} + \cdots + P(1 + i)^{-n} \]

\[ = P(1 + i)^{-n}[1 + (1 + i) + \cdots + (1 + i)^{n-1}] \]

The expression in brackets is the sum of the first \( n \) terms of a geometric sequence, whose ratio is \( 1 + i \). As a result,

\[ V = P(1 + i)^{-n} \frac{1 - (1 + i)^n}{1 - (1 + i)} = P \frac{(1 + i)^{-n} - 1}{i} = P \frac{1 - (1 + i)^{-n}}{i} \]

**Theorem**

**Present Value of an Annuity**

Suppose an annuity earns interest at the rate of \( i \) per payment period. If \( n \) withdrawals of \( P \) are made at each payment period, the amount \( V \) required is

\[ V = P \cdot \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \]

(1)

Here \( V \) is called the **present value of the annuity**.
EXAMPLE 2  Getting By in College

A student requires $200 each month for the next 10 months to cover miscellaneous expenses at school. A money market fund will pay her interest of 2% per annum compounded monthly. How much money should she ask for from her parents now so that she can withdraw $200 each month for the next 10 months?

SOLUTION  We seek the present value of an annuity. The monthly withdrawal is $200, the interest rate is $i = \frac{0.02}{12}$ per month, and the number of withdrawals is $n = 10$. The money required for this is given by Formula (1):

$$V = P \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = \$200 \left[ \frac{1 - \left( 1 + \frac{0.02}{12} \right)^{-10}}{\frac{0.02}{12}} \right] = \$200(9.90894) = $1981.79$$

She should ask her parents for $1981.79.

NOW WORK PROBLEM 1.

EXAMPLE 3  Determining the Cost of a Car

A man agrees to pay $300 per month for 48 months to pay off a used car loan with no money down. If interest of 12% per annum is charged monthly, how much did the car originally cost? How much interest was paid?

SOLUTION  The cost of the car is the same as the present value $V$ of an annuity of $\$300$ per month at 12% for 48 months. The original cost of the car is

$$V = P \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = \$300 \left[ \frac{1 - \left( 1 + \frac{0.12}{12} \right)^{-48}}{\frac{0.12}{12}} \right] = \$300(37.9739595) = $11,392.19$$

The total payment is $(\$300)(48) = $14,400. The interest paid is

$$\$14,400 - \$11,392.19 = \$3007.81$$

NOW WORK PROBLEM 15.
EXAMPLE 4 Lease or Purchase

A company may obtain a copying machine either by leasing it for 4 years (the useful life) at an annual cost of $1000 or by purchasing the machine for $3000.

(a) Which alternative is preferable if the company can invest money at 5% per annum?

(b) What if it can invest at 7% per annum?

SOLUTION

(a) The company can invest money and receive 5% per annum. The cost of leasing the copying machine is $1000 each year for 4 years. The money needed now to make these payments is the present value \( V \) of an annuity with \( P = 1000 \), interest \( i = 0.05 \), and \( n = 4 \). Now

\[
V = P \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = 1000 \left[ \frac{1 - (1 + 0.05)^{-4}}{0.05} \right] = 1000(3.54595) = 3545.95
\]

The company would need $3545.95 now to lease the copier and can purchase it now for $3000. It is better to purchase.

(b) The company can invest money and receive 7% per annum. The cost of leasing the copying machine is $1000 each year for 4 years. The money needed now to make these payments is the present value \( V \) of an annuity with \( P = 1000 \), interest \( i = 0.07 \), and \( n = 4 \). Now

\[
V = P \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = 1000 \left[ \frac{1 - (1 + 0.07)^{-4}}{0.07} \right] = 1000(3.38721) = 3387.21
\]

The company would need $3387.21 now to lease the copier. It is better to purchase for $3000 than to lease.

NOW WORK PROBLEM 45.

Solve Problems Involving Amortization

Look again at Example 3. What it also says is that if the man pays $300 per month for 48 months with an interest of 12% compounded monthly, then the car is his. In other words, he amortized the debt in 48 equal monthly payments. (The Latin word mort means “death.” Paying off a loan is regarded as “killing” it.) A loan with a fixed rate of interest is said to be amortized if both principal and interest are paid by a sequence of equal payments made over equal periods of time.

When a loan of \( V \) dollars is amortized at a rate of interest \( i \) per payment period over \( n \) payment periods, the customary question is, “What is the payment \( P \)?” In other words, in amortization problems, we want to find the payment \( P \) that, after \( n \) payment periods at the rate of interest \( i \) per payment period, gives us a present value equal to the amount of the loan. We need to find \( P \) in the formula

\[
V = P \left[ \frac{1 - (1 + i)^{-n}}{i} \right]
\]

Solving for \( P \), we find

\[
P = V \left[ \frac{1 - (1 + i)^{-n}}{i} \right]^{-1} = V \left[ \frac{i}{1 - (1 + i)^{-n}} \right]
\]
Theorem

Amortization

The payment $P$ required to pay off a loan of $V$ dollars borrowed for $n$ payment periods at a rate of interest $i$ per payment period is

$$P = \frac{Vi}{1 - (1 + i)^{-n}}$$

(2)

EXAMPLE 5 Finding the Payment for an Amortized Loan

What monthly payment is necessary to pay off a loan of $800 at 10% per annum (a) In 2 years? (b) In 3 years? (c) What total amount is paid out for each loan?

SOLUTION (a) For the 2-year loan, $V = 800$, $n = 24$, and $i = \frac{0.10}{12}$. The monthly payment $P$ is

$$P = \frac{800 \left(\frac{0.10}{12}\right)}{1 - \left(1 + \frac{0.10}{12}\right)^{-24}} = 36.92$$

(b) For the 3-year loan, $V = 800$, $n = 36$, and $i = \frac{0.10}{12}$. The monthly payment $P$ is

$$P = \frac{800 \left(\frac{0.10}{12}\right)}{1 - \left(1 + \frac{0.10}{12}\right)^{-36}} = 25.81$$

(c) For the 2-year loan, the total amount paid out is $(36.92)(24) = 886.08$; for the 3-year loan, the total amount paid out is $(25.81)(36) = 929.16$. 

NOW WORK PROBLEM 9.

EXAMPLE 6 Home Mortgage Payments

Mr. and Mrs. Corey have just purchased a $300,000 house and have made a down payment of $60,000. They can amortize the balance ($300,000 - $60,000 = $240,000) at 6% for 30 years.

(a) What are the monthly payments?

(b) What is their total interest payment?

(c) After 20 years, what equity do they have in their house (that is, what is the sum of the down payment and the amount paid on the loan)?
SOLUTION  (a) The monthly payment \( P \) needed to pay off the loan of $240,000 at 6% for 30 years (360 months) is

\[
P = \sqrt[360]{\frac{i}{1 - (1 + i)^{-n}}} = \frac{0.06}{12} \left( \frac{1}{1 - \left(1 + \frac{0.06}{12}\right)^{-360}} \right) = \frac{240,000(0.0059955)}{1438.92} \]

(b) The total paid out for the loan is

\[
(1438.92)(360) = 518,011.20
\]

The interest on this loan amounts to

\[
518,011.20 - 240,000 = 278,011.20
\]

(c) After 20 years (240 months) there remains 10 years (or 120 months) of payments. The present value of the loan is the present value of a monthly payment of $1438.92 for 120 months at 6%, namely,

\[
V = P \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = \frac{1438.92}{1 - \left(1 + \frac{0.06}{12}\right)^{-120}} = \frac{1438.92(90.073453)}{129,608.49}
\]

The amount paid on the loan is

(Original loan amount) - (Present value) = $240,000 - $129,608.49 = $110,391.51

The equity after 20 years is

(Down payment) + (Amount paid on loan) = $60,000 + $110,391.51 = $170,391.51

NOTE  This equity does not include any appreciation in the value of the house over the 20 year period.

Table 7 gives a partial schedule of payments for the loan in Example 5. It is interesting to observe how slowly the amount paid on the loan increases early in the payment schedule, with very little of the payment used to reduce principal, and how quickly the amount paid on the loan increases during the last 5 years.

**TABLE 7**

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Monthly Payment</th>
<th>Principal</th>
<th>Interest</th>
<th>Amount Paid on Loan</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1,438.92</td>
<td>$238.92</td>
<td>$1,200.00</td>
<td>$238.92</td>
</tr>
<tr>
<td>60</td>
<td>$1,438.92</td>
<td>$320.67</td>
<td>$1,118.26</td>
<td>$16,669.54</td>
</tr>
<tr>
<td>120</td>
<td>$1,438.92</td>
<td>$432.53</td>
<td>$1,006.39</td>
<td>$39,154.26</td>
</tr>
<tr>
<td>180</td>
<td>$1,438.92</td>
<td>$583.42</td>
<td>$855.50</td>
<td>$69,482.77</td>
</tr>
<tr>
<td>240</td>
<td>$1,438.92</td>
<td>$786.94</td>
<td>$651.98</td>
<td>$110,391.39</td>
</tr>
<tr>
<td>300</td>
<td>$1,438.92</td>
<td>$1,061.47</td>
<td>$377.45</td>
<td>$165,570.99</td>
</tr>
<tr>
<td>360</td>
<td>$1,438.92</td>
<td>$1,431.76</td>
<td>$7.16</td>
<td>$240,000.00</td>
</tr>
</tbody>
</table>

NOW WORK PROBLEM 17.
USING TECHNOLOGY

EXAMPLE 7 Using Excel

Use Excel to solve Example 6, parts (a) and (c).

**SOLUTION (a)** Use the Excel function PMT to find the monthly payments. The function has the following syntax:

\[
PMT(rate, nper, pv, fv, type)
\]

- *rate* is the interest rate per period.
- *nper* is the total number of payment periods
- *pv* is the initial deposit or present value
- *fv* is the future value
- *type* is the number 0 or 1: use 0 for an ordinary annuity

The values of the parameters of the PMT function are:

- *rate* is \( \frac{0.06}{12} \)
- *nper* is \( 30 \times 12 = 360 \)
- *pv* is $240,000
- *fv* is 0
- *type* is 0

The Excel spreadsheet is given below.

In cell B4:

\[ =PMT(C2, D2, A2, E2, 0) \]

The final Excel spreadsheet for part (a) is given below.

Mr. and Mrs. Corey’s monthly payments are $1438.92.
(c) To create an amortization table, use the Excel function PPMT to compute the monthly principal.

- Set up the table headings in row 7.
- In A8 enter 1 for payment number 1.
- In B8 enter the formula \( = -1*SB\$.4 \). (Recall the $ keeps the cell constant.)
- In C8 enter the formula \( = -1*PPMT(SCS2,A8,SDS2,SA$S2,A$E2) \).
- In D8 enter the formula \( = B8 - C8 \).
- In E8 enter the formula \( = 240,000 + FV(SCS2, A8, SB\$.4, SA$S2, 0) \).
- Highlight row 8, click and drag 2 rows (filling rows 9 and 10).
- Change A9 to 60 and A10 to 120.
- Highlight rows 9 and 10; click and drag the two rows to row 14 to get the rest of the table.

The completed table is given below.

**NOW WORK PROBLEM 17 USING EXCEL.**

**EXAMPLE 8 Determining the Amount of an Inheritance**

When Mr. Nicholson died, he left an inheritance of $15,000 for his family to be paid to them over a 10-year period in equal amounts at the end of each year. If the $15,000 is invested at 4% per annum, what is the annual payout to the family?
6.4 Present Value of an Annuity: Amortization

EXAMPLE 9 Determining Retirement Income

Joan is 20 years away from retiring and starts saving $100 a month in an account paying 6% compounded monthly. When she retires, she wishes to withdraw a fixed amount each month for 25 years. What will this fixed amount be?

SOLUTION After 20 years the amount $A$ accumulated in her account is the amount of an annuity with $n = 240$ monthly payments of $P = 100$ at an interest rate of $i = \frac{0.06}{12}$.

$$A = P \left[ \frac{(1 + i)^n - 1}{i} \right] = 100 \left[ \frac{(1 + \frac{0.06}{12})^{240} - 1}{\frac{0.06}{12}} \right]$$

$$= 100 \cdot (462.0408952) = 46,204.09$$

The amount $W$ she can withdraw each month for 25 years (300 months) at 6% compounded monthly is

$$W = 46,204.09 \times \frac{0.06}{12} \left[ \frac{1 - \left(1 + \frac{0.06}{12}\right)^{-300}}{1 - \left(1 + \frac{0.06}{12}\right)^{-1}} \right] = 297.69$$

NOW WORK PROBLEM 23.

EXAMPLE 10 Capital Expenditure

A corporation is faced with a choice between two machines, both of which are designed to improve operations by saving on labor costs. Machine A costs $8000 and will generate an annual labor savings of $2000. Machine B costs $6000 and will save $1800 in labor annually. Machine A has a useful life of 7 years while machine B has a useful life of only 5 years. Assuming that the time value of money (the investment opportunity rate) of the corporation is 10% per annum, which machine is preferable? (Assume annual compounding and that the savings is realized at the end of each year.)
The company can invest money and receive 10% per annum. Machine A costs $8000 and has a useful life of 7 years. The cost per year of Machine A is the annual payment $P$ required to amortize a loan of $8000 for 7 years at 10% per annum. We use Formula (2) with $V = 8000$, $i = 0.10$, and $n = 7$.

$$P = \frac{V}{1 - (1 + i)^{-n}} = \frac{8000}{1 - (1 + 0.1)^{-7}} = 8000(0.205405) = 1643.24$$

The annual cost of Machine A to the company is $1643.24.

Similarly, Machine B, with a cost of $6000 and a useful life of 5 years, will cost the company each year an amount equal to the annual payment $P$ required to amortize a loan of $6000 for 5 years at 10% per annum. We use Formula (2) with $V = 6000$, $i = 0.1$, and $n = 5$.

$$P = \frac{V}{1 - (1 + i)^{-n}} = \frac{6000}{1 - (1 + 0.1)^{-5}} = 6000(0.263797) = 1582.78$$

The annual cost of Machine B is $1582.78.

Now we take into account the savings to the company in labor generated by each machine. See Table 8.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual labor savings</td>
<td>$2000.00</td>
<td>$1800.00</td>
</tr>
<tr>
<td>Annual cost</td>
<td>$1643.24</td>
<td>$1582.78</td>
</tr>
<tr>
<td>Net savings</td>
<td>$ 356.76</td>
<td>$ 217.12</td>
</tr>
</tbody>
</table>

Machine A will generate more annual savings to the company than Machine B. Machine A is preferable to Machine B.

**NOW WORK PROBLEM 47.**

**3 Application: Pricing Corporate Bonds**

We begin with some definitions of terms related to corporate bonds.

**Definition**  
**Face Amount (Face Value or Par Value)**

The face amount or denomination of a bond (normally $1000) is the amount paid to the bondholder at maturity. It is also the amount usually paid by the bondholder when the bond is originally issued.

**Definition**  
**Nominal Interest (Coupon Rate)**

The nominal interest or coupon rate is the contractual interest paid on the bond.
Nominal interest is normally quoted as an annual percentage of the face amount. Nominal interest payments are conventionally made semiannually, so semiannual periods are used for compound interest calculations. For example, if a bond has a face amount of $1000 and a coupon rate of 8%, then every 6 months the owner of the bond would receive $(1000)(0.08)\left(\frac{1}{2}\right) = $40.

But, because of market conditions, such as the current prime rate of interest, or the discount rate set by the Federal Reserve Board, or changes in the credit rating of the company issuing the bond, the price of a bond will fluctuate. When the bond price is higher than the face amount, it is trading at a **premium**; when it is lower, it is trading at a **discount**. For example, a bond with a face amount of $1000 and a coupon rate of 8% may trade in the marketplace at a price of $1100, which means the **true yield** is less than 8%.

To obtain the **true interest rate** of a bond, we view the bond as a combination of an annuity of semiannual interest payments plus a single future amount payable at maturity. The price of a bond is therefore the sum of the present value of the annuity of semiannual interest payments plus the present value of the single future payment at maturity. This present value is calculated by discounting at the true interest rate and assuming semiannual payment periods.

### EXAMPLE 11 Pricing Bonds

A bond has a face amount of $1000 and matures in 10 years. The nominal interest rate is 8.5%. What is the price of the bond to yield a true interest rate of 8%?

**SOLUTION**

**STEP 1** Calculate the amount of each semiannual interest payment using the Simple Interest Formula:

\[
(1000)\left(\frac{1}{2}\right)(0.085) = $42.50
\]

**STEP 2** Calculate the present value of the annuity of semiannual payments:

Amount of each payment $P$ from Step 1: $42.50$
Number $n$ of payments ($2 \times 10$ years): 20
True interest rate $i$ per period (half of stated true rate): 4%
Present value $PV_1$ of the annuity for $V$

\[
PV_1 = \frac{P \left[ 1 - (1 + i)^{-n} \right]}{i} = \frac{42.50 \left[ 1 - (1 + 0.04)^{-20} \right]}{0.04} = $577.59
\]

The present value $PV_1$ of the interest payments is $577.59.

**STEP 3** Calculate the present value of the amount payable at maturity:

Amount $A$ payable at maturity: $1000$
Number $n$ of semiannual compounding periods before maturity: 20
True interest rate $i$ per period: 4%
The present value $PV_2$ of the amount payable at maturity:

\[
PV_2 = A(1 + i)^{-n} = 1000 \left(1 + 0.04\right)^{-20} = $456.39
\]

The present value $PV_2$ of the maturity value is $456.39.
STEP 4 The price of the bond is the sum of the two present values:

Present value of the interest payments $577.59
Present value of the maturity value 456.39
Price of bond (Add) $1033.98

To yield 8%, the price of the bond should be $1033.98.

NOW WORK PROBLEM 49.

USING TECHNOLOGY

EXAMPLE 12 Using Excel

Use Excel to Solve Example 11.

SOLUTION

STEP 1 Calculate the amount of each semiannual interest payment:

\[(1000)(.5)(0.085) = 42.50\]

STEP 2 Calculate the present value of the annuity of semiannual payments. Use the Excel function

\[\text{PV}(\text{rate, nper, pmt, fv, type})\]

rate is the interest rate per period, 4%.
nper is the total number of payments, 20.
pmt is the payment made each period, $42.50.
fv is the future value of the annuity or loan, 0.
type is set to 0 for an ordinary annuity.

The result is given in the Excel spreadsheet below.

STEP 3 Calculate the present value of the amount payable at maturity. Use the Excel function

\[\text{PV}(\text{rate, nper, pmt, fv, type})\]

rate is the interest rate per period, 4%.
nper is the total number of payments, 20.
pmt is the payment made each period, 0.00.
fv is the future value of the annuity or loan, 1000.
type is set to 0 for an ordinary annuity.
STEP 4  Determine the price of the bond. Add the two present values. The entire problem's Excel spreadsheet is given below.

![Microsoft Excel](image)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><strong>STEP 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td><strong>Annual interest rate</strong></td>
<td><strong>Period interest rate</strong></td>
<td><strong>Number of periods</strong></td>
<td><strong>Future Value</strong></td>
</tr>
<tr>
<td>3</td>
<td>8.00%</td>
<td>4.00%</td>
<td>20</td>
<td>$0.00</td>
<td>$42.50</td>
</tr>
<tr>
<td>4</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td><em>Present Value of Interest Payments</em></td>
<td>($577.59)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td><em>Step 3</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td><em>Future Value</em></td>
<td>$1,000.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td><em>Payments</em></td>
<td>$0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td><em>Present Value of Maturity Value</em></td>
<td>($456.39)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td><em>Step 4</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td><em>Bond Price</em></td>
<td>-$1,033.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**NOW WORK PROBLEM 49 USING EXCEL.**

### EXERCISE 6.4  Answers Begin on Page AN–33.

**Skill Building**

In Problems 1–6, find the present value of each annuity:

1. The withdrawal is to be $500 per month for 36 months at 4% compounded monthly.
2. The withdrawal is to be $1000 per year for 3 years at 3% compounded annually.
3. The withdrawal is to be $100 per month for 9 months at 2% compounded monthly.
4. The withdrawal is to be $400 per month for 18 months at 5% compounded monthly.
5. The withdrawal is to be $10,000 per year for 20 years at 5% compounded annually.
6. The withdrawal is to be $2000 per month for 3 years at 4% compounded monthly.

In Problems 7–12, find the monthly payment of each loan.

7. A loan of $10,000 amortized over 48 months at 8%
8. A loan of $50,000 amortized over 120 months at 6%
9. A loan of $500,000 amortized over 360 months at 10%
10. A loan of $5,000 amortized over 36 months at 12%
11. A loan of $1,000,000 amortized over 360 months at 8%
12. A loan of $100,000 amortized over 60 months at 6%
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Applications

13. Loan Payments What monthly payment is needed to pay off a loan of $10,000 amortized at 12% compounded monthly for 2 years?
14. Loan Payments What monthly payment is needed to pay off a loan of $500 amortized at 12% compounded monthly for 2 years?
15. Retirement Planning Mr. Doody, at age 65, can expect to live for 20 years. If he can invest at 5% per annum compounded monthly, how much does he need now to guarantee himself $250 every month for the next 20 years?
16. Retirement Planning Sharon, at age 65, can expect to live for 25 years. If she can invest at 4% per annum compounded monthly, how much does she need now to guarantee herself $300 every month for the next 25 years?
17. Used Car Loan In January 2011, the interest rate on 36-month used car loans was 7.4%. Suppose you have a contract to buy a used car for $8000.
   (a) If a bank will lend you $8000 amortized at 7.4% for 36 months, what is your monthly payment?
   (b) What is the total amount you pay?
   (c) What is the interest you pay?
18. New Car Loan In January 2011, the interest rate on 60-month new car loans was 6.93%. Suppose you have a contract to buy a new car for $15,000.
   (a) If a bank will lend you $15,000 amortized at 6.93% for 60 months, what is your monthly payment?
   (b) What is the total amount you pay?
   (c) What is the interest you pay?
19. House Mortgage A couple wishes to purchase a house for $200,000 with a down payment of $40,000. They can amortize the balance either at 8% for 20 years or at 9% for 25 years.
   (a) Which monthly payment is greater?
   (b) For which loan is the total interest paid greater?
   (c) After 10 years, which loan provides the greater equity?
20. House Mortgage A couple has decided to purchase a $250,000 house using a down payment of $20,000. They can amortize the balance at 6% for 25 years.
   (a) What is their monthly payment?
   (b) What is the total interest paid?
   (c) What is their equity after 5 years?
   (d) What is the equity after 20 years?
21. Retirement Planning John is 45 years old and wants to retire at 65. He wishes to make monthly deposits in an account paying 4% compounded monthly so when he retires he can withdraw $300 a month for 30 years. How much should John deposit each month?
22. Cost of a Lottery The grand prize in an Illinois lottery is $6,000,000, which will be paid out in 20 equal annual payments of $300,000 each. Assume the first payment of $300,000 is made, leaving the state with the obligation to pay out $5,700,000 in 19 equal yearly payments of $300,000 each. How much does the state need to deposit in an account paying 6% compounded annually to achieve this?
23. College Expenses Dan works during the summer to help with expenses at school the following year. He is able to save $100 each week for 12 weeks, and he invests it at 1% compounded weekly.
   (a) How much does he have after 12 weeks?
   (b) When school starts, Dan will begin to withdraw equal amounts from this account each week. What is the most Dan can withdraw each week for 34 weeks?
24. Value of an IRA A husband and wife contribute $4000 per year to an IRA paying 5% compounded annually for 20 years.
   (a) What is the value of their IRA?
   (b) How much can they withdraw each year for 25 years at 5% compounded annually?
25. Analyzing a Town House Purchase Mike and Yolanda have just purchased a town house for $200,000. They obtain financing with the following terms: a 20% down payment and the balance to be amortized over 30 years at 9%.
   (a) What is their down payment?
   (b) What is the loan amount?
   (c) How much is their monthly payment on the loan?
   (d) How much total interest do they pay over the life of the loan?
   (e) If they pay an additional $100 each month toward the loan, when will the loan be paid?
   (f) With the $100 additional monthly payment, how much total interest is paid over the life of the loan?
26. House Mortgage Mr. Smith obtained a 25-year mortgage on a house. The monthly payments are $2247.57 (principal and interest) and are based on a 7% interest rate.
   (a) How much did Mr. Smith borrow?
   (b) How much interest will be paid?
27. Car Payments A car costs $12,000. You put 20% down and amortize the rest with equal monthly payments over a 3-year period at 15% to be compounded monthly. What will the monthly payment be?
28. Cost of Furniture Jay pays $320 per month for 36 months for furniture, making no down payment.
   (a) If the interest charged is 0.5% per month on the unpaid balance, what was the original cost of the furniture?
   (b) How much interest did he pay?
29. Paying for Restaurant Equipment A restaurant owner buys equipment costing $20,000.
   (a) If the owner pays 10% down and amortizes the rest with equal monthly payments over 4 years at 12% compounded monthly, what will be the monthly payments?
   (b) How much interest is paid?
30. **Refinancing a Mortgage** A house that was bought 12 years ago for $150,000 is now worth $300,000. Originally, the house was purchased by paying 20% down with the rest financed through a 25-year mortgage at 6.5% interest. The owner (after making 144 equal monthly payments) is in need of cash, and would like to refinance the house. The finance company is willing to loan 80% of the new value of the house amortized over 25 years with the same interest rate. How much cash will the owner receive after paying the balance of the original loan?

31. **Comparing Mortgages** A home buyer is purchasing a $140,000 house. The down payment will be 20% of the price of the house, and the remainder will be financed by a 30-year mortgage at a rate of 6% interest compounded monthly. What will the monthly payment be? Compare the monthly payments and the total amounts of interest paid if a 15-year mortgage is chosen instead of a 30-year mortgage.

32. **Mortgage Payments** Mr. and Mrs. Hoch are interested in building a summer home that will cost $180,000. They intend to use the $60,000 equity in their present house as a down payment on the new one and will finance the rest with a 15-year mortgage at an interest rate of 5.1% compounded monthly. How large will their monthly payment be on the new house?

33. **IRA Withdrawals** How long will it take to exhaust an IRA of $100,000 if you withdraw $2000 every month? Assume a rate of interest of 5% compounded monthly.

34. **IRA Withdrawals** How long will it take to exhaust an IRA of $200,000 if you withdraw $3000 every month? Assume a rate of interest of 4% compounded monthly.

35. **Paying for College** In 2009–2010 the average total tuition, fees, and room and board charges for in-state students at public institutions was $15,213 or approximately $1690 per month for 9 months. Suppose Jeremy has a college fund that pays 4.8% compounded monthly. How much money must be in his college fund on August 1 in order to make 9 equal monthly withdrawals of $1690 beginning September 1 and leave a zero balance?

36. **First New Car** Nick recently graduated from college and began his first job. As a reward for these recent successes, he plans to purchase his first new car, a Ford Mustang GT. The base list price for a 2010 Ford Mustang GT Premium Coupe is $32,845. Since Nick is a first-time buyer, Ford will give him a $500 rebate. Nick will also receive $2500 for trading in his current car. Ford Credit will finance Nick’s purchase for 6.9% annual interest compounded monthly for 72 months. What will Nick’s monthly payment be?

**Source:** www.fordvehicles.com

37. **Research Endowment** A private foundation plans to fund a five-year $1,000,000 research project at a university. How much money must the foundation deposit into an account paying 7.8% annual interest compounded quarterly so that the university can make equal withdrawals at the end of each quarter totaling $1,000,000 over the five years?

38. **Scholarship Fund** An alumnus donates $100,000 to a university in order to fund a merit scholarship. The university invests the donation into an account paying 5.4% annual interest compounded annually. After one year the university will use this fund to award one scholarship per year for 20 years, at which time the fund is fully depleted. What will be the dollar amount of each scholarship?

39. **Refinancing a Home Loan** Suppose that a couple decides to refinance a 30-year home loan, after making monthly payments for 5 years. The original loan amount was $312,000 with an annual interest rate of 6.825%, compounded monthly. This couple accepts an offer from a California bank to refinance their loan with no closing costs. The new loan will be a 25-year loan and will have a lower interest rate, 6.125%, compounded monthly. With this refinancing, by how much will the monthly payments be reduced? Over the full term of the new loan, how much total interest will be saved?

40. **Refinancing a Home Loan** Suppose that a couple decides to refinance a 30-year home loan, after making monthly payments for 3 years. The original loan amount was $285,000 with an annual interest rate of 6.75%, compounded monthly. This couple accepts an offer from a California bank to refinance their loan with no closing costs. The new loan will be a 25-year loan but will have a lower interest rate, 5.5%, compounded monthly. With this refinancing, by how much will the monthly payments be increased? Over the full term of the new loan, how much total interest will be saved?

41. **Prepaying a Home Loan** After making minimum payments for 4 years on a 30-year home loan, a couple decides to pay an additional $150 per month toward the principal. The original loan amount was $235,000 with an annual interest rate of 6.125%, compounded monthly. By how many months (rounded to the nearest tenth) will the term of this loan be reduced with this additional monthly payment? Over the life of this loan, how much interest will be saved?
42. Adjustable Rate Mortgages A couple decides to accept an adjustable rate, 30-year mortgage, in which the interest rate is fixed for the first 5 years and then may be adjusted annually beginning with year 6. Suppose that this couple borrows $305,000 with the annual interest rate, compounded monthly, for the first 5 years set at 5.5%. Suppose that the interest rate is increased to 6.75% for the remaining term of the loan. Calculate the monthly payment during the first 5 years and after the first 5 years. Find the amount of interest that would be paid over the term of this loan.

43. Buying versus Leasing The state of Alaska negotiated multiple-award term contracts for the purchase and lease-purchase of copiers for state agencies. Among the options, a state office can lease a Xerox WC74289 copier for 36 months at $196.88 per month. The state office can also purchase the same copier outright for $5682. If the state office can invest money at 5.4% compounded monthly, which alternative is preferable?


44. Leasing Problem A corporation may obtain a machine either by leasing it for 5 years (the useful life) at an annual rent of $2000 or by purchasing the machine for $8100.
(a) If the corporation can borrow money at 10% per annum, which alternative is preferable?
(b) If the corporation can borrow money at 14% per annum, which alternative is preferable?

45. Leasing Decision A company can lease 10 trucks for 4 years (their useful life) for $50,000 a year, or it can purchase them for $175,000. If the company can earn 5% per annum on its money, which choice is preferable, leasing or purchasing?

46. Capital Expenditure Analysis In an effort to increase productivity, a corporation decides to purchase a new piece of equipment. Two models are available; both reduce labor costs. Model A costs $50,000, saves $12,000 per year in labor costs, and has a useful life of 10 years. Model B costs $42,000, saves $10,000 annually in labor costs, and has a useful life of 8 years. If the time value of money is 10% per annum, what piece of equipment provides a better investment?

47. Capital Expenditure Analysis Machine A costs $10,000 and has a useful life of 8 years, and machine B costs $8000 and has a useful life of 6 years. Suppose machine A generates an annual labor savings of $2000 while machine B generates an annual labor savings of $1800.

(a) Assuming the time value of money (investment opportunity rate) is 10% per annum, which machine is preferable?
(b) If the time value of money is 14% per annum, which machine is preferable?

Problems 48–53 involve pricing corporate bonds and Treasury notes.

48. Corporate Bonds A bond has a face amount of $1000 and matures in 15 years. The nominal interest rate is 9%.
(a) What is the price of the bond that will yield an effective interest rate of 8%?
(b) What is the price of the bond to yield an effective interest rate of 10%?

49. Corporate Bonds A bond has a face amount of $10,000 and matures in 8 years. The nominal rate of interest on the bond is 6.25%. At what price would the bond yield a true rate of interest of 6.5%?

Source: U.S. Treasury (This is an actual U.S. Treasury note issued on May 17, 2010.)

50. Treasury Notes A bond has a face amount of $1000 and matures in 15 years. The nominal interest rate is 3.500%, and a true interest rate of 3.548%.
Source: U.S. Treasury (This is an actual U.S. Treasury note issued on June 1, 2010.)

51. Treasury Notes Determine the selling price of a 5-year Treasury note, with a maturity value of $1000, a nominal interest rate of 2.125%, and a true interest rate of 2.130%.
Source: U.S. Treasury (This is an actual U.S. Treasury note issued on February 16, 2010.)

52. Treasury Notes Determine the selling price of a 10-year Treasury note with a maturity value of $1000, a nominal interest rate of 3.625%, and a true interest rate of 3.692%.
Source: U.S. Treasury (This is an actual U.S. Treasury note issued on December 31, 2009.)

53. Treasury Notes Determine the selling price of a 5-year Treasury note with a maturity value of $1000, a nominal interest rate of 2.625%, and a true interest rate of 2.665%.
Source: U.S. Treasury (This is an actual U.S. Treasury note issued on December 31, 2009.)

*Treasury notes pay interest semiannually.
6.5 Annuities and Amortization Using Recursive Sequences

PREPARING FOR THIS SECTION
Before getting started, review the following:
• Recursive Sequences (Appendix A, Section A.4, pp. A–34 to A–37)

NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 348.

OBJECTIVES
1. Use sequences and a graphing utility to solve annuity problems (p. 345)
2. Use sequences and a graphing utility to solve amortization problems (p. 347)

1 Use Sequences and a Graphing Utility to Solve Annuity Problems

In Section 6.2 we developed the Compound Interest Formula, which gives the future value when a fixed amount of money is deposited in an account that pays interest compounded periodically. Often, though, money is invested in equal amounts at periodic intervals. An annuity is a sequence of equal periodic deposits. The periodic deposits may be made annually, quarterly, monthly, or daily.

When deposits are made at the same time that the interest is credited, the annuity is called ordinary. We will only deal with ordinary annuities here. The amount of an annuity is the sum of all deposits made plus all interest paid.

Suppose that the initial amount deposited in an annuity is $M$, the periodic deposit is $P$, and the per annum rate of interest is $r$ compounded $N$ times per year. The periodic deposit is made at the same time that the interest is credited, so $N$ deposits are made per year. The amount $A_n$ of the annuity after $n$ deposits will equal the amount of the annuity after $n-1$ deposits, $A_{n-1}$, plus the interest earned on this amount plus the periodic deposit $P$. That is,

$$A_n = A_{n-1} + \frac{r}{N}A_{n-1} + P = \left(1 + \frac{r}{N}\right)A_{n-1} + P$$

We have established the following result:

Theorem

Annuity Formula

If $A_0 = M$ represents the initial amount deposited in an annuity that earns a rate of $r$ per annum compounded $N$ times per year, and if $P$ is the periodic deposit made at each payment period, then the amount $A_n$ of the annuity after $n$ deposits is given by the recursive sequence

$$A_0 = M, \quad A_n = \left(1 + \frac{r}{N}\right)A_{n-1} + P, \quad n \geq 1$$

*We use $N$ to represent the number of times interest is compounded per annum instead of $n$, since $n$ is the traditional symbol used with sequences to denote the term of the sequence.
Formula (1) may be explained as follows: the money in the account initially, $A_0$, is $M$; the money in the account after $n − 1$ payments, $A_{n−1}$, earns interest $\frac{r}{N}$ during the $n$th period; so when the periodic payment of $P$ dollars is added, the amount after $n$ payments, $A_n$, is obtained.

EXAMPLE 1 Saving for Spring Break

A trip to Cancun during spring break will cost $450 and full payment is due March 2. To have the money, a student, on September 1, deposits $100 in a savings account that pays 4% per annum compounded monthly. Beginning October 1, on the first of each month, the student deposits $50 in this account.

(a) Find a recursive sequence that explains how much is in the account after $n$ months.

(b) Use the TABLE feature to list the amounts of the annuity for the first 6 months.

(c) After the deposit on March 1 is made, is there enough in the account to pay for the Cancun trip?

(d) If the student deposits $60 each month, will there be enough for the trip?

SOLUTION

(a) The initial amount deposited in the account is $A_0 = 100$. The monthly deposit is $P = 50$, and the per annum rate of interest is $r = 0.04$ compounded $N = 12$ times per year. The amount $A_n$ in the account after $n$ monthly deposits is given by the recursive sequence

$$A_0 = 100, \quad A_n = \left(1 + \frac{r}{N}\right)A_{n−1} + P = \left(1 + \frac{0.04}{12}\right)A_{n−1} + 50$$

(b) In SEQuence mode on a TI-84 Plus, enter the sequence $\{A_n\}$ and create Table 8. On September 1 ($n = 0$), there is $100 in the account. After the first payment on October 1, the value of the account is $150.33. After the second payment on November 1, the value of the account is $200.83. After the third payment on December 1, the value of the account is $251.50, and so on.

(c) On March 1 ($n = 6$), there is only $404.53, not enough to pay for the trip to Cancun.

(d) If the periodic deposit, $P$, is $60, then on March 1, there is $465.03 in the account, enough for the trip. See Table 9.

<table>
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<tr>
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<td>1</td>
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<tr>
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<tr>
<td>6</td>
<td>546.03</td>
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</table>

NOW WORK PROBLEM 7.
Recursive sequences can also be used to compute information about loans. When equal periodic payments are made to pay off a loan, the loan is said to be **amortized**.

### Theorem

**Amortization Formula**

If $B$ is borrowed at an interest rate of $r$ per annum compounded monthly, the balance $A_n$ due after $n$ monthly payments of $P$ is given by the recursive sequence

$$A_0 = B, \quad A_n = \left(1 + \frac{r}{12}\right)A_{n-1} - P, \quad n \geq 1 \quad (2)$$

Formula (2) may be explained as follows:

The initial loan balance is $B$. The balance due after $n$ payments will equal the balance due previously, $A_{n-1}$, plus the interest charged on that amount reduced by the periodic payment $P$.

### EXAMPLE 2 Home Mortgage Payments

John and Wanda borrowed $180,000 at 7\% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be $1197.54.

(a) Find a recursive formula that represents their balance after each payment of $1197.54 has been made.

(b) Determine their balance after the first payment is made.

(c) When will their balance be below $170,000?

### SOLUTION

(a) We use Formula (2) with $A_0 = 180,000$, $r = 0.07$, and $P = $1197.54. Then

$$A_0 = 180,000, \quad A_n = \left(1 + \frac{0.07}{12}\right)A_{n-1} - 1197.54$$

(b) In SEQuence mode on a TI-84 Plus, enter the sequence \{A_n\} and create Table 10. After the first payment is made, the balance is $A_1 = $179,852.

(c) Scroll down until the balance is below $170,000. See Table 11. After the 58th payment is made ($n = 58$), the balance is below $170,000.

### TABLE 10

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### TABLE 11

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<tr>
<td>58</td>
<td>$170442$</td>
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<tr>
<td>59</td>
<td>$169350$</td>
</tr>
<tr>
<td>60</td>
<td>$168257$</td>
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</table>

NOW WORK PROBLEM 3.
Applications

3. **Credit Card Debt** John has a balance of $3000 on his credit card that charges 1\% interest per month on any unpaid balance. John can afford to pay $100 toward the balance each month. His balance each month after making a $100 payment is given by the recursively defined sequence

\[ B_n = 3000, \quad B_{n+1} = 1.01B_n - 100 \]

(a) Determine John's balance after making the first payment. That is, determine \( B_1 \).

(b) Using a graphing utility, determine when John's balance will be below $2000. How many payments of $100 have been made?

(c) Using a graphing utility, determine when John will pay off the balance. What is the total of all the payments?

(d) What was John's interest expense?

4. **Car Loans** Phil bought a car by taking out a loan for $18,500 at 0.5\% interest per month. Phil's normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month after making a $100 payment is given by the recursively defined sequence

\[ B_0 = 18500, \quad B_{n+1} = 1.005B_n - 534.47 \]

(a) Determine Phil's balance after making the first payment. That is, determine \( B_1 \).

(b) Using a graphing utility, determine when Phil's balance will be below $10,000. How many payments of $534.47 have been made?

(c) Using a graphing utility, determine when Phil will pay off the balance. What is the total of all the payments?

(d) What was Phil's interest expense?

5. **Trout Population** A pond currently contains 2000 trout. A fish hatchery decides to add an additional 20 trout each month. In addition, it is known that the trout population is growing 3\% per month. The size of the population after \( n \) months is given by the recursively defined sequence

\[ p_0 = 2000, \quad p_{n+1} = 1.03p_n + 20 \]

(a) How many trout are in the pond at the end of the second month? That is, what is \( p_2 \)?

(b) Using a graphing utility, determine how long it will be before the trout population reaches 5000.

6. **Environmental Control** The Environmental Protection Agency (EPA) determines that Maple Lake has 250 tons of pollutants as a result of industrial waste and that 10\% of the pollutants present are neutralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 tons of new pollutants entering the lake each year. The amount of pollutants in the lake at the end of each year is given by the recursively defined sequence

\[ p_0 = 250, \quad p_{n+1} = 0.9p_n - 15 \]

(a) Determine the amount of pollutants in the lake at the end of the second year. That is, determine \( p_2 \).

(b) Using a graphing utility, provide pollutant amounts for the next 20 years.

(c) What is the equilibrium level of pollution in Maple Lake? That is, what is \( \lim_{n \to \infty} p_n \)?

7. **Roth IRA** On January 1, 2011, Bob decides to place $500 at the end of each quarter into a Roth Individual Retirement Account. The amount of new pollutants entering the lake each year. The amount of pollutants in the lake at the end of each year is given by the recursively defined sequence

\[ p_0 = 250, \quad p_{n+1} = 0.9p_n - 15 \]

(a) Determine the amount of pollutants in the lake at the end of the second year. That is, determine \( p_2 \).

(b) Using a graphing utility, provide pollutant amounts for the next 20 years.

(c) What is the equilibrium level of pollution in Maple Lake? That is, what is \( \lim_{n \to \infty} p_n \)?

7. **Roth IRA** On January 1, 2011, Bob decides to place $500 at the end of each quarter into a Roth Individual Retirement Account.

(a) Find a recursive formula that represents Bob's balance at the end of each quarter if the rate of return is assumed to be 8\% per annum compounded quarterly.

(b) How long will it be before the value of the account exceeds $100,000?

(c) What will be the value of the account in 25 years when Bob retires?

8. **Education IRA** On January 1, 2011, John's parents decide to place $45 at the end of each month into an Education IRA.

(a) Find a recursive formula that represents the balance at the end of each month if the rate of return is assumed to be 6\% per annum compounded monthly.

(b) How long will it be before the value of the account exceeds $4000?

(c) What will be the value of the account in 16 years when John goes to college?

9. **Home Loan** Bill and Laura borrowed $150,000 at 6\% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be $899.33.

(a) Find a recursive formula for their balance after each monthly payment has been made.

(b) Determine Bill and Laura's balance after the first payment.

(c) Using a graphing utility, create a table showing Bill and Laura's balance after each monthly payment.

(d) Using a graphing utility, determine when Bill and Laura's balance will be below $140,000.

(e) Using a graphing utility, determine when Bill and Laura will pay off the balance.
(f) Determine Bill and Laura’s interest expense when the loan is paid.

(g) Suppose that Bill and Laura decide to pay an additional $100 each month on their loan. Answer parts (a) to (f) under this scenario.

(h) Is it worthwhile for Bill and Laura to pay the additional $100? Explain.

10. Home Loan  Jodi and Jeff borrowed $120,000 at 6.5% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be $758.48.

(a) Find a recursive formula for their balance after each monthly payment has been made.

(b) Determine Jodi and Jeff’s balance after the first payment.

(c) Using a graphing utility, create a table showing Jodi and Jeff’s balance after each monthly payment.

(d) Using a graphing utility, determine when Jodi and Jeff’s balance will be below $100,000.

(e) Using a graphing utility, determine when Jodi and Jeff will pay off the balance.

(f) Determine Jodi and Jeff’s interest expense when the loan is paid.

(g) Suppose that Jodi and Jeff decide to pay an additional $100 each month on their loan. Answer parts (a) to (f) under this scenario.

(h) Is it worthwhile for Jodi and Jeff to pay the additional $100? Explain.

‘Are You Prepared?’ Answers

1. $s_3 = 5$

2. $s_1 = -2, s_{n+1} = s_n + 3$

CHAPTER 6 REVIEW  OBJECTIVES

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<td>1</td>
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<td>26, 27, 28, 34, 36, 37, 43, 45, 46</td>
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<td>2</td>
<td>Use sequences and a graphing utility to solve amortization problems (p. 347)</td>
<td>29, 30, 31, 39, 47</td>
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</table>
Chapter 6  Finance

IMPORTANT FORMULAS

Simple Interest Formula (p. 295) \( I = Prt \)

Discounted Loans (p. 297) \( R = L - Lrt \)

Compound Interest Formula (pp. 303 and 307) 
\[
A = P\left(1 + \frac{r}{n}\right)^{nt}
\]
\[
A = Pe^{rt}
\]

Present Value of an Annuity (p. 330) 
\[
V = P\left[1 - \left(1 + \frac{i}{n}\right)^{-n}\right]
\]

Amortization (p. 333) 
\[
P = \frac{i}{1 - \left(1 + \frac{i}{n}\right)^{-n}}
\]

Amount of an Annuity (p. 318) 
\[
A = P\left[\frac{(1 + i)^n - 1}{i}\right]
\]

REVIEW EXERCISES

Answers to odd-numbered problems begin on page AN-34.

Blue Problem numbers indicate the author’s suggestions for a practice test.

In Problems 1–9, calculate the indicated quantity.

1. 3% of 500
2. 20% of 1200
3. 140% of 250
4. What percent of 200 is 40?
5. What percent of 350 is 75?
6. What percent of 50 is 125?
7. 12 is 15% of what number?
8. 25 is 6% of what number?
9. 11 is 0.5% of what number?
10. Sales Tax The sales tax in New York City is 8.875%. If the total charge for a DVD player (including tax) is $162.75, what does the DVD player itself cost?
11. Sales Tax Indiana has a sales tax of 7%. Dan purchased gifts for his family worth $330.00. How much sales tax will Dan have to pay on his purchase?
12. Find the interest \( I \) charged and amount \( A \) due, if $400 is borrowed for 9 months at 12% simple interest.
13. Dan borrows $500 at 9% per annum simple interest for 1 year and 2 months. What is the interest charged, and what is the amount due?
14. Interest Due Jim borrows $14,000 for a period of 4 years at 6% simple interest. Determine the interest due on the loan.
15. Loan Amount Warren needs $15,000 for a new machine for his auto repair shop. He obtains a 2-year discounted loan at 12% interest. How much must he repay to settle his debt?
16. Treasury Bills How much should a bank bid for a 15-month, $5,000 Treasury bill in order to earn 2.5% simple interest?
17. Find the amount of an investment of $100 after 2 years and 3 months at 3% compounded monthly.
18. Mike places $200 in a savings account that pays 4% per annum compounded monthly. How much is in his account after 9 months?
19. Choosing a Car Loan A car dealer offers Mike the choice of two loans:
   (a) $3000 for 3 years at 12% per annum simple interest
   (b) $3000 for 3 years at 10% per annum compounded monthly
Which loan costs Mike the least?
20. A mutual bond fund pays 9% per annum compounded monthly. How much should I invest now so that 2 years from now I will have $100 in the account?
21. Saving for a Bicycle Katy wants to buy a bicycle that costs $75 and will purchase it in 6 months. How much should she put in her savings account for this if she can get 3% per annum compounded monthly?
22. Doubling Money Marcia has $220,000 saved for her retirement. How long will it take for the investment to double in value if it earns 6% compounded semiannually?
23. Doubling Money What annual rate of interest will cause an investment to double in 12 years?
24. Effective Rate of Interest A bank advertises that it pays 3.5% interest compounded monthly. What is the effective rate of interest?
25. Effective Rate of Interest What interest rate compounded quarterly has an effective interest rate of 6%?
26. Saving for a Car Mike decides he needs $500 1 year from now to buy a used car. If he can invest at 3% compounded monthly, how much should he save each month to buy the car?
27. Saving for a House Mr. and Mrs. Corey are newlyweds and want to purchase a home, but they need a down payment of $40,000. If they want to buy their home in 2 years, how much should they save each month in their savings account that pays 3% per annum compounded monthly?
28. **True Cost of a Car**  Mike has just purchased a used car and will make equal payments of $50 per month for 18 months at 12% per annum charged monthly. How much did the car actually cost? Assume no down payment.

29. **House Mortgage**  Mr. and Mrs. Ostedt have just purchased a $400,000 home and made a 25% down payment. The balance can be amortized at 10% for 25 years.
   (a) What are the monthly payments?
   (b) How much interest will be paid?
   (c) What is their equity after 5 years?

30. **Inheritance Payouts**  An inheritance of $25,000 is to be paid in equal amounts over a 5-year period at the end of each year. If the $25,000 can be invested at 5% per annum, what is the annual payment?

31. **House Mortgage**  A mortgage of $125,000 is to be amortized at 9% per annum for 25 years. What are the monthly payments? What is the equity after 10 years?

32. **Paying Off Construction Bonds**  A state has $8,000,000 worth of construction bonds that are due in 25 years. What annual sinking fund deposit is needed if the state can earn 10% per annum on its money?

33. **Depletion Problem**  How much should Mr. Graff pay for a gold mine expected to yield an annual return of $20,000 and to have a life expectancy of 20 years, if he wants to have a 15% annual return on his investment and he can set up a sinking fund that earns 10% a year?

34. **Depletion Problem**  An oil well is expected to yield an annual net return of $25,000 for the next 15 years, after which it will run dry. An investor wants a return on his investment of 20%. He can establish a sinking fund earning 5% annually. How much should he pay for the oil well?

35. **Retirement Income**  Mr. Doody, at age 70, is expected to live for 15 years. If he can invest at 4% per annum compounded monthly, how much does he need now to guarantee himself $300 every month for the next 15 years?

36. **Saving for the Future**  Hal deposited $100 in an account paying 4% per annum compounded monthly for 25 years. At the end of the 25 years Hal retires. What is the largest amount he may withdraw monthly for the next 35 years?

37. **Saving for College**  Mr. Jones wants to save for his son's college education. If he deposits $500 every 6 months at 3% compounded semiannually, how much will he have on hand at the end of 8 years?

38. **Saving for the Future**  How much money should be invested at 2% compounded quarterly in order to have $20,000 in 6 years?

39. **Loan Payments**  What monthly payment will amortize a loan of $3000 borrowed at 12% compounded monthly for 2 years?

40. **Paying Off School Bonds**  A school board issues bonds in the amount of $20,000,000 to be retired in 25 years. After 20 years, how much must be paid each year into a sinking fund at 6% compounded annually to pay off the $20,000,000?

41. **What effective rate of interest corresponds to a rate of 9% compounded monthly?**

42. **If an amount was borrowed 5 years ago at 6% compounded quarterly, and $6000 is owed now, what was the original amount borrowed?**

43. **Trust Fund Payouts**  John is the beneficiary of a trust fund set up for him by his grandparents. If the trust fund amounts to $20,000 earning 8% compounded semiannually and he is to receive the money in equal semiannual installments for the next 15 years, how much will he receive each 6 months?

44. **IRA**  Mike has $4000 in his IRA. At 6% annual interest compounded monthly, how much will he have at the end of 20 years?

45. **Retirement Funds**  An employee gets paid at the end of each month and $60 is withheld from her paycheck for a retirement fund. The fund pays 1% per month (equivalent to 12% annually compounded monthly). What amount will be in the fund at the end of 30 months?

46. **Retirement Income**  A man has $50,000 invested at 6% compounded quarterly at the time he retires. If he wishes to withdraw money every 3 months for the next 7 years, what will be the size of his withdrawals?

47. **Buying a Car**  A student borrowed $4000 from a credit union toward purchasing a car. The interest rate on such a loan is 14% compounded quarterly, with payments due every quarter. The student wants to pay off the loan in 4 years. Find the quarterly payment.

48. **Leasing Decision**  A firm can lease office furniture for 5 years (its useful life) for $20,000 a year, or it can purchase the furniture for $80,000. Which is a better choice if the firm can invest its money at 10% per annum?
You’ve made a decision; you want to buy a new Toyota Prius. The cash purchase price of the vehicle you have selected is $23,700. Here are some options the Toyota dealer has suggested.

1. **Option A** For this option a down payment of $3,000 is required with monthly payments at a 5% annual rate for 5 years. What is the monthly payment? What is the total amount of interest paid over the term of this loan?

2. **Option B** A bigger down payment will lower the monthly payments. With a down payment of $6,000, the dealer offers you a loan with monthly payments at a 4.8% annual rate for 6 years. What is the monthly payment? What is the total amount of interest paid over the term of this loan?

3. **Option C** Your bank offers a car loan with 20% down and monthly payments at a 5.5% annual rate for 7 years. What would be the principal amount for this loan? What would be the monthly payment for this loan? What would be the total amount of interest paid over the term of this loan?

4. Based on minimizing the total amount of interest paid, which option would you select? Which option would you select to minimize monthly payments?

5. One way to compare the three options is to compute the total present value of each option. To do this, we will use an annual rate of return of 8% per year, compounded monthly. For each of the above options, find the present value of the monthly payments and add this present value to the down payment for that option to find the total present value for that option. Which option minimizes total present value?

6. **Option D** A local credit union offers car loans that require a down payment of 15%, no monthly payments for 12 months, and then monthly payments for 5 years of $409. Find the total present value of Option D and compare to the total present values found in Problem 5. Hint: First find the present value, $P_1$, of the 5 years of payments, using an annual rate of return of 8%, compounded monthly. $P_1$ is the present value of the payments at the time when the payments begin (one year from purchase). Then find the principal $P_0$ that would generate $P_1$ in 1 year at an annual rate of 8%, compounded monthly. The total present value for Option D would be the down payment plus $P_0$.

7. **Option E** The Toyota dealer also offers a lease with purchase option for this Prius, which requires an initial payment of $1,500 and monthly lease payments for up to 4 years of $225. At any time within these 4 years you would have the option to purchase the Toyota Prius for a depreciated price, determined as follows: Each year the price of this Prius depreciates by an amount equal to 10% of the price in the previous year. Hint: Given an initial purchase price $P$, the depreciated price after 1 year would be $0.9P$; the depreciated price after 2 years would be $0.9^2P$, and so on. Find the depreciated price after 3 years and after 4 years.

8. The total present value for Option E will be the initial payment, plus the present value of the lease payments, plus the present value of the depreciated price. Using an annual rate of return of 8%, compounded monthly, find (a) the total present value for Option E with purchase after 3 years of leasing and (b) the total present value for Option E with purchase after 4 years of leasing. In order to minimize total present value, should the purchase be made after 3 years of leasing or after 4 years of leasing?

9. Compare Options, A, B, C, D, E (with purchase after 3 years), and E (with purchase after 4 years). Which of these options has the lowest total present value?

10. For Option E with purchase after 4 years, it would be practical to set up a sinking fund to accumulate the amount needed to purchase the Prius after the 4-year leasing period. Determine the annual interest rate for Option E, with purchase after 4 years, as follows: Determine the required monthly payment into a sinking fund that earns interest at an annual rate of 8%, compounded monthly, to accumulate the purchase price at the end of 4 years of leasing. The monthly lease payment plus the monthly sinking fund payment provide a monthly payment that would finance the purchase of the Prius after 4 years. These total monthly payments would pay off an amount equal to the purchase price minus the $1,500 initial payment for the lease agreement. Round this annual interest rate to the nearest hundredth of a percent. Also compute the total present value of Option E (with purchase after 4 years) by using the present value of the monthly sinking fund payments needed to accumulate the purchase price in 4 years.

11. Use the method described in Problem 10 to compute the annual interest rate for Option E with purchase after 3 years. Also compute the total present value of Option E (with purchase after 3 years) by using the present value of the monthly sinking fund payments needed to accumulate the purchase price in 3 years.
Mathematical Questions from Professional Exams

1. CPA Exam Which of the following should be used to calculate the amount of the equal periodic payments that could be equivalent to an outlay of $3000 at the time of the last payment?
   (a) Amount of 1
   (b) Amount of an annuity of 1
   (c) Present value of an annuity of 1
   (d) Present value of 1

2. CPA Exam A businessman wants to withdraw $3000 (including principal) from an investment fund at the end of each year for 5 years. How should he compute his required initial investment at the beginning of the first year if the fund earns 6% compounded annually?
   (a) $3000 times the amount of an annuity of $1 at 6% at the end of each year for 5 years
   (b) $3000 divided by the amount of an annuity of $1 at 6% at the end of each year for 5 years
   (c) $3000 times the present value of an annuity of $1 at 6% at the end of each year for 5 years
   (d) $3000 divided by the present value of an annuity of $1 at 6% at the end of each year for 5 years

3. CPA Exam A businesswoman wants to invest a certain sum of money at the end of each year for 5 years. The investment will earn 6% compounded annually. At the end of 5 years, she will need a total of $30,000 accumulated. How should she compute the required annual investment?
   (a) $30,000 times the amount of an annuity of $1 at 6% at the end of each year for 5 years
   (b) $30,000 divided by the amount of an annuity of $1 at 6% at the end of each year for 5 years
   (c) $30,000 times the present value of an annuity of $1 at 6% at the end of each year for 5 years
   (d) $30,000 divided by the present value of an annuity of $1 at 6% at the end of each year for 5 years

4. CPA Exam What amount should be deposited in a bank today to grow to $1000 3 years from today?
   (a) $1000
   (b) $1000 \times 0.926 \times 3
   (c) $(1000 \times 0.926) + (1000 \times 0.857) + (1000 \times 0.794)
   (d) $1000 \times 0.794

5. CPA Exam What amount should an individual have in her bank account today before withdrawal if she needs $2000 each year for 4 years with the first withdrawal to be made today and each subsequent withdrawal at 1-year intervals? (She is to have exactly a zero balance in her bank account after the fourth withdrawal.)
   (a) $2000 + (2000 \times 0.926) + (2000 \times 0.857) + (2000 \times 0.794) 
   (b) $2000 \times 0.735 \times 4 
   (c) $(2000 \times 0.926) + (2000 \times 0.857) + (2000 \times 0.794) + (2000 \times 0.735) 
   (d) $2000 \times 0.926 \times 4

6. CPA Exam If an individual put $3000 in a savings account today, what amount of cash would be available 2 years from today?
   (a) $3000 \times 0.857 
   (b) $3000 \times 0.857 \times 2 
   (c) $3000 \times 0.857 
   (d) $3000 \times 0.926 \times 2

Items 4–6 apply to the appropriate use of present value tables. Given below are the present value factors for $1.00 discounted at 8% for 1 to 5 periods. Each of the following items is based on 8% interest compounded annually from day of deposit to day of withdrawal.

<table>
<thead>
<tr>
<th>Periods</th>
<th>Present Value of $1 Discounted at 8% per Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.926</td>
</tr>
<tr>
<td>2</td>
<td>0.857</td>
</tr>
<tr>
<td>3</td>
<td>0.794</td>
</tr>
<tr>
<td>4</td>
<td>0.735</td>
</tr>
<tr>
<td>5</td>
<td>0.681</td>
</tr>
</tbody>
</table>

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Probability

Take a look at the back cover of your book. Do you see the bar code and the ISBN number? Have you ever wondered what all the numbers and hyphens mean? You know it identifies your book, but what else does it do? And what about the bar code on that bottle of water you just bought? You know, the UPC code. What do those numbers mean? Every product you buy, from food to drugs to clothing—just about everything—has a bar code that identifies the item purchased. How is this possible using just 12 digits? In this chapter we discuss counting techniques that, together with the Chapter Project, will explain these everyday codes.

OUTLINE

7.1 Sets
7.2 The Number of Elements in a Set
7.3 The Multiplication Principle
7.4 Sample Spaces and the Assignment of Probabilities
7.5 Properties of the Probability of an Event
7.6 Expected Value

• Chapter Review
• Chapter Project
• Mathematical Questions from Professional Exams

A Look Back, A Look Forward

In Chapters 1–5 we investigated applications of linear equations and inequalities. In Chapter 6 we introduced a new topic, financial models. In this chapter we begin yet another new topic, probability. As the Outline above shows, we begin by discussing sets and techniques for counting the number of elements in a set.
Set Notation

To treat a collection of distinct objects as a whole, we use the idea of a set. For example, the set of digits consists of the collection of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. If we use the symbol $D$ to denote the set of digits, then we can write

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

In this notation the braces $\{ \}$ are used to enclose the objects, or elements, in the set. This method of denoting a set is called the roster method. A second way to denote a set is to use set-builder notation, where the set $D$ of digits is written as

$$D = \{x \mid x \text{ is a digit}\}$$

Read as “$D$ is the set of all $x$ such that $x$ is a digit.”

**EXAMPLE 1** Examples of Set Notation

(a) $E = \{x \mid x \text{ is an even digit}\} = \{0, 2, 4, 6, 8\}$

(b) $O = \{x \mid x \text{ is an odd digit}\} = \{1, 3, 5, 7, 9\}$

**EXAMPLE 2** Using Set Notation

(a) Let $A$ denote the set that consists of all possible outcomes resulting from tossing a coin two times and observing whether a head or tail occurs on each toss. If we let $H$ denote heads and $T$ denote tails, then the set $A$ can be written as

$$A = \{HH, HT, TH, TT\}$$

where, for instance, $TH$ means the first toss resulted in tails and the second toss resulted in heads.

(b) Let $B$ denote the set consisting of all possible arrangements of the digits without repetition. Some typical elements of $B$ are

$$1478906532 \quad 4875326019 \quad 3214569870$$

The number of elements in $B$ is very large, so listing all of them is impractical. Here set-builder notation can be used to write

$$B = \{x \mid x \text{ is a ten-digit number in which no digit is repeated}\}$$
The elements of a set are never repeated. That is, we never write \{3, 2, 2\}; the correct listing is \{3, 2\}. Finally, the order in which the elements of a set are listed does not make any difference. The three sets
\[
\{3, 2, 4\} \quad \{2, 3, 4\} \quad \{4, 3, 2\}
\]
are different listings of the same set. The elements of a set distinguish the set—not the order in which the elements are written.

A set that has no elements is called the **empty set** or **null set** and is denoted by the symbol \(\emptyset\).

### Identify Relations between Pairs of Sets

We begin to develop an algebra for sets by defining what we mean by two sets being *equal*.

#### Definition: Equality of Sets

Let \(A\) and \(B\) be two sets. We say that \(A\) is **equal to** \(B\), written as
\[
A = B
\]
if and only if \(A\) and \(B\) have the same elements. If \(A\) and \(B\) are not equal, we write
\[
A \neq B
\]

For example, \(\{1, 2, 3\} = \{1, 3, 2\}\) and \(\{1, 2\} \neq \{1, 2, 3\}\).

Two sets can also be compared using the ideas of **subset** and **proper subset**.

#### Definition: Subset

Let \(A\) and \(B\) be two sets. We say that \(A\) is a **subset** of \(B\) or that \(A\) is **contained in** \(B\), written as
\[
A \subseteq B
\]
if and only if every element of \(A\) is also an element of \(B\). If \(A\) is not a subset of \(B\), we write
\[
A \not\subseteq B
\]

For example, \(\{1, 2\} \subseteq \{1, 2, 3\}\) and \(\{1, 2, 3, 4\} \not\subseteq \{1, 2\}\).

**Now Work Problems 5 and 7.**

Subset can also be defined as follows: \(A \subseteq B\) if and only if whenever \(x\) is an element of \(A\), then \(x\) is an element of \(B\) for all \(x\). This way of defining \(A \subseteq B\) is useful for obtaining
various laws that sets obey. For example, it follows that for any set \( A \), \( A \subseteq A \); that is, every set is a subset of itself. Do you see why? Whenever \( x \) is in \( A \), then \( x \) is in \( A \! \) 

The statement “\( A \) is a subset of \( B \)” is equivalent to saying “there are no elements in set \( A \) that are not also elements in set \( B \).” In particular, suppose \( A = \emptyset \). Since the empty set \( \emptyset \) has no elements, there is no element of the set \( \emptyset \) that is not also in \( B \). That is,

\[
\emptyset \subseteq B \quad \text{for any set } B
\]

**Definition: Proper Subset**

Let \( A \) and \( B \) be two sets. We say that \( A \) is a **proper subset of** \( B \) or that \( A \) is **properly contained in** \( B \), written as

\[
A \subset B
\]

if and only if every element in the set \( A \) is also in the set \( B \), but there is at least one element in set \( B \) that is **not** in set \( A \). If \( A \) is not a proper subset of \( B \), we write

\[
A \nsubseteq B
\]

For example, \( \{1, 2\} \subset \{1, 2, 3, 4\} \), \( \{1, 2\} \nsubseteq \{1, 2\} \), and \( \{1, 2\} \nsubseteq \{1, 2\} \). Do you see why? The set to the right of the \( \subset \) symbol must have an element not found in the set to the left of the \( \subset \) symbol.

Notice that “\( A \) is a proper subset of \( B \)” means that there are **no** elements of \( A \) that are not also elements of \( B \), but there is at least one element of \( B \) that is not in \( A \). For example, if \( B \) is any nonempty set, that is, any set having at least one element, then

\[
\emptyset \subset B
\]

Also, \( A \nsubseteq A \); that is, a set is never a proper subset of itself.

The following example illustrates some uses of the three relationships, \( = \), \( \subseteq \), and \( \nsubseteq \), just defined.

**EXAMPLE 3** Identifying Relationships between Sets

Consider three sets \( A \), \( B \), and \( C \) given by

\[
A = \{1, 2, 3\} \quad B = \{1, 2, 3, 4, 5\} \quad C = \{3, 2, 1\}
\]

Some of the relationships between pairs of these sets are

(a) \( A = C \) \quad (b) \( A \subseteq B \) \quad (c) \( A \nsubseteq C \) \quad (d) \( A \subset B \) \quad (e) \( C \subseteq A \) \quad (f) \( \emptyset \nsubseteq A \)

Notice that if a set \( A \) is a subset of a set \( B \), then either \( A \) is a proper subset of \( B \) or else \( A \) equals \( B \). That is,

\[
A \subseteq B \quad \text{if and only if either} \quad A \subseteq B \text{ or } A = B
\]

Also, if \( A \) is a proper subset of \( B \), we can infer that \( A \) is a subset of \( B \), but \( A \) does not equal \( B \). That is,

\[
A \subset B \quad \text{if and only if} \quad A \subseteq B \text{ and } A \neq B
\]
In applications the elements that may be considered are usually limited to some specific all-encompassing set, a universal set. For example, in discussing students eligible to graduate from Midwestern University, the discussion would be limited to students enrolled at the university.

**Definition Universal Set**
The **universal set** \( U \) is defined as the set consisting of all elements under consideration.

If \( A \) is any set and if \( U \) is the universal set, then every element in \( A \) must be in \( U \) (since \( U \) consists of all elements under consideration). As a result,

\[
A \subseteq U
\]

for any set \( A \).

It is convenient to represent a set as the interior of a circle. Two or more sets may be depicted as circles enclosed in a rectangle, which represents the universal set. The circles may or may not overlap, depending on the situation. Such diagrams of sets are called **Venn diagrams**. See Figure 1.

2 **Find the Union and Intersection of Two Sets**

Continuing to develop an algebra of sets, we introduce two operations that may be performed on sets, union and intersection.

**Definition Union of Two Sets**
Let \( A \) and \( B \) be any two sets. The **union** of \( A \) with \( B \), written as

\[
A \cup B
\]

and read as “\( A \) union \( B \)” or as “\( A \) or \( B \)” is defined to be the set consisting of those elements either in \( A \) or in \( B \) or in both \( A \) and \( B \). That is,

\[
A \cup B = \{ x | x \text{ is in } A \text{ or } x \text{ is in } B \}
\]

For example, if \( A = \{ 1, 2, 4 \} \) and \( B = \{ 2, 3, 5 \} \), then \( A \cup B = \{ 1, 2, 3, 4, 5 \} \).

In the Venn diagram in Figure 2, the area with color corresponds to \( A \cup B \).

**Definition Intersection of Two Sets**
Let \( A \) and \( B \) be any two sets. The **intersection** of \( A \) with \( B \), written as

\[
A \cap B
\]

and read as “\( A \) intersect \( B \)” or as “\( A \) and \( B \)” is defined as the set consisting of those elements that are in both \( A \) and \( B \). That is,

\[
A \cap B = \{ x | x \text{ is in } A \text{ and } x \text{ is in } B \}
\]
For example, if \( A = \{1, 2, 4\} \) and \( B = \{2, 3, 5\} \), then \( A \cap B = \{2\} \).

In other words, to find the intersection of two sets \( A \) and \( B \) means to find the elements common to \( A \) and \( B \). In the Venn diagram in Figure 3 the region with color is \( A \cap B \).

### Example 4 Finding the Union and Intersection of Two Sets

Use the sets
\[
A = \{1, 3, 5\} \quad B = \{3, 4, 5, 6\} \quad C = \{6, 7\}
\]
to find
(a) \( A \cup B \) \hspace{1cm} (b) \( A \cap B \) \hspace{1cm} (c) \( A \cap C \)

**Solution**

(a) \( A \cup B = \{1, 3, 5\} \cup \{3, 4, 5, 6\} = \{1, 3, 4, 5, 6\} \)

(b) \( A \cap B = \{1, 3, 5\} \cap \{3, 4, 5, 6\} = \{3, 5\} \)

(c) \( A \cap C = \{1, 3, 5\} \cap \{6, 7\} = \emptyset \)

---

**Example 5 Interpreting the Intersection of Two Sets**

Let \( T \) be the set of all taxpayers and let \( S \) be the set of all people over 65 years of age. Describe \( T \cap S \).

**Solution** \( T \cap S \) is the set of all taxpayers who are also over 65 years of age.

### Definition Disjoint Sets

If two sets \( A \) and \( B \) have no elements in common, that is, if
\[
A \cap B = \emptyset
\]
then \( A \) and \( B \) are called **disjoint sets**.

For example, if \( A = \{1, 2\} \) and \( B = \{3, 4, 5\} \), then \( A \cap B = \emptyset \) so \( A \) and \( B \) are disjoint sets.

Two disjoint sets \( A \) and \( B \) are illustrated in the Venn diagram in Figure 4. Notice that the circles corresponding to \( A \) and \( B \) do not overlap.

### Example 6 Tossing a Die*

Suppose that a die is tossed. What is the universal set? Let \( A \) be the set of outcomes in which an even number turns up; let \( B \) be the set of outcomes in which an odd number shows.

(a) Find \( A \). \hspace{1cm} (b) Find \( B \). \hspace{1cm} (c) Find \( A \cap B \).

*A die (plural dice) is a cube with 1, 2, 3, 4, 5, or 6 dots showing on the six faces.*
**SOLUTION** Since only a 1, 2, 3, 4, 5, or 6 can result when a die is tossed, the universal set is $U = \{1, 2, 3, 4, 5, 6\}$. It follows that

(a) $A = \{2, 4, 6\}$
(b) $B = \{1, 3, 5\}$

(c) Notice that $A$ and $B$ have no elements in common. As a result, $A \cap B = \emptyset$ so the sets $A$ and $B$ are disjoint sets.

3 Find the Complement of a Set

Consider all the employees of a company as the universal set $U$. Let $A$ be the subset of employees who smoke. Then all the nonsmokers will make up the subset of $U$ that is called the complement of the set of smokers.

**Definition**

Complement of a Set

Let $A$ be any set. The complement of $A$, written as $A'$, is defined as the set consisting of elements in the universal set $U$ that are not in $A$. That is,

$$\bar{A} = \{x | x \text{ is not in } A\}$$

The region in color in Figure 5 illustrates the complement, $\bar{A}$.

For any set $A$ it follows that

$$A \cup \bar{A} = U \quad A \cap \bar{A} = \emptyset \quad \bar{A} = A$$

**Example 7** Finding the Complement of a Set

Use the sets

$$U = \{a, b, c, d, e, f\}, \quad A = \{a, b, c\}, \quad B = \{a, c, f\}$$

to list the elements of the following sets:

(a) $\bar{A}$
(b) $\bar{B}$
(c) $\bar{A} \cup \bar{B}$
(d) $\bar{A} \cap \bar{B}$
(e) $A \cap \bar{B}$
(f) $\bar{A} \cup B$

**SOLUTION**

(a) $\bar{A}$ consists of all the elements in $U$ that are not in $A$:

$$\bar{A} = \{d, e, f\}$$

(b) Similarly,

$$\bar{B} = \{b, d, e\}$$

*Some books use $A'$ or $A^c$ for complement.*
EXAMPLE 9 Using a Venn Diagram to Illustrate the Equality of Two Sets

SOLUTION We begin with Figure 7. There $A \cap \overline{B}$ is the purple region. Now shade the regions $A \cap \overline{B}$, $A \cap B$, and $\overline{A} \cap B$, as shown in Figure 8. The union of the three regions in Figure 8 is the set $A \cup B$. See Figure 9.

NOW WORK PROBLEMS 29 AND 35.
Look back at Example 7. The answers to parts (c) and (d) are the same, and the answers to parts (e) and (f) are the same. This is no coincidence. It is a consequence of two important properties involving intersections and unions of complements of sets, known as De Morgan’s properties:

**Theorem**

**De Morgan’s Properties**

Let $A$ and $B$ be any two sets. Then

\[
\begin{align*}
(a) \quad \overline{A \cup B} &= \overline{A} \cap \overline{B} \\
(b) \quad \overline{A \cap B} &= \overline{A} \cup \overline{B}
\end{align*}
\]

**EXAMPLE 10** Using a Venn Diagram to Illustrate De Morgan’s Properties

Use a Venn diagram to illustrate that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

**SOLUTION**

First draw two diagrams, as shown in Figures 10(a) and 10(b).

**FIGURE 10**

\[
\begin{align*}
\text{(a)} \quad & \quad \text{ } \quad \text{ } \quad \text{ } \quad \text{ } \quad \text{ } \quad \text{ } \quad \text{ } \\
\text{(b)} & \quad \text{ } \quad \text{ } \quad \text{ } \quad \text{ } \quad \text{ } \quad \text{ } \\
\end{align*}
\]

Use the diagram on the left for $\overline{A \cup B}$ and the one on the right for $\overline{A} \cap \overline{B}$. Figure 11 illustrates the completed Venn diagrams of these sets.

**FIGURE 11**

\[
\begin{align*}
\text{(a)} & \quad \overline{A \cup B} \\
\text{(b)} & \quad \overline{A} \cap \overline{B}
\end{align*}
\]

In Figure 11(a), $\overline{A \cup B}$ is represented by the region in color, and in Figure 11(b), $\overline{A} \cap \overline{B}$ is represented by the same region. This illustrates that the two sets $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$ are equal.

**EXERCISE 7.1** Answers Begin on Page AN–35.

**Concepts and Vocabulary**

1. Insert $\subseteq$, $\subsetneq$, or $=$ to make a true statement. More than one answer is possible. $\{1, 2, 3, 4\} \quad \{1, 2, 3, 4\}$
2. Insert $\cup$ or $\cap$ to make a true statement. $\{1, 2, 3, 4\} \quad \{1, 3, 5, 7\} = \{1, 2, 3, 4, 5, 7\}$
3. Insert $\cup$ or $\cap$ to make a true statement. $\{1, 2, 3\} \quad \{2, 3, 4, 5\} = \{2, 3\}$
4. If $U = \{1, 2, 3, 4, 5\}$, then $\overline{\{1, 2\}} = \{3, 4, 5\}$. 

Skill Building

In Problems 5–16, determine whether the given statement is true or false.

5. \( \{1, 3, 2\} = \{2, 1, 3\} \)
6. \( \{2, 3\} \subseteq \{1, 2, 3\} \)
7. \( \{6, 7, 9\} \subseteq \{1, 6, 9\} \)
8. \( \{4, 8, 2\} = \{2, 4, 6, 8\} \)
9. \( \{6, 7, 9\} \subset \{1, 6, 9\} \)
10. \( \{4, 8, 2\} \subset \{2, 4, 6, 8\} \)
11. \( \{1, 2\} \cap \{2, 3, 4\} = \{2\} \)
12. \( \{2, 3\} \cap \{2, 3, 4\} = \{2, 3\} \)
13. \( \{\{4, 5\} \cap \{1, 2, 3, 4\}\} \subset \{4\} \)
14. \( \{\{1, 4\} \cup \{2, 3\}\} \subseteq \{1, 2, 3, 4\} \)
15. \( \{1, 2, 3\} \cap \{3, 4, 5\} \subseteq \{3, 4, 5\} \)
16. \( \{1, 2, 3\} \cup \{3, 4, 5\} \subseteq \{3, 4, 5\} \)

In Problems 17–24, write each expression as a single set.

17. \( \{1, 2, 3\} \cap \{2, 3, 4, 5\} \)
18. \( \{0, 1, 3\} \cup \{2, 3, 4\} \)
19. \( \{1, 2, 3\} \cup \{2, 3, 4, 5\} \)
20. \( \{0, 1, 2\} \cap \{2, 3, 4\} \)
21. \( \{2, 4, 6, 8\} \cap \{1, 3, 5, 7\} \)
22. \( \{2, 4, 6\} \cup \{1, 3, 5\} \)
23. \( \{a, b, e\} \cup \{d, e, f, g\} \)
24. \( \{a, c, m\} \cup \{p, o, m\} \)

25. If \( U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) and if \( A = \{0, 1, 2, 3, 4, 5\} \), \( B = \{2, 3, 5, 8\} \), \( C = \{5, 6, 9\} \), find

   (a) \( A \cup B \)
   (b) \( A \cap C \)
   (c) \( A \cap B \)
   (d) \( \bar{A} \cap B \)
   (e) \( \bar{A} \cap \bar{B} \)
   (f) \( A \cup \overline{B} \)
   (g) \( (C \cap A) \cap (\overline{A}) \)
   (h) \( (A \cap B) \cup (B \cap C) \)

26. If \( U = \{0, 1, 2, 3, 4, 5\} \) and if \( A = \{3, 5\} \), \( B = \{1, 2, 3\} \), \( C = \{2, 3, 4\} \), find

   (a) \( A \cap \overline{B} \)
   (b) \( A \cup B \cap C \)
   (c) \( A \cup (B \cap C) \)
   (d) \( (A \cup B) \cap (A \cup C) \)
   (e) \( \bar{A} \cap \bar{C} \)
   (f) \( \bar{A} \cup \bar{B} \)
   (g) \( \bar{A} \cup \bar{B} \)
   (h) \( (A \cap B) \cap C \)

27. Let \( U = \{\text{All letters of the alphabet}\} \), \( A = \{b, c, d\} \), and \( B = \{c, e, f, g\} \). List the elements of the sets:

   (a) \( A \cup B \)
   (b) \( A \cap B \)
   (c) \( \bar{A} \cap B \)
   (d) \( \bar{A} \cup B \)

28. Let \( U = \{a, b, c, d, e, f\} \), \( A = \{b, c\} \), and \( B = \{c, d, e\} \). List the elements of the sets:

   (a) \( A \cup B \)
   (b) \( A \cap B \)
   (c) \( \bar{A} \)
   (d) \( B \)
   (e) \( A \cap B \)
   (f) \( A \cup B \)

In Problems 29–36, use a Venn diagram to illustrate each set.

29. \( \bar{A} \cap B \)
30. \( \overline{A \cap B} \cup C \)
31. \( A \cap (A \cup B) \)
32. \( A \cup (A \cap B) \)
33. \( (A \cup B) \cap (A \cup C) \)
34. \( A \cup (B \cap C) \)
35. \( A = (A \cap B) \cup (A \cap \overline{B}) \)
36. \( B = (A \cup B) \cup (\overline{A} \cap B) \)

In Problems 37–40, use a Venn diagram to illustrate each property.

37. \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \) (Distributive property)
38. \( A \cap (A \cup B) = A \) (Absorption property)
39. \( \bar{A} \cap \bar{B} = \bar{A} \cap \bar{B} \) (De Morgan’s property)
40. \( (A \cup B) \cup C = A \cup (B \cup C) \) (Associative property)

Applications

In Problems 41–46, describe each set in words.

\( A = \{x | x \text{ is a customer of IBM}\} \)
\( B = \{x | x \text{ is a secretary employed by IBM}\} \)
\( C = \{x | x \text{ is a computer operator at IBM}\} \)
\( D = \{x | x \text{ is a stockholder of IBM}\} \)
\( E = \{x | x \text{ is a member of the Board of Directors of IBM}\} \)

41. \( A \cap E \)
42. \( B \cap D \)
43. \( A \cup D \)
44. \( C \cap E \)
45. \( \bar{A} \cap D \)
46. \( A \cup \overline{D} \)

In Problems 47–52, describe each set in words.

\( U = \{\text{All college students}\} \)
\( M = \{\text{All male students}\} \)
\( S = \{\text{All students who smoke}\} \)
\( F = \{\text{All freshmen}\} \)

47. \( M \cap S \)
48. \( M \cup S \)
49. \( \overline{M} \cup \overline{F} \)
50. \( \overline{M} \cap \overline{S} \)
51. \( F \cap S \cap M \)
52. \( F \cup S \cup M \)

53. **U.S. Senate Judiciary Committee** The 111th U.S. Senate Committee on the Judiciary is composed of the following set of senators:

\( J = \{\text{Leahy, Kohl, Feinstein, Feingold, Sessions, Hatch, Grassley, Kyl, Schumer, Durbin, Cardin, Whitehouse, Klobuchar, Kaufman, Graham, Cornyn, Coburn, Specter, Franken}\} \).

The Administrative Oversight subcommittee is composed of the subset

\( O = \{\text{Whitehouse, Feinstein, Feingold, Schumer, Cardin, Kaufman, Franken, Sessions, Grassley, Kyl, Graham}\} \).

The Antitrust, Competition Policy, and Consumer Rights subcommittee is composed of the subset
7.2 The Number of Elements in a Set

**OBJECTIVES**

1. Use the counting formula (p. 367)
2. Use Venn diagrams to analyze survey data (p. 367)

When we count objects, what we are actually doing is taking each object to be counted and matching each of these objects exactly once to the counting numbers 1, 2, 3, and so on, until no objects remain. Even before numbers had names and symbols assigned to them, this method of counting was used. Prehistoric peoples used rocks to determine how many cattle did not return from pasture. As each cow left, a rock was placed aside. As each cow returned, a rock was removed from the pile. If rocks remained after all the cows returned, it was then known that some cows were missing.

---

**A = {Kohl, Schumer, Whitehouse, Klobuchar, Kaufman, Specter, Franken, Hatch, Grassley, Cornyn}**.

(a) Find $O \cap A$ and explain what it represents.
(b) Find $O \cup A$ and explain what it represents.
(c) Find $O \cup A$ and explain what it represents.
(d) Find $O \cup A$ and explain what it represents.

**Source:** http://judiciary.senate.gov

**54. 10 Largest Companies** America’s 10 largest corporations in 2009, according to *Fortune* magazine, are listed in the following set:

$L = \{\text{Royal Dutch Shell, Exxon Mobil, Wal-Mart, BP, Chevron, Total, ConocoPhillips, ING Group, Sinopec, Toyota Motor}\}.$

America’s 10 most admired companies in 2009, also according to *Fortune* magazine, are listed in the following set:

$A = \{\text{Apple, Google, Berkshire Hathaway, Johnson & Johnson, Amazon.com, Procter & Gamble, Toyota Motor, Goldman Sachs, Wal-Mart, Coca-Cola}\}.$

---

**55. Investments** A financial advisor maintains a database of clients and the stocks they own. A search of the database for clients who own stock in Intel Corp. (INTC) yields the following subset:

$I = \{\text{Black, Bodden, Forbes, Gallaher, Murphy, Petiev, Russell, Smith, Stein, Sutton}\}.$

Another search of the database for clients who own stock in Hewlett Packard (HPO) yields the subset

$H = \{\text{Bodden, Brown, Earnest, Gallaher, Johnson, Randolph, Rhoades, Sutton}\}.$

(a) Find $I \cup H$ and explain what it represents.
(b) Find $I \cap H$ and explain what it represents.

**56. Used Cars** A used car dealer maintains a database of customers and the vehicles they purchase. A search of the database for customers who purchased a minivan within the past year yields the subset

$M = \{\text{Ahuja, Bachman, Crawford, Eilers, Martin, McKinney, Pogue, Sheets, Tripp, Yost}\}.$

Another search of the database for customers who purchased a four-door sedan within the past year yields

$S = \{\text{Arnold, Bachman, Carpenter, Godfrey, McKinney, Nicholson, Sheets, Simpson, Wyatt, Yost}\}.$

(a) Find $M \cup S$ and explain what it represents.
(b) Find $M \cap S$ and explain what it represents.

**57. List all the subsets of \{a, b, c\}.
58. List all the subsets of \{a, b, c, d\}.

**Discussion and Writing**

59. Explain in your own words the concept of subset.

60. How would you explain to a fellow student the meaning of union and intersection of sets?
If $A$ is any set, the notation $n(A)$ denotes the number of elements in $A$. For example, for the set $L$ of letters in the alphabet,

$$L = \{a, b, c, d, e, f, \ldots, x, y, z\}$$

we have $n(L) = 26$. Also, for the set

$$N = \{1, 2, 3, 4, 5\}$$

we have $n(N) = 5$. The empty set $\emptyset$ has no elements, so

$$n(\emptyset) = 0$$

If the number of elements in a set is zero or a positive integer, we say that the set is finite. Otherwise, the set is said to be infinite. The area of mathematics that deals with the study of finite sets is called finite mathematics.

**EXAMPLE 1** Analyzing Survey Data

A survey of a group of children indicated there were 25 with brown eyes and 15 with black hair.

(a) If 10 children had both brown eyes and black hair, how many children interviewed had either brown eyes or black hair?

(b) If 23 children had neither brown eyes nor black hair, how many children in all were interviewed?

**SOLUTION**

Let $A$ denote the set of children with brown eyes and $B$ the set of children with black hair. Then the data given tell us

$$n(A) = 25 \quad n(B) = 15$$

(a) Since 10 children had both brown eyes and black hair, we know that $n(A \cap B) = 10$. The number of children with either brown eyes or black hair is $n(A \cup B)$. But $n(A \cup B)$ cannot be $n(A) + n(B)$, since those with both characteristics would then be counted twice. To obtain $n(A \cup B)$, we need to subtract those with both characteristics from $n(A) + n(B)$ to avoid counting them twice. So,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 25 + 15 - 10 = 30$$

There are 30 children with either brown eyes or black hair.

(b) The sum of the number of children either in $A$ or in $B$ (30) and the number of children neither in $A$ nor in $B$ (23) is the total interviewed. The total number of children interviewed is

$$30 + 23 = 53$$

**FIGURE 12**

Figure 12 illustrates how a Venn diagram can be used to represent this situation. In constructing Figure 12, since $n(A \cap B) = 10$, we begin by placing 10 in $A \cap B$. Since $n(A) = 25$, $A$ requires 15 more. Similarly, since $n(B) = 15$, $B$ requires 5 more. Now we can see that $n(A \cup B) = 15 + 10 + 5 = 30$. Since 23 are in neither $A$ nor $B$, we have $n(U) = 15 + 10 + 5 + 23 = 53$. In Example 1 we discovered the following important relationship:
Theorem 7.2 The Number of Elements in a Set

**Counting Formula**

Let $A$ and $B$ be two finite sets. Then

\[
 n(A \cup B) = n(A) + n(B) - n(A \cap B)
\]  

(1)

1. **Use the Counting Formula**

**EXAMPLE 2 Using the Counting Formula**

Let $A = \{a, b, c, d, e\}$, $B = \{a, e, g, u, w, z\}$. Find $n(A)$, $n(B)$, $n(A \cap B)$, and $n(A \cup B)$.

**SOLUTION** $n(A) = 5$ and $n(B) = 6$. To find $n(A \cap B)$, first find $A \cap B = \{a, e\}$. Then $n(A \cap B) = 2$. Since $A \cup B = \{a, b, c, d, e, g, u, w, z\}$, then $n(A \cup B) = 9$. This checks with the Counting Formula

\[
 n(A \cup B) = n(A) + n(B) - n(A \cap B) = 5 + 6 - 2 = 9
\]

Now work Problem 9.

2. **Use Venn Diagrams to Analyze Survey Data**

**EXAMPLE 3 Analyzing a Consumer Survey Using Venn Diagrams**

In a survey of 75 consumers, 12 indicated they were going to buy a new car, 18 said they were going to buy a new refrigerator, and 24 said they were going to buy a new stove. Of these, 6 were going to buy both a car and a refrigerator, 4 were going to buy a car and a stove, and 10 were going to buy a stove and refrigerator. One person indicated he was going to buy all three items.

(a) How many were going to buy none of these items?
(b) How many were going to buy only a car?
(c) How many were going to buy only a stove?
(d) How many were going to buy only a refrigerator?
(e) How many were going to buy a stove and refrigerator but not a car?

**SOLUTION** Denote the sets of people buying cars, refrigerators, and stoves by $C$, $R$, and $S$, respectively. $U$ will denote the universal set. Then we know from the data given that

- $n(C) = 12$ buy cars.
- $n(R) = 18$ buy refrigerators.
- $n(S) = 24$ buy stoves.
- $n(C \cap R) = 6$ buy both a car and a refrigerator.
- $n(C \cap S) = 4$ buy both a car and a stove.
- $n(S \cap R) = 10$ buy both a stove and a refrigerator.
- $n(C \cap R \cap S) = 1$ buy all three.
- $n(U) = 75$ 75 consumers are surveyed.
Use the information above and put it into a Venn diagram using three interlocking circles labeled C, R, and S. Beginning with the fact that \( n(C \cap R \cap S) = 1 \), place a 1 in that set, as shown in Figure 13(a). Now we work our way out.

**FIGURE 13**

(a) ![Venn Diagram (a)](image1)
(b) ![Venn Diagram (b)](image2)
(c) ![Venn Diagram (c)](image3)
(d) ![Venn Diagram (d)](image4)

Since \( n(C \cap R) = 6 \), \( n(C \cap S) = 4 \), and \( n(S \cap R) = 10 \), we place 6 - 1 = 5 in the proper region (giving a total of 6 in the set \( C \cap R \)). Similarly, we place 3 and 9 in the proper regions for the sets \( C \cap S \) and \( S \cap R \). See Figure 13(b).

Now \( n(C) = 12 \), and 9 of these 12 are already accounted for, so 3 remain. Also, \( n(R) = 18 \) with 15 accounted for so 3 remain, and \( n(S) = 24 \) with 13 accounted for so 11 remain. See Figure 13(c).

Finally, the number in \( C \cup R \cup S \) is the total of 75 less those accounted for in \( C \), \( R \), and \( S \), namely, \( 3 + 5 + 1 + 3 + 3 + 9 + 11 = 35 \). That is,

\[
n(C \cup R \cup S) = 75 - 35 = 40
\]

See Figure 13(d).

From this figure, we can conclude that

(a) 40 were going to buy none of the items;
(b) 3 were going to buy only a car;
(c) 11 were going to buy only a stove;
(d) 3 were going to buy only a refrigerator;
(e) 9 were going to buy a stove and refrigerator, not a car.

■

**NOW WORK PROBLEM 19.**

**EXAMPLE 4** Analyzing Data

In a survey of 10,281 people, the following data were obtained:

<table>
<thead>
<tr>
<th></th>
<th>Single: 3490</th>
<th>Single males: 1745</th>
<th>Single male over 18: 239</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male:</td>
<td>5822</td>
<td>Over and male: 859</td>
<td></td>
</tr>
<tr>
<td>Over 18:</td>
<td>4722</td>
<td>Over 18 and single: 1341</td>
<td></td>
</tr>
</tbody>
</table>

The data are inconsistent. Why?

**SOLUTION** Denote the set of people who were single by \( S \), male by \( M \), and over 18 by \( H \). Then we know that
Let $n(S) = 3490$, $n(M) = 5822$, $n(H) = 4722$, $n(S \cap M) = 1745$, $n(H \cap M) = 859$, $n(H \cap S) = 1341$, $n(H \cap M \cap S) = 239$, and $n(U) = 10,281$.

Construct the Venn diagram shown in Figure 14. Then
\[ n(S \cup H \cup M) = 239 + 1102 + 620 + 1506 + 3457 + 643 + 2761 = 10,328 \]

But this number exceeds the number of people surveyed, 10,281. This means the data are inconsistent.

\[ \text{NOW WORK PROBLEM 27.} \]

### 7.2 The Number of Elements in a Set

- $n(S) = 3490$ are single.
- $n(M) = 5822$ are male.
- $n(H) = 4722$ are over 18.
- $n(S \cap M) = 1745$ are single and male.
- $n(H \cap M) = 859$ are males over 18.
- $n(H \cap S) = 1341$ are single over 18.
- $n(H \cap M \cap S) = 239$ are single, male, and over 18.
- $n(U) = 10,281$ are surveyed.

### Concepts and Vocabulary

#### Numerical Expressions

1. **True or False** If $A$ and $B$ are sets, then $n(A \cup B) = n(A) + n(B) + n(A \cap B)$.

2. $n([1, 2, 3, 4]) = ____$

### Skill Building

*In Problems 3–8, use the sets $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$ to find the number of elements in each set.*

3. $A$

4. $B$

5. $A \cap B$

6. $A \cup B$

7. $(A \cap B) \cup A$

8. $(B \cap A) \cup B$

### 9.

Find $n(A \cup B)$, given that $n(A) = 4$, $n(B) = 3$, and $n(A \cap B) = 2$.

### 10.

Find $n(A \cup B)$, given that $n(A) = 14$, $n(B) = 11$, and $n(A \cap B) = 6$.

### 11.

Find $n(A \cap B)$, given that $n(A) = 5$, $n(B) = 4$, and $n(A \cup B) = 7$.

### 12.

Find $n(A \cap B)$, given that $n(A) = 8$, $n(B) = 9$, and $n(A \cup B) = 16$.

### 13.

Find $n(A)$, given that $n(B) = 8$, $n(A \cap B) = 4$, and $n(A \cup B) = 14$.

### 14.

Find $n(B)$, given that $n(A) = 10$, $n(A \cap B) = 5$, and $n(A \cup B) = 29$.

### 15. **Car Options**

Motors, Inc., manufactured 325 cars with a GPS system, 216 with satellite radio, and 89 with both of these options. How many cars were manufactured if every car has at least one of these options?

### 16. **College Classes**

Suppose that out of 1500 first-year students at a certain college, 350 are taking history, 300 are taking mathematics, and 270 are taking both history and mathematics. How many first-year students are taking history or mathematics?

*In Problems 17–26, use the data in the figure to answer each question.*

### 17.

How many elements are in set $A$?

### 18.

How many elements are in set $B$?

### 19.

How many elements are in $A$ or $B$?

### 20.

How many elements are in $B$ or $C$?
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21. How many elements are in $A$ but not $B$?
22. How many elements are in $B$ but not $C$?
23. How many elements are in $A$ or $B$ or $C$?

24. How many elements are in neither $A$ nor $B$ nor $C$?
25. How many elements are in $A$ and $B$ and $C$?
26. How many elements are in $U$?

Applications

27. Single-Parent Households  The table shows the number of single-parent households (in thousands) by marital status of the parent and geographic region in 2009.

<table>
<thead>
<tr>
<th>Maintained by</th>
<th>Northeast</th>
<th>Midwest</th>
<th>South</th>
<th>West</th>
</tr>
</thead>
<tbody>
<tr>
<td>Father</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Never married</td>
<td>99</td>
<td>132</td>
<td>174</td>
<td>105</td>
</tr>
<tr>
<td>Divorced</td>
<td>117</td>
<td>202</td>
<td>310</td>
<td>196</td>
</tr>
<tr>
<td>Separated</td>
<td>58</td>
<td>52</td>
<td>138</td>
<td>70</td>
</tr>
<tr>
<td>Widowed</td>
<td>19</td>
<td>18</td>
<td>33</td>
<td>20</td>
</tr>
<tr>
<td>Maintained by</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mother</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Never married</td>
<td>854</td>
<td>1001</td>
<td>1782</td>
<td>790</td>
</tr>
<tr>
<td>Divorced</td>
<td>436</td>
<td>72</td>
<td>1279</td>
<td>684</td>
</tr>
<tr>
<td>Separated</td>
<td>355</td>
<td>301</td>
<td>833</td>
<td>459</td>
</tr>
<tr>
<td>Widowed</td>
<td>57</td>
<td>89</td>
<td>156</td>
<td>74</td>
</tr>
</tbody>
</table>

(a) How many households maintained by a mother are in the South?
(b) How many households in the Midwest are maintained by a divorced parent?
(c) How many households are maintained by a father who was never married or widowed?

28. The Venn diagram illustrates the number of seniors ($S$), female students ($F$), and students on the dean’s list ($D$) at a small western college. Describe each number in terms of the sets $S$, $F$, or $D$.


29. Civilian Employment  The table shows seasonally adjusted employment figures (in thousands) by gender for U.S. civilians ages 20 years and older in March 2010.

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>Employed</td>
<td>70,913</td>
<td>63,495</td>
</tr>
<tr>
<td>Unemployed</td>
<td>7882</td>
<td>5532</td>
</tr>
<tr>
<td>Not in Labor Force</td>
<td>27,403</td>
<td>44,947</td>
</tr>
</tbody>
</table>

Source: Bureau of Labor Statistics, March 2010

30. Military Personnel  The table shows the number of active-duty military personnel by branch of service, officer/enlisted status, and gender in 2008.

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>Army</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Officer</td>
<td>74,000</td>
<td>13,500</td>
</tr>
<tr>
<td>Enlisted</td>
<td>392,000</td>
<td>59,700</td>
</tr>
<tr>
<td>Navy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Officer</td>
<td>44,000</td>
<td>7700</td>
</tr>
<tr>
<td>Enlisted</td>
<td>235,000</td>
<td>41,400</td>
</tr>
<tr>
<td>Marine Corps</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Officer</td>
<td>19,000</td>
<td>1200</td>
</tr>
<tr>
<td>Enlisted</td>
<td>167,000</td>
<td>11,100</td>
</tr>
<tr>
<td>Air Force</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Officer</td>
<td>53,000</td>
<td>11,900</td>
</tr>
<tr>
<td>Enlisted</td>
<td>207,000</td>
<td>51,400</td>
</tr>
</tbody>
</table>

Find
(a) The number of U.S. civilians 20 years or older who are unemployed.
(b) The number of U.S. civilians 20 years or older who are unemployed or not in the labor force.
(c) The number of U.S. civilians 20 years or older who are female or employed.
(d) The number of U.S. civilians 20 years or older who are male but not in the labor force.
31. **Employment Analysis** In its January 1, 2010 Employment Analysis Summary, a community college provided the following table, which shows the number of layoffs by employment category, gender, and race (8/01/2004–7/31/2005).

<table>
<thead>
<tr>
<th>Employment Category</th>
<th>White Male</th>
<th>White Female</th>
<th>Black Male</th>
<th>Black Female</th>
<th>Hispanic Male</th>
<th>Hispanic Female</th>
<th>Other Male</th>
<th>Other Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>Executive Managerial</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Faculty</td>
<td>30</td>
<td>31</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>Professional Nonfaculty</td>
<td>54</td>
<td>76</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>Secretarial/Clerical</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Technical/Paraprofessional</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Qualified Crafts</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Service/Maintenance</td>
<td>20</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Find
(a) The number of layoffs who were executive managerial or faculty or professional nonfaculty.
(b) The number of layoffs who were female.
(c) The number of layoffs who were black female or faculty.
(d) The number of layoffs who were white male or executive managerial.

32. **Industry Type and Location** Suppose 250 domestic CEOs were surveyed about their company’s industry type and geographic location in the United States. The CEOs were allowed to choose only one industry type (manufacturing, communications, or finance) and one location (Northeast, Southeast, Midwest, or West). The results are given below.

<table>
<thead>
<tr>
<th>Industry Type</th>
<th>North-east</th>
<th>Southeast</th>
<th>Midwest</th>
<th>West</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturing</td>
<td>37</td>
<td>8</td>
<td>27</td>
<td>15</td>
</tr>
<tr>
<td>Communications</td>
<td>35</td>
<td>23</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Finance</td>
<td>30</td>
<td>12</td>
<td>10</td>
<td>18</td>
</tr>
</tbody>
</table>

Find
(a) The number of CEOs whose response was not Southeast.
(b) The number of CEOs whose response was Communications or West.
(c) The number of CEOs whose response was Northeast but not Manufacturing.
(d) The number of CEOs whose response was Manufacturing or Communications or Midwest or West.

33. **Fitness Center** A fitness club is considering the purchase of additional exercise bikes and stair steppers for its cardiovascular room. Before deciding on its purchases, the club surveyed 250 of its members about the types of equipment they use regularly. The results follow:

- 170 use an exercise bike regularly
- 119 use a stair stepper regularly
- 67 use both regularly

(a) Draw a Venn diagram to illustrate this application.
(b) Of the 250 surveyed, how many members use an exercise bike or a stair stepper regularly?
(c) How many members use an exercise bike regularly but not a stair stepper?
(d) How many members use a stair stepper regularly but not an exercise bike?
(e) How many members use neither an exercise bike nor a stair stepper regularly?

34. **Appetizers** A restaurant is considering an update to its list of appetizers by adding mozzarella sticks and onion rings to its menu. Before making the additions, the restaurant surveys 325 patrons about the appetizers they enjoy. The results follow:

- 218 enjoy mozzarella sticks
- 173 enjoy onion rings
- 126 enjoy both

(a) Draw a Venn diagram to illustrate this application.
(b) Of the 325 surveyed, how many patrons enjoy mozzarella sticks or onion rings?
35. Investments A financial advisor maintains a database on his 963 clients and their investments. A search of the database for clients who own stocks in the companies General Electric (GE), Ford Motor (F), and 3M (MMM) yields the following results:

- 142 clients own GE
- 123 clients own F
- 107 clients own MMM
- 31 clients own GE and F
- 25 clients own GE and MMM
- 19 clients own F and MMM
- 5 clients own all three

(a) How many clients own only GE?
(b) How many clients own only F?
(c) How many clients own only MMM?
(d) How many clients own GE or MMM but not F?
(e) How many clients own GE and MMM but not F?
(f) How many clients own none of these three stocks?

36. Tax Returns One year, an accountant prepared individual income tax returns for 439 clients. For these individual tax returns, completion of the following schedules was required:

- 298 required schedule A (itemized deductions)
- 256 required schedule B (interest and ordinary dividends)
- 167 required schedule C (business income)
- 212 required schedules A and B
- 87 required schedules A and C
- 61 required schedules B and C
- 54 required schedules A, B, and C

(a) How many clients required only schedule A?
(b) How many individual tax returns required only schedule B?
(c) How many individual tax returns required only schedule C?
(d) How many individual tax returns required only schedule A or C but not B?
(e) How many individual tax returns required schedules A or C but not B?
(f) How many clients required none of these three schedules?

37. At a small midwestern college:

- 31 female seniors were on the dean’s list
- 62 women were on the dean’s list who were not seniors
- 45 male seniors were on the dean’s list
- 87 female seniors were not on the dean’s list
- 96 male seniors were not on the dean’s list
- 275 women were not seniors and were not on the dean’s list
- 89 men were on the dean’s list who were not seniors
- 227 men were not seniors and were not on the dean’s list

(a) How many were seniors?
(b) How many were women?

(c) How many patrons enjoy mozzarella sticks but not onion rings?
(d) How many patrons enjoy onion rings but not mozzarella sticks?
(e) How many patrons enjoy neither mozzarella sticks nor onion rings?

38. Survey Analysis In a survey of 75 business travelers, it was found that of three daily newspapers, New York Times, Wall Street Journal, and USA Today,

- 23 read New York Times
- 18 read Wall Street Journal
- 14 read USA Today
- 10 read New York Times and Wall Street Journal
- 9 read New York Times and USA Today
- 8 read Wall Street Journal and USA Today
- 5 read all three

(a) How many read none of these three magazines?
(b) How many read New York Times alone?
(c) How many read Wall Street Journal alone?
(d) How many read USA Today?
(e) How many read neither New York Times nor Wall Street Journal?
(f) How many read New York Times or Wall Street Journal or both?

39. Car Sales Of the cars sold during the month of July, 90 had heated seats, 100 had GPS, and 75 had satellite radio. Five cars had all three of these extras. Twenty cars had none of these extras. Twenty cars had only heated seats; 60 cars had only GPS; and 30 cars had only satellite radio. Ten cars had both GPS and satellite radio.

(a) How many cars had both satellite radio and heated seats?
(b) How many had both GPS and heated seats?
(c) How many had neither satellite radio nor GPS?
(d) How many cars were sold in July?
(e) How many had GPS or heated seats or both?

40. Incorrect Information A staff member at a large engineering school was presenting data to show that the students there received a liberal education as well as a scientific one. “Look at our record,” she said. “Out of our senior class of 500 students, 281 are taking English, 196 are taking history and a foreign language, 143 are taking all of these. “She was fired. Why?

41. Blood Classification Blood is classified as being either Rh-positive or Rh-negative and according to type. If blood contains an A antigen, it is type A; if it has a B antigen, it is type B; if it has both A and B antigens, it is type AB; and if it has neither antigen, it is type O. Use a Venn diagram to illustrate these possibilities. How many different possibilities are there?

42. Survey Analysis A survey of 52 families from a suburb of Chicago indicated that there was a total of 241 children below the age of 18. Of these, 109 were male; 132 were below the age of 11; 143 had played Little League; 69 males were below the age of 11; 45 females under 11 had played Little League; and 30 males under 11 had played Little League.

(a) How many children over 11 and under 18 had played Little League?
(b) How many females under 11 did not play Little League?
43. **Survey Analysis** Of 100 personal computer users surveyed:
27 use HPs; 35 use Macs, 35 use Dell, 10 use both HPs and Macs, 10 use both HPs and Dell, 10 use both Macs and Dell, 3 use all three; and 30 use another computer brand. How many people exclusively use one of the three brands mentioned, that is, only HPs or only Macs or only Dell?

44. List all the subsets of \{a, b, c\}. How many are there?

45. List all the subsets of \{a, b, c, d\}. How many are there?

### 7.3 The Multiplication Principle

**OBJECTIVES**

1. Use the Multiplication Principle (p. 373)

**Example 1** Counting the Number of Possible Meals

The fixed-price dinner at Mabenka Restaurant provides the following choices:
- Appetizer: soup or salad
- Entree: baked chicken, broiled beef patty, baby beef liver, or roast beef au jus
- Dessert: ice cream or cheesecake

How many different meals can be ordered?

**Solution**

Ordering such a meal requires three separate decisions:

<table>
<thead>
<tr>
<th>Choose an Appetizer</th>
<th>Choose an Entree</th>
<th>Choose a Dessert</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 choices</td>
<td>4 choices</td>
<td>2 choices</td>
</tr>
</tbody>
</table>

Look at the tree diagram in Figure 15. We see that, for each choice of appetizer, there are 4 choices of entrees. And for each of these 2 \cdot 4 = 8 choices, there are 2 choices for dessert. A total of

\[ 2 \cdot 4 \cdot 2 = 16 \]

different meals can be ordered.

**Figure 15**

NOW WORK PROBLEM 1.
Example 1 demonstrates a general principle of counting.

**Theorem**

**Multiplication Principle of Counting**

If a task consists of a sequence of choices in which there are $p$ selections for the first choice, $q$ selections for the second choice, $r$ selections for the third choice, and so on, then the task of making these selections can be done in

\[ p \cdot q \cdot r \cdots \]

different ways.

**EXAMPLE 2** Forming Codes

How many two-symbol code words can be formed if the first symbol is a letter (uppercase) and the second symbol is a digit?

**SOLUTION**

It sometimes helps to begin by listing some of the possibilities. The code consists of a letter (uppercase) followed by a digit, so some possibilities are A1, A2, B3, X0, and so on. The task consists of making two selections. The first selection requires choosing an uppercase letter (26 choices) and the second task requires choosing a digit (10 choices). By the Multiplication Principle, there are

\[ 26 \cdot 10 = 260 \]

different code words of the type described.

NOW WORK PROBLEM 3.

**EXAMPLE 3** Combination Locks

A particular type of combination lock has 100 numbers on it.

(a) How many sequences of three numbers can be formed to open the lock?

(b) How many sequences can be formed if no number is repeated?

**SOLUTION**

(a) Each of the three numbers can be chosen in 100 ways. By the Multiplication Principle, there are

\[ 100 \cdot 100 \cdot 100 = 1,000,000 \]

different sequences.

(b) If no number can be repeated, then there are 100 choices for the first number, only 99 for the second number, and 98 for the third number. By the Multiplication Principle, there are

\[ 100 \cdot 99 \cdot 98 = 970,200 \]

different sequences.
**EXAMPLE 4 Counting the Number of Ways Four Offices Can Be Filled**

In a city election there are four candidates for mayor, three candidates for vice-mayor, six candidates for treasurer, and two for secretary. In how many ways can these four offices be filled?

**SOLUTION** The task of filling an office can be divided into four choices:

- Select a mayor
- Select a vice-mayor
- Select a treasurer
- Select a secretary

Corresponding to each of the four possible mayors, there are three vice-mayors. These two offices can be filled in \(4 \cdot 3 = 12\) different ways. Also, corresponding to each of these 12 possibilities, we have six different choices for treasurer—giving \(12 \cdot 6 = 72\) different possibilities. Finally, to each of these 72 possibilities there correspond two choices for secretary. In all, these offices can be filled in \(4 \cdot 3 \cdot 6 \cdot 2 = 144\) different ways. A partial illustration is given by the tree diagram in Figure 16.

**FIGURE 16**

A partial illustration is given by the tree diagram in Figure 16.

**NOW WORK PROBLEM 11.**

**EXAMPLE 5 License Plates in South Carolina**

Standard license plates in the state of South Carolina consist of three letters of the alphabet followed by three digits.

(a) The South Carolina system will allow how many possible license plates?

(b) Of these, how many will have all their digits distinct?

(c) How many will have distinct digits and distinct letters?

**SOLUTION**

(a) There are six positions on the license plate to be filled, the first three by letters and the last three by digits. Each of the first three positions can be filled in any one of 26 ways, while each of the remaining three positions can each be filled in any of 10 ways. The total number of license plates, by the Multiplication Principle, is then

\[26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000\]

(b) Here the tasks involved in filling the digit positions are slightly different. The first digit can be any one of 10, but the second digit can be only any one of 9 (we cannot duplicate the first digit); there are only 8 choices for the third digit (we cannot duplicate either the first or the second). By the Multiplication Principle, there are

\[26 \cdot 26 \cdot 26 \cdot 10 \cdot 9 \cdot 8 = 12,654,720\]

plates with no repeated digit.
1. **Shirts and Ties**  A man has 5 shirts and 3 ties. How many different shirt and tie arrangements can he wear?

2. **Blouses and Skirts**  A woman has 5 blouses and 8 skirts. How many different outfits can she wear?

3. **Four-Digit Numbers**  How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0? Repeated digits are allowed.

4. **Five-Digit Numbers**  How many five-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0 or 1? Repeated digits are allowed.

5. There are 2 roads between towns A and B. There are 4 roads between towns B and C. How many different routes may one travel from town A to town C through town B?

6. REIT, Inc. wants to build a complex consisting of a factory, office building, and warehouse. If the building contractor has 3 different kinds of factories, 2 different office buildings, and 4 different warehouses, how many models must be built to show all possibilities to REIT, Inc.?

7. **Car Displays**  Cars, Inc., has 3 different car models and 8 color schemes. If you are one of the dealers, how many cars must you display to show each possibility?

8. **Choosing an Outfit**  A man has 3 pairs of shoes, 8 pairs of socks, 4 pairs of slacks, and 9 sweaters. How many outfits can he wear?

9. A house has 3 outside doors and 12 windows. In how many ways can a person enter the house through a window and exit through a door?

10. **License Plates**  How many license plates consisting of 2 letters (uppercase) followed by 2 digits are possible?

11. **Lunch Selections**  A restaurant offers 3 different salads, 8 different main courses, 10 different desserts, and 4 different drinks. How many different lunches—each consisting of a salad, a main course, a dessert, and a drink—are possible?

12. **Arranging Books**  Five different mathematics books are to be arranged on a student’s desk. How many arrangements are possible?

13. How many ways can 6 people be seated in a row of 6 seats?

14. **Codes**  How many 4-letter code words are possible using the first 6 letters of the alphabet with no letters repeated? How many codes are there when letters are allowed to repeat?

---

(c) If the letters and digits are each to be distinct, then the total number of possible license plates is

\[
26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11,232,000
\]

**NOW WORK PROBLEM 21.**

**EXERCISE 7.3 Answers Begin on Page AN–36.**

### Skill Building

1. **Shirts and Ties**  A man has 5 shirts and 3 ties. How many different shirt and tie arrangements can he wear?

2. **Blouses and Skirts**  A woman has 5 blouses and 8 skirts. How many different outfits can she wear?

3. **Four-Digit Numbers**  How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0? Repeated digits are allowed.

4. **Five-Digit Numbers**  How many five-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0 or 1? Repeated digits are allowed.

5. There are 2 roads between towns A and B. There are 4 roads between towns B and C. How many different routes may one travel from town A to town C through town B?

6. REIT, Inc. wants to build a complex consisting of a factory, office building, and warehouse. If the building contractor has 3 different kinds of factories, 2 different office buildings, and 4 different warehouses, how many models must be built to show all possibilities to REIT, Inc.?

7. **Car Displays**  Cars, Inc., has 3 different car models and 8 color schemes. If you are one of the dealers, how many cars must you display to show each possibility?

8. **Choosing an Outfit**  A man has 3 pairs of shoes, 8 pairs of socks, 4 pairs of slacks, and 9 sweaters. How many outfits can he wear?

---

**Applications**

In Problems 15–18, use the following discussion.

Students log on to the California Virtual Campus with a user name consisting of eight characters: four uppercase letters of the alphabet followed by four digits.

**Source:** California Virtual Campus.

15. **User Names**  How many user names are theoretically possible for this system?

16. **User Names**  How many user names have no repeated letters or digits?

17. **User Names**  How many user names have no matching adjacent letters or digits?

18. **User Names**  How many user names begin with A and end with 9?

19. **Baseball**  In the World Series, the National League champion plays the American League champion. There are 16 teams in the National League and 14 teams in the American League. How many different theoretical match-ups of two teams are possible in the World Series?

**Source:** Major League Baseball, 2010
20. **New Car Options** Marjorie is purchasing a new 2010 Honda Accord LX sedan and must choose her exterior accessories. There are four categories of accessories she may choose from (wheels, cargo handling, spoilers, and exterior protection) and she will only select one option from each category. If there are 3 types of wheels, 3 types of cargo handling, 2 types of spoilers, and 7 types of exterior protection, how many different accessory packages are possible?  

**Source:** American Honda Motor Co.

21. **Car Insurance** Adam is purchasing car insurance for his new car and needs to decide on the protection for his vehicle. His insurance company offers coverage for comprehensive, collision, rental reimbursement, and towing/labor and Adam must select a level for each coverage. There are 9 levels of comprehensive, 9 levels for collision, 3 levels for rental reimbursement, and 2 levels for towing/labor. How many different types of vehicle coverage can Adam get?

22. **Car Insurance** Shawn is purchasing a used car for college and needs to obtain liability insurance. His insurance company offers 8 levels for bodily injury coverage and 6 levels for property damage coverage. If Shawn must carry both types of coverage, how many different liability policies does he have to choose from?

23. **Pharmaceutical Sales** A pharmaceutical sales rep will be responsible for 3 different products. He must select 1 pain reliever, 1 blood pressure medicine, and 1 antifungal medicine. If his company manufactures 5 types of pain relievers, 2 types of blood pressure medicine, and 3 types of antifungal medicine, how many different sales portfolios are possible?

24. **Pharmaceutical Sales** A pharmaceutical sales rep will be responsible for 4 different products. She must select 1 pain reliever, 1 antidepressant, 1 allergy medicine, and 1 high cholesterol medicine. If her company manufactures 7 types of pain relievers, 3 types of antidepressants, 6 types of allergy medicine, and 2 types of cholesterol medicine, how many different sales portfolios are possible?

25. **Ordering Pizza** After a night of studying for exams, David decides to order a pizza through the Domino’s® Web site. He must choose among 4 different sizes, 2 different crusts, and 25 different toppings. How many different one-topping pizzas are possible?

**Source:** www.dominos.com

26. **Ordering Pizza** Betty and Dorothy plan to order pizza, soda, and a side item for dinner on movie night. They can choose from 9 specialty pizzas, 3 types of soda, and 11 side items. How many different meals are possible?

27. **Television Schedule** A television network had 13 drama pilots to fill 5 remaining time slots in the Fall 2010 lineup: 10 P.M. Monday, 10 P.M. Wednesday, 10 P.M. Thursday, 9 P.M. Friday, 10 P.M. Sunday. How many different drama lineups were possible?

28. **Television Schedule** A television network had 19 comedy pilots to fill 3 remaining time slots in the Fall 2010 lineup: 9 P.M. Tuesday, 9:30 P.M. Tuesday, and 8 P.M. Thursday. How many different comedy lineups were possible?

29. **Itineraries** A third-party auditor is hired by a fast-food chain to conduct audits at its 10 franchise stores. How many different ways can the audits be scheduled?

30. **Itineraries** The auditor in the previous problem must visit the store that is closest first and decides to visit the store that is furthest away last. If the remaining stores can be visited in any order, how many different ways can the audits be scheduled?

31. **Telephone Numbers** Find the number of 7-digit telephone numbers
   (a) With no repeated digits (lead 0 is allowed)
   (b) With no repeated digits (lead 0 not allowed)
   (c) With repeated digits allowed including a lead 0

32. **Ranking Candidates** How many ways are there to rank 7 candidates who apply for a job?

33. **Arranging Letters**
   (a) How many different ways are there to arrange the 7 letters in the word PROBLEM?
   (b) If we insist that the letter P comes first, how many ways are there?
   (c) If we insist that the letter P comes first and the letter M last, how many ways are there?

34. **Testing** On a math test there are 10 multiple-choice questions with 4 possible answers and 15 true–false questions. In how many possible ways can the 25 questions be answered?

35. **License Plate Possibilities** How many different license plate numbers can be made using 2 letters followed by 4 digits, if
   (a) Letters and digits may be repeated?
   (b) Letters may be repeated, but digits are not repeated?
   (c) Neither letters nor digits may be repeated?

36. **Security** A system has 7 switches, each of which may be either open or closed. The state of the system is described by indicating for each switch whether it is open or closed. How many different states of the system are there?

37. **Product Choice** An automobile manufacturer produces 3 different models. Models A and B can come in any of 3 body styles; model C can come in only 2 body styles. Each car also comes in either black or green. How many distinguishable car types are there?
38. **Telephone Numbers** How many 7-digit numbers can be formed if the first digit cannot be 0 or 9 and if the last digit is greater than or equal to 2 and less than or equal to 3? Repeated digits are allowed.

39. **Home Choices** A contractor constructs homes with 5 different choices of exterior finish, 3 different roof arrangements, and 4 different window designs. How many different types of homes can be built?

40. **License Plate Possibilities** A license plate consists of 1 letter, excluding O and I, followed by a 4-digit number that cannot have a 0 in the lead position. How many different plates are possible?

41. **Bytes** Using only the digits 0 and 1, how many different numbers consisting of 8 digits can be formed?

42. **Stock Portfolios** As a financial planner, you are asked to select one stock from each of the following groups: 8 DOW stocks, 15 NASDAQ stocks, and 4 global stocks. How many different portfolios are possible?

43. **Combination Locks** A combination lock has 50 numbers on it. To open it, you turn counterclockwise to a number, then rotate clockwise to a second number, and then counterclockwise to the third number. How many different lock combinations are there?

44. **Opinion Polls** An opinion poll is to be conducted among college students. Eight multiple-choice questions, each with 3 possible answers, will be asked. In how many different ways can a student complete the poll if exactly one response is given to each question?

45. **Path Selection in a Maze** The maze below is constructed so that a rat must pass through a series of one-way doors. How many different paths are there from start to finish?

46. **Codes** How many 3-letter code words are possible using the first 10 letters of the alphabet if
   (a) No letter can be repeated?
   (b) Letters can be repeated?
   (c) Adjacent letters cannot be the same?

### 7.4 Sample Spaces and the Assignment of Probabilities

**OBJECTIVES**

1. Find a sample space (p. 380)
2. Assign probabilities (p. 382)
3. Construct a probability model (p. 382)
4. Find probabilities involving equally likely outcomes (p. 383)

**Introduction**

Certain phenomena in the real world may be considered chance phenomena. These phenomena do not always produce the same observed outcome, and the outcome of any given observation of the phenomena may not be predictable. But they have a long-range behavior known as statistical regularity. Some examples of such cases, called random events, are given next.

- Tossing a fair coin gives a result that is either heads or tails. For any one throw, we cannot predict the result, although it is obvious that it is determined by definite causes (such as the initial velocity of the coin, the initial angle of the throw, and the surface on which the coin rests). Even though some of these causes can be controlled, we cannot predetermine the result of any particular toss. The result of tossing a coin is a random event.

  Although we cannot predict the result of any particular toss of the coin, if we perform a long sequence of tosses, we expect that the number of heads is approximately equal to the number of tails. That is, it seems reasonable to say that in any toss of this fair coin, heads or tails is equally likely to occur. As a result, we might assign a probability of \( \frac{1}{2} \) for obtaining a head (or tail) on a particular toss.
• In throwing an ordinary die, we cannot predict the result with certainty. The result of throwing a die is a random event. We do know that one of the faces 1, 2, 3, 4, 5, or 6 will occur.

The appearance of any particular face of the die is an outcome. If we perform a long series of tosses, any face is as likely to occur as any other, provided the die is fair. Here we might assign a probability of \( \frac{1}{6} \) for obtaining a particular face.

• The sex of a newborn baby is either male or female. This, too, is an example of a random event.

Our intuition tells us that a boy baby and a girl baby are equally likely to occur.

If we follow this reasoning, we might assign a probability of \( \frac{1}{2} \) to having a girl baby. However, if we consult the data about births in the United States found in Table 1, we see that it might be more accurate to assign a probability of 0.488 to having a girl baby.

### TABLE 1

<table>
<thead>
<tr>
<th>Year of Birth</th>
<th>Number of Births (in thousands)</th>
<th>Total Number of Births (in thousands)</th>
<th>Ratio of Births</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Boys ( b )</td>
<td>Girls ( g )</td>
<td>( \frac{b}{b+g} )</td>
</tr>
<tr>
<td>2006</td>
<td>2184</td>
<td>2081</td>
<td>0.512</td>
</tr>
<tr>
<td>2005</td>
<td>2119</td>
<td>2019</td>
<td>0.512</td>
</tr>
<tr>
<td>2004</td>
<td>2105</td>
<td>2007</td>
<td>0.512</td>
</tr>
<tr>
<td>2003</td>
<td>2094</td>
<td>1996</td>
<td>0.512</td>
</tr>
<tr>
<td>2002</td>
<td>2058</td>
<td>1964</td>
<td>0.512</td>
</tr>
<tr>
<td>2001</td>
<td>2058</td>
<td>1968</td>
<td>0.511</td>
</tr>
<tr>
<td>2000</td>
<td>2077</td>
<td>1982</td>
<td>0.512</td>
</tr>
<tr>
<td>1999</td>
<td>2027</td>
<td>1933</td>
<td>0.512</td>
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<tr>
<td>1998</td>
<td>2016</td>
<td>1925</td>
<td>0.512</td>
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<td>1997</td>
<td>1986</td>
<td>1895</td>
<td>0.512</td>
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<tr>
<td>1996</td>
<td>1990</td>
<td>1901</td>
<td>0.511</td>
</tr>
<tr>
<td>1995</td>
<td>1996</td>
<td>1903</td>
<td>0.512</td>
</tr>
<tr>
<td>1994</td>
<td>2023</td>
<td>1930</td>
<td>0.512</td>
</tr>
<tr>
<td>1993</td>
<td>2049</td>
<td>1951</td>
<td>0.512</td>
</tr>
<tr>
<td>1992</td>
<td>2082</td>
<td>1983</td>
<td>0.512</td>
</tr>
<tr>
<td>1991</td>
<td>2102</td>
<td>2009</td>
<td>0.511</td>
</tr>
<tr>
<td>1990</td>
<td>2129</td>
<td>2029</td>
<td>0.512</td>
</tr>
</tbody>
</table>

*Source:* National Center for Health Statistics

These examples demonstrate that in studying a sequence of random experiments it is not possible to forecast individual results. These are subject to irregular, random fluctuations that cannot be exactly predicted. However, if the number of observations is large—that is, if we deal with a mass phenomenon—some regularity appears. This leads to the study of probability.
Chapter 7  Probability

Probability is concerned with experiments, real or conceptual, and their outcomes. In this study we try to formulate in a precise manner a mathematical model that closely resembles the experiment in question. The first stage of development of such a mathematical model is the building of a probability model. This model is then used to analyze and predict outcomes of the experiment. The purpose of this section is to learn how a probability model can be constructed.

1 Find a Sample Space

Definition

A sample space of an experiment is the set of all possibilities that can occur as a result of the experiment. Each element of a sample space is called an outcome.

A sample space of an experiment plays the same role in probability as the universal set does in set theory. In this chapter we discuss only finite sample spaces, that is, sample spaces that have only a finite number of outcomes.

Notice in the definition we say a sample space, rather than the sample space, since an experiment can often be described in different ways.

EXAMPLE 1 Finding a Sample Space

Flip a coin. Find a sample space for this experiment.

SOLUTION

The experiment consists of flipping a coin. The only possible outcomes are heads (H) and tails (T). Therefore, a sample space for the experiment is the set \{H, T\}.

EXAMPLE 2 Finding a Sample Space

Consider an experiment in which two dice are rolled, one green and the other red. The set of outcomes consists of all the different ways that the dice may come to rest. Find a sample space for this experiment.

SOLUTION

Since there are 6 possible ways for the green die to come up and 6 ways for the red die to come up, the number of outcomes of rolling the two dice is 6 \times 6 = 36.

We use a tree diagram to obtain a list of the 36 outcomes. See Figure 17. Figure 18 gives a graphical representation of the 36 outcomes. The sample space consists of the 36 ordered pairs listed.
Other examples of experiments and their sample spaces are given below.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Sample Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) A spinner is marked from 1 to 8. An experiment consists of spinning the dial once.</td>
<td>{1, 2, 3, 4, 5, 6, 7, 8}</td>
</tr>
<tr>
<td>(b) An experiment consists of tossing two coins, a penny and a nickel, and observing whether the coins match (M) or do not match (D).</td>
<td>{M, D}</td>
</tr>
<tr>
<td>(c) An experiment consists of tossing a coin twice and observing the number of heads that appear.</td>
<td>{0, 1, 2}</td>
</tr>
<tr>
<td>(d) An experiment consists of tossing a coin twice and observing whether the coin falls heads (H) or tails (T) on each toss.</td>
<td>{HH, HT, TH, TT}</td>
</tr>
<tr>
<td>(e) An experiment consists of tossing three coins and observing whether the coins fall heads (H) or tails (T).</td>
<td>{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}</td>
</tr>
</tbody>
</table>

NOW WORK PROBLEM 7.

EXAMPLE 3 Finding a Sample Space

Describe a sample space for the experiment of selecting one family from the set of all possible three-child families.

SOLUTION

One way of describing a sample space for this experiment is to list the number of girls in the family. The only possibilities are members of the set

\[\{0, 1, 2, 3\}\]

That is, a three-child family can have 0, 1, 2, or 3 girls. This sample space has four outcomes.

Another way of describing a sample space for this experiment is to denote B as “boy” and G as “girl.” Then the sample space can be given as

\[\{BBB, BBG, BGB, GBB, GGB, GBBGG\}\]

where BBB means first child is a boy, second child is a boy, third child is a boy, and so on. This sample space has \(2 \cdot 2 \cdot 2 = 8\) outcomes. See the tree diagram in Figure 19.

NOW WORK PROBLEM 15.
Definition

Probability of an Outcome

Suppose the sample space \( S \) of an experiment has \( n \) outcomes given by

\[
S = \{ e_1, e_2, \ldots, e_n \}
\]

To each outcome of \( S \), we assign a real number, \( P(e) \), called the probability of the outcome \( e \), so that

\[
P(e_1) \geq 0, \ P(e_2) \geq 0, \ldots, \ P(e_n) \geq 0 \quad (1)
\]

and

\[
P(e_1) + P(e_2) + \cdots + P(e_n) = 1 \quad (2)
\]

Condition (1) states that each probability assignment must be nonnegative. Condition (2) states that the sum of all the probability assignments must equal one. Only assignments of probabilities satisfying both conditions (1) and (2) are valid.

EXAMPLE 4  Valid Probability Assignments

Let a die be thrown. A sample space \( S \) is then

\[
S = \{ 1, 2, 3, 4, 5, 6 \}
\]

There are six outcomes in \( S \): 1, 2, 3, 4, 5, 6.

One valid assignment of probabilities is

\[
P(1) = \frac{1}{6} \quad P(2) = \frac{1}{6} \quad P(3) = \frac{1}{6} \quad P(4) = \frac{1}{6} \quad P(5) = \frac{1}{6} \quad P(6) = \frac{1}{6}
\]

This choice is consistent with the definition since the probability assigned to each outcome is nonnegative and their sum is 1. This assignment is made when the die is fair.

Another assignment that is valid is

\[
P(1) = 0 \quad P(2) = 0 \quad P(3) = \frac{1}{3} \quad P(4) = \frac{2}{3} \quad P(5) = 0 \quad P(6) = 0
\]

This assignment, although unnatural, is made when the die is “loaded” in such a way that only a 3 or a 4 can occur and the 4 is twice as likely as the 3 to occur.

Many other assignments can also be made that are valid.

NOW WORK PROBLEM 23.

Construct a Probability Model

The sample space and the assignment of probabilities to each outcome of an experiment constitute a probability model for the experiment.
Constructing a Probability Model

To construct a probability model requires two steps:

**STEP 1** Find a sample space. List all the possible outcomes of the experiment, or, if this is not easy to do, determine the number of outcomes of the experiment.

**STEP 2** Assign to each outcome \( e \) a probability \( P(e) \) so that

\[
\begin{align*}
(a) & \quad P(e) \geq 0 \\
(b) & \quad \text{The sum of all the probabilities assigned to the outcomes equals 1.}
\end{align*}
\]

Sometimes a probability model is referred to as a **stochastic model**.

**EXAMPLE 5 Constructing a Probability Model**

A coin is tossed. The coin is weighted so that heads \((H)\) is 5 times more likely to occur than tails \((T)\). Construct a probability model for this experiment.

**SOLUTION**

**STEP 1** A sample space \( S \) for this experiment is

\[ S = \{ T, H \} \]

**STEP 2** Let \( x \) denote the probability that tails occurs. Then

\[ P(T) = x \quad \text{and} \quad P(H) = 5x \quad \text{Heads is 5 times more likely than tails.} \]

Since the sum of all the probability assignments equals 1, we have

\[ P(H) + P(T) = 5x + x = 1 \quad \text{Sum of the probabilities equals 1.} \]

\[ 6x = 1 \quad \text{Simplify.} \]

\[ x = \frac{1}{6} \quad \text{Solve for } x. \]

As a result, we assign the probabilities

\[ P(T) = \frac{1}{6} \quad P(H) = \frac{5}{6} \]

The above discussion constitutes the construction of a probability model for the experiment.

**NOW WORK PROBLEM 49.**

4 Find Probabilities Involving Equally Likely Outcomes

When the same probability is assigned to each outcome of a sample space, the outcomes are termed **equally likely outcomes**.
Equally likely outcomes often occur when items are selected randomly. For example, in randomly selecting 1 person from a group of 10, the probability of selecting a particular individual is $\frac{1}{10}$. If a card is chosen randomly from a deck of 52 cards, the probability of drawing a particular card is $\frac{1}{52}$. In general, if a sample space $S$ has $n$ equally likely outcomes, the probability assigned to each outcome is $\frac{1}{n}$.

**Definition**

**Event $E$ in a Sample Space**

Let a sample space $S$ be given by

$$S = \{e_1, e_2, \ldots, e_n\}$$

An event $E$ in $S$ is any subset of $S$.

**Theorem**

**Probability of an Event $E$ in a Sample Space with Equally Likely Outcomes**

If the sample space $S$ of an experiment has $n$ equally likely outcomes, and the event $E$ in $S$ contains $m$ outcomes, then the probability of event $E$, written as $P(E)$, is

$$P(E) = \frac{\text{Number of outcomes in the event } E}{\text{Number of outcomes in } S} = \frac{m}{n}$$  \hspace{1cm} (3)

To compute the probability of an event $E$ in which the outcomes are equally likely, count the number of outcomes in $E$, and divide by the total number of outcomes in the sample space.

**EXAMPLE 6** **Finding Probabilities Involving Equally Likely Outcomes**

If a person is chosen randomly from a group of 100 people, 60 females and 40 males, the probability of the event $E$, ‘a male is chosen,’ is:

$$P(E) = \frac{\text{Number of males}}{\text{Number of people}} = \frac{40}{100} = 0.4$$

**EXAMPLE 7** **Government Bonds**

At the beginning of April 2010, the most recent issues of government bonds consisted of T-bills, bonds and notes, and Treasury inflation-protected securities (TIPS) bonds. See Table 2.
7.4 Sample Spaces and the Assignment of Probabilities

### SOLUTION

(a) The sample space $S$ consists of 10 government bonds. Since the selection of a bond is random, the outcomes in $S$ are equally likely. If $E$ is the event “Treasury inflation-protected securities,” then

$$P(E) = \frac{\text{Number of Treasury inflation-protected securities bonds}}{\text{Number of government bonds}} = \frac{4}{10} = 0.4$$

(b) If $F$ is the event “30-year bond,” then $n(F) = 2$ and

$$P(F) = \frac{n(F)}{n(S)} = \frac{2}{10} = 0.2$$

(c) If $G$ is the event “yield is more than 3%,” then $n(G) = 2$ and

$$P(G) = \frac{n(G)}{n(S)} = \frac{2}{10} = 0.2$$

(d) If $H$ is the event “maturity is 4 years or more,” then $n(H) = 7$ and

$$P(H) = \frac{n(H)}{n(S)} = \frac{7}{10} = 0.7$$

NOW WORK PROBLEMS 27 AND 59.

### EXERCISE 7.4 Answers Begin on Page AN–36.

**Concepts and Vocabulary**

1. **True or False** In assigning a probability $P(e)$ to an outcome $e$ of a sample space $S$, $P(e)$ must be a positive number less than or equal to 1.

2. A box contains 3 red and 2 blue marbles. If a marble is chosen at random, the probability it is red is __________.

3. A __________ __________ of an experiment is the set of all possibilities that can result from performing the experiment.

4. **True or False** For a sample space $S = \{a, b, c\}$, a valid assignment of probabilities is $P(a) = 0.3$, $P(b) = 0.4$, $P(c) = 0.3$.

5. A(n) __________ is a subset of a sample space.

6. **True or False** In a sample space $S$ with equally likely outcomes, the probability of an event $E$ is $P(E) = n(S) - n(E)$.  

### TABLE 2 GOVERNMENT BONDS

<table>
<thead>
<tr>
<th>T-Bills</th>
<th>Bonds and Notes</th>
<th>Treasury Inflation-Protected Securities</th>
</tr>
</thead>
<tbody>
<tr>
<td>13 week, 0.178% yield</td>
<td>3 year, 1.776% yield</td>
<td>4 year 6 month, 1.25% yield</td>
</tr>
<tr>
<td>26 week, 0.269% yield</td>
<td>5 year, 2.405% yield</td>
<td>9 year 9 month, 1.709% yield</td>
</tr>
<tr>
<td></td>
<td>10 year, 3.692% yield</td>
<td>20 year, 1.43% yield</td>
</tr>
<tr>
<td></td>
<td>30 year, 4.72% yield</td>
<td>30 year, 2.229% yield</td>
</tr>
</tbody>
</table>

Source: U.S. Treasury
In Problems 7–14, use the pictured spinners to list the outcomes of a sample space associated with each experiment.

- **Spinner 1**: Red, Green
- **Spinner 2**: A, B
- **Spinner 3**: 1, 2, 3, 4

7. First spinner 1 is spun and then spinner 2 is spun.
8. First spinner 2 is spun and then spinner 3 is spun.
9. Spinner 2 is spun twice.
10. Spinner 3 is spun twice.
11. Spinner 2 is spun twice and then spinner 3 is spun.
12. Spinner 3 is spun once and then spinner 2 is spun twice.
13. Spinners 1, 2, and 3 are each spun once in this order.
14. Spinners 3, 2, and 1 are each spun once in this order.

In Problems 15–22, find the number of outcomes of a sample space associated with each experiment.

- 15. Tossing a coin 4 times
- 16. Tossing a coin 5 times
- 17. Tossing 3 dice
- 18. Tossing 2 dice and then a coin
- 19. Selecting 2 cards in order (without replacement) from a regular deck of 52 cards*
- 20. Selecting 3 cards in order (without replacement) from a regular deck of 52 cards
- 21. Picking two letters from the alphabet (Repetitions allowed; assume order is important.)
- 22. Picking two letters from the alphabet (No repetitions allowed; assume order is important.)

In Problems 23–26, consider the experiment of tossing a coin twice. The table lists six possible assignments of probabilities for this experiment:

23. Which of the assignments of probabilities are valid?
24. Which of the assignments of probabilities should be used if the coin is known to be fair, that is, the outcomes are equally likely?
25. Which of the assignments of probabilities should be used if the coin is known to always come up tails?
26. Which of the assignments of probabilities should be used if tails is twice as likely as heads to occur?

<table>
<thead>
<tr>
<th></th>
<th>HH</th>
<th>HT</th>
<th>TH</th>
<th>TT</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>[\frac{1}{4}]</td>
<td>[\frac{1}{4}]</td>
<td>[\frac{1}{4}]</td>
<td>[\frac{1}{4}]</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>[\frac{3}{16}]</td>
<td>[\frac{5}{16}]</td>
<td>[\frac{5}{16}]</td>
<td>[\frac{3}{16}]</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>[\frac{1}{2}]</td>
<td>[\frac{1}{2}]</td>
<td>[\frac{-1}{2}]</td>
<td>[\frac{1}{2}]</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>[\frac{1}{8}]</td>
<td>[\frac{1}{4}]</td>
<td>[\frac{1}{4}]</td>
<td>[\frac{1}{8}]</td>
</tr>
<tr>
<td><strong>F</strong></td>
<td>[\frac{1}{9}]</td>
<td>[\frac{2}{9}]</td>
<td>[\frac{2}{9}]</td>
<td>[\frac{4}{9}]</td>
</tr>
</tbody>
</table>

*A regular deck of cards has 52 cards. There are four suits of 13 cards each. The suits are called clubs (black), diamonds (red), hearts (red), and spades (black). In each suit the 13 cards are labeled 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and A (ace), in order from lowest (2) to highest (A). Sometimes A is considered to be 1.*
7.4 Sample Spaces and the Assignment of Probabilities

In Problems 27–34, a ball is picked at random from a box containing 3 white, 5 red, 8 blue, and 7 green balls. Find the probability of each event.

27. White ball is picked.
28. Blue ball is picked.
29. Green ball is picked.
30. Red ball is picked.
31. White or red ball is picked.
32. Green or blue ball is picked.
33. Neither red nor green ball is picked.
34. Red or white or blue ball is picked.

In Problems 35–44, a card is drawn at random from a regular deck of 52 cards. Find the probability of each event.

35. The ace of hearts is drawn.
36. An ace is drawn.
37. A spade is drawn.
38. A red card is drawn.
39. A picture card (J, Q, K) is drawn.
40. A number card (A, 2, 3, 4, 5, 6, 7, 8, 9, 10) is drawn.
41. A card with a number less than 6 is drawn (count A as 1).
42. A card with a value of 10 or higher is drawn (count A as high).
43. A card that is not an ace is drawn.
44. A card that is either a queen or king of any suit is drawn.

Applications and Extensions

45. Shampoo Survey A marketing firm sends a new shampoo along with a survey to a test market. The survey asks for the user’s gender and hair type—dry, oily, or regular. Find a sample space. How many outcomes are in the sample space?

46. Driving Habits A marketing firm surveys driving habits; the researcher asks the respondents their gender, age group (under 35 years, 35 years or older), and whether they prefer driving a car, SUV, or pickup truck. Find a sample space. How many outcomes are in the sample space?

47. Four-Child Families Families with 4 children are surveyed and the gender of the children according to birth order is recorded.

(a) Construct a probability model describing the experiment.
(b) Find the probability that
(i) the first 2 children are girls.
(ii) all children are boys.
(iii) at least one child is a girl.
(iv) the first child and the last child are girls.

48. Dividends and the Dow The Dow Jones Industrial Average consists of 30 blue-chip stocks. On April 8, 2010, 18 of these stocks pay annual dividends of more than $1.00 per share. The remaining 12 stocks pay annual dividends of less than $1.00 per share. Construct a probability model that describes the dividends paid by the stocks in the Dow.

Family Income According to the U.S. Census Bureau, the 2008 income of U.S. families is as described in the table.

<table>
<thead>
<tr>
<th>Income Level</th>
<th>Number of Families (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; $25,000</td>
<td>28,943.7</td>
</tr>
<tr>
<td>$25,000–$49,999</td>
<td>29,178.1</td>
</tr>
<tr>
<td>$50,000–$74,999</td>
<td>20,975.4</td>
</tr>
<tr>
<td>$75,000–$99,999</td>
<td>13,944.5</td>
</tr>
<tr>
<td>≥ $100,000</td>
<td>24,022.1</td>
</tr>
</tbody>
</table>


In Problems 49–54, use the information in the table above. Round your answers to three decimal places.

49. Construct a probability model that describes family income in the United States.
50. What is the probability a randomly selected family has an income below $25,000?
51. What is the probability a randomly selected family has an income between $50,000 and $99,999?
52. What is the probability a randomly selected family has an income of $75,000 or more?
53. What is the probability a randomly selected family has an income of less than $50,000?
54. What is the probability a randomly selected family has an income of $100,000 or more?
Credit and Signature Debit Cards  At the end of 2009, the four major U.S. credit card networks had credit and signature debit cards in force as follows:

<table>
<thead>
<tr>
<th>Visa</th>
<th>MasterCard</th>
<th>American Express</th>
<th>Discover</th>
</tr>
</thead>
<tbody>
<tr>
<td>270.1 million</td>
<td>203 million</td>
<td>48.9 million</td>
<td>54.4 million</td>
</tr>
</tbody>
</table>

Source: Card Data

In Problems 55–58, use the information in the table above. Round your answers to three decimal places.

55. What is the probability a randomly selected card is a MasterCard?
56. What is the probability a randomly selected card is a Visa card?
57. What is the probability a randomly selected card is an American Express or a Discover card?
58. What is the probability a randomly selected card is a MasterCard or a Visa card?

Dow Jones Industrial Average  The 30 stocks of the Dow Jones Industrial Average and their closing price on April 7, 2010, are given below.

<table>
<thead>
<tr>
<th>Company</th>
<th>Close</th>
</tr>
</thead>
<tbody>
<tr>
<td>1  Alcoa Inc.</td>
<td>14.74</td>
</tr>
<tr>
<td>2  American Express Co.</td>
<td>42.37</td>
</tr>
<tr>
<td>3  Boeing Co.</td>
<td>72.10</td>
</tr>
<tr>
<td>4  Bank of America</td>
<td>18.62</td>
</tr>
<tr>
<td>5  Caterpillar Inc.</td>
<td>64.47</td>
</tr>
<tr>
<td>6  Cisco Systems, Inc.</td>
<td>26.34</td>
</tr>
<tr>
<td>7  Chevron Corp.</td>
<td>77.37</td>
</tr>
<tr>
<td>8  E.I. Du Pont de Nemours &amp; Co.</td>
<td>38.79</td>
</tr>
<tr>
<td>9  Walt Disney Co.</td>
<td>35.32</td>
</tr>
<tr>
<td>10 General Electric Co.</td>
<td>18.50</td>
</tr>
<tr>
<td>11 Home Depot Inc.</td>
<td>32.61</td>
</tr>
<tr>
<td>12 Hewlett-Packard Co.</td>
<td>53.29</td>
</tr>
<tr>
<td>13 International Business Machines Corp.</td>
<td>128.48</td>
</tr>
<tr>
<td>14 Intel Corp.</td>
<td>22.45</td>
</tr>
<tr>
<td>15 Johnson &amp; Johnson</td>
<td>65.22</td>
</tr>
<tr>
<td>16 JPMorgan Chase &amp; Co.</td>
<td>45.32</td>
</tr>
<tr>
<td>17 Kraft Foods Inc. Cl A</td>
<td>30.07</td>
</tr>
<tr>
<td>18 Coca-Cola Co.</td>
<td>53.82</td>
</tr>
<tr>
<td>19 McDonald’s Corp.</td>
<td>67.70</td>
</tr>
<tr>
<td>20 3M Co.</td>
<td>83.66</td>
</tr>
<tr>
<td>21 Merck &amp; Co. Inc.</td>
<td>36.79</td>
</tr>
<tr>
<td>22 Microsoft Corp.</td>
<td>29.35</td>
</tr>
<tr>
<td>23 Pfizer Inc.</td>
<td>17.07</td>
</tr>
<tr>
<td>24 Procter &amp; Gamble Co.</td>
<td>62.50</td>
</tr>
<tr>
<td>25 AT&amp;T Inc.</td>
<td>25.65</td>
</tr>
<tr>
<td>26 Travelers Cos. Inc.</td>
<td>52.50</td>
</tr>
<tr>
<td>27 United Technologies Corp.</td>
<td>73.99</td>
</tr>
<tr>
<td>28 Verizon Communications Inc.</td>
<td>30.24</td>
</tr>
<tr>
<td>29 Wal-Mart Stores Inc.</td>
<td>55.28</td>
</tr>
<tr>
<td>30 Exxon Mobil Corp.</td>
<td>67.34</td>
</tr>
</tbody>
</table>

Source: Dow Jones Indexes

In Problems 59–62, use the information in the table above. Round your answers to three decimal places.

59. What is the probability the price of a randomly selected stock is greater than $50?
60. What is the probability the price of a randomly selected stock is less than $40?
61. What is the probability the price of a randomly selected stock is between $40 and $50?
62. What is the probability the price of a randomly selected stock is less than $40 or greater than $50?
New Home Sales  The number of new home sales (in thousands) in the United States as of March 24, 2010, is given in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Northeast</th>
<th>Midwest</th>
<th>South</th>
<th>West</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feb–Mar 2009</td>
<td>47</td>
<td>94</td>
<td>402</td>
<td>143</td>
</tr>
<tr>
<td>April–Jun 2009</td>
<td>82</td>
<td>607</td>
<td>607</td>
<td>278</td>
</tr>
<tr>
<td>Jul–Sep 2009</td>
<td>114</td>
<td>179</td>
<td>623</td>
<td>302</td>
</tr>
<tr>
<td>Oct–Dec 2009</td>
<td>102</td>
<td>174</td>
<td>578</td>
<td>253</td>
</tr>
<tr>
<td>Jan–Feb 2010</td>
<td>67</td>
<td>91</td>
<td>299</td>
<td>170</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau

In Problems 63–70, use the information in the table above. Round your answers to three decimal places.

63. What is the probability a home selected at random was sold from Feb–Mar 2009?
64. What is the probability a home selected at random was sold in the South?
65. What is the probability a home selected at random was sold from Oct–Dec 2009?
66. What is the probability a home selected at random was sold in the Midwest?
67. What is the probability a home selected at random was sold in the West from Jul–Sep 2009?
68. What is the probability a home selected at random was sold in the South from Apr–Jun 2009?
69. What is the probability a home selected at random was sold in the Northeast from Jan–Feb 2010?

70. What is the probability a home selected at random was sold in the Midwest from Oct–Dec 2009?

71. Health Insurance Coverage  The U.S. Census Bureau reported that 255,143,000 Americans were covered by health insurance in the year 2008 and 46,340,000 Americans were not covered by health insurance that year.
   (a) What is the probability a randomly selected American in the year 2008 was covered by health insurance? Write the answer as a decimal rounded to three decimal places.
   (b) What is the probability a randomly selected American in the year 2008 was not covered by health insurance?

72. Tossing a Fair Coin  Consider the experiment of tossing a fair coin. Simulate the experiment using a random number function, considering a toss to be tails (T) if the result is less than 0.5, and considering a toss to be heads (H) if the result is greater than or equal to 0.5. [Note: Most utilities repeat the action of the last entry if you simply press the ENTER, or EXE, key again.] Repeat the experiment 10 times. Using these 10 outcomes of the experiment you can estimate the probabilities \( P(H) \) and \( P(T) \) by the ratios
   \[
   P(H) \approx \frac{\text{Number of times } H \text{ occurred}}{10}
   \]
   \[
   P(T) \approx \frac{\text{Number of times } T \text{ occurred}}{10}
   \]
   What are the actual probabilities? How close are the results of the experiment to the actual values?

73. Urn and Balls  Consider the experiment of choosing a ball from an urn containing 18 red and 12 white balls. Simulate the experiment using a random number function, considering a selection to be a red ball (R) if the result is less than 0.6, and considering a selection to be a white ball (W) if the result is greater than or equal to 0.6. [Note: Most utilities repeat the action of the last entry if you simply press the ENTER, or EXE, key again.] Repeat the experiment 10 times. Using these 10 outcomes of the experiment you can estimate the probabilities \( P(R) \) and \( P(W) \) by the ratios
   \[
   P(R) \approx \frac{\text{Number of times } R \text{ occurred}}{10}
   \]
   \[
   P(W) \approx \frac{\text{Number of times } W \text{ occurred}}{10}
   \]
   What are the actual probabilities? How close are the results of the experiment to the actual values?

* Most graphing utilities have a random number function (usually RAND or RND) for generating numbers between 0 and 1. Check your user’s manual to see how to use this function.
74. **Tossing a Loaded Coin** Consider the experiment of tossing a loaded coin in which heads is three times as likely as tails to occur. Simulate the experiment using a random number function, considering a toss to be tails (T) if the result is less than 0.25, and considering a toss to be heads (H) if the result is greater than or equal to 0.25. [Note: Most utilities repeat the action of the last entry if you simply press the ENTER, or EXE, key again.] Repeat the experiment 20 times. Using these 20 outcomes of the experiment you can estimate the probabilities \( P(H) \) and \( P(T) \) by the ratios:

\[
P(H) \approx \frac{\text{Number of times } H \text{ occurred}}{20}
\]

\[
P(T) \approx \frac{\text{Number of times } T \text{ occurred}}{20}
\]

What are the actual probabilities? How close are the results of the experiment to the actual values?

75. **Urn and Balls** Consider the experiment of choosing a ball from an urn containing 15 red and 35 white balls. Simulate the experiment using a random number function, considering a selection to be a red ball (R) if the result is less than 0.3, and considering a selection to be a white ball (W) if the result is greater than or equal to 0.3. [Note: Most utilities repeat the action of the last entry if you simply press the ENTER, or EXE, key again.] Repeat the experiment 10 times. Using these 10 outcomes of the experiment you can estimate the probabilities \( P(R) \) and \( P(W) \) by the ratios:

\[
P(R) \approx \frac{\text{Number of times } R \text{ occurred}}{10}
\]

\[
P(W) \approx \frac{\text{Number of times } W \text{ occurred}}{10}
\]

What are the actual probabilities? How close are the results of the experiment to the actual values?

76. **Jar and Marbles** Consider an experiment of choosing a marble from a jar containing 5 red, 2 yellow, and 8 white marbles. Simulate the experiment using a random number function on your calculator, considering a selection to be a red marble (R) if the result is less than or equal to 0.33, a yellow marble (Y) if the result is between 0.33 and 0.47, and a white marble (W) if the result is greater than or equal to 0.47. [Note: Most calculators repeat the action of the last entry if you simply press the ENTER, or EXE, key again.] Repeat the experiment 20 times. Using these 20 outcomes of the experiment you can estimate the probabilities \( P(R) \), \( P(Y) \), and \( P(W) \) by the ratios:

\[
P(R) \approx \frac{\text{Number of times } R \text{ occurred}}{10}
\]

\[
P(Y) \approx \frac{\text{Number of times } Y \text{ occurred}}{10}
\]

\[
P(W) \approx \frac{\text{Number of times } W \text{ occurred}}{10}
\]

What are the actual probabilities? How close are the results of the experiment to the actual values?

77. **Poker Game** The probabilities in a game of poker that Adam, Beatrice, or Cathy win are 0.22, 0.60, and 0.18, respectively. Simulate the game using a random number function on your calculator, considering that Adam won (A) if the result is less than or equal to 0.22, that Beatrice won (B) if the result is between 0.22 and 0.82, and that Cathy won (C) if the result is greater than or equal to 0.82. [Note: Most calculators repeat the action of the last entry if you simply press the ENTER, or EXE, key again.] Repeat the experiment 20 times; that is, simulate 20 games. Using these 20 outcomes of the experiment you can estimate the probabilities \( P(A) \), \( P(B) \), and \( P(C) \) by the ratios:

\[
P(A) \approx \frac{\text{Number of times } A \text{ won}}{20}
\]

\[
P(B) \approx \frac{\text{Number of times } B \text{ won}}{20}
\]

\[
P(C) \approx \frac{\text{Number of times } C \text{ won}}{20}
\]

What are the actual values? How close are the results of the experiment to the actual values?
In Section 7.4 we defined the probability of an event \( E \) in a sample space \( S \) with equally likely outcomes. We begin this section with a definition for the probability of an event \( E \) in a sample space \( S \) whose outcomes have been assigned valid probabilities, not necessarily all the same.

### 1 Find the Probability of an Event

**Definition**

**Event; Simple Event**

An event is any subset of a sample space. If an event has exactly one element, that is, if it consists of only one outcome, it is called a simple event.

For example, consider the experiment discussed in Example 3 of Section 7.4: selecting one family from the set of three-child families. A sample space \( S \) for this experiment is

\[
S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}
\]

where, for example, \( BGB \) means that the family selected had a boy as a first child, a girl as second child, and a boy as third child.

The event \( E \) that the family has two boys followed by a girl, \( E = \{BBG\} \), consists of only one outcome and is a simple event.

The event \( F \) that the family consists of exactly two boys is

\[
F = \{BBG, BGB, GBB\}
\]

Notice that the event \( F \) is not a simple event but can be expressed as the union of the three simple events \( \{BBG\}, \{BGB\}, \{GBB\} \). That is,

\[
F = \{BBG\} \cup \{BGB\} \cup \{GBB\}
\]
In fact, every event can be written as the union of simple events. Since a sample space \( S \) is also an event, we can express a sample space \( S \) as the union of simple events. If the sample space \( S \) consists of \( n \) outcomes, 
\[
S = \{e_1, e_2, \ldots, e_n\}
\]
then
\[
S = \{e_1\} \cup \{e_2\} \cup \cdots \cup \{e_n\}
\]
Suppose valid probabilities have been assigned to each outcome of \( S \). Let \( E \) be any event of \( S \). Then either \( E = \emptyset \) or \( E \) is the union of two or more simple events. We give the following definition for the probability of \( E \).

**Definition**

**Probability of an Event**

If \( E = \emptyset \), the event \( E \) is **impossible**. We define the probability of \( \emptyset \) as
\[
P(\emptyset) = 0
\]
If \( E = \{e\} \) is a simple event, then \( P(E) = P(e) \); that is, \( P(E) \) equals the probability assigned to the outcome \( e \).
\[
P(E) = P(e)
\]
If \( E \) is the union of \( r \) simple events \( \{e_1\}, \{e_2\}, \ldots, \{e_r\} \), we define the **probability of \( E \)** to be the sum of the probabilities assigned to each simple event in \( E \). That is,
\[
P(E) = P(e_1) + P(e_2) + \cdots + P(e_r) \quad (1)
\]
If the sample space \( S \) is given by
\[
S = \{e_1, e_2, \ldots, e_n\}
\]
then the probability of \( S \) is
\[
P(S) = P(e_1) + \cdots + P(e_n) = 1
\]
That is, the probability of \( S \), the sample space, is 1.

**EXAMPLE 1** **Finding the Probability of an Event**

Each of the eight possible blood types is listed in Table 3 along with the percent of the U.S. population having that type. What is the probability that a randomly selected person in the United States has a blood type that is Rh-negative?

**SOLUTION**

The sample space \( S \) consists of the eight blood types, and the percents, when given as decimals, are the probability assignments for each element of the sample space. For example, the simple event “O positive” is assigned the probability 0.39.

Let \( E \) be the event that a randomly selected person in the United States has a blood type that is Rh-negative. We seek \( P(E) \). Since \( E \) is the union of the simple events \( \{O \text{ negative}\}, \{A \text{ negative}\}, \{AB \text{ negative}\}, \text{ and } \{B \text{ negative}\} \), we have
\[
E = \{O \text{ negative, A negative, AB negative, B negative}\}
\]
The probabilities assigned each of these simple events are

\[
P(O \text{ negative}) = 0.09 \quad P(A \text{ negative}) = 0.06 \\
P(B \text{ negative}) = 0.02 \quad P(AB \text{ negative}) = 0.01
\]

Then,

\[
P(E) = P\{O \text{ negative}\} + P\{A \text{ negative}\} + P\{AB \text{ negative}\} + P\{B \text{ negative}\} \\
= 0.09 + 0.06 + 0.01 + 0.02 = 0.18
\]

The probability is 0.18, or 18%, that a randomly selected person in the United States has Rh-negative blood.

\[\square\]

NOW WORK PROBLEM 35.

Find the Probability of \(E\) or \(F\) When \(E\) and \(F\) Are Mutually Exclusive

**Definition**

**Mutually Exclusive Events**

Two or more events of a sample space \(S\) are said to be **mutually exclusive** if and only if they have no outcomes in common.

Since events are sets, two events \(E\) and \(F\) are mutually exclusive if they are disjoint, that is, if \(E \cap F = \emptyset\).

**Theorem**

**Probability of \(E\) or \(F\) If \(E\) and \(F\) Are Mutually Exclusive**

Let \(E\) and \(F\) be two events of a sample space \(S\). If \(E\) and \(F\) are mutually exclusive, that is, if \(E \cap F = \emptyset\), then the probability of \(E\) or \(F\) is the sum of their probabilities. That is,

\[
P(E \cup F) = P(E) + P(F) \quad \text{if} \quad E \cap F = \emptyset
\]

(2)

This result follows since \(E\) and \(F\) can be written as a union of simple events in which no simple event of \(E\) appears in \(F\) and no simple event of \(F\) appears in \(E\).

**EXAMPLE 2**

**Finding the Probability of \(E\) or \(F\) When \(E\) and \(F\) Are Mutually Exclusive**

Refer back to Table 3 on blood types in the United States. What is the probability a person chosen at random is type \(O\) or type \(A\)?

**SOLUTION**

Define the events \(E\) and \(F\) as:

\[
E: \text{ person selected is type } O \quad F: \text{ person selected is type } A
\]
Then
\[ E = \{ \text{O positive, O negative}\} \quad F = \{ \text{A positive, A negative}\} \]

Now
\[ P(E) = P(\text{O positive}) + P(\text{O negative}) = 0.39 + 0.09 = 0.48 \]
\[ P(F) = P(\text{A positive}) + P(\text{A negative}) = 0.31 + 0.06 = 0.37 \]

Since \( E \cap F = \emptyset \), the events \( E \) and \( F \) are mutually exclusive. Based on Formula (2), the probability of \( E \) or \( F \), namely, is
\[ P(E \cup F) = P(E) + P(F) = 0.48 + 0.37 = 0.85 \]

So 85% of the U.S. population is either type O or type A.

Now work problems 15(a) and 15(b).

**3 Use the Additive Rule**

The following result, called the *Additive Rule*, gives a formula for finding the probability of the union of two events whether they are mutually exclusive or not.

**Theorem**

**Additive Rule**

For any two events \( E \) and \( F \) of a sample space \( S \), the probability of \( E \) or \( F \) is

\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) \quad (3) \]

A proof of this result is outlined in Problem 63.

**EXAMPLE 3 Using the Additive Rule**

If \( P(E) = 0.30 \), \( P(F) = 0.20 \), and the probability of \( E \) or \( F \) is 0.40, what is the probability of \( E \) and \( F \)?

**SOLUTION** We are given \( P(E) = 0.30 \) and \( P(F) = 0.20 \). The probability of \( E \) or \( F \) is \( P(E \cup F) = 0.40 \). We seek the probability of \( E \) and \( F \), namely, \( P(E \cap F) \).

Use the Additive Rule, Formula (3).
\[
\begin{align*}
P(E \cup F) &= P(E) + P(F) - P(E \cap F) \\
0.40 &= 0.30 + 0.20 - P(E \cap F) \\
P(E \cap F) &= 0.30 + 0.20 - 0.40 = 0.10
\end{align*}
\]

\[ P(E \cap F) = 0.30 + 0.20 - 0.40 = 0.10 \]

Now work Problem 5.

**EXAMPLE 4 Using the Additive Rule**

Refer back to Table 3 on blood types in the United States. What is the probability a person chosen at random is type O or Rh-negative?
Define the events \( E \) and \( F \) as

\( E: \) person selected is type O  \( F: \) person selected is Rh-negative

Then

\[
E = \{ \text{O positive, O negative} \} \\
F = \{ \text{A negative, B negative, AB negative, O negative} \}
\]

\[
P(E) = P(\text{O positive}) + P(\text{O negative}) = 0.39 + 0.09 = 0.48
\]

\[
P(F) = P(\text{A negative}) + P(\text{B negative}) + P(\text{AB negative}) + P(\text{O negative})
\]

\[
= 0.06 + 0.02 + 0.01 + 0.09 = 0.18
\]

Since \( E \cap F = \{ \text{O negative} \} \), then \( P(E \cap F) = 0.09 \). Using Formula (3), the probability of \( E \) or \( F \), namely \( P(E \cup F) \), is

\[
P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.48 + 0.18 - 0.09 = 0.57
\]

So 57% of the U.S. population is type O or Rh-negative.

---

**Find the Probability of the Complement of an Event**

Let \( E \) be an event of a sample space \( S \). The complement of \( E \) is the event “Not \( E \)” in \( S \).

**Theorem**

**Probability of the Complement of an Event**

Let \( E \) be an event of a sample space \( S \). Then the probability that \( E \) does not occur is

\[
P(\bar{E}) = 1 - P(E) \tag{4}
\]

where \( \bar{E} \) is the complement of \( E \).

**Proof**

We know that

\[
S = E \cup \bar{E} \quad E \cap \bar{E} = \emptyset
\]

Since \( E \) and \( \bar{E} \) are mutually exclusive,

\[
P(S) = P(E) + P(\bar{E})
\]

Since \( P(S) = 1 \), it follows that

\[
1 = P(E) + P(\bar{E}) \quad P(S) = 1
\]

\[
P(\bar{E}) = 1 - P(E) \quad \text{Solve for } P(\bar{E}).
\]

Formula (4) gives us a tool for finding the probability that an event does not occur if we know the probability that it does occur. The probability \( P(\bar{E}) \) that \( E \) does not occur is obtained by subtracting the probability \( P(E) \) that \( E \) does occur from 1. We will see shortly that it is sometimes easier to find \( P(E) \) by finding \( P(\bar{E}) \) and using Formula (4) than it is to proceed directly.
EXAMPLE 5 Using Formula (4)

A study of people over age 40 with an M.B.A. degree shows that it is reasonable to assign a probability of 0.756 that such a person will have annual earnings in excess of $140,000. The probability that such a person will earn $140,000 or less is then

\[1 - 0.756 = 0.244\]

NOW WORK PROBLEMS 15(c), 15(d), 15(e), AND 15(f).

EXAMPLE 6 Using a Venn Diagram to Find Probabilities

Suppose \(E\) and \(F\) are two events for which

\[P(E) = 0.25 \quad P(F) = 0.45 \quad P(E \cap F) = 0.10\]

(a) Construct a Venn diagram to represent this situation.
(b) Find the probability of neither \(E\) nor \(F\).
(c) Find the probability of \(E\) but not \(F\).

SOLUTION

(a) Begin with Figure 20(a) and the fact that \(P(E \cap F) = 0.10\). Then, since 

\[P(E) = 0.25, \quad \text{insert} \quad 0.25 - 0.10 = 0.15 \quad \text{in the remaining part of} \quad E. \quad \text{Similarly, insert} \quad 0.45 - 0.10 = 0.35 \quad \text{in the remaining part of} \quad F. \quad \text{See Figure 20(b).}\]

From Figure 20(b) we have

\[P(E \cup F) = 0.15 + 0.10 + 0.35 = 0.60\]

Alternatively use the Additive Rule. Then

\[P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.25 + 0.45 - 0.10 = 0.60\]

which checks.

Finally, since \(P(S) = 1\), enter \(1 - 0.60 = 0.40\) in the part outside the sets \(E\) and \(F\). See Figure 20(c).

(b) From Figure 20(c), the probability of neither \(E\) nor \(F\) is 0.40.
(c) From Figure 20(c), the probability of \(E\) but not \(F\) is 0.15.

NOW WORK PROBLEM 13.
Use the Multiplication Principle to Find Probabilities

**EXAMPLE 7** Finding Probabilities Using the Multiplication Principle

What is the probability that a four-digit telephone extension has one or more repeated digits? Assume no one digit is more likely than another to be used.

**SOLUTION** By the Multiplication Principle, there are \(10^4 = 10,000\) distinct four-digit telephone extensions, so the number of outcomes in the sample space \(S\) is

\[ n(S) = 10,000 \]

We wish to find the probability that a four-digit telephone extension has one or more repeated digits. Define the event \(E\) as follows:

\[ E: \text{The extension has one or more repeated digits} \]

Since it is difficult to count the outcomes of this event, we examine the complement of \(E\) to see if it is easier to count.

\[ E^c: \text{No repeated digits in four-digit extension} \]

We can find \(n(E^c)\) by using the Multiplication Principle. The number of four-digit extensions that have no repeated digits is

\[ n(E^c) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040 \]

So,

\[ P(E^c) = \frac{n(E^c)}{n(S)} = \frac{5040}{10,000} = 0.504 \]

The probability of one or more repeated digits is \(P(E)\) and

\[ P(E) = 1 - P(E^c) = 1 - 0.504 = 0.496 \]

**NOW WORK PROBLEM 53.**

**EXAMPLE 8** The Birthday Problem

An interesting problem, called the birthday problem, is to find the probability that in a group of \(r\) people there are at least two people who have the same birthday (the same month and day of the year). Assume a person is as likely to be born on one day as another.

**SOLUTION** First determine the number of outcomes in the sample space \(S\). There are 365 possibilities for each person’s birthday (we exclude February 29 for simplicity). Since there are \(r\) people in the group, there are \(365^r\) possibilities for the birthdays. [For one person in the group, there are 365 days on which his or her birthday can fall; for two people, there are \((365)(365) = 365^2\) pairs of days; and, in general, using the Multiplication Principle, for \(r\) people there are \(365^r\) possibilities.] That is,

\[ n(S) = 365^r \]

These outcomes are equally likely.
We wish to find the probability of the event $E$:

$$E: \text{At least two people have the same birthday}$$

It is difficult to count the elements in this set; it is much easier to count the elements of the complement:

$$\bar{E}: \text{No two people have the same birthday}$$

We proceed to find $n(\bar{E})$ as follows:

Choose one person at random. There are 365 possibilities for his or her birthday. Choose a second person. There are 364 possibilities for this birthday, if no two people are to have the same birthday. Choose a third person. There are 363 possibilities left for this birthday. Finally, we arrive at the $r$th person. There are possibilities left for this birthday. By the Multiplication Principle, the total number of possibilities is

$$n(\bar{E}) = 365 \cdot 364 \cdot 363 \cdots (365 - r + 1)$$

The probability of event $\bar{E}$ is

$$P(\bar{E}) = \frac{n(\bar{E})}{n(S)} = \frac{365 \cdot 364 \cdot 363 \cdots (365 - r + 1)}{365^r}$$

The probability that two or more people have the same birthday is then

$$P(E) = 1 - P(\bar{E})$$

For example, in a group of eight people, $r = 8$ and the probability that two or more will have the same birthday is

$$P(E) = 1 - \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361 \cdot 360 \cdot 359 \cdot 358}{365^8}$$

$$= 1 - 0.93$$

$$= 0.07$$

Table 4 gives the probabilities for two or more people having the same birthday. Notice that the probability is greater than $\frac{1}{2}$ for any group of 23 or more people.

<table>
<thead>
<tr>
<th>Number of People</th>
<th>Probability That 2 or More Have Same Birthday</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.027</td>
</tr>
<tr>
<td>10</td>
<td>0.117</td>
</tr>
<tr>
<td>15</td>
<td>0.253</td>
</tr>
<tr>
<td>20</td>
<td>0.411</td>
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<tr>
<td>21</td>
<td>0.444</td>
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<td>22</td>
<td>0.476</td>
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<td>30</td>
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<td>0.99916</td>
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<td>80</td>
<td>0.99991</td>
</tr>
<tr>
<td>90</td>
<td>0.99999</td>
</tr>
</tbody>
</table>

Now work Problem 59.

7 Compute Odds

In many instances the probability of an event is expressed as odds—either odds for an event or odds against an event.
7.5 Properties of the Probability of an Event

**Definition**

If $E$ is an event:
The odds for $E$ are $\frac{P(E)}{P(\overline{E})}$ read as $P(E)$ to $P(\overline{E})$.
The odds against $E$ are $\frac{P(\overline{E})}{P(E)}$ read as $P(\overline{E})$ to $P(E)$.

**EXAMPLE 9 Computing Odds Given a Probability**

Suppose the probability of the event $E$: It will rain
is 0.3. Then $P(E) = 0.3$ and $P(\overline{E}) = 0.7$.
The odds for rain are
$$\frac{P(E)}{P(\overline{E})} = \frac{0.3}{0.7} = \frac{3}{7} \text{ read as 3 to 7}$$
The odds against rain are
$$\frac{P(\overline{E})}{P(E)} = \frac{0.7}{0.3} = \frac{7}{3} \text{ read as 7 to 3}$$

**NOW WORK PROBLEM 23.**

To obtain the probability of the event $E$ when either the odds for $E$ or the odds against $E$ are known, use the following formulas:

**Theorem**

If the odds for $E$ are $a$ to $b$, then
$$P(E) = \frac{a}{a + b} \quad (5)$$

If the odds against $E$ are $a$ to $b$, then
$$P(\overline{E}) = \frac{b}{a + b} \quad (6)$$

A proof of Equation (5) is outlined in Problem 64. You are asked to prove Equation (6) in Problem 66.

**EXAMPLE 10 Computing a Probability from Odds**

(a) The odds for a Republican victory in the next presidential election are 7 to 5. What is the probability that a Republican victory occurs?
(b) The odds against the Chicago Cubs winning the league pennant are 200 to 1. What is the probability that the Cubs win the pennant?
SOLUTION  (a) The event E is “A Republican victory occurs.” The odds for E are 7 to 5. Then

\[ P(E) = \frac{7}{7 + 5} = \frac{7}{12} \approx 0.583 \]

(b) The event F is “The Cubs win the pennant.” The odds against F are 200 to 1. Then

\[ P(F) = \frac{1}{200 + 1} = \frac{1}{201} \approx 0.00498 \]

NOW WORK PROBLEM 51.

EXERCISE 7.5 Answers Begin on Page AN–37.

Concepts and Vocabulary

1. A(n) ________ is a subset of a sample space.

2. True or False If E and F are events, sometimes \( P(E \cup F) = P(E) + P(F) \).

Skill Building

In Problems 5–12, E and F are events of a sample space S.

5. \( P(E) = 0.4 \), \( P(F) = 0.5 \), \( P(E \cap F) = 0.2 \). Find \( P(E \cup F) \).

6. \( P(E) = 0.6 \), \( P(F) = 0.3 \), \( P(E \cap F) = 0.2 \). Find \( P(E \cup F) \).

7. \( P(E) = 0.7 \), \( P(F) = 0.5 \), \( P(E \cup F) = 0.8 \). Find \( P(E \cap F) \).

8. \( P(E) = 0.6 \), \( P(F) = 0.8 \), \( P(E \cup F) = 0.9 \). Find \( P(E \cap F) \).

9. \( P(E) = 0.4 \), \( P(E \cap F) = 0.1 \), \( P(E \cup F) = 0.6 \). Find \( P(F) \).

10. \( P(E \cap F) = 0.4 \), \( P(F) = 0.5 \), \( P(E \cup F) = 0.7 \). Find \( P(E) \).

11. \( P(E) = 0.4 \). Find \( P(\overline{E}) \).

12. \( P(F) = 0.5 \). Find \( P(F) \).

13. Let A and B be events of a sample space S and let \( P(A) = 0.5 \), \( P(B) = 0.4 \), and \( P(A \cap B) = 0.2 \). Find the probability of each of the following events:
   (a) A or B  (b) A but not B  (c) B but not A  (d) Neither A nor B

14. Let A and B be events of a sample space S and let \( P(A) = 0.7 \), \( P(B) = 0.4 \), and \( P(A \cap B) = 0.2 \). Find the probability of each of the following events:
   (a) A or B  (b) A but not B  (c) B but not A  (d) Neither A nor B

15. Let A and B be two mutually exclusive events of a sample space S. If \( P(A) = 0.6 \) and \( P(B) = 0.2 \), find each of the following probabilities:
   (a) \( P(A \cap B) \)  (b) \( P(A \cup B) \)  (c) \( P(\overline{A} \cup \overline{B}) \)
   (d) \( P(\overline{B}) \)  (e) \( P(\overline{A}) \)  (f) \( P(A \cap B) \)

16. Let A and B be two mutually exclusive events of a sample space. If \( P(A) = 0.35 \) and \( P(B) = 0.60 \), find each of the following probabilities:
   (a) \( P(A \cap B) \)  (b) \( P(A \cup B) \)  (c) \( P(\overline{A} \cup \overline{B}) \)
   (d) \( P(\overline{B}) \)  (e) \( P(\overline{A}) \)  (f) \( P(A \cap \overline{B}) \)

In Problems 17–22, determine the probability of E for the given odds.

17. 3 to 1 for E

18. 4 to 1 against E

19. 7 to 5 against E

20. 2 to 9 for E

21. 1 to 1 for E (even)

22. 50 to 1 for E

In Problems 23–26, determine the odds for and against each event for the given probability.

23. \( P(E) = 0.6 \)

24. \( P(H) = \frac{1}{4} \)

25. \( P(F) = \frac{3}{4} \)

26. \( P(G) = 0.1 \)
27. **Chicago Bears** The Chicago Bears football team has a probability of winning of 0.65 and of tying of 0.05. What is the probability of losing?

28. **Chicago Black Hawks** The Chicago Black Hawks hockey team has a probability of winning of 0.6 and a probability of losing of 0.25. What is the probability of a tie?

29. **Likelihood of Passing** Anne is taking courses in both mathematics and English. She estimates her probability of passing mathematics at 0.4 and English at 0.6, and she estimates her probability of passing at least one of them at 0.8. What is her probability of passing both courses?

30. **Dropping a Course** After midterm exams, Anne (see Problem 29) reassesses her probability of passing mathematics to 0.7. She feels her probability of passing at least one of these courses is still 0.8 but feels she has a probability of only 0.1 of passing both courses. If her probability of passing English is less than 0.4, she will drop English. Should Anne drop English?

31. **Car Repair** At the Milex tune-up and brake repair shop, the manager has found that a car will require a tune-up with a probability of 0.6, a brake job with a probability of 0.1, and both with a probability of 0.02.

(a) What is the probability that a car requires either a tune-up or a brake job?

(b) What is the probability that a car requires a tune-up but not a brake job?

(c) What is the probability that a car requires neither type of repair?

32. **Factory Shortages** A factory needs two raw materials, say, E and F. The probability of not having an adequate supply of material E is 0.06, whereas the probability of not having an adequate supply of material F is 0.04. A study shows that the probability of a shortage of both E and F is 0.02. What is the probability of the factory being short of either material E or F?

33. **TV Sets** In a survey of the number of TV sets in a house, the following probability table was constructed:

<table>
<thead>
<tr>
<th>Number of TV sets</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.05</td>
<td>0.24</td>
<td>0.33</td>
<td>0.21</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Find the probability of a house having

(a) 1 or 2 TV sets

(b) 1 or more TV sets

(c) 3 or fewer TV sets

(d) 3 or more TV sets

(e) Fewer than 2 TV sets

(f) Not even 1 TV set

(g) 1, 2, or 3 TV sets

(h) 2 or more TV sets

34. **Supermarket Lines** Through observation it has been determined that the probability for a given number of people waiting in line at a particular checkout register of a supermarket is as shown in the table:

<table>
<thead>
<tr>
<th>Number waiting in line</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.10</td>
<td>0.15</td>
<td>0.20</td>
<td>0.24</td>
<td>0.31</td>
</tr>
</tbody>
</table>

Find the probability of

(a) At most 2 people in line

(b) At least 2 people in line

(c) At least 1 person in line

Problems 35–40 require the following discussion:

**Blood Types** Each of the eight possible blood types is listed in the table below along with the percent of the U.S. population having that type.

<table>
<thead>
<tr>
<th>Blood Type</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>O Positive</td>
<td>39%</td>
</tr>
<tr>
<td>A Negative</td>
<td>6%</td>
</tr>
<tr>
<td>A Positive</td>
<td>31%</td>
</tr>
<tr>
<td>AB Positive</td>
<td>3%</td>
</tr>
<tr>
<td>B Positive</td>
<td>9%</td>
</tr>
<tr>
<td>B Negative</td>
<td>2%</td>
</tr>
<tr>
<td>O Negative</td>
<td>9%</td>
</tr>
<tr>
<td>AB Negative</td>
<td>1%</td>
</tr>
</tbody>
</table>

**Source:** AABB Facts about Blood, 2010

35. What is the probability a randomly selected person has a blood type that is Rh-positive?

36. What is the probability a randomly selected person has a blood type that is type O?

37. What is the probability a randomly selected person has a blood type that contains the A antigen? [Hint: AB blood contain both the A antigen and the B antigen.]

38. What is the probability a randomly selected person has a blood type that contains both the A and the B antigens?

39. What is the probability a randomly selected person has a blood type that is type O or Rh-positive?

40. What is the probability a randomly selected person has a blood type that is type AB or Rh-negative?

41. **Childbirth** In 2006, the probability a U.S. woman aged 15–44 years, who gave birth in the previous year, gave birth to her first child, was 0.457. What is the probability a U.S. woman aged 15–44 years who gave birth in the previous year had previously given birth?

**Source:** U.S. Census Bureau, *Fertility of American Women: June 2006*, Issued August 2008

42. **Tax Returns** As of October 31, 2008, 62% of tax refunds for the 2008 tax season were issued through direct deposit. If two refunds are selected at random, the probability that both were issued by direct deposit is 0.49. What is the probability at least one of the refunds is issued by direct deposit?

**Source:** Internal Revenue Service
43. **Bridge Inventory**  In 2009, there were about 603,000 bridges in the United States, of which 150,000 were either deficient or obsolete. What are the odds that a randomly selected bridge in the United States is either deficient or obsolete?

*Source:* U.S. Federal Highway Administration

44. **Trademarks**  In 2009, there were roughly 352,000 trademark applications filed in the United States. From these applications, about 320,000 trademarks were issued. What are the odds that a trademark application will result in a trademark being issued?


45. **Apple iPhone Market Share**  According to the February 2010 Mobile Metrics Report issued by AdMob, the Apple iPhone accounted for 14.4% of the worldwide smart phone market. What is the probability that a smart phone owner chosen at random will not have an iPhone?

46. **Family Cars**  A family owns two vehicles, a 12-year-old Geo Prizm and a 1-year-old Toyota Sienna. On any given morning, the probabilities that the vehicles will start are 0.98 and 0.84 for the Sienna and Prizm, respectively. If the probability at least one of the vehicles will start is 0.9968, what is the probability that they both start?

47. **Patents**  The number and type of U.S. patents (in thousands) issued in 2009 are given in the table. What is the probability that a randomly selected patent issued in 2009 was either for a design or a botanical plant?

<table>
<thead>
<tr>
<th>Type of Patent</th>
<th>Number Issued</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invention</td>
<td>165,212</td>
</tr>
<tr>
<td>Designs</td>
<td>23,915</td>
</tr>
<tr>
<td>Botanical plants</td>
<td>1096</td>
</tr>
<tr>
<td>Reissues</td>
<td>398</td>
</tr>
</tbody>
</table>


48. **Small Business Loans**  The number of small business loans awarded to minority-owned businesses in 2007 is given in the table. What is the probability that a randomly selected small business loan awarded to a minority-owned business in 2007 was awarded to a Hispanic-American small business?

<table>
<thead>
<tr>
<th>Minority Group</th>
<th>Number of Loans</th>
</tr>
</thead>
<tbody>
<tr>
<td>African American</td>
<td>6,632</td>
</tr>
<tr>
<td>Asian American</td>
<td>13,455</td>
</tr>
<tr>
<td>Hispanic American</td>
<td>8,795</td>
</tr>
<tr>
<td>Native American</td>
<td>835</td>
</tr>
</tbody>
</table>

*Source:* U.S. Census Bureau, 2009

49. **Mutual Funds**  A financial consultant estimates that there is a 12% chance a mutual fund will outperform the market during any given year. She also estimates that there is a 5% chance that the mutual fund will outperform the market for the next two years. What is the probability that the mutual fund will outperform the market in at least one of the next two years?

50. **Airline Travel**  A poll asked respondents how many air trips they had taken on a commercial airliner in the past 12 months. The results are given in the table.

<table>
<thead>
<tr>
<th>Number of air trips</th>
<th>Percent of respondents</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>52</td>
</tr>
<tr>
<td>1–2</td>
<td>29</td>
</tr>
<tr>
<td>3–4</td>
<td>8</td>
</tr>
<tr>
<td>5 or more</td>
<td>11</td>
</tr>
</tbody>
</table>

(a) What is the probability a respondent selected at random did not travel by air in the last 12 months?
(b) What is the probability a respondent selected at random made more than 2 air trips in the past 12 months?
(c) What is the probability a respondent selected at random made fewer than 3 air trips in the past 12 months?

51. **Track**  In a track contest the odds that A will win are 1 to 2, and the odds that B will win are 2 to 3. Find the probability and the odds that A or B wins the race, assuming a tie is impossible.

52. **DUI**  It has been estimated that in 70% of the fatal accidents involving two cars, at least one of the drivers is drunk. If you hear of a two-car fatal accident, what odds should you give a friend for the event that at least one of the drivers was drunk?

53. **Picking Numbers**  If 30 students are asked to pick a number between 1 and 60, what is the probability at least two will choose the same number?

54. **Picking Numbers**  If 40 students are asked to pick a number between 1 and 80, what is the probability at least two will choose the same number?
59. **Birthday Problem**  What is the probability that, in a group of 3 people, at least 2 were born in the same month (disregard day and year)?

60. **Birthday Problem**  What is the probability that, in a group of 6 people, at least 2 were born in the same month (disregard day and year)?

61. **Birthday Problem**  Find the probability 2 or more U.S. senators have the same birthday. (There are 100 senators.)

62. **Birthday Problem**  Find the probability 2 or more members of the House of Representatives have the same birthday. (There are 435 representatives.)

63. Prove the Additive Rule, Formula (3) on page 394. 
   
   [Hint: From Example 9 in Section 7.1 (page 362) we have 
   \[ P(E \cap F) = P(E) + P(F) - P(E \cup F) \]  
   Since \( E \cap F \) and \( E \cap \overline{F} \) are disjoint and \( E \cap F \) and \( \overline{E} \cap F \) are disjoint, we have 
   \[ P(E) = P(E \cap F) + P(E \cap \overline{F}) \] 
   and 
   \[ P(F) = P(E \cap F) + P(\overline{E} \cap F) \]  
   Now combine (1) and (2).] 

64. Prove Equation (5) on page 399. 
   [Hint: If the odds for \( E \) are \( \frac{a}{b} \), then, by the definition of odds, 
   \[ \frac{P(E)}{P(\overline{E})} = \frac{a}{b} \]  
   But \( P(\overline{E}) = 1 - P(E) \). So 
   \[ \frac{P(E)}{1 - P(E)} = \frac{a}{b} \]  
   Now solve for \( P(E) \).]

65. Generalize the Additive Rule by showing the probability of the occurrence of at least one of the three events \( A, B, C \) is given by 
   \[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \]

66. Prove Equation (6) on page 399.

**Discussion and Writing**

67. Thirty students are asked to pick a number between 1 and 60. Ask some friends what they think the probability is that at least 2 will choose the same number. Now find the probability using the birthday problem as a guide. Did your friends answer correctly? If not, write a paragraph to convince them of the correct answer.

**7.6 Expected Value**

**OBJECTIVES**

1. Find the expected value (p. 405)
2. Solve applied problems involving expected value (p. 407)

**EXAMPLE 1**  Examples of Expected Value

(a) Suppose that 1000 tickets are sold for a raffle that has the following prizes: one $300 prize, two $100 prizes, and one hundred $1 prizes. Then, of the 1000 tickets, 1 ticket has a cash value of $300, 2 are worth $100, and 100 are worth $1, while the remaining are worth $0. The expected value \( E \) of a ticket is then

\[
E = \frac{100 \times 300}{1000} + \frac{2 \times 100}{1000} + \frac{100 \times 1}{1000} + \frac{897 \times 0}{1000} = \frac{30000 + 200 + 100 + 0}{1000} = \frac{31100}{1000} = 31.10
\]

(b) The expected value of the tickets is $31.10.
If the raffle is to be nonprofit to all, the charge for each ticket should be $0.60.

The situation above can also be viewed as follows: If we entered such a raffle many times, \(\frac{1}{1000}\) of the time we would win $300, \(\frac{2}{1000}\) of the time we would win $100, and so on, with our winnings in the long run averaging $0.60 per ticket.

(b) Suppose that you are to receive $3.00 each time you obtain two heads when flipping a fair coin two times and $0 otherwise. Then the expected value \(E\) is

\[
E = 3.00 \cdot \frac{1}{4} + 0 \cdot \frac{3}{4} = 0.75
\]

This means, if the game is to be fair, that you should be willing to pay $0.75 each time you play the game.

(c) A game consists of flipping a fair coin. If a head shows, the player loses $1; but if a tail shows, the player wins $2. That is, half the time the player loses $1 and the other half the player wins $2. The expected value \(E\) of the game is

\[
E = \frac{1}{2} \cdot 2 + (-\frac{1}{2}) \cdot 1 = 0.50
\]

The player is expected to win an average of $0.50 on each play.

In each of the above examples, we arrive at the expected value \(E\) by multiplying the amount earned for a given outcome times the probability for that outcome to occur, and adding all the products.

Look back at Example 1(a). In the expression for the expected value in the raffle problem, the term \(300 \cdot \frac{1}{1000}\) is the product of the value $300 with its corresponding probability \(\frac{1}{1000}\), namely, the probability of having a $300 ticket. Likewise, the probability of getting a $1 ticket is \(\frac{100}{1000}\) and this produces the term \(1 \cdot \frac{100}{1000}\) in the expression for the expected value, \(E\). So the expression for \(E\) is obtained by multiplying each ticket value by the probability of its occurrence and adding the results.

In Example 1(b), the term \(3 \cdot \frac{1}{4}\) is the product of the value $3.00 with its corresponding probability \(\frac{1}{4}\), namely, the probability of obtaining \(HH\). Likewise, the probability of getting \(HT, TH,\) and \(TT\) is \(\frac{3}{4}\) with payoff $0, and this produces \(0 \cdot \frac{3}{4}\) in the expression for the expected value.

Finally, in Example 1(c) the term \(2 \cdot \frac{1}{2}\) is the product of the value $2.00 with its corresponding probability \(\frac{1}{2}\), namely, the probability of obtaining \(T\). Likewise, the probability of getting \(H\) is \(\frac{1}{2}\), with payoff $-1.

This leads to the following definition.
Definition

Expected Value

Let \( S \) be a sample space and let \( A_1, A_2, \ldots, A_n \) be \( n \) events of \( S \) that form a partition of \( S \). That is, the union of the \( n \) events is \( S \) and the \( n \) events are pair-wise disjoint:

\[
A_1 \cup A_2 \cup \ldots \cup A_n = S \quad \text{and} \quad A_i \cap A_j = \emptyset, \quad i \neq j, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n
\]

Let \( p_1, p_2, \ldots, p_n \) be the probabilities, respectively, of the events \( A_1, A_2, \ldots, A_n \). If each event \( A_1, A_2, \ldots, A_n \) is assigned, respectively, the payoffs \( m_1, m_2, \ldots, m_n \), the expected value \( E \) corresponding to these payoffs is

\[
E = m_1 \cdot p_1 + m_2 \cdot p_2 + \cdots + m_n \cdot p_n \quad (1)
\]

The term \textit{expected value} should not be interpreted as a value that actually occurs in the experiment. In Example 1(a) there was no raffle ticket costing $0.60. Rather, this number represents the average value of a raffle ticket.

In gambling, \( E \) is interpreted as the average winnings expected for the player in the long run. If \( E \) is positive, we say that the game is \textit{favorable} to the player; if \( E = 0 \), we say the game is \textit{fair}; and if \( E \) is negative, we say the game is \textit{unfavorable} to the player.

When the payoff assigned to an outcome of an experiment is positive, it can be interpreted as a profit, winnings, gain, and so on. When it is negative, it represents losses, penalties, deficits, and so on.

1 Find the Expected Value

Steps for Finding the Expected Value

\textbf{STEP 1} Partition a sample space \( S \) into \( n \) events \( A_1, A_2, \ldots, A_n \).

\textbf{STEP 2} Determine the probability \( p_1, p_2, \ldots, p_n \) of each event \( A_1, A_2, \ldots, A_n \). Since these events constitute a partition of \( S \), it follows that

\[
p_1 + p_2 + \cdots + p_n = 1.
\]

\textbf{STEP 3} Assign payoff values \( m_1, m_2, \ldots, m_n \) to each event \( A_1, A_2, \ldots, A_n \).

\textbf{STEP 4} Calculate the expected value \( E = m_1 \cdot p_1 + m_2 \cdot p_2 + \cdots + m_n \cdot p_n \).

\textbf{EXAMPLE 2} Find the Expected Value

Consider the experiment of rolling a fair die. The player recovers an amount of dollars equal to the number of dots on the face that turns up, except when face 5 or 6 turns up, in which case the player will lose $5 or $6, respectively. What is the expected value of the game? Is the game fair?
Chapter 7  Probability

**SOLUTION**

**STEP 1**  
The sample space is \(S = \{1, 2, 3, 4, 5, 6\}\). Each of these 6 outcomes constitutes an event that forms a partition of \(S\).

**STEP 2**  
Since each outcome (event) is equally likely, the probability of each one is \(\frac{1}{6}\).

**STEP 3**  
Since the player wins $1 for the outcome 1, $2 for the outcome 2, $3 for a 3, and $4 for a 4, and the player loses $5 for a 5 or equivalently wins −$5 for a 5 and loses $6 for a 6 or equivalently wins −$6 for a 6, we assign payoffs (the winnings of the player) as follows:

<table>
<thead>
<tr>
<th>Event</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff (winnings)</td>
<td>$1</td>
<td>$2</td>
<td>$3</td>
<td>$4</td>
<td>−$5</td>
<td>−$6</td>
</tr>
</tbody>
</table>

**STEP 4**  
The expected value of the game is

\[
E = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \left(- \frac{1}{6}\right) \cdot (-5) + \left(- \frac{1}{6}\right) \cdot (-6)
\]

\[
= -\frac{1}{6} = -0.167
\]

The player would expect to lose an average of 16.7 cents on each throw.  

The game is not fair; it is unfavorable to the player.

**NOW WORK PROBLEM 3.**

**EXAMPLE 3  Find the Expected Value**

What is the expected number of heads in tossing a fair coin three times?

**SOLUTION**

**STEP 1**  
The sample space is

\[S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}\]

Since we are interested in the number of heads and since a coin tossed three times will result in 0, 1, 2, or 3 heads, we partition \(S\) as follows:

\[A_1 = \{TTT\} \quad A_2 = \{HHT, THT, TTH\} \quad A_3 = \{HHT, HTH, THH\} \quad A_4 = \{HHH\}\]

\[0 \text{ heads} \quad 1 \text{ head} \quad 2 \text{ heads} \quad 3 \text{ heads}\]

**STEP 2**  
The probability of each of these events is

\[
P(A_1) = \frac{1}{8} \quad P(A_2) = \frac{3}{8} \quad P(A_3) = \frac{3}{8} \quad P(A_4) = \frac{1}{8}
\]

**STEP 3**  
Since we are interested in the expected number of heads, we assign payoffs of 0, 1, 2, and 3, respectively, to \(A_1, A_2, A_3,\) and \(A_4\).

**STEP 4**  
The expected number of heads can now be found by multiplying each payoff by its corresponding probability and finding the sum of these values.

\[
\text{Expected number of heads} = E = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2} = 1.5
\]

On average, tossing a coin three times will result in 1.5 heads.

**NOW WORK PROBLEM 13.**
2 Solve Applied Problems Involving Expected Value

**EXAMPLE 4** Bidding for Oil Wells

An oil company may bid for only one of two contracts for oil drilling in two different locations. If oil is discovered at location I, the profit to the company will be $3,000,000. If no oil is found, the company’s loss will be $250,000. If oil is discovered at location II, the profit will be $4,000,000. If no oil is found, the loss will be $500,000. The probability of discovering oil at location I is 0.7, and at location II it is 0.6. Which field should the company bid on; that is, for which location is expected profit higher?

**SOLUTION**

In the first location the company expects to discover oil 0.7 of the time at a profit of $3,000,000. It would not discover oil 0.3 of the time at a loss of $250,000. The expected profit $E_I$ is therefore

$$E_I = (3,000,000)(0.7) + (-250,000)(0.3) = 2,025,000$$

For the second location, the expected profit $E_{II}$ is

$$E_{II} = (4,000,000)(0.6) + (-500,000)(0.4) = 2,200,000$$

Since the expected profit for the second location exceeds that for the first, the oil company should bid on the second location.

**NOW WORK PROBLEM 27.**

**EXAMPLE 5** Evaluating Insurance

Mr. Richmond is producing an outdoor concert. He estimates that he will make $300,000 if it does not rain and make $60,000 if it does rain. The weather bureau predicts that the chance of rain is 0.34 for the day of the concert.

(a) What are Mr. Richmond’s expected earnings from the concert?

(b) An insurance company is willing to insure the concert for $150,000 against rain for a premium of $30,000. If he buys this policy, what are his expected earnings from the concert?

(c) Based on the expected earnings, should Mr. Richmond buy an insurance policy?

**SOLUTION**

The sample space consists of two outcomes: $R$: Rain or $N$: No rain. The probability of rain is $P(R) = 0.34$; the probability of no rain is $P(N) = 0.66$.

(a) If it rains, the earnings are $60,000; if there is no rain, the earnings are $300,000. The expected earnings $E$ are

$$E = 60,000 \cdot P(R) + 300,000 \cdot P(N)$$

$$= 60,000(0.34) + 300,000(0.66)$$

$$= 218,400$$

(b) With insurance, if it rains, the earnings are

$$150,000 + 60,000 - 30,000 = 180,000$$

The concert profit and cost of insurance are

<table>
<thead>
<tr>
<th>Insurance payout</th>
<th>Concert profit</th>
<th>Cost of insurance</th>
</tr>
</thead>
<tbody>
<tr>
<td>150,000</td>
<td>60,000</td>
<td>30,000</td>
</tr>
<tr>
<td>$180,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
With insurance, if it does not rain, the earnings are 

\[
\text{with insurance, the expected earnings } E \text{ are}
\]

\[
E = 180,000(0.34) + 270,000(0.66)
\]

\[
= 239,400
\]

(c) Since the expected earnings are higher with insurance, it would be better to obtain the insurance.

NOW WORK PROBLEM 25.

**EXAMPLE 6** Car Rentals

A national car rental agency rents luxury cars for $80 per day (gasoline is an additional expense to the customer). The daily cost per car (for example, lease costs and overhead) is $15 per day. The daily profit depends on two factors: the demand for cars and the number of cars the company has available to rent. Previous rental records show that the daily demand for luxury cars is as given in Table 5:

<table>
<thead>
<tr>
<th>Number of Customers</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.10</td>
<td>0.10</td>
<td>0.30</td>
<td>0.30</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Find the optimal number of cars the company should have available for rental. (This is the number that yields the largest expected profit.)

**SOLUTION** We seek the expected profit for each possible number of cars. The largest expected profit will tell us how many cars to have on hand.

For example, if there are 10 cars available, the expected profit for 8, 9, or 10 customers is found as follows:

For 8 customers, the revenue generated is \(8(80) = 640\) and the cost of the 10 cars is \(10(15) = 150\), for a profit of \$490\). The probability of 8 customers is 0.1.

For 9 customers, the revenue generated is \(9(80) = 720\) and the cost of the 10 cars is \$150\) for a profit of \$570\). The probability of 9 customers is 0.1.

For 10 customers, the revenue generated is \(10(80) = 800\) and the cost of the 10 cars is \$150\) for a profit of \$650\). The probability of 10 customers is 0.8, because 10 cars are rented when 10, 11, or 12 customers show up.

The expected profit \(E\) is

\[
E = 490(0.1) + 570(0.1) + 650(0.8) = 626
\]

Table 6 lists the expected profit for 8 to 12 available cars. The optimal stock size is 12 cars, since this number of cars maximizes expected profit.
Table 6

<table>
<thead>
<tr>
<th>Number of Available Cars</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Profit</td>
<td>$520</td>
<td>$577</td>
<td>$626</td>
<td>$651</td>
<td>$652</td>
</tr>
</tbody>
</table>

Now work problem 23.

Exercise 7.6 Answers Begin on Page 37.

Concepts and Vocabulary

1. True or False Sometimes the expected value is a negative number.

2. If the expected value of a game is 0, the game is _________.

Skill Building

3. For the data given below, compute the expected value.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.4</td>
<td>0.2</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>Payoff</td>
<td>2</td>
<td>3</td>
<td>−2</td>
<td>0</td>
</tr>
</tbody>
</table>

4. For the data below, compute the expected value.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Payoff</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>−2</td>
</tr>
</tbody>
</table>

Applications and Extensions

5. Weather and Attendance Attendance at a football game in a certain city results in the following pattern. If it is extremely cold, the attendance will be 30,000; if it is cold, it will be 40,000; if it is moderate, 60,000; and if it is warm, 80,000. If the probabilities for extremely cold, cold, moderate, and warm are 0.08, 0.42, 0.42, and 0.08, respectively, how many fans are expected to attend the game?

6. Analyzing a Game A player rolls a fair die and receives a number of dollars equal to the number of dots appearing on the face of the die. What is the least the player should expect to pay in order to play the game?

7. Analyzing a Game Mary will win $8 if she draws an ace from a set of 10 different cards from ace (1) to 10. How much should she pay for one draw?

8. Analyzing a Game Thirteen playing cards, ace through king, are placed randomly with faces down on a table. The prize for guessing correctly the face of any given card is $1. What would be a fair price to pay for a guess?

9. Analyzing a Game David gets $12 if he throws a double on a single throw of a pair of dice. How much should he pay for a throw?

10. Analyzing a Game You pay $1 to toss 2 coins. If you toss 2 heads, you get $2 (including your $1); if you toss only 1 head, you get back your $1; and if you toss no heads, you lose your $1. Is this a fair game to play?

11. Raffles In a raffle 1000 tickets are being sold at $1.00 each. The first prize is $100. There are 2 second prizes of $50 each, and 3 third prizes of $10 each. Jenny buys 1 ticket. How much more than the expected value of the ticket does she pay?

12. Raffles In a raffle 1000 tickets are being sold at $1.00 each. The first prize is $100. There are 2 second prizes of $50 each, and 3 third prizes of $10 each. Jenny buys 1 ticket. How much more than the expected value of the ticket does she pay?

13. Analyzing a Game A fair coin is tossed 3 times, and a player wins $3 if 3 tails occur, wins $2 if 2 tails occur, and loses $3 if no tails occur. If 1 tail occurs, no one wins.
   (a) What is the expected value of the game?
   (b) Is the game fair?
   (c) If the answer to part (b) is “No,” how much should the player win or lose for a toss of exactly 1 tail to make the game fair?

14. Analyzing a Game Colleen bets $1 on a 2-digit number. She wins $75 if she draws her number from the set of all 2-digit numbers, \{00, 01, 02, \ldots, 99\}; otherwise, she loses her $1.
   (a) Is this game fair to the player?
   (b) How much is Colleen expected to lose in a game?

15. Sports Two teams have played each other 14 times. Team A won 9 games, and team B won 5 games. They will play again next week. Bob offers to bet $6 on team A while you bet $4 on team B. The winner gets the $10. Is the bet fair to you in view of the past records of the two teams? Explain your answer.

16. Department Store Sales A department store wants to sell 11 purses that cost the store $40 each and 32 purses that cost the store $10 each. If all purses are wrapped in 43 identical boxes and if each customer picks a box randomly, find
   (a) Each customer’s expectation.
   (b) The department store’s expected profit if it charges $15 for each box.
17. **Cards** Sarah draws a card from a deck of 52 cards. She receives 40¢ for a heart, 50¢ for an ace, and 90¢ for the ace of hearts. If the cost of a draw is 15¢, should she play the game? Explain.

18. **Family Size** The following data give information about family size in the United States in 2009 for a household containing a husband and wife where the husband is in the 30–34 age bracket. A family is chosen at random. Find the expected number of children in the family.

<table>
<thead>
<tr>
<th>Number of Children</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportion of Families</td>
<td>19.9%</td>
<td>28.4%</td>
<td>30.7%</td>
<td>14.5%</td>
<td>6.5%</td>
</tr>
</tbody>
</table>

**Source:** U.S. Census Bureau

19. **Corporate Receipts** The following data give the number of tax returns (in thousands) filed in 2006 by S-corporations with total receipts under $1 million. Use the midpoint to find the expected receipts for a corporation with receipts of less than $1,000,000.

<table>
<thead>
<tr>
<th>Size-class of Receipts</th>
<th>under $25,000</th>
<th>$25,000–$99,999</th>
<th>$100,000–$249,999</th>
<th>$250,000–$499,999</th>
<th>$500,000–$999,999</th>
</tr>
</thead>
<tbody>
<tr>
<td>Midpoint</td>
<td>$12,500</td>
<td>$62,500</td>
<td>$175,000</td>
<td>$375,000</td>
<td>$750,000</td>
</tr>
<tr>
<td>Number of Returns</td>
<td>954,123</td>
<td>650,138</td>
<td>707,229</td>
<td>545,457</td>
<td>443,573</td>
</tr>
</tbody>
</table>

**Source:** Internal Revenue Service

20. **Number of Employees** The following data give the distribution of business establishments in 2006 by employment-size class for businesses with less than 1,000 employees. Use the midpoint to find the expected number of employees for a business in this category.

<table>
<thead>
<tr>
<th>Number of Employees</th>
<th>0 to 19</th>
<th>20 to 99</th>
<th>100 to 499</th>
<th>500 to 999</th>
</tr>
</thead>
<tbody>
<tr>
<td>Midpoint</td>
<td>10</td>
<td>60</td>
<td>300</td>
<td>750</td>
</tr>
<tr>
<td>Probability</td>
<td>0.483</td>
<td>0.504</td>
<td>0.012</td>
<td>0.001</td>
</tr>
</tbody>
</table>

**Source:** U.S. Census Bureau, *County Business Patterns*

21. **Housing Sales** The following data give the number of new houses sold (in thousands) in the United States in 2009 for prices under $750,000. Use the midpoint to find the expected sale price of a new house.

<table>
<thead>
<tr>
<th>Price</th>
<th>0–$149,999</th>
<th>$150,000–$199,999</th>
<th>$200,000–$299,999</th>
<th>$300,000–$399,999</th>
<th>$400,000–$499,999</th>
<th>$500,000–$749,999</th>
</tr>
</thead>
<tbody>
<tr>
<td>Midpoint</td>
<td>$75,000</td>
<td>$175,000</td>
<td>$250,000</td>
<td>$350,000</td>
<td>$450,000</td>
<td>$625,000</td>
</tr>
<tr>
<td>Number Sold</td>
<td>66</td>
<td>96</td>
<td>114</td>
<td>46</td>
<td>23</td>
<td>20</td>
</tr>
</tbody>
</table>

**Source:** U.S. Census Bureau

22. **Kentucky Derby** Of the 19 horses in the 2009 Kentucky Derby, the top two finishers (in order) were Mine That Bird and Pioneer of the Nile. On a $2 exacta bet (picking the top two horses in order), the house paid out $2074.80. That is, for every $2 bet, you either lost $2 or gained $2072.80. What was the house’s expected earnings per $2 bet?

**Source:** www.thespread.com

23. **Market Assessment** A truck rental agency has fixed costs of $20 per truck per day and the revenue for each truck rented is $90 per day. The daily demand is given in the table:

<table>
<thead>
<tr>
<th>Number of Customers</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.10</td>
<td>0.20</td>
<td>0.40</td>
<td>0.20</td>
<td>0.10</td>
</tr>
</tbody>
</table>

(a) Find the expected number of customers.
(b) Determine the optimal number of trucks the company should have on hand each day.
(c) What is the expected revenue in this case?

24. **Ski Rentals** Swen’s Ski Rental rents skis, boots, and poles for $20 a day. The daily cost per set of skis is $6. It includes maintenance, storage, and overhead. Daily profits depend on daily demand for skis and the number of sets available. Swen knows that on a typical day the demand for skis is given in the table.
### Number of Customers and Probability

<table>
<thead>
<tr>
<th>Number of Customers</th>
<th>90</th>
<th>91</th>
<th>92</th>
<th>93</th>
<th>94</th>
<th>95</th>
<th>96</th>
<th>97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.01</td>
<td>0.10</td>
<td>0.20</td>
<td>0.20</td>
<td>0.30</td>
<td>0.10</td>
<td>0.05</td>
<td>0.04</td>
</tr>
</tbody>
</table>

How many sets of skis should Swen have ready for rental to maximize his profit?

25. **Life Insurance** A 20-year-old male purchases a 1-year life insurance policy worth $250,000. The insurance company determines that he will survive the policy period with probability 0.9986.

(a) If the premium for the policy is $450, what is the expected profit for the insurance company?

(b) At what value should the company set its premium so its expected profit will be $250 per policy for 20-year-old males?

**Source:** National Center for Health Statistics

26. **Life Insurance** A 20-year-old female purchases a 1-year life insurance policy worth $250,000. The insurance company determines that she will survive the policy period with probability 0.9995.

(a) If the premium for the policy is $300, what is the expected profit for the insurance company?

(b) At what value should the company set its premium so its expected profit will be $250 per policy for 20-year-old females?

**Source:** National Center for Health Statistics

27. **Site Selection** A company operating a chain of supermarkets plans to open a new store in 1 of 2 locations. It conducted a survey of the 2 locations and estimated that the first location will show an annual profit of $15,000 if it is successful and a $3000 loss otherwise. For the second location, the estimated annual profit is $20,000 if successful and a $6000 loss otherwise.

(a) If the probability of success at each location is \( \frac{1}{2} \), what location should the management decide on in order to maximize its expected profit?

(b) If the probability of success at the first location is \( \frac{2}{3} \) and at the second location is \( \frac{1}{3} \), what location should be chosen?

28. **Roulette** In roulette there are 38 equally likely possibilities: the numbers 1–36, 0, and 00 (double zero). See the figure. What is the expected value for a gambler who bets $1 on number 15 if she wins $35 each time the number 15 turns up and loses $1 if any other number turns up? If the gambler plays the number 15 for 200 consecutive times, what is the total expected gain?

### Discussion and Writing

31. Make up a game different than any given in the text that is favorable to the player.

32. Look up the rules for a lottery and discuss the expected value of a ticket.

33. Refer to Example 8. As manager, would you have 11 or 12 luxury cars available to rent? Provide a rationale for your decision. Notice that the 12th car only generates $1 more in profit than the 11 cars do.
## CHAPTER 7 REVIEW  OBJECTIVES

<table>
<thead>
<tr>
<th>Section</th>
<th>Examples</th>
<th>You should be able to</th>
<th>Review Exercises</th>
</tr>
</thead>
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<td>7.1</td>
<td>3, 4, 5, 6</td>
<td>1 Identify relations between pairs of sets (p. 357)</td>
<td>1–16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 Find the union and intersection of two sets (p. 359)</td>
<td>3, 4, 6, 9, 11–13, 15–18</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3 Find the complement of a set (p. 361)</td>
<td>23–28, 37, 38</td>
</tr>
<tr>
<td></td>
<td>8, 9, 10</td>
<td>4 Use Venn diagrams (p. 362)</td>
<td>18 a, c–f, 24, 26, 28</td>
</tr>
<tr>
<td>7.2</td>
<td>1, 2</td>
<td>1 Use the Counting Formula (p. 367)</td>
<td>19–22</td>
</tr>
<tr>
<td></td>
<td>3, 4</td>
<td>2 Use Venn diagrams to analyze survey data (p. 367)</td>
<td>29–34</td>
</tr>
<tr>
<td>7.3</td>
<td>1–5</td>
<td>1 Use the Multiplication Principle (p. 373)</td>
<td>35, 36</td>
</tr>
<tr>
<td>7.4</td>
<td>1, 2, 3</td>
<td>1 Find a sample space (p. 380)</td>
<td>63–66</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2 Assign probabilities (p. 382)</td>
<td>39–42</td>
</tr>
<tr>
<td></td>
<td>6, 7</td>
<td>3 Construct a probability model (p. 382)</td>
<td>43–46</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4 Find probabilities involving equally likely outcomes (p. 383)</td>
<td>47a, 48a</td>
</tr>
<tr>
<td>7.5</td>
<td>1</td>
<td>1 Find the probability of an event (p. 391)</td>
<td>47b, 48b, 49, 50</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2 Find the probability of $E$ or $F$ when $E$ and $F$ are mutually exclusive (p. 393)</td>
<td>50, 53, 54, 55, 61, 62</td>
</tr>
<tr>
<td></td>
<td>3, 4</td>
<td>3 Use the Additive Rule (p. 394)</td>
<td>58, 59</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4 Find the probability of the complement of an event (p. 395)</td>
<td>51a, 54b, 55b, 56b, 57a, 60</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>5 Use a Venn diagram to find probabilities (p. 396)</td>
<td>51b, 54a, 55a, 56a, 56c, 57b</td>
</tr>
<tr>
<td></td>
<td>7, 8</td>
<td>6 Use the Multiplication Principle to find probabilities (p. 397)</td>
<td>57c, 58a, 58b, 59a, 59b</td>
</tr>
<tr>
<td></td>
<td>9, 10</td>
<td>7 Compute odds (p. 398)</td>
<td>56–59</td>
</tr>
<tr>
<td>7.6</td>
<td>1, 2, 3</td>
<td>1 Find the expected value (p. 405)</td>
<td>67, 68</td>
</tr>
<tr>
<td></td>
<td>4, 5, 6</td>
<td>2 Solve applied problems involving expected value (p. 407)</td>
<td>69–71</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>72–78</td>
</tr>
</tbody>
</table>

## THINGS TO KNOW

**Set** (p. 356)<br>Well-defined collection of distinct objects, called elements

| Empty set (p. 357) | $\emptyset$ | Set that has no elements |
| Equality (p. 357) | $A = B$ | $A$ and $B$ have the same elements. |
| Subset (p. 357) | $A \subseteq B$ | Each element of $A$ is an element of $B$. |
| Proper subset (p. 358) | $A \subset B$ | Each element of $A$ is an element of $B$, but there is at least one element in $B$ not in $A$. |
| Universal set (p. 359) | $U$ | Set consisting of all the elements that we wish to consider |
| Union (p. 359) | $A \cup B$ | Set consisting of elements that belong to either $A$ or $B$, or both |
| Intersection (p. 359) | $A \cap B$ | Set consisting of elements that belong to both $A$ and $B$ |
| Complement (p. 361) | $\bar{A}$ | Set consisting of elements of the universal set that are not in $A$ |
Finite set (p. 366)
Infinite set (p. 366)

**Counting Formula** (p. 367)

**Multiplication Principle** (p. 374)

**Probability of an Event**

**Equally Likely Outcomes** (p. 384)

**Mutually Exclusive Events** (p. 393)

**Additive Rule** (p. 394)

**Probability of the Complement** of an Event (p. 395)

**Odds for** $E$ **Are** $a$ **to** $b$ (p. 399)

---

**REVIEW problem exercises**

Answers to odd-numbered problems begin on page AN-38.

**Blue** problem numbers indicate the author’s suggestions for a practice test.

In Problems 1–16, replace the blank by any of the symbol(s) that result in a true statement. If none result in a true statement, write “None of these.” More than one answer may be possible.

1. \( \emptyset \) \{ 0 \}  
2. \{ 0 \} \{ 1, 0, 3 \}  
3. \{ 5, 6 \} \{ 2, 6 \} \{ 8 \}  
4. \{ 3 \} \{ 2, 3 \} \{ 4, 3 \}  
5. \{ 8, 9 \} \{ 9, 10, 11 \}  
6. \{ 1 \} \{ 1, 3, 5 \} \{ 3, 4 \}  
7. \{ 5 \} \{ 0, 5 \}  
8. \emptyset \{ 1, 2, 3 \}  
9. \emptyset \{ 1, 2 \} \{ 3, 4, 5 \}  
10. \{ 2, 3 \} \{ 3, 4 \}  
11. \{ 1, 2 \} \{ 1 \} \{ 2 \}  
12. \{ 5 \} \{ 1 \} \{ 2, 3 \}  
13. \{ 4, 5 \} \{ 5, 6 \} \{ 4, 5 \}  
14. \{ 6, 8 \} \{ 8, 9, 10 \}  
15. \{ 6, 7, 8 \} \{ 6 \} \{ 6 \}  
16. \{ 4 \} \{ 6, 8 \} \{ 4, 8 \}  

---

17. For the sets
   \( A = \{ 1, 3, 5, 6, 8 \} \quad B = \{ 2, 3, 6, 7 \} \quad C = \{ 6, 8, 9 \} \)
   find:
   (a) \( (A \cap B) \cup C \)  
   (b) \( (A \cap B) \cap C \)  
   (c) \( (A \cup B) \cap B \)  
   (d) \( B \cup \emptyset \)  
   (e) \( A \cup \emptyset \)  

18. For the sets \( U = \text{universal set} = \{ 1, 2, 3, 4, 5, 6, 7 \} \) and
   \( A = \{ 1, 3, 5, 6 \} \quad B = \{ 2, 3, 6, 7 \} \quad C = \{ 4, 6, 7 \} \)
   find:
   (a) \( \overline{A} \cap \overline{B} \)  
   (b) \( (B \cap C) \cap A \)  
   (c) \( \overline{B} \cup \overline{A} \)  
   (d) \( C \cup \overline{C} \)  
   (e) \( \overline{B} \)  

19. Use a Venn diagram to illustrate the following sets:
   (a) \( A \cup \overline{B} \)  
   (b) \( (A \cap B) \cup \overline{B} \)  
   (c) \( B \cap \overline{A} \)  
   (d) \( (A \cup B) \cap C \)  
   (e) \( (A \cap B) \cap C \)  
   (f) \( \overline{B} \cup \overline{C} \)  

20. Use a Venn diagram to illustrate the following properties:
   (a) \( (A \cap B) \cap C = A \cap (B \cap C) \)  
   (b) \( \overline{A \cap B} = \overline{A} \cup \overline{B} \)  
   (c) \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)  
   (d) \( \overline{A} = A \)  

21. Draw a Venn diagram that illustrates \( A \cap B = \emptyset \).  
22. Draw a Venn diagram that illustrates \( A \subset B \).

---

In Problems 23–28, use \( U = \text{Universal set} = \{ x \mid x \text{ is a state in the United States} \} \)

\( A = \{ a \mid a \text{ is a state whose name begins with the letter } A \} \)

\( V = \{ v \mid v \text{ is a state whose name ends with a vowel} \} \)

\( E = \{ e \mid e \text{ is a state that lies east of the Mississippi River} \} \)

to describe each set in words.

23. \( A \cup V \)  
24. \( A \cap \overline{V} \)  
25. \( V \cap E \)  
26. \( \overline{E} \)  
27. \( (A \cup V) \cap E \)  
28. \( A \cup \overline{V} \)

29. If \( A \) and \( B \) are sets and if \( n(A) = 24 \), \( n(A \cup B) = 33 \), and \( n(B) = 12 \), find \( n(A \cap B) \).
414 Chapter 7  Probability

30. If \( n(A) = 10, n(B) = 8 \), and \( n(A \cap B) = 2 \), find \( n(A \cup B) \).
31. (a) If \( n(A \cap B) = 0, n(A) = 3 \), and \( n(B) = 17 \), find \( n(A \cup B) \).
   (b) What can you conclude about the sets \( A \) and \( B \)?
32. (a) If \( n(A) = 14, n(B) = 8 \), and \( n(A \cap B) = 8 \), find \( n(A \cup B) \).
   (b) What can you conclude about the relation between \( A \) and \( B \)?

33. Suppose a die is tossed. Let \( A \) denote that the outcome is an even number; let \( B \) denote that the outcome is a number less than or equal to 3. How many elements are in \( A \cap B \)? How many elements are in \( A \cup B \)?
34. Suppose a die is tossed. Let \( A \) denote the outcome is an even number; let \( B \) denote the outcome is divisible by 3. How many elements are in \( A \cap B \)? How many elements are in \( A \cup B \)?

35. Car Options  During June, Colleen’s Motors sold 75 cars with heated seats, 95 with GPS, and 100 with a sunroof. Twenty cars had all three options. 10 cars had none of these options, and 10 cars were sold that had only heated seats. In addition, 50 cars had both a sunroof and a GPS, and 60 cars had both a sunroof and heated seats.
   (a) How many cars were sold in June?
   (b) How many cars had only a GPS?

36. Student Survey  In a survey of 125 college students, it was found that of three newspapers, the Wall Street Journal, New York Times, and Chicago Tribune:
   - 60 read the Chicago Tribune
   - 40 read the New York Times
   - 15 read the Wall Street Journal
   - 25 read the Chicago Tribune and New York Times
   - 8 read the New York Times and Wall Street Journal
   - 3 read the Chicago Tribune and Wall Street Journal
   - 1 read all three
   (a) How many read none of these papers?
   (b) How many read only the Chicago Tribune?
   (c) How many read neither the Chicago Tribune nor the New York Times?

37. If \( U = \) universal set = \{1, 2, 3, 4, 5\} and \( B = \{1, 4, 5\} \), find all sets \( A \) for which \( A \cap B = \{1\} \).
38. If \( U = \) universal set = \{1, 2, 3, 4, 5\} and \( B = \{1, 4, 5\} \), find all sets \( A \) for which \( A \cup B = \{1, 2, 4, 5\} \).

39. Tossing a Coin  A coin is tossed 5 times. We are interested in the number of heads that show. List the outcomes of the sample space.

40. Rolling a Die  A die is rolled 10 times. We are interested in the number of times an even number shows. List the outcomes of the sample space.

41. Two-Child Families  A survey of families with 2 children is made, and the gender of the children is recorded. Describe the sample space and draw a tree diagram of this experiment.

42. The spinner pictured is spun 3 times. Each time the color is noted. List the outcomes of the sample space. Draw a tree diagram of the experiment.

43. Jars and Coins  A jar has 15 coins: 4 pennies, 5 dimes, and 6 quarters. A coin is selected from the jar. Assign valid probabilities to the outcomes of this experiment.

44. Choosing a Ball  A bag has 12 golf balls, 5 of which are cracked. A golf ball is chosen from the bag at random. Assign valid probabilities to this experiment.

45. Tossing a Weighted Die  A die is weighted so that 2 and 5 appear twice as often as 1, 3, 4, and 6. The die is tossed. Assign valid probabilities to this experiment.

46. Tossing an Unfair Coin  A coin is not fair. When tossed, heads shows up 4 times more often than tails. Assign valid probabilities to this experiment.

47. Four-Child Families  Families with 4 children were surveyed.
   (a) Construct a probability model describing the possible number of girls in the family. (Assume the probability a child is a girl is 0.5.)
   (b) Find the probability that
      (i) No children are girls.
      (ii) Exactly 2 children are girls.
      (iii) At least one boy and one girl.
      (iv) At least one child is a boy.

48. Tossing a Fair Coin  A fair coin is tossed three times.
   (a) Construct a probability model corresponding to this experiment.
   (b) Find the probabilities of the following events:
      (i) The first toss is tails.
      (ii) The first toss is heads.
      (iii) Either the first toss is tails or the third toss is heads.
      (iv) At least one of the tosses is heads.
      (v) There are at least 2 tails.
      (vi) No tosses are heads.
49. **Selecting Marbles** A jar contains 3 white marbles, 2 yellow marbles, 4 red marbles, and 5 blue marbles. Two marbles are picked at random. What is the probability that
(a) Both are blue?
(b) Exactly 1 is blue?
(c) At least 1 is blue?

50. **Drawing Cards** Two cards are drawn from a 52-card deck. What is the probability that
(a) 1 card is black and 1 card is red?
(b) Both cards are the same color?
(c) Both cards are red?

51. Let A and B be events with \( P(A) = 0.3 \), \( P(B) = 0.5 \), and \( P(A \cap B) = 0.2 \). Find the probability that
(a) A or B happens.
(b) A does not happen.
(c) Neither A nor B happens.
(d) Either A does not happen or B does not happen.

52. **Loaded Die** A loaded die is rolled 400 times, and the following outcomes are recorded:

<table>
<thead>
<tr>
<th>No. Times Showing</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>45</td>
<td>84</td>
<td>74</td>
<td>92</td>
<td>73</td>
<td></td>
</tr>
</tbody>
</table>

Estimate the probability of rolling a
(a) 3 (b) 5 (c) 6

53. **Tossing an Unfair Coin** An unfair coin is tossed 300 times and outcomes are recorded

<table>
<thead>
<tr>
<th>Outcomes</th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. Times Occurring</td>
<td>243</td>
<td>57</td>
</tr>
</tbody>
</table>

Estimate the probability of tossing a
(a) head (b) tail

54. A survey of a group of criminals shows that 65% came from low-income families, 40% from broken homes, and 30% came from low-income families and broken homes. Define
\[
E: \text{Criminal came from low-income family} \\
F: \text{Criminal came from broken home}
\]
A criminal is selected at random.
(a) Find the probability that the criminal is not from a low-income family.
(b) Find the probability that the criminal comes from a broken home or a low-income family.
(c) Are E and F mutually exclusive?

55. **Working Students** A survey of a group of 18- to 22-year-olds revealed that 63% were college students, 50% held jobs, and 35% did both. Define
\[
E: \text{The respondent is a college student.} \\
F: \text{The respondent has a job.}
\]
A respondent is selected at random.
(a) Find the probability that the respondent does not have a job.
(b) Find the probability that the respondent is a student or holds a job.
(c) Are E and F mutually exclusive?

56. If E and F are events with \( P(E \cup F) = \frac{5}{8} \), \( P(E \cap F) = \frac{1}{3} \), and \( P(E) = \frac{1}{2} \), find
(a) \( P(\bar{E}) \) (b) \( P(F) \) (c) \( P(\bar{F}) \)

57. If \( P(E) = 0.2 \), \( P(F) = 0.6 \), and \( P(E \cap F) = 0.1 \), find
(a) \( P(E \cup F) \) (b) \( P(\bar{E}) \) (c) \( P(\bar{E} \cap F) \)

58. If E and F represent mutually exclusive events, \( P(E) = 0.30 \), and \( P(F) = 0.45 \), find each of the probabilities:
(a) \( P(\bar{E}) \) (b) \( P(\bar{F}) \) (c) \( P(E \cap F) \)
(d) \( P(E \cup F) \) (e) \( P(\bar{E} \cap F) \) (f) \( P(E \cup \bar{F}) \)
(g) \( P(\bar{E} \cup F) \) (h) \( P(\bar{E} \cap \bar{F}) \)

59. If \( P(E) = 0.25 \), \( P(F) = 0.3 \), and \( P(E \cup F) = 0.55 \), find each of the probabilities:
(a) \( P(\bar{E}) \) (b) \( P(\bar{F}) \) (c) \( P(E \cap F) \)
(d) \( P(E \cup \bar{F}) \) (e) \( P(\bar{E} \cap \bar{F}) \) (f) \( P(E \cup F) \)

60. **Choosing a Card** A card is drawn from a 52-card deck. Define
\[
E: \text{The card is red.} \\
F: \text{The card is a face card (J, Q, or K).}
\]
Find the probability that it is either red or a face card.

61. Consider the experiment of spinning the spinner shown in the figure 3 times. (Assume the spinner cannot fall on a line.)

(a) Are all outcomes equally likely?
(b) If not, which of the outcomes has the highest probability?
(c) Let F be the event, “Each digit will occur exactly once.” Find \( P(F) \).
62. **M&M Candies** A bowl contains 22 M&M candies; 4 are red, 6 are green, and 12 are blue. A piece of candy is chosen, and its color is noted.
   (a) Is each color equally likely to be picked?
   (b) Which outcome has the highest probability of occurring?
   (c) Suppose you took 3 pieces of candy without looking. Define the event 
   $$E:$$ Each color is represented. 
   What is $P(E)$?

63. How many house styles are possible if a contractor offers 3 choices of roof designs, 4 choices of window designs, and 6 choices of brick?

64. In a cafeteria the $6.95 lunch menu lets you choose 1 salad, 1 entree, 1 dessert, and 1 beverage. If today’s menu features 3 different salads, 5 different entrees, 6 different desserts, and 10 different beverages, how many distinct meals could one order?

65. How many different answers are possible in a true-false test consisting of 10 questions?

66. **License Plate Numbers** An automobile license number contains 1 or 2 letters followed by a 4-digit number. Compute the maximum number of different licenses.

67. **Birthday Problem** Fifteen people are in a room. Each is asked the day of the month (e.g., 1–31) he or she was born. What is the probability each person was born on a different day of the month?

68. **Birthday Problem** There are 5 people in the Smith family. What is the probability at least 2 of them have birthdays in the same month?

69. **Gender of Children** A family chosen at random has 4 children. What are the odds that all four are boys?

70. **Bears Win!** A bettor is willing to give 7 to 6 odds that the Bears will win the NFL title. What is the probability of the Bears winning?

71. **Football Pool** You want to enter a football pool. The daily paper states the odds the Giants will win Sunday's game are 5:3. What is the probability the Giants will win Sunday's game? What is the probability they will lose?

72. What is the expected number of girls in families having exactly 3 children?

73. **Evaluating a Game** Frank pays $0.70 to play a certain game. He draws 2 balls (together) from a bag containing 2 red balls and 4 green balls. He receives $1 for each red ball that he draws. Has he paid too much? By how much?

74. **Playing the Lottery** In a lottery 1000 tickets are sold at $0.25 each. There are 3 cash prizes: $100, $50, and $30. Alice buys 8 tickets.
   (a) What would have been a fair price for a ticket?
   (b) How much extra did Alice pay?

75. **Evaluating a Game** The figure shows a spinning game for which a person pays $0.30 to purchase an opportunity to spin the dial. The numbers in the figure indicate the amount of payoff and its corresponding probability in parentheses. Find the expected value of this game. Is the game fair?

76. **Evaluating a Game** Consider the 3 boxes in the figure. The game is played in 2 stages. The first stage is to choose a ball from box A. If the result is a ball marked I, then we go to box I and select a ball from there. If the ball is marked II, then we select a ball from box II. The number drawn on the second stage is the gain. Find the expected value of this game.

77. **European Roulette** A European roulette wheel has only 37 compartments, 18 red, 18 black, and 1 green. A player will be paid $2 (including his $1 bet) if he picks correctly the color of the compartment in which the ball finally rests. Otherwise, he loses $1. Is the game fair to the player?

78. **Insuring against Weather** The profits from an outdoor sporting event are affected by inclement weather. The promoter of the event expects it to net a profit of $750,000 if the weather is dry but only $20,000 if the weather is wet. The historical probability of rain is 5% for the day of the event.
   (a) What is the expected profit from the event?
   (b) The promoter can buy $500,000 weather insurance for a premium of $50,000. Based on expected profit, would it be wise to buy insurance? Explain your reasoning.
   (c) Suppose the chance of rain is 20% and the insurance costs $70,000. Based on the new information, what is the expected profit from the event?
   (d) Based on expected profit, would it be wise to buy insurance? Explain your reasoning.
IDENTIFICATION NUMBERS: HOW MANY ARE ENOUGH?

If you look carefully around you, you will see batches of numbers and letters on almost everything: on products in every store, on books, and certainly on the credit cards in your wallet. Even though they may contain letters of the alphabet, these numbers are called identification numbers. They serve to identify the product, the book, or the person who owns the credit card.

Producing identification numbers is a lot more difficult than just deciding to print numbers on products or books. Every system of identification numbers has a plan that tells exactly how to assign those numbers. These plans must confront several issues. First, there must be enough numbers for all of the things that you expect to number. Since giving the same identification number to different items would defeat the purpose of having an identification number, it is important that the plan have a way to give a unique number to each item. It is also necessary to be able to check for errors in the identification number. This check helps to prevent misidentifying items. As a result, most identification number plans include a number, called a check digit, for this purpose. We consider two examples of identification numbers that you have probably already seen many times.

The Universal Product Code

The Universal Product Code, or UPC, appears on just about every item you buy from a major store. It is designed to help the store track inventory and to make price changes easier. Figure 1 shows three examples. The first is from a box of Bigelow tea, the second is from a box of Xerox toner cartridges, and the third is from the front page of a newspaper.

There are two parts to the code: a bar code and a UPC number. Notice that the UPC number has 12 digits. The first 6 digits identify the producer and the next 5 digits identify the exact product. The final digit is the check digit. Of the 6 digits identifying the producer, the first digit is particularly important. This digit, which appears at the far left of the code and is called the number system character, tells the type of product. Table 1 lists the number system characters currently in use. The numbers 1, 8, and 9 have been reserved for future use. The 5 digits that identify the product can be any digits whatsoever. Once the first 11 digits are known, the check digit is calculated from them.

![FIGURE 1 Universal Product Codes](image)

<table>
<thead>
<tr>
<th>TABLE 1 UPC NUMBER SYSTEM CHARACTERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
</tbody>
</table>

1. Under the current system, how many different producer numbers are possible?
2. If all digits were allowed for the number system character, how many more producer numbers would be created?
3. How many possible correct UPC codes are there? Assume all possible producer numbers are allowed.
The International Standard Book Number  The International Standard Book Number, or ISBN, is used to identify works published anywhere in the world. The number is used to track and identify books quickly. Table 2 shows some examples.

<table>
<thead>
<tr>
<th>Table 2 ISBN’s for Various Books</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Last Song by Nicholas Sparks</td>
</tr>
<tr>
<td>Cien Años de Soledad by Gabriel García Marquez</td>
</tr>
<tr>
<td>Probability and Statistics for Computer Science by James Johnson</td>
</tr>
<tr>
<td>The Very Hungry Caterpillar by Eric Carle</td>
</tr>
<tr>
<td>Les Trois Mousquetaires by Alexandre Dumas</td>
</tr>
<tr>
<td>Le Véritable Amour by Jacqueline Harpman</td>
</tr>
<tr>
<td>Solstices by Ananda Devi</td>
</tr>
</tbody>
</table>

The 13-digit ISBN is a numerical code unique to books but aligns with number identification systems used globally for most consumer products. The first block of numbers designates the type of product or industry. That is, 978 indicates book publishing.

Notice that each ISBN has 13 digits, which are divided into 5 blocks with 4 hyphens. The first block always has 3 digits, 978. The final block always contains just 1 digit: the check digit. The other four blocks can contain anywhere from 1 to 7 numbers, but the total number of digits in the four blocks must always be 12. Only digits are used for these groups; no letters are allowed.

The second block tells the language or country in which the book was published. Books from English-speaking areas are assigned the numbers 0 or 1, those from French-speaking areas are assigned 2, those from German-speaking areas are assigned 3, and so on. Since a smaller country will likely produce fewer books, these countries get longer country codes. For example, the code for Spain is 84, and the code for Mauritius (an island nation in the Indian Ocean) is 99903. The third block identifies the publisher. For example, John Wiley and Sons has publisher number 470, so we see that this company published *Probability and Statistics for Computer Science* as well as this book. The fourth block identifies the individual book. Notice that the number of digits in this block limits the number of different titles a publisher can publish without needing to get another publisher number. Thus Ancrage Press, which published *Le Véritable Amour*, only has two digits (or 100 titles) for its use. Publishers that produce large numbers of titles are given shorter publisher numbers to accommodate more titles.

4. How many different titles may John Wiley and Sons publish using the publisher number 470?

5. How many different titles may be published in Mauritius?  
   [Hint: In this case, four digits define the publisher and the title. First consider how many different four-digit blocks are possible. Then consider how many different ways a hyphen may be inserted into the four-digit block.]

6. How many correct ISBN codes are possible for a book from the English-speaking area?

7. In practice, the number of correct ISBN codes is limited by restrictions on both the format of the country code and the format of the publisher number. For the English-speaking area, the publisher number must be in the following ranges: 00–19, 200–699, 7000–8499, 85000–89999, 900000–949999, and 9500000–9999999. With these restrictions on the publisher number, how many correct ISBN codes are possible for a book from the English-speaking area?

8. Research the use of a check digit in the ISBN. Write a brief report on what you find.
1. **CPA Exam**  
The Stat Company wants more information on the demand for its products. The following data are relevant:

<table>
<thead>
<tr>
<th>Units Demanded</th>
<th>Probability of Unit Demand</th>
<th>Total Cost of Units Demanded</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.10</td>
<td>$0</td>
</tr>
<tr>
<td>1</td>
<td>0.15</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>0.20</td>
<td>2.00</td>
</tr>
<tr>
<td>3</td>
<td>0.40</td>
<td>3.00</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>4.00</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
<td>5.00</td>
</tr>
</tbody>
</table>

What is the total expected value or payoff with perfect information?  
(a) $2.40  
(b) $7.40  
(c) $9.00  
(d) $9.15

2. **CPA Exam**  
Your client wants your advice on which of 2 alternatives he should choose. One alternative is to sell an investment now for $10,000. Another alternative is to hold the investment 3 days, after which he can sell it for a certain selling price based on the following probabilities:

<table>
<thead>
<tr>
<th>Selling Price</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ 5,000</td>
<td>0.4</td>
</tr>
<tr>
<td>$ 8,000</td>
<td>0.2</td>
</tr>
<tr>
<td>$12,000</td>
<td>0.3</td>
</tr>
<tr>
<td>$30,000</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Using probability theory, which of the following is the most reasonable statement?  
(a) Hold the investment 3 days because the expected value of holding exceeds the current selling price.  
(b) Hold the investment 3 days because of the chance of getting $30,000 for it.  
(c) Sell the investment now because the current selling price exceeds the expected value of holding.  
(d) Sell the investment now because there is a 60% chance that the selling price will fall in 3 days.

3. **CPA Exam**  
The ARC Radio Company is trying to decide whether to introduce as a new product a wrist "radiowatch" designed for shortwave reception of exact time as broadcast by the National Bureau of Standards. The "radiowatch" would be priced at $60, which is exactly twice the variable cost per unit to manufacture and sell it. The incremental fixed costs necessitated by introducing this new product would amount to $240,000 per year. Subjective estimates of the probable demand for the product are shown in the following probability distribution:

<table>
<thead>
<tr>
<th>Annual Demand</th>
<th>6,000</th>
<th>8,000</th>
<th>10,000</th>
<th>12,000</th>
<th>14,000</th>
<th>16,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

The expected value of demand for the new product is  
(a) 11,000 units  
(b) 10,200 units  
(c) 9000 units  
(d) 10,600 units  
(e) 9800 units

4. **CPA Exam**  
In planning its budget for the coming year, King Company prepared the following payoff probability distribution describing the relative likelihood of monthly sales volume levels and related contribution margins for product A:

<table>
<thead>
<tr>
<th>Monthly Sales Volume</th>
<th>Contribution Margin</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,000</td>
<td>$ 80,000</td>
<td>0.20</td>
</tr>
<tr>
<td>6,000</td>
<td>120,000</td>
<td>0.25</td>
</tr>
<tr>
<td>8,000</td>
<td>160,000</td>
<td>0.30</td>
</tr>
<tr>
<td>10,000</td>
<td>200,000</td>
<td>0.15</td>
</tr>
<tr>
<td>12,000</td>
<td>240,000</td>
<td>0.10</td>
</tr>
</tbody>
</table>

What is the expected value of the monthly contribution margin for product A?  
(a) $140,000  
(b) $148,000  
(c) $160,000  
(d) $180,000

5. **CPA Exam**  
A decision tree has been formulated for the possible outcomes of introducing a new product line. Branches related to alternative 1 reflect the possible payoffs from introducing the product without an advertising campaign. The branches for alternative 2 reflect the possible payoffs with an advertising campaign costing $40,000.
The expected values of alternatives 1 and 2, respectively, are
(a) #1: \((0.7 \times \$100,000) + (0.3 \times \$70,000)\)
    #2: \((0.8 \times \$170,000) + (0.2 \times \$80,000)\)
(b) #1: \((0.7 \times \$100,000) + (0.3 \times \$70,000)\)
    #2: \((0.8 \times \$130,000) + (0.2 \times \$40,000)\)
(c) #1: \((0.7 \times \$100,000) + (0.3 \times \$70,000)\)
    #2: \((0.8 \times \$170,000) + (0.2 \times \$80,000) - \$40,000\)
(d) #1: \((0.7 \times \$100,000) + (0.3 \times \$70,000) - \$40,000\)
    #2: \((0.8 \times \$170,000) + (0.2 \times \$80,000) - \$40,000\)

### 6. CPA Exam
A battery manufacturer warrants its automobile batteries to perform satisfactorily for as long as the owner keeps the car. Auto industry data show that only 20% of car buyers retain their cars for 3 years or more. Historical data suggest the following:

<table>
<thead>
<tr>
<th>Number of Years Owned</th>
<th>Probability of Battery Failure</th>
<th>Battery Exchange Costs</th>
<th>Percentage of Failed Batteries Returned</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 3 years</td>
<td>0.4</td>
<td>$50</td>
<td>75%</td>
</tr>
<tr>
<td>3 years or more</td>
<td>0.6</td>
<td>$20</td>
<td>50%</td>
</tr>
</tbody>
</table>

If 50,000 batteries were sold this year, what is the estimated warranty cost?
(a) $375,000  (b) $435,000
(c) $500,000  (d) $660,000
Additional Probability Topics

Have you ever participated in a survey? You know, someone asks you a question and you give an answer. But what if the question is one you’d rather not answer. Suppose you were asked to answer “Yes” or “No” to the question “Do you feel you are overweight?” If you’re not overweight, you’d probably answer honestly. But what would you do if you were overweight? Would you lie? If you could be assured of the confidentiality of your response, would that make a difference? For example, suppose the person asking the question said to you, “Flip a coin and don’t tell me the face. If it’s heads, check ‘Yes’; if it’s tails, answer the question honestly. Then drop the paper in the box.” This process guarantees the confidentiality of your response. But how does the surveyor get the information he wants? As it turns out, conditional probability can be used to complete the survey. After you study Section 8.1, look at the Chapter Project for details.

A Look Back, A Look Forward

In Chapter 7 we developed the Multiplication Principle as a counting tool and used it to find probabilities. In this chapter we continue our study of probability with the notions of conditional probability and independent events. These lead to Bayes’ Theorem, a formula used in many applications. Then we develop additional counting techniques, permutations and combinations, that lead us to the binomial probability model, which is also used in many applications.
**8.1 Conditional Probability**

**PREPARING FOR THIS SECTION** Before getting started, review the following:

- Properties of the Probability of an Event (Section 7.5, pp. 391–400)

NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 429.

**OBJECTIVES**
1. Find a conditional probability (p. 423)
2. Find probabilities using the Product Rule (p. 425)
3. Find probabilities using a tree diagram (p. 426)
4. Find conditional probabilities from a data table (p. 428)

Whenever we compute the probability of an event, we do it relative to the entire sample space. When we ask for the probability \( P(E) \) of the event \( E \), this probability \( P(E) \) represents the likelihood that a chance experiment will produce an outcome in the set \( E \) relative to the sample space \( S \).

However, sometimes we want to compute the probability of an event \( E \) of a sample space relative to another event \( F \) of the same sample space. That is, if we have prior information that the outcome must be in a set \( F \), this information should be used to reappraise the likelihood that the outcome will also be in \( E \). Called **conditional probability**, this reappraised probability is denoted by \( P(E|F) \), and is read as the **probability of \( E \) given \( F \)**. It represents the answer to the question, “How probable is \( E \), given that \( F \) has occurred?”

**EXAMPLE 1** Example of Conditional Probability

Consider the experiment of flipping two fair coins. A sample space \( S \) is

\[
S = \{HH, HT, TH, TT\}
\]

Figure 1 illustrates the sample space and the probability of each outcome.

Suppose the experiment is performed by another person and we have no knowledge of the result, but we are informed that at least one tail was tossed. This information means the outcome \( HH \) could not have occurred. But the remaining outcomes \( HT, TH, TT \) are still possible. How does this alter the probabilities of the remaining outcomes?

The three outcomes \( TH, HT, TT \) were each assigned the probability \( \frac{1}{4} \) before we knew the information that at least one tail occurred, so it is not reasonable to assign them this same probability now. Since only three equally likely outcomes are now possible, we assign to each of them the probability \( \frac{1}{3} \). See Figure 2.

**EXAMPLE 2** Example of Conditional Probability

Suppose a population of 1000 people includes 70 accountants and 520 females. There are 40 females who are accountants. A person is chosen at random, and we are told the person is female. Since there are 520 females, 40 of whom are accountants, the probability the person is an accountant, given that the person is female, is \( \frac{40}{520} = \frac{1}{13} \). That is, the **conditional probability** of the event \( E \) (accountant) assuming the event \( F \) (the person chosen is female) is \( \frac{40}{520} = \frac{1}{13} \).
8.1 Conditional Probability

We use the symbol \( P(E|F) \), read “the probability of \( E \) given \( F \),” to denote conditional probability. For Example 2, if \( E \) is the event “A person chosen at random is an accountant” and \( F \) is the event “A person chosen at random is female,” we would write

\[
P(E|F) = \frac{40}{520} = \frac{1}{13}
\]

Figure 3 illustrates a Venn diagram for Example 2. Notice that the quotient of the number of entries in \( E \) and \( F \), 40, with the number that are in \( F \), 520, is That is,

\[
P(E|F) = \frac{40}{520} = \frac{n(E \cap F)}{n(F)}
\]

Since

\[
P(E|F) = \frac{n(E \cap F)}{n(F)} = \frac{n(S)}{n(S)} = \frac{P(E \cap F)}{P(F)}
\]

we define conditional probability as follows:

**Definition**

**Conditional Probability**

Let \( E \) and \( F \) be events of a sample space \( S \) and suppose \( P(F) > 0 \). The conditional probability of the event \( E \), assuming the event \( F \), denoted by \( P(E|F) \), is defined as

\[
P(E|F) = \frac{P(E \cap F)}{P(F)} \quad (1)
\]

**EXAMPLE 3** Finding a Conditional Probability

Consider the experiment of selecting one family from the set of three-child families. A sample space \( S \) for this experiment is

\[
S = \{ \text{BBB, BBG, BGB, GBB, GBG, GGB, GGG} \}
\]

We assume that each outcome is equally likely, so that each outcome is assigned a probability of \( \frac{1}{8} \). Let \( E \) be the event, “The family has exactly two boys” and let \( F \) be the event “The first child is a boy.” What is the probability that the family has exactly two boys, given that the first child is a boy?

**SOLUTION** We want to find \( P(E|F) \). The events \( E \) and \( F \) are

\[
E = \{ \text{BBG, BGB, GBB} \} \quad F = \{ \text{BBB, BBG, BGB, GBB} \}
\]

Since \( E \cap F = \{ \text{BBG, BGB} \} \), we have

\[
P(E \cap F) = \frac{2}{8} = \frac{1}{4} \quad P(F) = \frac{4}{8} = \frac{1}{2}
\]
Now use Formula (1) to get

\[ P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1}{4} = \frac{1}{2} \]

In a three-child family, the probability the family exactly has two boys, given that the first child is a boy, is \( \frac{1}{2} \).

Since the sample space \( S \) of Example 3 consists of equally likely outcomes, we can also compute \( P(E|F) \) using the Venn diagram in Figure 4. Then

\[ P(E|F) = \frac{n(E \cap F)}{n(F)} = \frac{2}{4} = \frac{1}{2} \]

Let’s analyze the situation in Example 3 more carefully. Look again at Figure 4. The event \( E \) is “the family has exactly 2 boys.” We have computed the probability of event \( E \), knowing event \( F \) has occurred. This means we are computing a probability relative to a new sample space \( F \). That is, \( F \) is treated as the universal set and we should consider only the part of \( E \) that is included in \( F \), namely, \( E \cap F \). So if \( E \) is any subset of the sample space \( S \), then \( P(E|F) \) provides the reappraisal of the likelihood that an outcome of the experiment will be in the set \( E \) if we have prior information that it must be in the set \( F \).

NOW WORK PROBLEM 7.

**EXAMPLE 4** Finding Probabilities

Suppose \( E \) and \( F \) are events of a sample space \( S \) for which 

\[ P(E) = 0.7 \quad P(F) = 0.8 \quad P(E \cap F) = 0.6 \]

Find

(a) \( P(E \cup F) \)  
(b) \( P(E|F) \)  
(c) \( P(F|E) \)  
(d) \( P(E|F) \)

**SOLUTION** (a) To find \( P(E \cup F) \), we use the Additive Rule.

\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.7 + 0.8 - 0.6 = 0.9 \]

(b) To find the conditional probability \( P(E|F) \), we use Formula (1).

\[ P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.6}{0.8} = 0.75 \]

(c) To find the conditional probability \( P(F|E) \), we use a variation of Formula (1), replacing \( E \) by \( F \) and \( F \) by \( E \). The result is

\[ P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{0.6}{0.7} = 0.857 \]
To find the conditional probability $P(E|F)$, we use a variation of Formula (1), replacing $E$ by $\bar{E}$ and $F$ by $\bar{F}$. The result is

$$P(\bar{E}|\bar{F}) = \frac{P(\bar{E} \cap \bar{F})}{P(\bar{F})}$$

Since $\bar{E} \cap \bar{F} = E \cup F$ (De Morgan’s property), we have

$$P(\bar{E}|\bar{F}) = \frac{P(E \cup F)}{P(\bar{F})} = \frac{1 - P(E \cup F)}{1 - P(F)} = \frac{1 - 0.9}{1 - 0.8} = \frac{0.1}{0.2} = 0.50$$

Use the probability of a complement.

NOW WORK PROBLEMS 27 AND 29.

**Finding Probabilities Using the Product Rule**

In Formula (1), multiply both sides of the equation by $P(F)$.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \quad \text{Formula (1)}$$

$$P(F) \cdot P(E|F) = P(E \cap F) \quad \text{Multiply by } P(F).$$

This formula is referred to as the **Product Rule**:

**Theorem**

**Product Rule**

For two events $E$ and $F$, the probability of the event $E$ and $F$, namely, $P(E \cap F)$, is given by

$$P(E \cap F) = P(F) \cdot P(E|F)$$ (2)

**EXAMPLE 5**

Finding Probabilities Using the Product Rule

Two cards are drawn in order (without replacement) from a regular deck of 52 cards. What is the probability that the first card is a diamond and the second is red?

**SOLUTION**

Define the events

- $E$: The first card is a diamond
- $F$: The second card is red

We seek the probability of $E$ and $F$, namely, $P(E \cap F)$.

Since there are 52 cards in the deck, of which 13 are diamonds, it follows that

$$P(E) = \frac{13}{52} = \frac{1}{4}$$

If $E$ occurred, it means that there are only 51 cards left in the deck, of which 25 are red, so

$$P(F|E) = \frac{25}{51}$$
By the Product Rule,

\[ P(E \cap F) = P(F \cap E) = P(E) \cdot P(F|E) = \frac{1}{4} \cdot \frac{25}{51} = \frac{25}{204}. \]

### 3 Find Probabilities Using a Tree Diagram

A tree diagram is helpful for problems like that of Example 5. See Figure 5.

- **FIGURE 5**

<table>
<thead>
<tr>
<th>First draw</th>
<th>Second draw</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond</td>
<td>Red</td>
</tr>
<tr>
<td>Heart</td>
<td>Black</td>
</tr>
<tr>
<td>Club</td>
<td>Red</td>
</tr>
<tr>
<td>Spade</td>
<td>Black</td>
</tr>
</tbody>
</table>

The branch leading to $E$: “Diamond” has probability $\frac{13}{52} = \frac{1}{4}$; the branch leading to “Heart” also has probability $\frac{13}{52} = \frac{1}{4}$; and the branch leading to “Club” (or “Spade”) has probability $\frac{13}{52} = \frac{1}{4}$. These must add up to 1 since no other possibilities (branches) are possible. The branch from “Diamond” to “Red” is the conditional probability of drawing a red card on the second draw after a diamond on the first draw, $P(\text{Red}|\text{Diamond})$, which is $\frac{25}{51}$.

The branch from “Diamond” to “Black” is the conditional probability $P(\text{Black}|\text{Diamond})$, which is $\frac{26}{51}$ (26 black cards and 51 total cards remain after a diamond on the first draw). The remaining entries are obtained similarly. Notice that the probability the first card is a diamond and the second is red corresponds to tracing the top branch of the tree. The Product Rule then tells us to multiply the branch probabilities.

A further advantage of using a tree diagram is that it enables us to easily answer other questions about the experiment. For example, based on Figure 5, we see that

- **(a)** Probability the first card is a heart and the second is black equals $\frac{1}{4} \cdot \frac{26}{51} = \frac{13}{102}$

- **(b)** Probability the first card is a club and the second is red equals $\frac{1}{4} \cdot \frac{26}{51} = \frac{13}{102}$

- **(c)** Probability the first card is a spade and the second is black equals $\frac{1}{4} \cdot \frac{25}{51} = \frac{25}{204}$

**NOW WORK PROBLEM 35.**
From a box containing four white, three yellow, and one green ball, two balls are drawn one at a time without replacing the first ball before the second is drawn. Use a tree diagram to find the probability that one white and one yellow ball are drawn.

**SOLUTION**

To fix our ideas, define the events

- \( W \): White ball drawn
- \( Y \): Yellow ball drawn
- \( G \): Green ball drawn

The tree diagram for this experiment is given in Figure 6.

The event of drawing one white ball and one yellow ball can occur in two ways: drawing a white ball first and then a yellow ball (path 2 of the tree diagram in Figure 6), or drawing a yellow ball first and then a white ball (path 4).

Consider path 2. Since four of the eight balls are white, on the first draw we have

\[
P(W) = P(W \text{ on } 1st) = \frac{4}{8} = \frac{1}{2}
\]

Since one white ball has been removed, leaving seven balls in the box, of which three are yellow, on the second draw we have

\[
P(Y|W) = P(Y \text{ on } 2nd|W \text{ on } 1st) = \frac{3}{7}
\]

For path 2 we have

\[
P(W) \cdot P(Y|W) = P(W \text{ on } 1st) \cdot P(Y \text{ on } 2nd|W \text{ on } 1st) = \frac{1}{2} \cdot \frac{3}{7} = \frac{3}{14}
\]

Similarly, for path 4 we have

\[
P(Y) \cdot P(W|Y) = P(Y \text{ on } 1st) \cdot P(W \text{ on } 2nd|Y \text{ on } 1st) = \frac{3}{8} \cdot \frac{4}{7} = \frac{3}{14}
\]

Since the two events are mutually exclusive, the probability of drawing one white ball and one yellow ball is the sum of these two probabilities:

\[
\frac{3}{14} + \frac{3}{14} = \frac{6}{14} = \frac{3}{7}
\]

**NOW WORK PROBLEM 37.**
EXAMPLE 7  Using a Tree Diagram to Find Conditional Probabilities

Motors, Inc., has two plants to manufacture cars. Plant I manufactures 80% of the cars and plant II manufactures 20%. At plant I, 85 out of every 100 cars are rated standard quality or better. At plant II, only 65 out of every 100 cars are rated standard quality or better.

(a) What is the probability that a customer obtains a standard quality car if he buys a car from Motors, Inc.?

(b) What is the probability that the car came from plant I if it is known that the car is of standard quality?

SOLUTION  Begin with the tree diagram in Figure 7.

Define the following events:

I: Car came from plant I
II: Car came from plant II
A: Car is of standard quality

(a) There are two ways a standard car can be obtained: either it is standard and came from plant I, or else it is standard and came from plant II. By the Product Rule,

\[
P(A \cap I) = P(I) \cdot P(A|I) = 0.80 \cdot 0.85 = 0.68
\]

\[
P(A \cap II) = P(II) \cdot P(A|II) = 0.20 \cdot 0.65 = 0.13
\]

Since \( A \cap I \) and \( A \cap II \) are mutually exclusive, we have

\[
P(A) = P(A \cap I) + P(A \cap II) = 0.68 + 0.13 = 0.81
\]

(b) To compute \( P(I|A) \), use the definition of conditional probability.

\[
P(I|A) = \frac{P(I \cap A)}{P(A)} = \frac{0.68}{0.81} = 0.8395
\]

NOW WORK PROBLEM 73.

EXAMPLE 8  Finding Conditional Probabilities from a Data Table

On April 30, 2010, the market breadth for stocks traded consisted of the data shown in Table 1. What is the probability an issue selected at random advanced given it is High Yield?

<table>
<thead>
<tr>
<th>TABLE 1  MARKET BREADTH</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>All Issues</strong></td>
</tr>
<tr>
<td>Advances</td>
</tr>
<tr>
<td>Declines</td>
</tr>
<tr>
<td>Unchanged</td>
</tr>
<tr>
<td>52 Week High</td>
</tr>
<tr>
<td>52 Week Low</td>
</tr>
</tbody>
</table>
8.1 Conditional Probability

**SOLUTION** Define the events $A$ and $H$ as

- $A$: Issue advanced
- $H$: Issue is high yield

We seek the probability of $A$ given $H$, that is, $P(A|H)$. Now by Formula (1)

$$P(A|H) = \frac{P(A \cap H)}{P(H)}$$

To use this formula, we need to find $P(A \cap H)$ and $P(H)$. Since the selection is random, we count the elements in the sample space $S$, the event $A \cap H$, and the event $H$.

The total number of issues in the sample space $S$ is $n(S) = 2958 + 2194 + 201 = 5353$. Of these, the number of issues that advanced and are high yield is $n(A \cap H) = 732$ and the number of issues that are high yield is $n(H) = 732 + 533 + 107 = 1372$. Then

$$P(A \cap H) = \frac{n(A \cap H)}{n(S)} = \frac{732}{5353} = 0.137$$

$$P(H) = \frac{n(H)}{n(S)} = \frac{1372}{5353} = 0.256$$

$$P(A|H) = \frac{P(A \cap H)}{P(H)} = \frac{0.137}{0.256} = 0.535$$

The probability an issue selected at random advanced given it is high yield is 0.535. ■

**NOW WORK PROBLEM 77.**

**EXERCISE 8.1 Answers Begin on Page AN–39.**

‘Are You Prepared?’ Problems Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Two cards are drawn in order (without replacement) from a regular deck of 52 cards. What is the probability the first card is a diamond? (pp. 391–400)
2. If $E$ and $F$ are events of a sample space for which $P(E) = 0.7, P(F) = 0.8$, find $P(E \cap F) = 0.6$, find $P(E \cup F)$. (pp. 391–400)

Concepts and Vocabulary

3. True or False The conditional probability of the event $E$ given the event $F$ is denoted by $P(E|F)$.

Skill Building

In Problems 5–12, use the Venn diagram below to find each probability:

- $P(E)$
- $P(F)$
- $P(E|F)$
- $P(F|E)$

5. $P(E)$
6. $P(F)$
7. $P(E|F)$
8. $P(F|E)$
9. $P(E \cap F)$
10. $P(E \cup F)$
11. $P(\overline{E})$
12. $P(\overline{F})$

13. If $E$ and $F$ are events with $P(E) = 0.2, P(F) = 0.4$, and $P(E \cap F) = 0.1$, find the probability of $E$ given $F$. Also find $P(F|E)$.

14. If $E$ and $F$ are events with $P(E) = 0.5, P(F) = 0.6$, and $P(E \cap F) = 0.3$, find the probability of $E$ given $F$. Also find $P(F|E)$.

15. If $E$ and $F$ are events with $P(E \cap F) = 0.2$ and $P(E|F) = 0.4$, find $P(F)$.

16. If $E$ and $F$ are events with $P(E \cap F) = 0.2$ and $P(E|F) = 0.6$, find $P(F)$.

17. If $E$ and $F$ are events with $P(F) = \frac{5}{13}$ and $P(E|F) = \frac{4}{5}$, find $P(E \cap F)$.

18. If $E$ and $F$ are events with $P(F) = 0.38$ and $P(E|F) = 0.46$, find $P(E \cap F)$.

19. If $E$ and $F$ are events with $P(E \cap F) = \frac{1}{3}$, $P(E|F) = \frac{1}{2}$, and $P(F|E) = \frac{2}{3}$, find
   - (a) $P(E)$
   - (b) $P(F)$

20. If $E$ and $F$ are events with $P(E \cap F) = 0.1$, $P(E|F) = 0.25$, and $P(F|E) = 0.125$, find
   - (a) $P(E)$
   - (b) $P(F)$
Applications and Extensions

In Problems 21–26, find each probability by referring to the following tree diagram:

\[ \begin{align*}
0.7 & \quad A \\
0.3 & \quad B \\
0.2 & \quad C \\
0.1 & \quad D \\
0.9 & \quad C \\
0.8 & \quad D 
\end{align*} \]

21. \( P(C) \)  
22. \( P(D) \)  
23. \( P(C|A) \)  
24. \( P(D|A) \)  
25. \( P(C|B) \)  
26. \( P(D|B) \)

In Problems 27–32, E and F are events in a sample space \( S \) for which \( P(E) = 0.5 \), \( P(F) = 0.4 \), and \( P(E \cup F) = 0.8 \). Find each probability.

27. \( P(E \cap F) \)  
28. \( P(E|F) \)  
29. \( P(F|E) \)  
30. \( P(E|F) \)  
31. \( P(E) \)  
32. \( P(F) \)

In Problems 41–52, use the table to obtain probabilities for events in a sample space \( S \).

| \( E \) \( F \) \( G \) \( \text{Totals} \) |
|---|---|---|---|
| \( H \) | 0.10 | 0.06 | 0.08 | 0.24 |
| \( I \) | 0.30 | 0.14 | 0.32 | 0.76 |
| \( \text{Totals} \) | 0.40 | 0.20 | 0.40 | 1.00 |

For Problems 41–48, read each probability directly from the table:

41. \( P(E) \)  
42. \( P(G) \)  
43. \( P(H) \)  
44. \( P(I) \)  
45. \( P(E \cap H) \)  
46. \( P(E \cap I) \)  
47. \( P(G \cap H) \)  
48. \( P(G \cap I) \)

For Problems 49–52, use Formula (1) to find each conditional probability.

49. \( P(E|H) \)  
50. \( P(E|I) \)  
51. \( P(G|H) \)  
52. \( P(G|I) \)

Surveys In Problems 53–60, use the table below, which shows the result of a survey conducted by a deodorant producer.

<table>
<thead>
<tr>
<th>( \text{Group I} )</th>
<th>( \text{Group II} )</th>
<th>( \text{Group III} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Like the Deodorant} )</td>
<td>( \text{Did Not Like the Deodorant} )</td>
<td>( \text{No Opinion} )</td>
</tr>
<tr>
<td>180</td>
<td>60</td>
<td>20</td>
</tr>
<tr>
<td>110</td>
<td>85</td>
<td>12</td>
</tr>
<tr>
<td>55</td>
<td>65</td>
<td>7</td>
</tr>
</tbody>
</table>

Define the events \( E \), \( F \), \( G \), \( H \), and \( K \) as follows:

\( E \): Customer likes the deodorant
\( F \): Customer does not like the deodorant
\( G \): Customer is from group I
\( H \): Customer is from group II
\( K \): Customer is from group III

Find each conditional probability and state in words the meaning of each one.

53. \( P(E|G) \)  
54. \( P(G|E) \)  
55. \( P(H|E) \)  
56. \( P(K|E) \)  
57. \( P(F|G) \)  
58. \( P(G|F) \)  
59. \( P(H|F) \)  
60. \( P(K|F) \)
A respondent is selected at random. Find the probability that
(a) the respondent does not drink beer.
(b) the respondent is a female.
(c) the respondent is a female who prefers regular beer.
(d) the respondent prefers regular beer, given that the respondent is male.
(e) the respondent is male, given that the respondent prefers regular beer.
(f) the respondent is female, given that the respondent prefers regular beer or does not drink beer.

Blood Types In Problems 69–72, use the information given below. Round your answers to three decimal places.

Source: www.aabb.org/blood, April 2010

8.1 Conditional Probability

A respondent is selected at random. Find the probability that
(a) the respondent does not drink beer.
(b) the respondent is a female.
(c) the respondent is a female who prefers regular beer.
(d) the respondent prefers regular beer, given that the respondent is male.
(e) the respondent is male, given that the respondent prefers regular beer.
(f) the respondent is female, given that the respondent prefers regular beer or does not drink beer.

Blood Types In Problems 69–72, use the information given below. Round your answers to three decimal places.

Source: www.aabb.org/blood, April 2010
432 Chapter 8 Additional Probability Topics

75. Internet Sales A September 2010 marketing survey of Internet purchases resulted in the probabilities in the table.

<table>
<thead>
<tr>
<th>Age</th>
<th>Purchase Value</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0–$99.99</td>
<td>$100–$499.99</td>
<td>$500.00</td>
<td></td>
</tr>
<tr>
<td>&lt;30</td>
<td>0.20</td>
<td>0.07</td>
<td>0.18</td>
<td></td>
</tr>
<tr>
<td>30–50</td>
<td>0.12</td>
<td>0.21</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>&gt;50</td>
<td>0.06</td>
<td>0.04</td>
<td>0.02</td>
<td></td>
</tr>
</tbody>
</table>

(a) What is the probability a purchase was made by a person younger than 30 years of age?
(b) If a purchase was $500.00 or more, what is the probability that it was made by a person over 50 years old?
(c) Given that the purchase was between $100 and $499.99, what is the probability it was made by a consumer between the ages of 30 and 50?

76. Multiple Jobs According to the U.S. Bureau of Labor Statistics, there is a 5.84% chance that a randomly selected employed individual has more than one job. There is also a 52.6% probability that a randomly selected employed individual is male, given that he has more than one job.
(a) What is the probability that a randomly selected employed individual is male and has more than one job?
(b) Would it be unusual to randomly select such an individual?

Market Breadth April 30, 2010 In Problems 77–80, use the table below. Round your answers to three decimal places.

<table>
<thead>
<tr>
<th>Age</th>
<th>&lt;18</th>
<th>18–44</th>
<th>45–64</th>
<th>≥65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Health Insurance</td>
<td>74,403</td>
<td>110,676</td>
<td>77,237</td>
<td>36,790</td>
</tr>
<tr>
<td>No Health Insurance</td>
<td>8,149</td>
<td>26,037</td>
<td>10,784</td>
<td>686</td>
</tr>
</tbody>
</table>

Source: Income, Poverty and Health Insurance Coverage in the United States, 2008, U.S. Census Bureau
(a) What is the probability that a randomly selected individual who is less than 18 years old has no health insurance?
(b) What is the probability that a randomly selected individual who has no health insurance is less than 18 years old?
(c) What is the probability that a randomly selected individual who has health insurance is less than 18 years old?

82. Private vs Public Schools Of the first-year students in a certain college, it is known that 40% attended private secondary schools and 60% attended public schools. The registrar reports that 30% of all students who attended private schools maintain an A average in their first year at college and that 24% of all first-year students had an A average. At the end of the year, one student is chosen at random from the class. If the student has an A average, what is the conditional probability that the student attended a private school? [Hint: Use a tree diagram.]

Problems 83–84 require the following discussion.

Craps In a popular dice game, the player rolls a pair of fair dice. If the total number of spots is 7 or 11, the player wins. If the total number of spots is 2, 3, or 12, the player loses. If the player rolls any other total, the total rolled becomes the player’s “point” and the player continues to roll the pair of dice. To win the game after establishing the player’s “point,” the player must re-roll a total equal to the “point” that has been established before rolling a total of 7. The player loses at this stage if a total of 7 is rolled while attempting to re-roll the player’s “point.”

Source: The International Bone Rollers Guild, Games with Ordinary Dice, from According to Hoyle, by Richard L. Frey.

83. Let E be the event that the player rolls a total of 8 on the first roll. (Therefore, the player’s “point” is 8.) Let W be the event that the player wins the game. Find \( P(W|E) \).
84. Let E be the event that the player rolls a total of 5 on the first roll. (Therefore, the player’s “point” is 5.) Let W be the event that the player wins the game. Find \( P(W|E) \).
85. Job Applications “Temp Help” uses a preemployment test to screen applicants for the job of programmer. The test is passed by 70% of the applicants. Among those who pass the test, 85% complete training successfully. In an experiment a random sample of applicants who do not pass the test is also
Republicans also voted for the Democratic candidate so that the Democrat won by a ratio of 5 to 4.

(a) If a voter is selected at random, what is the probability he or she is Republican?

(b) What is the probability a voter is Republican, if it is known that he or she voted for the Democratic candidate?

89. If \( E \) and \( F \) are two events with \( P(E) > 0 \) and \( P(F) > 0 \), show that
\[
P(F) \cdot P(E|F) = P(E) \cdot P(F|E)
\]

90. Show that \( P(E|E) = 1 \) when \( P(E) \neq 0 \).

91. Show that \( P(E|F) + P(F|E) = 1 \).

92. If \( S \) is the sample space, show that \( P(E|S) = P(E) \).

93. If \( P(E) > 0 \) and \( P(E|F) = P(E) \), show that \( P(F|E) = P(F) \).

8.2 Independent Events

**OBJECTIVES**

1. Show two events are independent (p. 435)
2. Find \( P(E \cap F) \) for independent events \( E \) and \( F \) (p. 437)
3. Find probabilities for several independent events (p. 438)
4. Model: Optimal testing size for components (p. 439)

**EXAMPLE 1**

**Example of Independent Events**

Consider a group of 36 students. Define the events \( E \) and \( F \) as

\( E: \) Student has blue eyes  \( F: \) Student is female

With regard to these two characteristics, suppose it is found that the 36 students are distributed as shown in Table 2. What is \( P(E|F) \)?
### SOLUTION

If we choose a student at random, the following probabilities can be obtained from the table:

\[
P(E) = \frac{18}{36} = \frac{1}{2}, \quad P(F) = \frac{24}{36} = \frac{2}{3}
\]

\[
P(E \cap F) = \frac{12}{36} = \frac{1}{3}
\]

Then we find that

\[
P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1}{2} = \frac{1}{2} = P(E)
\]

The probability of \(E\) given \(F\) equals the probability of \(E\). This situation can be described by saying that the information that the event \(F\) has occurred does not affect the probability of the event \(E\). If this is the case, we say that \(E\) is independent of \(F\).

---

### Definition

**\(E\) is Independent of \(F\)**

Let \(E\) and \(F\) be two events of a sample space \(S\) with \(P(F) > 0\). The event \(E\) is independent of the event \(F\) if and only if

\[
P(E|F) = P(E)
\]

---

### Theorem

Let \(E, F\) be events for which \(P(E) > 0\) and \(P(F) > 0\). If \(E\) is independent of \(F\), then \(F\) is independent of \(E\).

A proof is outlined in Problem 45.

This result forms the basis for the following definition.
Definition

**Independent Events**

If two events have positive probabilities and if either event is independent of the other, the events are called **independent events**.

The following result will be used frequently. A proof is outlined in Problem 44.

**Theorem**

**Criterion for Independent Events**

Two events $E$ and $F$ of a sample space $S$ are independent events if and only if

$$P(E \cap F) = P(E) \cdot P(F)$$  \hspace{1cm} (1)

**Theorem**

The Criterion for Independent Events actually consists of two statements:

1. If $P(E \cap F) = P(E) \cdot P(F)$ for two events $E$ and $F$, then the events $E$ and $F$ are independent.
2. If two events $E$ and $F$ are independent, then $P(E \cap F) = P(E) \cdot P(F)$.

Use statement 1 to show that two events are independent. Use statement 2 when we know or are told two events are independent.

**Show Two Events Are Independent**

**EXAMPLE 2** **Showing Two Events Are Independent**

Suppose $P(E) = \frac{1}{4}$, $P(F) = \frac{2}{3}$, and $P(E \cap F) = \frac{1}{6}$. Show that $E$ and $F$ are independent events.

**SOLUTION** We see if Equation (1) holds.

$$P(E) \cdot P(F) = \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$$

$$P(E \cap F) = \frac{1}{6}$$

Since Equation (1) holds, by the Criterion for Independence, $E$ and $F$ are independent events.

**NOW WORK PROBLEM 7**
EXAMPLE 3  Showing Two Events Are Independent

Suppose a red die and a green die are thrown. Let event $E$ be “Throw a 5 with the red die,” and let event $F$ be “Throw a 6 with the green die.” Show that $E$ and $F$ are independent events.

**SOLUTION**

In this experiment the events $E$ and $F$ are

$$E = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$$

$$F = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)\}$$

Since the sample space has 36 outcomes that are equally likely, we have

$$P(E) = \frac{6}{36} = \frac{1}{6} \quad P(F) = \frac{6}{36} = \frac{1}{6}$$

Also, the event $E$ and $F$ is

$$E \cap F = \{(5, 6)\}$$

so that

$$P(E \cap F) = \frac{1}{36}$$

Since $P(E) \cdot P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(E \cap F)$, $E$ and $F$ are independent events.

---

EXAMPLE 4  Testing for Independent Events

In a T-maze a mouse may run to the right, $R$, or to the left, $L$. Suppose its behavior in making such “choices” is random so that $R$ and $L$ are equally likely outcomes. A mouse is put through the T-maze three times. Define events $E$, $G$, and $H$ as

$E$: Run to the right 2 or more consecutive times

$G$: Run to the left on first trial

$H$: Run to the right on second trial

(a) Show that $E$ and $G$ are not independent.

(b) Show that $G$ and $H$ are independent.

**SOLUTION**

A sample space $S$ for this experiment is

$$S = \{LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR\}$$

where, for example, $LLR$ means run to the left the first and second time and run to the right the third time. $S$ has eight outcomes, and the outcomes are equally likely.

(a) The events $E$ and $G$ are

$$E = \{RRL, LRR, RRR\}$$

$$G = \{LLL, LLR, LRL, LRR\}$$

Then

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8} \quad P(G) = \frac{n(G)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

(b) Since

$$P(E) \cdot P(G) = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16} \neq \frac{1}{8} = P(E \cap G),$$

$E$ and $G$ are not independent.

Since

$$P(G) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16} = P(G \cap H),$$

$G$ and $H$ are independent.
8.2 Independent Events

Also, the event $E$ and $G$ is
\[ E \cap G = \{LRR\} \]
so
\[ P(E \cap G) = \frac{1}{8} \]
Since $P(E) \cdot P(G) = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16}$, we find that $P(E \cap G) \neq P(E) \cdot P(G)$. The events $E$ and $G$ are not independent.

(b) The event $H$ is
\[ H = \{RRL, RRR, LRL, LRR\} \]
so
\[ P(H) = \frac{1}{2} \]
The event $G$ and $H$ and its probability are
\[ G \cap H = \{LRL, LRR\} \quad P(G \cap H) = \frac{1}{4} \]
Since $P(G) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, $P(G \cap H) = P(G) \cdot P(H)$. The events $G$ and $H$ are independent.

Example 4 illustrates that the question of whether two events are independent can be answered simply by determining whether Equation (1) is satisfied. Although we may often suspect two events $E$ and $F$ as being independent, our intuition must be checked by computing $P(E)$, $P(F)$, and $P(E \cap F)$ and determining whether $P(E \cap F) = P(E) \cdot P(F)$.

**NOW WORK PROBLEM 27.**

2 Find $P(E \cap F)$ for Independent Events $E$ and $F$

If $E$ and $F$ are two independent events and $P(E)$ and $P(F)$ are known, then we can use Equation (1) to find $P(E \cap F)$, the probability of the event $E$ and $F$.

**EXAMPLE 5 Finding $P(E \cap F)$ for Independent Events $E$ and $F$**

Suppose $E$ and $F$ are independent events with $P(E) = 0.4$ and $P(F) = 0.3$. Find $P(E \cap F)$.

**SOLUTION** Since $E$ and $F$ are independent, we can use Equation (1) to find $P(E \cap F)$.
\[
P(E \cap F) = P(E) \cdot P(F) \quad \text{Equation (1)}
\]
\[
= (0.4) \cdot (0.3)
\]
\[
= 0.12
\]

**NOW WORK PROBLEM 3.**
For some probability models, an assumption of independence is made.

**EXAMPLE 6 Planting Seeds**

In a group of seeds, \( \frac{1}{4} \) of which should produce white flowers, the best germination that can be obtained is 75%. If one seed is planted, what is the probability that it will grow into a white flower? Assume that germination and the production of a white flower are independent events.

**SOLUTION** Let \( G \) and \( W \) be the events

- \( W \): The seed will produce a white flower
- \( G \): The plant will grow

Then

\[
P(W) = \frac{1}{4}, \quad P(G) = 0.75 = \frac{3}{4}
\]

Since \( G \) and \( W \) are independent events, the probability that the plant grows and its flower is white, namely, \( P(W \cap G) \), is

\[
P(W \cap G) = P(W) \cdot P(G) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}
\]

A white flower will grow 3 out of 16 times.

**NOW WORK PROBLEM 21.**

There is a danger that mutually exclusive events and independent events may be confused. A source of this confusion is the common expression, “Events are independent if they have nothing to do with each other.” This expression provides a description of independence when applied to everyday events; but when it is applied to sets, it suggests nonoverlapping. Nonoverlapping sets are mutually exclusive but in general are not independent.

### 3 Find Probabilities for Several Independent Events

The concept of independent events can be applied to more than two events:

**Independent Events**

A set \( \{E_1, E_2, \ldots, E_n\} \) of \( n \) events is called independent if the occurrence of one or more of them does not change the probability of any of the others. It can be shown that, for such events,

\[
P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdot \cdots \cdot P(E_n)
\]
EXAMPLE 7 Finding Probabilities of Independent Events

A new skin cream can cure skin infection 90% of the time. If five randomly selected people with skin infections use this cream, what is the probability that
(a) All five are cured?
(b) All five still have the infection?
Assume that the choice of a person and the elimination of the infection are independent.

SOLUTION

(a) Let

\[ E_1: \text{First person does not have the infection} \]
\[ E_2: \text{Second person does not have the infection} \]
\[ E_3: \text{Third person does not have the infection} \]
\[ E_4: \text{Fourth person does not have the infection} \]
\[ E_5: \text{Fifth person does not have the infection} \]

Then

\[ P(E_1) = 0.9 \quad P(E_2) = 0.9 \quad P(E_3) = 0.9 \quad P(E_4) = 0.9 \quad P(E_5) = 0.9 \]

Since the events \( E_1, E_2, E_3, E_4, E_5 \), are independent, we have

\[
P(\text{all 5 are cured}) = P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5) \\
= P(E_1) \cdot P(E_2) \cdot P(E_3) \cdot P(E_4) \cdot P(E_5) \\
= (0.9)^5 \\
= 0.59
\]

The probability all five are cured is 0.59.

(b) Let

\[ \overline{E}_i: \text{ith person has the infection, } i = 1, 2, 3, 4, 5 \]

Then

\[
P(\overline{E}_i) = 1 - 0.9 = 0.1 \\
P(\text{all 5 are infected}) = P(\overline{E}_1 \cap \overline{E}_2 \cap \overline{E}_3 \cap \overline{E}_4 \cap \overline{E}_5) \\
= P(\overline{E}_1) \cdot P(\overline{E}_2) \cdot P(\overline{E}_3) \cdot P(\overline{E}_4) \cdot P(\overline{E}_5) \\
= (0.1)^5 \\
= 0.00001
\]

The probability none is cured is 0.00001.

4 Model: Optimal Testing Size for Components

A factory produces electronic components, and each component must be tested. If the component is good, it will allow the passage of current; if the component is defective, it will block the passage of current. Let \( p \) denote the probability that a component is good. See Figure 8. If we test each component, the number of tests required equals the number of components. If the number of components is large, this will increase the production cost of the electronic components since one test is required per component.

To reduce the number of tests, a quality control engineer proposes, instead, a new testing procedure: Connect the components pairwise in series, as shown in Figure 9.

If the current passes two components in series, then both components are good and only one test is required. Now, each component has probability \( p \) of being good. Further,
the events “the first component is good” and “the second component is good” are independent. By the Product Rule, the probability both components are good is \( p \cdot p = p^2 \).

If the current does not pass, the components must be tested separately. In this case three tests are required. The probability that three tests are needed is \( 1 - p^2 \) (1 minus probability both are good). The expected number of tests for a pair of components is

\[
E = 1 \cdot p^2 + 3 \cdot (1 - p^2) = p^2 + 3 - 3p^2 = 3 - 2p^2
\]

The number of tests saved for a pair is

\[
2 - (3 - 2p^2) = 2p^2 - 1
\]

The number of tests saved per component is

\[
\frac{2p^2 - 1}{2} = p^2 - \frac{1}{2} \text{ tests saved per component}
\]

The greater the probability \( p \) that the component is good, the greater the saving. For example, if \( p \) is almost 1, we have a saving of almost \( 1 - \frac{1}{2} \) or \( \frac{1}{2} \), which is 50% of the original number of tests needed. Of course, if \( p \) is small, say, less than 0.7, we do not save anything since \( (0.7)^2 \frac{1}{2} \) is less than 0, so the extra tests provide no benefit.

If the reliability of the components manufactured is very high, it might even be advisable to test larger groups. Suppose three components are connected in series. See Figure 10.

Individual testing requires three tests. For group testing we have

- 1 test needed with probability \( p^3 \)
- 4 tests needed with probability \( 1 - p^3 \)

The expected number of tests is

\[
E = 1 \cdot p^3 + 4 \cdot (1 - p^3) = 4 - 3p^3 \text{ tests}
\]

The number of tests saved per component is

\[
\frac{3 - (4 - 3p^3)}{3} = \frac{3p^3 - 1}{3} = p^3 - \frac{1}{3}
\]

If the components are arranged in groups of four connected in series, then the number of tests saved per component is

\[
p^4 - \frac{1}{4}
\]

In general, for groups of \( n \), the number of tests saved per component is

\[
p^n - \frac{1}{n}
\]

Notice from the above formula that as \( n \), the group size, gets very large, the number of tests saved per component gets very, very small.

Table 3 shows the expected tests saved per component for \( p = 0.9 \). From the table we see that the optimal group size is four, resulting in a substantial saving of approximately 41%.
8.2 Independent Events

<table>
<thead>
<tr>
<th>Group Size</th>
<th>Expected Tests Saved per Component $p = 0.9$</th>
<th>Percent Saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$p^2 - \frac{1}{2} = 0.81 - 0.50 = 0.31$</td>
<td>31</td>
</tr>
<tr>
<td>3</td>
<td>$p^3 - \frac{1}{3} = 0.729 - 0.333 = 0.396$</td>
<td>39.6</td>
</tr>
<tr>
<td>4</td>
<td>$p^4 - \frac{1}{4} = 0.6561 - 0.25 = 0.4061$</td>
<td>40.61</td>
</tr>
<tr>
<td>5</td>
<td>$p^5 - \frac{1}{5} = 0.59049 - 0.2 = 0.39049$</td>
<td>39.05</td>
</tr>
<tr>
<td>6</td>
<td>$p^6 - \frac{1}{6} = 0.531 - 0.167 = 0.364$</td>
<td>36.4</td>
</tr>
<tr>
<td>7</td>
<td>$p^7 - \frac{1}{7} = 0.478 - 0.143 = 0.335$</td>
<td>33.5</td>
</tr>
<tr>
<td>8</td>
<td>$p^8 - \frac{1}{8} = 0.430 - 0.125 = 0.305$</td>
<td>30.5</td>
</tr>
</tbody>
</table>

NOW WORK PROBLEM 35.


Concepts and Vocabulary

1. True or False If $E$ and $F$ are mutually exclusive, then they are also independent.

Skill Building

3. If $E$ and $F$ are independent events and if $P(E) = 0.4$ and $P(F) = 0.6$, find $P(E \cap F)$.

4. If $E$ and $F$ are independent events and if $P(E) = 0.6$ and $P(E \cap F) = 0.2$, find $P(F)$.

5. If $E$ and $F$ are independent events, find $P(F)$ if $P(E) = 0.2$ and $P(E \cup F) = 0.3$.

6. If $E$ and $F$ are independent events, find $P(E)$ if $P(F) = 0.3$ and $P(E \cup F) = 0.6$.

7. Suppose $E$ and $F$ are two events such that $P(E) = \frac{4}{21}$, $P(F) = \frac{7}{12}$, and $P(E \cap F) = \frac{2}{9}$. Are $E$ and $F$ independent?

8. If $E$ and $F$ are two events such that $P(E) = 0.25$, $P(F) = 0.36$, and $P(E \cap F) = 0.09$, are $E$ and $F$ independent?

9. If $E$ and $F$ are two independent events with $P(E) = 0.2$, and $P(F) = 0.4$, find

EXERCISE 8.2

10. If $E$ and $F$ are independent events with $P(E) = 0.3$ and $P(F) = 0.5$, find

11. If $E$, $F$, and $G$ are three independent events with $P(E) = \frac{2}{3}$, $P(F) = \frac{3}{7}$, and $P(G) = \frac{2}{21}$, find $P(E \cap F \cap G)$.

12. If $E_1$, $E_2$, $E_3$, and $E_4$ are four independent events with $P(E_1) = 0.6$, $P(E_2) = 0.3$, $P(E_3) = 0.5$, and $P(E_4) = 0.4$, find $P(E_1 \cap E_2 \cap E_3 \cap E_4)$.

13. If $P(E) = 0.3$, $P(F) = 0.2$, and $P(E \cup F) = 0.4$, what is $P(E\mid F)$? Are $E$ and $F$ independent?

14. If $P(E) = 0.4$, $P(F) = 0.6$, and $P(E \cup F) = 0.7$, what is $P(E\mid F)$? Are $E$ and $F$ independent?

Applications and Extensions

15. T-maze In a T-maze a mouse may turn to the right ($R$) and receive a mild shock, or to the left ($L$) and get a piece of cheese. Its behavior in making such “choices” is studied by psychologists. Suppose a mouse runs a T-maze 3 times. List the set of all possible outcomes and assign valid probabilities to each outcome under the assumption that the first two times the maze is run the mouse chooses equally between left and right, but on the third run, the mouse is twice as likely to choose cheese.
Assume the mouse has no memory so the trials are independent. Find the probability of each of the events listed.
(a) $E$: Run to the right exactly 2 consecutive times.
(b) $F$: Never run to the right.
(c) $G$: Run to the left on the first trial.
(d) $H$: Run to the right on the second trial.

16. Drawing Cards  A first card is drawn at random from a regular deck of 52 cards and is then put back into the deck. A second card is drawn. What is the probability that
(a) The first card is a club?
(b) The second card is a heart, given that the first is a club?
(c) The first card is a club and the second is a heart?
(d) The first card is an ace?
(e) The second card is a king, given that the first card is an ace?
(f) The first card is an ace and the second is a king?

17. Selecting Marbles  A box has 10 marbles in it, 6 red and 4 white. Suppose we draw a marble from the box, replace it, and then draw another. Find the probability that
(a) Both marbles are red.
(b) Just one of the two marbles is red.

18. Survey  In a survey of 100 people, categorized as drinkers or nondrinkers, with or without a liver ailment, the following data were obtained:

<table>
<thead>
<tr>
<th></th>
<th>Liver Ailment $F$</th>
<th>No Liver Ailment $\overline{F}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drinkers, $E$</td>
<td>52</td>
<td>18</td>
</tr>
<tr>
<td>Nondrinkers, $\overline{E}$</td>
<td>8</td>
<td>22</td>
</tr>
</tbody>
</table>

(a) Are the events $E$ and $F$ independent?
(b) Are the events $E$ and $\overline{F}$ independent?
(c) Are the events $E$ and $\overline{F}$ independent?

19. Cardiovascular Disease  Records show that a child of parents with heart disease has a probability of $\frac{3}{4}$ of inheriting the disease. Assuming independence, what is the probability that, for a couple with heart disease that have two children:
(a) Both children have heart disease.
(b) Neither child has heart disease.
(c) Exactly one child has heart disease.

20. Sex of Newborns  The probability of a newborn baby being a girl is 0.49. Assuming that the sex of one baby is independent of the sex of all other babies—that is, the events are independent—what is the probability that all four babies born in a certain hospital on one day are girls?

21. Germination  In a group of seeds, $\frac{1}{3}$ of which should produce violets, the best germination that can be obtained is 60%. If one seed is planted, what is the probability that it will grow into a violet? Assume independence.

22. Insurance  By examining the past driving records of 840 randomly selected drivers over a period of 1 year, the following data were obtained:

<table>
<thead>
<tr>
<th></th>
<th>Under 25 $U$</th>
<th>Over 25 $\overline{U}$</th>
<th>Totals $U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accident, $A$</td>
<td>40</td>
<td>5</td>
<td>45</td>
</tr>
<tr>
<td>No Accident, $\overline{A}$</td>
<td>285</td>
<td>510</td>
<td>795</td>
</tr>
<tr>
<td>Totals $U$</td>
<td>325</td>
<td>515</td>
<td>840</td>
</tr>
</tbody>
</table>

(a) What is the probability of a driver having an accident, given that the person is under 25?
(b) What is the probability of a driver having an accident, given that the person is over 25?
(c) Are events $U$ and $A$ independent?
(d) Are events $U$ and $\overline{A}$ independent?
(e) Are events $U$ and $A$ independent?
(f) Are events $U$ and $\overline{A}$ independent?

23. Quality Control  Efraim Furniture Manufacturing Company hires two people who work independently to identify defects in workmanship.
(a) If the probability an inspector misses a defective piece is 0.20, what is the probability both inspectors pass a defective piece of furniture?
(b) How many inspectors should be hired to ensure the probability of failing to identify a defective piece is less than 0.01?

24. AARP Life Insurance  Underwriters for AARP sell life insurance to anyone 55 years of age. They know that the probability a 55-year-old male will live to age 56 is 0.992225, and they make life insurance rates accordingly.
(a) What is the probability that 10 randomly chosen 55-year-old males all live to age 56?
(b) What is the probability that at least one of the ten 55-year-old males dies within the next year?

25. Stock Selection  As a stockbroker, you recommend two stocks to a client. Each stock has a 60% probability of increasing over the next year. Assuming that the performances of the two stocks are independent of each other, what is the probability that both stocks will increase in value over the next year? What is the probability that at least one stock will not increase in value?

26. Earnings and a Bachelor’s Degree  According to the U.S. Census Bureau, the probability that a randomly selected
individual in the United States earns more than $75,000 per year is 18.4%. The probability a randomly selected individual in the United States earns more than $75,000 given that the individual has earned a bachelor’s degree is 35%. Are the events “earn more than $75,000 per year” and “earned a bachelor’s degree” independent?

27. Health Insurance The contingency table representing the numbers, in thousands, of persons with and without health insurance coverage by age in the year 2007 is as follows:

<table>
<thead>
<tr>
<th>Age</th>
<th>Health Insurance</th>
<th>No Health Insurance</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;18</td>
<td>74,403</td>
<td>8,149</td>
</tr>
<tr>
<td>18–44</td>
<td>110,676</td>
<td>26,037</td>
</tr>
<tr>
<td>45–64</td>
<td>77,237</td>
<td>10,784</td>
</tr>
<tr>
<td>≥65</td>
<td>36,790</td>
<td>6,861</td>
</tr>
</tbody>
</table>

Source: Income, Poverty and Health Insurance Coverage in the United States, 2008, U.S. Census Bureau

(a) Determine $P(<18 \text{ years old})$ and $P(<18 \text{ years old} | \text{no health insurance})$.  
(b) Are the events “<18 years old” and “no health insurance” independent?

28. Women in Business The data below represent the number of women-owned businesses (in thousands) in the United States during 2002.

<table>
<thead>
<tr>
<th>Type of Business</th>
<th>With Paid Employees</th>
<th>Without Paid Employees</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construction</td>
<td>52</td>
<td>150</td>
<td>202</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>40</td>
<td>70</td>
<td>110</td>
</tr>
<tr>
<td>Trade</td>
<td>188</td>
<td>878</td>
<td>1066</td>
</tr>
<tr>
<td>Other</td>
<td>533</td>
<td>3694</td>
<td>4227</td>
</tr>
<tr>
<td>Total</td>
<td>813</td>
<td>4792</td>
<td>5605</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau

(a) What is the probability that a woman-owned business is in trade, given that the business has paid employees?  
(b) What is the probability that a woman-owned business has no paid employees, given that the business is in construction?  
(c) Are the events “Manufacturing” and “With Paid Employees” independent?

29. Pumping Station A pumping station at a hydroelectric plant operates two identical pumps. Each pump has a probability of failure of 0.05, and the probability that both pumps fail is 0.0025.

(a) Are failures in the two pumps mutually exclusive? Explain.  
(b) What is the probability that at least one of the pumps fails?  
(c) Are failures in the two pumps independent?

30. Ambulatory Care The following table represents the number of ambulatory care visits (in millions) to various locations in 2006 by gender.

<table>
<thead>
<tr>
<th>Gender</th>
<th>Physician Office</th>
<th>Outpatient Dept.</th>
<th>Emergency Room</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>368.5</td>
<td>40.2</td>
<td>54.2</td>
<td>462.9</td>
</tr>
<tr>
<td>Female</td>
<td>533.1</td>
<td>61.4</td>
<td>64.9</td>
<td>659.4</td>
</tr>
<tr>
<td>Total</td>
<td>901.6</td>
<td>102.1</td>
<td>119.1</td>
<td>1122.8</td>
</tr>
</tbody>
</table>

Source: U.S. National Center for Health Statistics

(a) What is the probability that an ambulatory visit will be to an emergency room, given that the patient is female?  
(b) What is the probability that the patient is male, given that the ambulatory visit is to a physician’s office?  
(c) Are the events “Male” and “Physician Office” independent?

31. Property Crime In a survey of 500 property crimes, the following data were obtained:

<table>
<thead>
<tr>
<th>Residence</th>
<th>Burglary</th>
<th>Vehicle Theft</th>
<th>Theft</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Urban</td>
<td>44</td>
<td>14</td>
<td>162</td>
<td>220</td>
</tr>
<tr>
<td>Suburban</td>
<td>30</td>
<td>10</td>
<td>106</td>
<td>146</td>
</tr>
<tr>
<td>Rural</td>
<td>26</td>
<td>4</td>
<td>104</td>
<td>134</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td>28</td>
<td>372</td>
<td>500</td>
</tr>
</tbody>
</table>

Source: U.S. National Center for Health Statistics

(a) What is the probability of being burglarized, given the residence is suburban?  
(b) What is the probability that the residence is rural, given that there has been a vehicle theft?  
(c) Are the events “Rural” and “Vehicle Theft” independent?  
(d) Are the events “Urban” and “Burglary” independent?

32. Voting Patterns The following data show the number of voters in a sample of 1000 from a large city, categorized by religion and their voting preference.

<table>
<thead>
<tr>
<th></th>
<th>Democrat</th>
<th>Republican</th>
<th>Independent</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Catholic, C</td>
<td>160</td>
<td>150</td>
<td>90</td>
<td>400</td>
</tr>
<tr>
<td>Protestant, P</td>
<td>220</td>
<td>220</td>
<td>60</td>
<td>500</td>
</tr>
<tr>
<td>Jewish, J</td>
<td>20</td>
<td>30</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>Totals</td>
<td>400</td>
<td>400</td>
<td>200</td>
<td>1000</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau

(a) Find the probability a person is a Democrat.  
(b) Find the probability a person is a Catholic.  
(c) Find the probability a person is Catholic, knowing the person is a Democrat.  
(d) Are the events $R$ and $D$ independent?  
(e) Are the events $P$ and $R$ independent?
Discussion and Writing

45. Give examples of two events that are
   (a) Independent but not mutually exclusive
   (b) Not independent but mutually exclusive
   (c) Not independent and not mutually exclusive
8.3 Bayes’ Theorem

PREPARING FOR THIS SECTION Before getting started, review the following:
• Sets (Section 6.1, pp. 356–363)

OBJECTIVES
1 Solve probability problems: sample space partitioned into two sets (p. 446)
2 Solve probability problems: sample space partitioned into three sets (p. 447)
3 Use Bayes’ Theorem to solve probability problems (p. 450)
4 Find a priori and a posteriori probabilities (p. 451)

NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 453.

In this section we consider experiments with sample spaces that are divided or partitioned into two (or more) mutually exclusive events. This study involves a further application of conditional probabilities and leads us to the famous Bayes’ Theorem, named after Thomas Bayes, who first published it in 1763.

EXAMPLE 1 Introduction to Bayes’ Theorem

Given two urns, suppose urn I contains four black and seven white balls. Urn II contains three black, one white, and four yellow balls. We select an urn at random and then draw a ball. What is the probability that a black ball is chosen?

SOLUTION Let \( U_I \) and \( U_{II} \) stand for the events “Urn I is selected” and “Urn II is selected,” respectively. Similarly, let \( B, W, Y \) stand for the event that “a black,” “a white,” or “a yellow” ball is chosen, respectively. Since the urn is selected at random and there are two of them, we have

\[
P(U_I) = \frac{1}{2} \quad P(U_{II}) = \frac{1}{2}
\]

If urn I is selected, the probability a black ball is chosen is \( \frac{4}{11} \) and the probability a white ball is chosen is \( \frac{7}{11} \).

If urn II is selected, the probability a black ball is chosen is \( \frac{3}{8} \), the probability a white ball is chosen is \( \frac{1}{8} \), and the probability a yellow ball is chosen is \( \frac{4}{8} \).

We seek the probability a black ball is chosen; that is, we seek \( P(B) \).

SOLUTION A Using a Tree Diagram

Construct the tree diagram shown in Figure 11.

\[
\begin{align*}
P(B \cap U_I) &= \frac{4}{11} \\
\text{Probability (Black)} &= \frac{2}{11} + \frac{2}{16} = 0.369
\end{align*}
\]
Then the probability a black ball is chosen is

\[ P(B) = \frac{2}{11} + \frac{3}{16} = \frac{65}{176} = 0.369 \]

**SOLUTION B** Using Conditional Probability

We know that

\[ P(U_I) = P(U_{II}) = \frac{1}{2} \]

\[ P(B|U_I) = \frac{4}{11}, \quad P(B|U_{II}) = \frac{3}{8} \]

The event \(B\) can be written as

\[ B = (B \cap U_I) \cup (B \cap U_{II}) \]

Since \(B \cap U_I\) and \(B \cap U_{II}\) are mutually exclusive, we add their probabilities. Then

\[ P(B) = P(B \cap U_I) + P(B \cap U_{II}) \quad (1) \]

Using the Product Rule, we have

\[ P(B \cap U_I) = P(U_I) \cdot P(B|U_I) \quad P(B \cap U_{II}) = P(U_{II}) \cdot P(B|U_{II}) \quad (2) \]

Combining (1) and (2), we have

\[ P(B) = P(U_I) \cdot P(B|U_I) + P(U_{II}) \cdot P(B|U_{II}) \]

\[ = \frac{1}{2} \cdot \frac{4}{11} + \frac{1}{2} \cdot \frac{3}{8} = \frac{2}{11} + \frac{3}{16} = 0.369 \]

\[ \square \]

**1 Solve Probability Problems: Sample Space Partitioned into Two Sets**

The preceding discussion leads to the following generalization.

Suppose \(A_1\) and \(A_2\) are two nonempty, mutually exclusive events of a sample space \(S\) and the union of \(A_1\) and \(A_2\) is \(S\); that is, suppose

\[ A_1 \neq \emptyset \quad A_2 \neq \emptyset \quad A_1 \cap A_2 = \emptyset \quad S = A_1 \cup A_2 \]

See Figure 12(a). The events \(A_1\) and \(A_2\) form a partition of \(S\).

Now if \(E\) is any event in \(S\), then \(E\) can be written in the form

\[ E = (E \cap A_1) \cup (E \cap A_2) \]

See Figure 12(b).

The sets \(E \cap A_1\) and \(E \cap A_2\) are disjoint since

\[ (E \cap A_1) \cap (E \cap A_2) = (E \cap E) \cap (A_1 \cap A_2) = E \cap \emptyset = \emptyset \]

Using the Product Rule, the probability of \(E\) is

\[ P(E) = P(E \cap A_1) + P(E \cap A_2) \]

\[ = P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) \quad (3) \]

Formula (3) is used to find the probability of an event \(E\) of a sample space when the sample space is partitioned into two sets \(A_1\) and \(A_2\).
8.3 Baye's Theorem

Figure 13 shows a tree diagram depicting Formula (3). You may find it easier to remember Formula (3) by constructing Figure 13.

**FIGURE 13**

\[ P(A_1) \] \[ \begin{align*} \text{A}_1 : & \ \text{Eligible} \\ \text{E} : & \ \text{pass admission test} \end{align*} \]

\[ P(A_2) \]

\[ \begin{align*} \text{A}_2 : & \ \text{Ineligible} \\ \text{E} : & \ \text{pass admission test} \end{align*} \]

\[ P(E) = P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) \]

**EXAMPLE 2 Admissions Tests for Medical School**

Of the applicants to a medical school, 80% are eligible to enter and 20% are not. To aid in the selection process, an admissions test is administered that is designed so that an eligible candidate will pass 90% of the time, while an ineligible candidate will pass only 30% of the time. What is the probability that an applicant will pass the admissions test?

**SOLUTION A**

Using a Tree Diagram

**FIGURE 14**

<table>
<thead>
<tr>
<th>Eligible</th>
<th>Pass</th>
<th>Fail</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>(0.8)(0.9) = 0.72</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>(0.2)(0.3) = 0.06</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Probability of passing = 0.72 + 0.06 = 0.78

**SOLUTION B**

Using Conditional Probability

The sample space \( S \) consists of the applicants for admission, and \( S \) can be partitioned into the following two events:

- \( A_1 \): Eligible applicant
- \( A_2 \): Ineligible applicant

These two events are mutually exclusive, and their union is \( S \). See Figure 15.

The event \( E \) is

- \( E \): Applicant passes admissions test

Now

\[
\begin{align*}
P(A_1) &= 0.8 \\ P(A_2) &= 0.2 \\ P(E|A_1) &= 0.9 \\ P(E|A_2) &= 0.3
\end{align*}
\]

Using Formula (3), we have

\[
P(E) = P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) = (0.8)(0.9) + (0.2)(0.3) = 0.78
\]

The probability that an applicant will pass the admissions test is 0.78.

NOW WORK PROBLEM 19.

2 Solve Probability Problems: Sample Space Partitioned into Three Sets

If a sample space \( S \) is partitioned into three sets \( A_1, A_2, \) and \( A_3 \) so that

\[
S = A_1 \cup A_2 \cup A_3 \\
A_1 \cap A_2 = \emptyset \\
A_2 \cap A_3 = \emptyset \\
A_1 \cap A_3 = \emptyset \\
A_1 \neq \emptyset \\
A_2 \neq \emptyset \\
A_3 \neq \emptyset
\]
then any event $E$ in $S$ can be written as

$$E = (E \cap A_1) \cup (E \cap A_2) \cup (E \cap A_3)$$

See Figure 16.

Since $E \cap A_1$, $E \cap A_2$, and $E \cap A_3$ are mutually exclusive events, the probability of event $E$ is

$$P(E) = P(E \cap A_1) + P(E \cap A_2) + P(E \cap A_3)$$

Formula (4) is used to find the probability of an event $E$ of a sample space when the sample space is partitioned into three sets $A_1$, $A_2$, and $A_3$.

See Figure 17 for a tree diagram depicting Formula (4). Again, you may find it easier to remember Formula (4) by constructing Figure 17.

### EXAMPLE 3 Quality Control

Three machines, I, II, and III, manufacture 0.4, 0.5, and 0.1 of the total production in a plant, respectively. The percentage of defective items produced by I, II, and III is 2%, 4%, and 1%, respectively. For an item chosen at random, what is the probability that it is defective?

#### SOLUTION Using Conditional Probability

The sample space $S$ is partitioned into three events $A_1$, $A_2$, and $A_3$ defined as follows:

- $A_1$: Item produced by machine I
- $A_2$: Item produced by machine II
- $A_3$: Item produced by machine III

The events $A_1$, $A_2$, and $A_3$ are mutually exclusive and their union is $S$. Define the event $E$ in $S$ to be

$$E: \text{Item is defective}$$
8.3 Bayes’ Theorem

Now

\[ P(A_1) = 0.4 \quad P(A_2) = 0.5 \quad P(A_3) = 0.1 \]
\[ P(E|A_1) = 0.02 \quad P(E|A_2) = 0.04 \quad P(E|A_3) = 0.01 \]

Using Formula (4), we find

\[
P(E) = (0.4)(0.02) + (0.5)(0.04) + (0.1)(0.01)
= 0.008 + 0.020 + 0.001 = 0.029
\]

NOW WORK PROBLEMS 11 AND 21.

To generalize Formulas (3) and (4) to a sample space \( S \) partitioned into \( n \) subsets, we require the following definition:

**Definition**

A sample space \( S \) is **partitioned** into \( n \) subsets \( A_1, A_2, \ldots, A_n \), provided:

- **(a)** Each subset is nonempty.
- **(b)** The intersection of any two of the subsets is empty.
- **(c)** \( A_1 \cup A_2 \cup \cdots \cup A_n = S \)

Let \( S \) be a sample space and let \( A_1, A_2, A_3, \ldots, A_n \) be \( n \) events that form a partition of the set \( S \). If \( E \) is any event in \( S \), then

\[ E = (E \cap A_1) \cup (E \cap A_2) \cup \cdots \cup (E \cap A_n) \]

Since \( E \cap A_1, E \cap A_2, \ldots, E \cap A_n \) are mutually exclusive events, we have

\[
P(E) = P(E \cap A_1) + P(E \cap A_2) + \cdots + P(E \cap A_n) \tag{5}
\]

In (5) replace \( P(E \cap A_1), P(E \cap A_2), \ldots, P(E \cap A_n) \) using the Product Rule. Then we obtain the formula

\[
P(E) = P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + \cdots + P(A_n) \cdot P(E|A_n) \tag{6}
\]

**EXAMPLE 4 Admissions Tests for Medical School**

Return to Example 2. Of the applicants to a medical school, 80% are eligible to enter and 20% are not. To aid in the selection process, an admissions test is administered that is designed so that an eligible candidate will pass 90% of the time, while an ineligible candidate will pass only 30% of the time. If an applicant passes the admissions test, what is the probability he or she is eligible?

**SOLUTION** We follow the solution to Example 2 using conditional probability. The sample space \( S \) is partitioned into the two events:

- \( A_1 \): Eligible applicant
- \( A_2 \): Ineligible applicant

The event \( E \) is

\[ E: \text{Applicant passes the admissions test} \]

The probability we seek is \( P(A_1|E) \).
Using the definition of conditional probability and the Product Rule,

\[ P(A_1|E) = \frac{P(A_1 \cap E)}{P(E)} = \frac{P(A_1) \cdot P(E|A_1)}{P(E)} \]  

Using the results obtained in Example 2, namely, \( P(A_1) = 0.8, P(E|A_1) = 0.9, \) and \( P(E) = 0.78, \) we have

\[ P(A_1|E) = \frac{P(A_1) \cdot P(E|A_1)}{P(E)} = \frac{(0.8)(0.9)}{0.78} = \frac{0.72}{0.78} = 0.923 \]

The admissions test is a reasonably effective device. If an applicant passes the admissions test, the probability is over 0.92 that the applicant is eligible for admission. In other words, less than 8% of the students passing the test are ineligible.

\[
\text{NOW WORK PROBLEM 23.}
\]

### 3 Use Bayes’ Theorem to Solve Probability Problems

Equation (7) is a special case of Bayes’ Theorem when the sample space is partitioned into two sets \( A_1 \) and \( A_2 \). The general formula is given below.

**Bayes’ Theorem**

Let \( S \) be a sample space partitioned into \( n \) events, \( A_1, \ldots, A_n \). Let \( E \) be any event of \( S \) for which \( P(E) > 0 \). The probability of the event \( A_j \) \((j = 1, 2, \ldots, n) \), given the event \( E \), is

\[
P(A_j|E) = \frac{P(A_j) \cdot P(E|A_j)}{P(E)} = \frac{P(A_j) \cdot P(E|A_j)}{P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + \cdots + P(A_n) \cdot P(E|A_n)}
\]

The proof is left as an exercise (see Problem 54).

### EXAMPLE 5 Quality Control: Source of Defective Cars

Motors, Inc. has three plants. Plant I produces 35% of the car output, plant II produces 20%, and plant III produces the remaining 45%. One percent of the output of plant I is defective, as is 1.8% of the output of plant II, and 2% of the output of plant III. The annual total output of Motors, Inc., is 1,000,000 cars. A car is chosen at random from the annual output and it is found to be defective.

(a) What is the probability the defective car came from plant I?
(b) What is the probability the defective car came from plant II?
(c) What is the probability the defective car came from plant III?
SOLUTION

Define the following events:

- **E**: Car is defective
- **A₁**: Car produced by plant I
- **A₂**: Car produced by plant II
- **A₃**: Car produced by plant III

The probabilities we seek are:

(a) \( P(A₁|E) \): the probability a car was produced by plant I, given that it was defective.
(b) \( P(A₂|E) \): the probability a car was produced by plant II, given that it was defective.
(c) \( P(A₃|E) \): the probability a car was produced by plant III, given that it was defective.

From information given in the problem we know the following probabilities:

\[
\begin{align*}
P(A₁) &= 0.35 & P(E|A₁) &= 0.010 \\
P(A₂) &= 0.20 & P(E|A₂) &= 0.018 \\
P(A₃) &= 0.45 & P(E|A₃) &= 0.020
\end{align*}
\]

Now use Bayes' Theorem to find \( P(A₁|E) \), \( P(A₂|E) \), and \( P(A₃|E) \). To apply Formula (8), begin by finding the denominator \( P(E) \).

\[
P(E) = P(A₁)P(E|A₁) + P(A₂)P(E|A₂) + P(A₃)P(E|A₃) = (0.35)(0.01) + (0.20)(0.018) + (0.45)(0.02) = 0.0161
\]

and

(a) \( P(A₁|E) = \frac{P(A₁)P(E|A₁)}{P(E)} = \frac{(0.35)(0.01)}{0.0161} = 0.217 \)

(b) \( P(A₂|E) = \frac{P(A₂)P(E|A₂)}{P(E)} = \frac{(0.2)(0.018)}{0.0161} = 0.224 \)

(c) \( P(A₃|E) = \frac{P(A₃)P(E|A₃)}{P(E)} = \frac{(0.45)(0.02)}{0.0161} = 0.559 \)

Given that a defective car is chosen, the probability it came from plant A₁ is 0.217, from plant A₂ is 0.224, and from plant A₃ is 0.559.

NOW WORK PROBLEM 29.

4 Find A Priori and A Posteriori Probabilities

In Bayes' Theorem the probabilities \( P(Aᵢ) \) are referred to as *a priori* probabilities, while the \( P(Aᵢ|E) \) are called *a posteriori* probabilities. We use Example 5 to explain the reason for this terminology. Knowing nothing else about a car, the probability that it was produced by plant I is given by \( P(Aᵢ) \), so \( P(Aᵢ) \) can be regarded as a “before the fact,” or *a priori*, probability. With the additional information that the car is defective, we reassess the likelihood of whether it came from plant I and compute \( P(Aᵢ|E) \). Then \( P(Aᵢ|E) \) can be viewed as an “after the fact,” or *a posteriori*, probability.

Note that \( P(Aᵢ) = 0.35 \), while \( P(Aᵢ|E) = 0.217 \). So the knowledge that a car is defective decreases the chance that it came from plant I.

**EXAMPLE 6** Testing for Cancer

The residents of a community are examined for cancer. The examination results are classified as positive (+) if a malignancy is suspected, and as negative (−) if there are no indications of a malignancy. If a person has cancer, the probability of a positive result
The manager of a car repair shop knows from past experience that when a call is received from a person whose car will not start, the probabilities for various troubles (assuming no two can occur simultaneously) are as given in Table 4.

<table>
<thead>
<tr>
<th>Event</th>
<th>Trouble</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>Flooded</td>
<td>0.3</td>
</tr>
<tr>
<td>$A_2$</td>
<td>Battery cable loose</td>
<td>0.2</td>
</tr>
<tr>
<td>$A_3$</td>
<td>Points bad</td>
<td>0.1</td>
</tr>
<tr>
<td>$A_4$</td>
<td>Out of gas</td>
<td>0.3</td>
</tr>
<tr>
<td>$A_5$</td>
<td>Something else</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The manager also knows that if the person will hold the gas pedal down and try to start the car, the probability that it will start ($E$) is

\[ P(E|A_1) = 0.9 \quad P(E|A_2) = 0 \quad P(E|A_3) = 0.2 \quad P(E|A_4) = 0 \quad P(E|A_5) = 0.2 \]

(a) If a person has called and is instructed to "hold the pedal down . . .," what is the probability that the car will start?

(b) If the car does start after holding the pedal down, what is the probability that the car was flooded?
SOLUTION  (a) We first find \( P(E) \). Using Formula (6) for \( n = 5 \) (the sample space is partitioned into five sets), we have

\[
P(E) = P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + P(A_3) \cdot P(E|A_3) + P(A_4) \cdot P(E|A_4) + P(A_5) \cdot P(E|A_5)
\]

\[
= (0.3)(0.9) + (0.2)(0) + (0.1)(0.2) + (0.3)(0) + (0.1)(0.2)
\]

\[
= 0.27 + 0.02 + 0.02 = 0.31
\]

See Figure 19 for the tree diagram.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig19}
\caption{Tree diagram for Example 8.3.}
\end{figure}

(b) We seek the probability \( P(A_1|E) \). To find the \textit{a posteriori} probability \( P(A_1|E) \), use Bayes’ Theorem, Formula (8).

\[
P(A_1|E) = \frac{P(A_1) \cdot P(E|A_1)}{P(E)} = \frac{(0.3)(0.9)}{0.31} = 0.87
\]

The probability that the car was flooded, after it is known that holding down the pedal started the car, is 0.87.

\[\boxed{\text{NOW WORK PROBLEM 35.}}\]

**EXERCISE 8.3 Answers Begin on Page AN–41.**

\begin{itemize}
\item **Are You Prepared?** Problem  Answer is given at the end of these exercises. If you get a wrong answer, read the pages listed in red.
\item 1. If \( A \) and \( B \) are two disjoint sets, then \( A \cap B = \) \text{__________}. (p. 356–363)
\item **Concepts and Vocabulary**
\item 2. If \( A_1 \) and \( A_2 \) are two nonempty, mutually exclusive events in a sample space \( S \) for which \( A_1 \cup A_2 = S \), then \( A_1 \) and \( A_2 \) form a \text{__________} of \( S \).
\item 3. True or False  If \( A_1 \) and \( A_2 \) form a partition of a sample space \( S \) and if \( E \) is an event in \( S \), then \( P(E) = P(A_1 \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) \).
\item 4. True or False  If \( A_1 \) and \( A_2 \) form a partition of a sample space \( S \) and if \( E \) is an event in \( S \), then \( P(A_1|E) = \frac{P(A_2 \cdot P(E|A_2)}{P(E)} \).
\end{itemize}

**Skill Building**

In Problems 5–18, find the indicated probabilities by referring to the tree diagram below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig19}
\caption{Tree diagram for Exercise 8.3.}
\end{figure}

\begin{itemize}
\item 5. \( P(E|A) \)
\item 6. \( P(\overline{E}|A) \)
\item 7. \( P(E|B) \)
\item 8. \( P(\overline{E}|B) \)
\item 9. \( P(E|C) \)
\item 10. \( P(\overline{E}|C) \)
\item 11. \( P(E) \)
\item 12. \( P(\overline{E}) \)
\item 13. \( P(A|E) \)
\item 14. \( P(B|\overline{E}) \)
\item 15. \( P(C|E) \)
\item 16. \( P(A|\overline{E}) \)
\item 17. \( P(B|E) \)
\item 18. \( P(C|\overline{E}) \)
\item 19. Events \( A_1 \) and \( A_2 \) form a partition of a sample space \( S \) with \( P(A_1) = 0.4 \) and \( P(A_2) = 0.6 \). If \( E \) is an event in \( S \) with \( P(E|A_1) = 0.03 \) and \( P(E|A_2) = 0.02 \), compute \( P(E) \).
\end{itemize}
20. Events $A_1$ and $A_2$ form a partition of a sample space $S$ with $P(A_1) = 0.3$ and $P(A_2) = 0.7$. If $E$ is an event in $S$ with $P(E|A_1) = 0.04$ and $P(E|A_2) = 0.01$, compute $P(E)$. 

21. Events $A_1, A_2,$ and $A_3$ form a partition of a sample space $S$ with $P(A_1) = 0.6$, $P(A_2) = 0.2$, and $P(A_3) = 0.2$. If $E$ is an event in $S$ with $P(E|A_1) = 0.01$, $P(E|A_2) = 0.03$, and $P(E|A_3) = 0.02$, compute $P(E)$. 

22. Events $A_1, A_2,$ and $A_3$ form a partition of a sample space $S$ with $P(A_1) = 0.3$, $P(A_2) = 0.2$, and $P(A_3) = 0.5$. If $E$ is an event in $S$ with $P(E|A_1) = 0.01$, $P(E|A_2) = 0.02$, and $P(E|A_3) = 0.02$, compute $P(E)$. 

23. Use the information in Problem 19 to find $P(A_1|E)$ and $P(A_2|E)$. 

24. Use the information in Problem 20 to find $P(A_1|E)$ and $P(A_2|E)$. 

25. Use the information in Problem 21 to find $P(A_1|E)$, $P(A_2|E)$, and $P(A_3|E)$. 

26. Use the information in Problem 22 to find $P(A_1|E)$, $P(A_2|E)$, and $P(A_3|E)$. 

Applications and Extensions

27. Three jars contain colored balls as follows:

<table>
<thead>
<tr>
<th>Jar</th>
<th>Red, R</th>
<th>White, W</th>
<th>Blue, B</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>5</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>II</td>
<td>3</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>III</td>
<td>7</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

One jar is chosen at random and a ball is withdrawn. 
(a) What is the probability the ball is red? 
(b) What is the probability the ball is white? 
(c) What is the probability the ball is blue? 
(d) If the ball is red, what is the probability it came from Jar I? 
(e) If the ball is blue, what is the probability it came from Jar II? 
(f) If the ball is white, what is the probability it came from Jar III?

28. Car Production Cars are being produced by two factories, but factory I produces twice as many cars as factory II in a given time. Factory I is known to produce 2% defectives and factory II produces 1% defectives. A car is examined and found to be defective. 
(a) What is the probability the car is defective if it came from factory I? 
(b) What is the probability the car is defective if it came from factory II? 
(c) What is the probability the car came from factory I if it is defective? 
(c) What is the probability the car came from factory II if it is defective?

29. Color Blindness According to the 2000 U.S. Census, 50.9% of the U.S. population is female. NBC news reported that 1 out of 12 males, but only 1 out of 250 females is color-blind. Given that a person randomly chosen from the U.S. population is color-blind, what is the probability that the person is a male? 
Source: U.S. Census Bureau.

30. Hospital Bills A hospital billing department knows that the probability patients 60 or older pay the balance of their bill after one billing is 80%, while for a person under the age of 60 the probability is 45%. Seventy percent of the hospital's patients are 60 years or older. 
(a) What is the probability the balance is paid after one billing? 
(b) The balance of the bill is not paid after one billing. What is the probability the patient was 60 years or older?

31. Advertising Castaway Cruise lines mails a promotional advertisement to 10,000 potential customers, 6500 males and 3500 females. Of the mailing, 80% of the males and 40% of the females have cruised on Castaway before. A customer who has received the promotion calls and books a cruise. 
(a) What is the probability the customer has never sailed with Castaway? 
(b) If the customer has never cruised with Castaway, what is the probability he is male?

32. Market Research From past experience Kave Jewelers know that 80% of their customers in the month before Christmas are male. Market research shows that 75% of adult males and 30% of adult females watch professional football. So Kave buys advertising spots during Sunday football games. An adult walks into a Kave Jewelers on December 15. 
(a) What is the probability the person has seen an ad? 
(b) If a customer who shops at Kave has seen an ad, what is the probability the customer is female? 
(c) If a customer has not seen an ad, what is the probability the customer is male?

33. On-Time Performance You are meeting a friend at Fort Myers Airport. According to the Bureau of Transportation Statistics, in 2009 flights into Fort Myers were on time 81.3% of the time, but Southwest had a better record and is on time 84.5% of the time. You cannot remember what airline she is taking but you know that in the past she has flown on Southwest 70% of the time. The plane is late. What is the probability she is taking Southwest?
Source: Bureau of Transportation Statistics (www.bts.gov)

34. Air Travel Liz flies regularly from Midway Airport in Chicago to Hartsfield-Jackson Airport in Atlanta using AirTran Airways 75% of the time and Delta Airlines 25% of the time. In 2009
AirTran’s on-time percentage for this flight was 73.3%. Delta’s on-time percentage for this flight was 71.3%. If a randomly selected flight arrives on time, what is the probability Liz is on a Delta Airlines flight?

Source: Bureau of Transportation Statistics

35. Customer Surveys Juan Morales is a restauranteur who owns three authentic Mexican restaurants in the towns of Riverside, Springfield, and Centerville. To receive customer feedback, Juan provides postage-paid survey cards at each establishment which can be completed by patrons and mailed directly to him. By keeping records on the returned surveys, Juan has learned that 40% are from patrons of the Riverside restaurant, 25% are from patrons of the Springfield restaurant, and 35% are from patrons of the Centerville restaurant. Juan has also learned that the percent of surveys returned with negative feedback are 8%, 12%, and 9% for the three restaurants, respectively.

(a) If Juan receives a survey in the mail, what is the probability that it will give negative feedback?
(b) If Juan receives a survey with negative feedback, what is the probability it is from a patron of the Riverside restaurant?
(c) If Juan receives a survey with negative feedback, what is the probability it is from a patron of the Springfield restaurant?

36. Female Unemployment Based on information from the Bureau of Labor Statistics, of women in the civilian labor force, 51.8% are married, 28.2% have never been married, 12.8% are divorced, 4.1% are separated, and 3.1% are widowed. Of the married women, 3.6% are unemployed; of the never-married women, 8.5% are unemployed; of the divorced women, 5.2% are unemployed; of the separated women, 8.3% are unemployed; of the widowed women, 5.7% are unemployed.

(a) What is the probability a randomly selected woman from the civilian labor force is unemployed?
(b) If an unemployed woman is selected, what is the probability she is married?
(c) If an unemployed woman is selected, what is the probability she never married?
(d) If an unemployed woman is selected, what is the probability she is divorced?
(e) If an unemployed woman is selected, what is the probability she is separated?
(f) If an unemployed woman is selected, what is the probability she is widowed?


37. Individual Tax Audits In 2008, the audit risk for the average individual filing a federal income tax return was about 1 in 100. For individuals with an adjusted gross income of $1 million or higher, the audit risk was about 1 in 16. About 1 in 250 individuals filing federal income tax returns have adjusted gross incomes of $1 million or higher. If an individual income tax return has been chosen to be audited, find the probability that the individual’s adjusted gross income is $1 million or higher.

Source: Internal Revenue Service

38. Promotional Coupon An office supply store desires to increase sales of its store brand toner cartridges over its sales of less profitable name brand cartridges. To do so, it mass mails a sales flyer to its entire database of current customers. Each flyer contains a personalized coupon for $30 off the purchase of any store brand toner cartridge. Of its current customers, 24% have previously purchased store brand cartridges. From past experience, store management knows that about 46% of customers who have made a previous purchase of an item will redeem a coupon for that item. Management also knows that about 15% of personalized coupons are redeemed.

(a) If a redeemed coupon is selected at random, what is the probability that it was used by a customer who had previously purchased a store brand cartridge?
(b) If a customer who has not previously purchased a store brand toner cartridge is randomly selected, what is the probability that he or she will redeem the personalized promotional coupon?

39. Occupations Based on data from the Bureau of Labor Statistics, 36.3% of employed persons in the United States have occupations classified as management, professional, and related; 16.8% have occupations classified as service; 24.5% have occupations classified as sales and office; 10.2% have occupations classified as natural resources, construction, and maintenance; 12.2% have occupations classified as production, transportation, and material moving. Women hold 50.8% of the management, professional, and related occupations, 57.2% of the service occupations, 63.2% of the sales and office occupations, 4.2% of the natural resources, construction, and maintenance occupations, 22.4% of the production, transportation, and material moving occupations.

(a) What percentage of employed persons in the United States are women?
(b) If an employed woman is selected, what is the probability her occupation is classified as management, professional, and related?
(c) If an employed woman is selected, what is the probability her occupation is classified as service?
(d) If an employed woman is selected, what is the probability her occupation is classified as sales and office?

Source: Bureau of Labor Statistics, Employed persons by detailed occupation, sex, race, and Hispanic or Latino ethnicity, 2008
40. Medical Diagnosis In a certain small town, 16% of the population developed lung cancer. If 45% of the population are smokers, and 85% of those developing lung cancer are smokers, what is the probability that a smoker in this population will develop lung cancer?

41. Voting Pattern In Cook County, 55% of the registered voters are Democrats, 30% are Republicans, and 15% are Independents. During a recent election, 35% of the Democrats voted, 65% of the Republicans voted, and 75% of the Independents voted.
   (a) What is the probability that someone who voted is a Democrat?
   (b) What is the probability someone who voted is a Republican?
   (c) What is the probability someone who voted is Independent?

42. Quality Control A computer manufacturer has three assembly plants. Records show that 2% of the computers shipped from plant A turn out to be defective, as compared to 3% of those that come from plant B and 4% of those that come from plant C. In all, 30% of the manufacturer’s total production comes from plant A, 50% from plant B, and 20% from plant C. If a customer finds that his computer is defective, what is the probability it came from plant A?

43. Oil Drilling An oil well is to be drilled in a certain location. The soil there is either rock (probability 0.53), clay (probability 0.21), or sand. If it is rock, a geological test gives a positive result with 35% accuracy; if it is clay, this test gives a positive result with 48% accuracy; and if it is sand, the test gives a positive result with 75% accuracy. The test is positive.
   (a) What is the probability that the soil is rock?
   (b) What is the probability that the soil is clay?
   (c) What is the probability that the soil is sand?

44. Oil Drilling A geologist is using seismographs to test for oil. It is found that if oil is present, the test gives a positive result 95% of the time, and if oil is not present, the test gives a positive result 2% of the time. Oil is actually present in 1% of the cases tested. If the test shows positive, what is the probability that oil is present?

45. Political Polls In conducting a political poll, a pollster divides the United States into four regions: Northeast (N), containing 25% of the population; Midwest (M), containing 20% of the population; South (S), containing 30% of the population; and West (W), containing 25% of the population. From the poll it is found that in the next election 40% of the people in the Northeast say they will vote for Republicans, in the South 56% will vote Republican, in the Midwest 48% will vote Republican, and in the West 52% will vote Republican.
   (a) What is the probability that a person chosen at random will vote Republican?
   (b) Assuming a person votes Republican, what is the probability he or she is from the Northeast?

46. TB Screening Suppose that if a person with tuberculosis is given a TB screening, the probability that his or her condition will be detected is 0.90. If a person without tuberculosis is given a TB screening, the probability that he or she will be diagnosed incorrectly as having tuberculosis is 0.3. Suppose, further, that 11% of the adult residents of a certain city have tuberculosis. If one of these adults is diagnosed as having tuberculosis based on the screening, what is the probability that he or she actually has tuberculosis? Interpret your result.

47. Detective Columbo An absent-minded nurse is to give Mr. Brown a pill each day. The probability that the nurse forgets to administer the pill is $\frac{2}{3}$. If he receives the pill, the probability that Mr. Brown will die is $\frac{1}{3}$. If he does not get his pill, the probability that he will die is $\frac{3}{4}$. Mr. Brown died. What is the probability that the nurse forgot to give Mr. Brown the pill?

48. Marketing To introduce a new beer, a company conducted a survey. It divided the United States into four regions: eastern, northern, southern, and western. The company estimates that 35% of the potential customers for the beer are in the eastern region, 30% are in the northern region, 20% are in the southern region, and 15% are in the western region. The survey indicates that 50% of the potential customers in the eastern region, 40% of the potential customers in the northern region, 36% of the potential customers in the southern region, and 42% of those in the western region will buy the beer. If a potential customer chosen at random indicates that he or she will buy the beer, what is the probability that the customer is from the southern region?

49. Car Insurance and Age Insurers know that young drivers between the ages of 16 and 24 have 2.6 times more accidents than older drivers and that 13% of all drivers are between the ages of 16 and 24. In 2007, the probability an adult was involved in an accident was 0.024 and resulted in 5,811,000 reported crashes.
   (a) What is the probability a young driver was involved in an accident?
   (b) Twenty percent of a company’s insureds are young drivers. A claim is made. What is the probability the driver was 25 or older?

Source: www.nhtsa.gov, www.iii.org
50. **Medical Test**  A scientist designed a medical test for a certain disease. Among 100 patients who have the disease, the test will show the presence of the disease in 97 cases out of 100, and will fail to show the presence of the disease in the remaining 3 cases out of 100. Among those who do not have the disease, the test will erroneously show the presence of the disease in 4 cases out of 100, and will show that there is no disease in the remaining 96 cases out of 100.

(a) What is the probability that a patient who tested positive on this test actually has the disease, if it is estimated that 20% of the population has the disease?

(b) What is the probability that a patient who tested positive on this test actually has the disease, if it is estimated that 4% of the population has the disease?

(c) What is the probability that a patient who took the test twice and tested positive both times actually has the disease, if it is estimated that 4% of the population has the disease?

51. **Testing for HIV**  An article in the *New York Times* some time ago reported that college students are beginning to routinely ask to be tested for the AIDS virus. The standard test for the HIV virus is the Elias test, which tests for the presence of HIV antibodies. It is estimated that this test has a 99.8% sensitivity and a 99.8% specificity. A 99.8% sensitivity means that, in a large-scale screening test, for every 1000 people tested who have the virus we can expect 998 people to test positive and 2 to have a false negative test. A 99.8% specificity means that, in a large-scale screening test, for every 1000 people tested who do not have the virus we can expect 998 people to have a negative test and 2 to have a false positive test.

(a) The *New York Times* article remarks that it is estimated that about 2 in every 1000 college students have the HIV virus. Assume that a large group of randomly chosen college students, say 100,000, are tested by the Elias test. If a student tests positive, what is the chance that this student has the HIV virus?

(b) What would this probability be for a population at high risk, where 5% of the population has the HIV virus?

(c) Suppose Jack tested positive on an Elias test. Another Elias test* is performed and the results are positive again. Assuming that the tests are independent, what is the probability that Jack has the HIV virus?

52. **Test for Tuberculosis**  A faster and simpler diagnostic test for the presence of tuberculosis bacilli, called “FASTPlaqueTB,” was evaluated in Cape Town, South Africa, and found to have the following reliability. For those subjects known to have the tuberculosis bacilli, the FASTPlaqueTB test was positive 70% of the time. For those subjects known not to have the tuberculosis bacilli, the FASTPlaqueTB test was negative 99% of the time. Statistics indicate that about 14.2% of the people in Cape Town have the tuberculosis bacilli.

(a) What is the probability that a randomly selected person from Cape Town will have a positive FASTPlaqueTB test result?

(b) What is the probability that a person who tests positive actually has tuberculosis bacilli?

53. Show that $P(E|F) = 1$ if $F$ is a subset of $E$ and $P(F) \neq 0$.

54. Prove Bayes’ Theorem, Formula (8).

---

**Are You Prepared?** Answer

1. Ø

8.4 Permutations

**PREPARING FOR THIS SECTION** Before getting started, review the following:

- The Multiplication Principle (Section 7.3, pp. 373–376)

**OBJECTIVES**

1. Evaluate factorials (p. 458)
2. Solve counting problems involving permutations [distinct, with repetition] (p. 459)
3. Solve counting problems involving permutations [distinct, without repetition] (p. 460)

In the next two sections we use the Multiplication Principle to discuss two general types of counting problems, called permutations and combinations. These concepts arise often in applications, especially in probability. Before discussing permutations, we introduce a useful shorthand notation—the factorial symbol.
Definition

Factorial

The symbol \( n! \), read as “\( n \) factorial,” is defined as

\[
\begin{align*}
0! &= 1 \\
1! &= 1 \\
2! &= 2 \cdot 1 = 2 \\
3! &= 3 \cdot 2 \cdot 1 = 6 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24
\end{align*}
\]

and, in general, for \( n \geq 1 \) an integer,

\[
 n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \quad (1)
\]

Evaluate Factorials

To compute \( n! \), find the product of all consecutive integers from \( n \) down to 1, inclusive, or from 1 up to \( n \), inclusive. Remember that, by definition, \( 0! = 1 \).

Look at Equation (1). It follows that

\[
 n! = n(n - 1)!
\]

Formula (2) can be useful for evaluating expressions containing factorials.

EXAMPLE 1 Evaluating Expressions Containing Factorials

(a) \( 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \)

(b) \( \frac{5!}{4!} = \frac{5 \cdot 4!}{4!} = 5 \)

(c) \( \frac{52!}{47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 47!} = 2,598,960 \)

(d) \( \frac{7!}{(7 - 5)!} = \frac{7!}{2!} = \frac{7 \cdot 6 \cdot 5!}{2 \cdot 1} = \frac{7 \cdot 6}{2} = 21 \)

(e) \( \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{50!} = \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45!} = \frac{1}{45!} \)

NOW WORK PROBLEM 7.

Factorials grow very quickly. Compare the following:

\[
\begin{align*}
5! &= 120 \\
10! &= 3,628,800 \\
15! &= 1,307,674,368,000
\end{align*}
\]

In fact, if your calculator has a factorial key, you will find that \( 69! = 1.71 \cdot 10^{98} \), while \( 70! \) produces an error message—indicating you have exceeded the range of the calculator.
Because of this, it is important to “cancel out” factorials whenever possible so that “out of range” errors may be avoided. For example, to calculate \( \frac{100!}{95!} \), we write

\[
\frac{100!}{95!} = \frac{100 \cdot 99 \cdot 98 \cdot 96 \cdot 95!}{95!} = 9.03 \cdot 10^9
\]

**Permutations**

**Definition**

A **permutation** is an ordered arrangement of \( r \) objects chosen from \( n \) objects.

We discuss three types of permutations:

1. The \( n \) objects are distinct (different), and repetition is allowed in the selection of \( r \) of them. [Distinct, with repetition]
2. The \( n \) objects are distinct (different), and repetition is not allowed in the selection of \( r \) of them, where \( r \leq n \). [Distinct, without repetition]
3. The \( n \) objects are not distinct, and we use all of them in the arrangement. [Not distinct]

We take up the first two types here and deal with the third type in the next section.

**Solve Counting Problems Involving Permutations [Distinct, with Repetition]**

**EXAMPLE 2** **Counting Airport Codes [Permutation: Distinct, with Repetition]**

The International Airline Transportation Association (IATA) assigns three-letter codes to represent airport locations. For example, the airport code for Ft. Lauderdale, Florida, is FLL. Notice that repetition is allowed in forming this code. How many airport codes are possible?

**SOLUTION** We are choosing 3 letters from 26 letters and arranging them in order. In the ordered arrangement a letter may be repeated. This is an example of a permutation with repetition in which 3 objects are chosen from 26 distinct objects.

The task of counting the number of such arrangements consists of making three selections. Each selection requires choosing a letter of the alphabet (26 choices). By the Multiplication Principle, there are

\[
26 \cdot 26 \cdot 26 = 17,576
\]

different airport codes.

The solution given to Example 2 can be generalized.

**Theorem** **Permutations: Distinct Objects, with Repetition**

The number of ordered arrangements of \( r \) objects chosen from \( n \) objects, in which the \( n \) objects are distinct and repetition is allowed, is \( n^r \).
Solve Counting Problems Involving Permutations  
[Distinct, without Repetition]

**EXAMPLE 3** Forming Codes [Permutation: Distinct, without Repetition]

Suppose that we wish to establish a three-letter code using any of the 26 uppercase letters of the alphabet, but we require that no letter be used more than once. How many different three-letter codes are there?

**SOLUTION** Some of the possibilities are: ABC, ABD, ABZ, ACB, CBA, and so on. The task consists of making three selections. The first selection requires choosing from 26 letters. Because no letter can be used more than once, the second selection requires choosing from 25 letters. The third selection requires choosing from 24 letters. (Do you see why?) By the Multiplication Principle, there are

\[ 26 \cdot 25 \cdot 24 = 15,600 \]

different three-letter codes with no letter repeated.

For this second type of permutation, we introduce the following notation:

The notation \( P(n, r) \) represents the number of ordered arrangements of \( r \) objects chosen from \( n \) distinct objects, where \( r \leq n \) and repetition is not allowed.

For example, the question posed in Example 3 asks for the number of ways that the 26 letters of the alphabet can be arranged in order using three nonrepeated letters. The answer is

\[ P(26, 3) = 26 \cdot 25 \cdot 24 = 15,600 \]

**EXAMPLE 4** Lining Up People

In how many ways can 5 people be lined up?

**SOLUTION** The 5 people are distinct. Once a person is in line, that person will not be repeated elsewhere in the line; and, in lining up people, order is important. We have a permutation of 5 objects taken 5 at a time. We can line up 5 people in

\[ P(5, 5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \text{ ways} \]

**NOW WORK PROBLEM 33.**

To arrive at a formula for \( P(n, r) \), we note that the task of obtaining an ordered arrangement of \( n \) objects in which only \( r \leq n \) of them are used, without repeating any of them, requires making \( r \) selections. For the first selection, there are \( n \) choices; for the second selection, there are \( n - 1 \) choices; for the third selection, there are \( n - 2 \) choices; \ldots; for the \( r \)th selection, there are \( n - (r - 1) \) choices. By the Multiplication Principle, we have

\[
P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot [n - (r - 1)] \\
= n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot (n - r + 1)
\]
This formula for \( P(n, r) \) can be compactly written using factorial notation.

\[
P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)
\]

\[
= n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1) \cdot \frac{(n - r) \cdots 3 \cdot 2 \cdot 1}{(n - r) \cdots 3 \cdot 2 \cdot 1} = \frac{n!}{(n - r)!}
\]

**Theorem**  
Number of Permutations of \( r \) Objects Chosen from \( n \) Distinct Objects without Repetition

The number of arrangements of \( n \) objects using \( r \leq n \) of them, in which

1. the \( n \) objects are distinct,
2. once an object is used it cannot be repeated, and
3. order is important,

is given by the formula

\[
P(n, r) = \frac{n!}{(n - r)!}
\]

**EXAMPLE 5** Computing Permutations

Evaluate:

(a) \( P(7, 3) \)  
(b) \( P(6, 1) \)  
(c) \( P(52, 5) \)

**SOLUTION**  
We shall work parts (a) and (b) in two ways.

(a) \[
P(7, 3) = 7 \cdot 6 \cdot 5 = 210
\]

3 factors

or

\[
P(7, 3) = \frac{7!}{(7 - 3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 210
\]

(b) \[
P(6, 1) = 6
\]

1 factor

or

\[
P(6, 1) = \frac{6!}{(6 - 1)!} = \frac{6!}{5!} = \frac{6 \cdot 5!}{5!} = 6
\]

(c) Figure 20 shows the solution using a TI-84 Plus graphing calculator:

\[
P(52, 5) = 311,875,200
\]

**NOW WORK PROBLEM 17.**

Here is a short list of problems with their solutions given in \( P(n, r) \) notation. Notice in each problem that the objects are distinct, no object is repeated, and order is important.
462 Chapter 8 Additional Probability Topics

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find the number of ways of choosing five people from a group of 10 and arranging them in a line.</td>
<td>(P(10, 5))</td>
</tr>
<tr>
<td>Find the number of six-letter ”words” that can be formed with no letter repeated.</td>
<td>(P(26, 6))</td>
</tr>
<tr>
<td>Find the number of seven-digit telephone numbers, with no repeated digit (allow 0 for a first digit).</td>
<td>(P(10, 7))</td>
</tr>
<tr>
<td>Find the number of ways of arranging eight people in a line.</td>
<td>(P(8, 8))</td>
</tr>
</tbody>
</table>

NOW WORK PROBLEM 25.

EXAMPLE 6 The Birthday Problem

All we know about Shannon, Patrick, and Ryan is that they have different birthdays. If we listed all the possible ways this could occur, how many would there be? Assume that there are 365 days in a year.

SOLUTION This is an example of a permutation in which 3 birthdays are selected from a possible 365 days, and no birthday may repeat itself. The number of ways that this can occur is

\[
P(365, 3) = \frac{365!}{(365 - 3)!} = \frac{365 \cdot 364 \cdot 363 \cdot 362!}{362!} = 365 \cdot 364 \cdot 363 = 48,228,180
\]

There are 48,228,180 ways in a group of three people that each can have a different birthday.

NOW WORK PROBLEM 49.

EXAMPLE 7 Answering Test Questions

A student has six questions on an examination and is allowed to answer the questions in any order. In how many different orders could the student answer these questions?

SOLUTION We seek the number of ordered arrangements of the six questions using all six of them. The number is given by

\[
P(6, 6) = \frac{6!}{(6 - 6)!} = \frac{6!}{0!} = \frac{6!}{1} = 720
\]

Example 7 leads us to formulate the next result.

Theorem

The number of permutations (arrangements) of \(n\) distinct objects using all \(n\) of them is given by

\[P(n, n) = n!\]
For example, in a class of \( n \) students, there are \( n! \) ways of positioning all the students in a line.

**Example 8: Arranging Books on a Shelf**

You own eight mathematics books and six computer science books and wish to fill seven positions on a shelf. If the first four positions are to be occupied by math books and the last three by computer science books, in how many ways can this be done?

**Solution** We think of the problem as consisting of two tasks. Task 1 is to fill the first four positions with four of the eight mathematics books. This can be done in \( P(8, 4) \) ways. Task 2 is to fill the remaining three positions with three of six computer books. This can be done in \( P(6, 3) \) ways. By the Multiplication Principle, the seven positions can be filled in

\[
P(8, 4) \cdot P(6, 3) = \frac{8!}{4!} \cdot \frac{6!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 6 \cdot 5 \cdot 4 = 201,600
\]

different ways.

NOW WORK PROBLEM 39.

**Exercise 8.4 Answers Begin on Page AN–41.**

‘Are You Prepared?’ Problems Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. How many different 4-digit numbers can be formed from the set \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) if no digit is to be repeated? (p. 373–376)

2. If a coin is tossed 5 times, how many different sequences of heads and tails are possible? (p. 373–376)

**Concepts and Vocabulary**

3. \( 0! = \) ; \( 3! = \)

4. \( P(n, r) = \)

5. True or False The number of ordered arrangements of \( r \) objects chosen from \( n \) objects, in which the \( n \) objects are distinct and repetition is allowed, is \( r^n \).

6. True or False \( P(n, n) = n \)

**Skill Building**

In Problems 7–24, evaluate each expression.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>( \frac{5!}{2!} )</td>
<td>8</td>
<td>( \frac{8!}{2!} )</td>
<td>9</td>
<td>( \frac{10!}{8!} )</td>
<td>10</td>
<td>( \frac{11!}{9!} )</td>
</tr>
<tr>
<td>11</td>
<td>( \frac{9!}{8!} )</td>
<td>12</td>
<td>( \frac{10!}{9!} )</td>
<td>13</td>
<td>( \frac{8!}{2!6!} )</td>
<td>14</td>
<td>( \frac{9!}{3!6!} )</td>
</tr>
<tr>
<td>15</td>
<td>( P(7, 2) )</td>
<td>16</td>
<td>( P(8, 1) )</td>
<td>17</td>
<td>( P(8, 7) )</td>
<td>18</td>
<td>( P(6, 6) )</td>
</tr>
<tr>
<td>19</td>
<td>( P(6, 0) )</td>
<td>20</td>
<td>( P(6, 4) )</td>
<td>21</td>
<td>( \frac{8!}{(8 - 3)!3!} )</td>
<td>22</td>
<td>( \frac{9!}{(9 - 5)!5!} )</td>
</tr>
<tr>
<td>23</td>
<td>( \frac{6!}{(6 - 6)!6!} )</td>
<td>24</td>
<td>( \frac{7!}{(0 - 0)!7!} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Applications

25. List all the ordered arrangements of 5 objects \(a, b, c, d,\) and \(e,\) choosing 3 at a time without repetition. What is \(P(5, 3)?\)

27. List all the ordered arrangements of 4 objects 1, 2, 3, and 4, choosing 3 at a time without repetition. What is \(P(4, 3)?\)

29. **Forming Codes** How many two-letter codes can be formed using the letters A, B, C, and D? Repeated letters are allowed.

30. **Forming Codes** How many two-letter codes can be formed using the letters A, B, C, D, and E? Repeated letters are allowed.

31. **Forming Numbers** How many three-digit numbers can be formed using the digits 0 and 1? Repeated digits are allowed.

32. **Forming Numbers** How many three-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9? Repeated digits are allowed.

33. **Lining Up People** In how many ways can 4 people be lined up?

34. **Stacking Boxes** In how many ways can 5 different boxes be stacked?

35. **Forming Codes** How many different three-letter codes are there if only the letters A, B, C, D, and E can be used and no letter can be used more than once?

36. **Forming Codes** How many different four-letter codes are there if only the letters A, B, C, D, E, and F can be used and no letter can be used more than once?

37. **Seating Arrangements** How many ways are there to seat 5 people in 8 chairs?

38. **Seating Arrangements** How many ways are there to seat 4 people in a 6-passenger automobile?

39. **Stocks on the NYSE** Companies whose stocks are listed on the New York Stock Exchange (NYSE) have their company name represented by either 1, 2, or 3 letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the NYSE?

40. **Stocks on the NASDAQ** Companies whose stocks are listed on the NASDAQ stock exchange have their company name represented by either 4 or 5 letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the NASDAQ?

41. **New Printers** An accounting firm is upgrading the network printers in each of its three branch offices and will select from among 7 different models, though each office may have different printers. How many ways can the firm purchase three printers for the offices?

42. **New Copiers** The college of arts and sciences at a university decides to replace the copiers in each of its 6 departments. If the dean can select from 4 different models and each department can have a different type of copier, how many ways can the dean purchase the new copiers?

43. **Television Schedule** A television network has four 30-minute time slots for comedies and has 14 shows to choose from for the fall lineup. How many different comedy lineups are possible?

44. **Television Schedule** A television network has eight 60-minute time slots for dramas and has 15 shows to choose from for the fall lineup. How many different drama lineups are possible?

45. **Promotion Schedule** A radio station has six promotional time slots available during a weekend and 11 businesses wishing to have the station broadcast from their location. How many different promotional schedules are possible?

46. **Comic Strips** A newspaper must select 9 daily comic strips from a list of 17 final candidates. If the strips are the same size and shape and position on the page is important, how many different layouts are possible for the comics page?

47. **Accounting Audits** A chief accounting officer must conduct audits at each of the company’s 21 facilities located in different states. If she can only visit 8 facilities each month, how many different travel itineraries are possible the first month?

48. **Call Letters** In the United States, all new radio stations west of the Mississippi River use four call letters, but the call letters must begin with "K." How many different sets of call letters are possible west of the Mississippi River if repeated letters are allowed? How many are allowed if repeated letters are not allowed?

49. **Birthday Problem** In how many ways can 2 people each have different birthdays? Assume that there are 365 days in a year.

50. **Birthday Problem** In how many ways can 5 people each have different birthdays? Assume that there are 365 days in a year.

51. **Arranging Letters**
   
   (a) How many different ways are there to arrange the 6 letters in the word SUNDAY?

   (b) If we insist that the letter S come first, how many ways are there?

   (c) If we insist that the letter S come first and the letter Y be last, how many ways are there?
52. Arranging Books There are 5 different French books and 5 different Spanish books. How many ways are there to arrange them on a shelf if
(a) Books of the same language must be grouped together, French on the left, Spanish on the right?
(b) French and Spanish books must alternate in the grouping, beginning with a French book?

53. Distributing Books In how many ways can 8 different books be distributed to 12 children if no child gets more than one book?

54. Networks A computer must assign each of 4 outputs to one of 8 different printers. In how many ways can it do this provided no printer gets more than one output?

55. Lottery Tickets From 1500 lottery tickets that are sold, 3 tickets are to be selected for first, second, and third prizes. How many possible outcomes are there?

56. Personnel Assignment A salesperson is needed in each of 7 different sales territories. If 10 equally qualified persons apply for the jobs, in how many ways can the jobs be filled?

57. Choosing Officers A club has 15 members. In how many ways can 4 officers consisting of a president, vice-president, secretary, and treasurer be chosen?

58. Psychology Testing In an ESP experiment a person is asked to select and arrange 3 cards from a set of 6 cards labeled A, B, C, D, E, and F. Without seeing the cards, a second person is asked to give the arrangement. Determine the number of possible responses by the second person if he simply guesses.

Discussion and Writing

59. Create a problem different from any found in the text that requires a permutation to solve. Give it to a friend to solve and critique.

‘Are You Prepared?’ Answers

1. \(10 \cdot 9 \cdot 8 \cdot 7 = 5040\)
2. \(2^5 = 32\)

8.5 Combinations

OBJECTIVES

1. Solve counting problems involving combinations (p. 466)
2. Solve counting problems involving permutations \([n \text{ objects, not all distinct}]\) (p. 469)
3. Find probabilities using counting techniques (p. 471)

Permutations focus on the order in which objects are arranged. However, in many cases, order is not important. For example, in a draw poker hand, the order in which you receive the cards is not relevant—all that matters is what cards are received. That is, with poker hands, we are concerned only with what combination of cards we have—not the particular order of the cards.

The following examples illustrate the distinction between selections in which order is important and those for which order is not important.

EXAMPLE 1 Arranging Letters

From the four letters \(a, b, c, d\), choose two without repeating any letter

(a) If order is important
(b) If order is not important

SOLUTION

(a) If order is important, there are \(P(4, 2) = 4 \cdot 3 = 12\) possible selections, namely,

\[ab \quad ac \quad ad \quad ba \quad bc \quad bd \quad ca \quad cb \quad cd \quad da \quad db \quad dc\]

See Figure 21.

(b) If order is not important, only 6 of the 12 selections found in part (a) are listed, namely,

\[ab \quad ac \quad ad \quad bc \quad bd \quad cd\]
Notice that the number of ordered selections, 12, is 2! = 2 times the number of unordered selections, 6. The reason is that each unordered selection consists of two letters that allow for 2! rearrangements. For example, the selection \(ab\) in the unordered list gives rise to \(ab\) and \(ba\) in the ordered list.

**EXAMPLE 2** Arranging Letters

From the four letters \(a, b, c, d\), choose three without repeating any letter

(a) If order is important

(b) If order is not important

**SOLUTION**

(a) If order is important, there are possible selections, namely,

\[
abc \quad abd \quad acb \quad acd \quad adb \quad adbc \quad bac \quad bad \quad bca \quad bcd \quad bda \quad bdc \\
cab \quad cad \quad cba \quad cbd \quad cda \quad cdb \quad dab \quad dac \quad dba \quad dbc \quad dca \quad dcb
\]

See Figure 22.

(b) If order is not important, only 4 of the 24 selections found in part (a) are listed, namely,

\(abc \quad abd \quad acd \quad bcd\)

Notice that the number of ordered selections, 24, is 3! = 6 times the number of unordered selections, 4. The reason is that each unordered selection consists of three letters that allow for 3! rearrangements. For example, the selection \(abc\) in the unordered list gives rise to \(abc, acb, bac, bca, cab, cba\) in the ordered list.

Unordered selections are called **combinations**.

1. Solve Counting Problems Involving Combinations

**Definition**

A **combination** is an arrangement, without regard to order, of \(r\) objects selected from \(n\) distinct objects without repetition, where \(r \leq n\). The notation \(C(n, r)\) represents the number of combinations of \(n\) distinct objects using \(r\) of them.

\(C(n, r)\) is also referred to as the number of **combinations of \(n\) objects taken \(r\) at a time**.

To obtain a formula for \(C(n, r)\), we first observe that each unordered selection of \(r\) objects will give rise to \(r!\) ordered selections. That is, the number of ordered selections, \(P(n, r)\), is \(r!\) times the number of unordered selections, \(C(n, r)\):

\[
P(n, r) = r!C(n, r)
\]

If we solve this equation for \(C(n, r)\), we find

\[
C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n - r)!r!}
\]

We have proved the following result:
Theorem 8.5

**Number of Combinations of \( n \) Distinct Objects Taken \( r \) at a Time**

The number of arrangements of \( n \) objects using \( r \leq n \) of them, in which

1. the \( n \) objects are distinct,
2. once an object is used, it cannot be repeated, and
3. order is not important,

is given by the formula

\[
C(n, r) = \frac{n!}{(n - r)!r!}
\]  

(1)

**EXAMPLE 3 Evaluating \( C(n, r) \)**

Use Formula (1) to find the value of each expression.

(a) \( C(3, 1) \)  
(b) \( C(6, 3) \)  
(c) \( C(n, n) \)  
(d) \( C(n, 0) \)  
(e) \( C(52, 5) \)

**SOLUTION**

(a) \( C(3, 1) = \frac{3!}{(3 - 1)!1!} = \frac{3!}{2!1!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3 \)

(b) \( C(6, 3) = \frac{6!}{(6 - 3)!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3} = 20 \)

(c) \( C(n, n) = \frac{n!}{(n - n)!n!} = \frac{n!}{0!n!} = \frac{1}{1} = 1 \)

(d) \( C(n, 0) = \frac{n!}{(n - 0)!0!} = \frac{n!}{n!0!} = \frac{1}{1} = 1 \)

(e) Figure 23 shows the solution using a TI-84 Plus graphing calculator:

\[ C(52, 5) = 2,598,960 \]

NOW WORK PROBLEM 5.

**EXAMPLE 4 Forming Committees**

From 5 faculty members, a committee of 2 is to be formed. In how many ways can this be done?

**SOLUTION**

The formation of committees is an example of a combination. The 5 faculty members are all different. The members of the committee are distinct. Order is not important. (On committees it is membership, not the order of selection, that is important.) For the situation described, we can form

\[
C(n, r) = C(5, 2) = \frac{5!}{3!2!} = 10 \quad n = 5, r = 2
\]

different committees.
Here are some other problems that are examples of combinations. The solutions are given in $C(n, r)$ notation.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find the number of ways of selecting 4 people from a group of 6</td>
<td>$C(6, 4)$</td>
</tr>
<tr>
<td>Find the number of committees of 6 that can be formed from the U.S. Senate (100 members)</td>
<td>$C(100, 6)$</td>
</tr>
<tr>
<td>Find the number of ways of selecting 5 courses from a catalog containing 200</td>
<td>$C(200, 5)$</td>
</tr>
</tbody>
</table>

**Example 5** Playing Cards

From a deck of 52 cards a hand of 5 cards is dealt. How many different hands are possible?

**Solution**

Such a hand is an unordered selection of 5 cards from a deck of 52. So the number of different hands is

$$C(52, 5) = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 47!} = 2,598,960$$

**Example 6** Six-Bit Strings

A bit is a 0 or a 1. A six-bit string is a sequence of length six consisting of 0’s and 1’s. How many six-bit strings contain

**(a)** Exactly one 1? 

**(b)** Exactly two 1’s?

**Solution**

**(a)** To form a six-bit string having one 1, we only need to specify where the single 1 is located (the other positions are 0s). The location for the 1 can be chosen in

$$C(6, 1) = 6 \text{ ways}$$

**(b)** Here, we must choose two of the six positions to contain 1’s. There are

$$C(6, 2) = 15 \text{ strings}$$

with exactly two 1’s and four 0’s.

**NOW WORK PROBLEM 23.**
EXAMPLE 7 Forming Committees

In how many ways can a committee consisting of 2 faculty members and 3 students be formed if 6 faculty members and 10 students are eligible to serve on the committee?

**SOLUTION**

The problem can be separated into two parts: the number of ways that the faculty members can be chosen, \( C(6, 2) \), and the number of ways that the student members can be chosen, \( C(10, 3) \). By the Multiplication Principle, the committee can be formed in different ways.

\[
C(6, 2) \cdot C(10, 3) = \frac{6!}{4!2!} \cdot \frac{10!}{7!3!} = \frac{6 \cdot 5 \cdot 4!}{4!2!} \cdot \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!3!} = \frac{30 \cdot 720}{2 \cdot 6} = 1800
\]

NOW WORK PROBLEM 29.

EXAMPLE 8 Forming Committees

From 6 women and 4 men a committee of 3 is to be formed. The committee must include at least 2 women. In how many ways can this be done?

**SOLUTION**

A committee of 3 that includes at least 2 women will contain either exactly 2 women and 1 man, or exactly 3 women and 0 men. Since no committee can contain exactly 2 women and simultaneously exactly 3 women, once we have counted the number of ways a committee with exactly 2 women and the number of ways a committee with exactly 3 women can be formed, their sum will give the number of ways exactly 2 women or exactly 3 women can be on the committee. [Refer to the Counting Formula (1) on page 367, noting that the sets are disjoint.]

Following the solution to Example 7, a committee of exactly 2 women and 1 man can be formed in \( C(6, 2) \cdot C(4, 1) \) ways, while a committee of exactly 3 women, 0 men can be formed in \( C(6, 3) \cdot C(4, 0) \) ways. A committee of 3 consisting of at least 2 women can be formed in

\[
C(6, 2) \cdot C(4, 1) + C(6, 3) \cdot C(4, 0) = \frac{6!}{4!2!} \cdot \frac{4!}{3!1!} + \frac{6!}{3!3!} \cdot \frac{4!}{4!0!} = 15 \cdot 4 + 20 \cdot 1 = 60 + 20 = 80
\]

**2 Solve Counting Problems Involving Permutations \([n \text{ objects, not all distinct}]\)**

Our previous discussion of permutations required that the objects we were arranging be distinct. We now examine what happens when some of the objects are the same.

EXAMPLE 9 Forming Different Words

How many different three-letter words (real or imaginary) can be formed from the letters in the word

(a) MAD?  
(b) DAD?

**SOLUTION**

(a) The three distinct letters in MAD can be rearranged in

\[
P(3, 3) = 3! = 6
\]

They are MAD, MDA, AMD, ADM, DAM, DMA.
(b) Straightforward listing shows that there are only three ways of rearranging the letters in the word DAD:

DAD, DDA, and ADD

The word DAD in Example 9 contains 2 Ds, and it is this duplication that results in fewer rearrangements for DAD than for MAD. In the next example we describe a way of dealing with the problem of duplication.

**EXAMPLE 10 Forming Different Words**

How many different “words” can be formed using all the letters of the six-letter word M A M M A L?

**SOLUTION** Any such word will have 6 letters formed from 3 Ms, 2 As, and 1 L. To form a word, think of six blank positions that will have to be filled by the above letters.

\[ T \quad T \quad T \quad T \quad T \quad T \]

We separate the construction of a word into three tasks.

**Task 1** Choose the positions for the 3 M's.

**Task 2** Choose the positions for the 2 A's.

**Task 3** Choose the position for the L.

Doing this sequence of tasks will result in a word and, conversely, every rearrangement of MAMMAL will be obtained from this sequence of tasks.

Task 1 can be done in \( C(6, 3) \) ways. There are now three positions left for the 2 A's, so Task 2 can be done in \( C(3, 2) \) ways. Five blanks have been filled, so that the L must go in the remaining blank. That is, Task 3 can be done in \( C(1, 1) \) way. The Multiplication Principle says that the number of rearrangements is

\[
C(6, 3) \cdot C(3, 2) \cdot C(1, 1) = \frac{6!}{3!3!} \cdot \frac{3!}{1!2!} \cdot \frac{1!}{0!1!} = \frac{6!}{3!2!1!}
\]

The form of the answer in Example 10 is suggestive of a general result. Had the letters in MAMMAL been distinct, there would have been \( P(6, 6) = 6! \) rearrangements possible. This is the numerator of the answer. The presence of 3 M's, 2 A's, and 1 L reduces the number of different words as shown in the denominator above. This reasoning can be used to derive the following general result.

**Theorem Permutation Involving n Objects That Are Not Distinct**

The number of permutations of \( n \) objects, of which \( n_1 \) are of one kind, \( n_2 \) are of a second kind, \( \ldots \), and \( n_k \) are of a \( k \)th kind, is given by

\[
\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}
\]

(2)

where \( n_1 + n_2 + \cdots + n_k = n \).
EXAMPLE 11 \textbf{Arranging Flags}

How many different vertical arrangements are possible for 10 flags if 2 are white, 3 are red, and 5 are blue?

**Solution**
Here we want the different arrangements of 10 objects, which are not all distinct. Using Formula (2), we have

\[
\frac{10!}{2!3!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{2 \cdot 3 \cdot 2 \cdot 5!} = 2520
\]

different arrangements.

\[\text{NOW WORK PROBLEM 27.}\]

### 8.5 \textbf{Combinations}

3 \textbf{Find Probabilities Using Counting Techniques}

EXAMPLE 12 \textbf{Finding Probabilities Using Counting Techniques}

From a box containing four white, three yellow, and one green ball, two balls are selected one at a time without replacing the first before the second is selected. Find the probability that one white and one yellow ball are selected.

**Solution**

The experiment consists of selecting 2 balls from 8 balls: 4 white, 3 yellow, 1 green. The number of ways that 2 balls can be selected from 8 balls without replacement is \(C(8, 2) = 28\). The number of elements in the sample \(S\) is \(n(S) = 28\).

Define the events \(E\) and \(F\) as

\[E: \text{A white ball is selected}\]
\[F: \text{A yellow ball is selected}\]

We seek the probability of \(E\) and \(F\); that is, \(P(E \cap F)\).

The number of ways a white ball can be selected is \(C(4, 1) = 4\) and the number of ways a yellow ball can be selected is \(C(3, 1) = 3\). By the Multiplication Principle, the number of ways a white and yellow ball can be selected is

\[C(4, 1) \cdot C(3, 1) = 4 \cdot 3 = 12\]

That is, \(n(E \cap F) = 12\). Since the outcomes are equally likely, the probability of selecting a white ball and a yellow ball is

\[P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{C(4, 1) \cdot C(3, 1)}{C(8, 2)} = \frac{12}{28} = \frac{3}{7}\]

Compare the solution given here to Example 12 with the solution given to Example 6 on page 427. Which solution do you prefer?
EXAMPLE 13 Finding Probabilities Using Counting Techniques

A box contains 12 lightbulbs, of which 5 are defective. All bulbs look alike and have equal probability of being chosen. Three lightbulbs are selected and placed in a box.

(a) What is the probability that all 3 are defective?

(b) What is the probability that exactly 2 are defective?

(c) What is the probability that at least 2 are defective?

SOLUTION

The number of elements in the sample space $S$ is equal to the number of combinations of 12 lightbulbs taken 3 at a time, namely,

$$C(12, 3) = \frac{12!}{3! 9!} = 220$$

So, $n(S) = 220$.

(a) Define $E$ as the event, “3 bulbs are defective.” Then $E$ can occur in $C(5, 3)$ ways, that is, the number of ways in which 3 defective bulbs can be chosen from 5 defective ones. So, $n(E) = C(5, 3)$. The probability $P(E)$ is

$$P(E) = \frac{n(E)}{n(S)} = \frac{C(5, 3)}{C(12, 3)} = \frac{\frac{5!}{3! 2!}}{220} = \frac{10}{220} = 0.045$$

(b) Define $F$ as the event, “2 bulbs are defective.” To obtain 2 defective bulbs when 3 are chosen requires that we select 2 defective bulbs from the 5 available defective ones and 1 good bulb from the 7 good ones. By the Multiplication Principle, this can be done in the following number of ways:

$$n(F) = C(5, 2) \cdot C(7, 1) = \frac{5!}{2! 3!} \cdot \frac{7!}{1! 6!} = 10 \cdot 7 = 70$$

Number of ways to select 2 defectives from 5 defectives

Number of ways to select 1 good bulb from 7 good ones

The probability of selecting exactly 2 defective bulbs is therefore

$$P(F) = P(\text{exactly 2 defectives}) = \frac{n(F)}{n(S)} = \frac{C(5, 2) \cdot C(7, 1)}{C(12, 3)} = \frac{70}{220} = 0.318$$

(c) Define $G$ as the event, “At least 2 are defective.” The event $G$ is equivalent to asking for the probability of selecting either exactly 2 or exactly 3 defective bulbs. Since these events are mutually exclusive, the sum of their probabilities will give the probability of $G$. Using the results found in parts (a) and (b), we find

$$P(G) = P(\text{Exactly 2 defectives}) + P(\text{Exactly 3 defectives})$$

$$= P(F) + P(E) = 0.318 + 0.045 = 0.363$$

NOW WORK PROBLEM 49.
EXAMPEL 14 Dogs of the Dow

“Dogs of the Dow” is a stock-picking strategy devoted to selecting the highest dividend-yielding stocks of the 30 blue chip stocks that make up the Dow Jones Industrial Average. The table below shows the top 10 stocks listed on the “Dogs of the Dow” website on December 31, 2009.

<table>
<thead>
<tr>
<th>Company</th>
<th>Yield on 12/31/2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>5.85</td>
</tr>
<tr>
<td>Verizon</td>
<td>5.73</td>
</tr>
<tr>
<td>DuPont</td>
<td>4.87</td>
</tr>
<tr>
<td>Kraft</td>
<td>4.27</td>
</tr>
<tr>
<td>Merck</td>
<td>4.16</td>
</tr>
<tr>
<td>Chevron</td>
<td>3.53</td>
</tr>
<tr>
<td>McDonald’s</td>
<td>3.52</td>
</tr>
<tr>
<td>Pfizer</td>
<td>3.52</td>
</tr>
<tr>
<td>Home Depot</td>
<td>3.11</td>
</tr>
<tr>
<td>Boeing</td>
<td>3.10</td>
</tr>
</tbody>
</table>

Source: www.dogsofthedow.com

(a) If two of these stocks are randomly selected, what is the probability both have yields above 4.25%?

(b) If three of these stocks are randomly selected, what is the probability at least one of them will have a yield above 4.25%?

SOLUTION

(a) We are selecting 2 of the 10 stocks, so the number of elements in the sample space \(S\) is equal to the number of combinations of 10 stocks taken 2 at a time, namely,

\[
C(10, 2) = \frac{10!}{2! \cdot 8!} = 45
\]

So, \(n(S) = 45\).

Define \(E\) as the event “both stocks have yields above 4.25%.” Of the stocks listed, 4 have yields above 4.25%, so \(E\) can occur in \(C(4, 2) = \frac{4!}{2! \cdot 2!} = 6\) ways. So, \(n(E) = 6\).

The probability \(P(E)\) is

\[
P(E) = \frac{n(E)}{n(S)} = \frac{C(4, 2)}{C(10, 2)} = \frac{6}{45} = 0.133
\]

(b) This time, 3 of the 10 stocks are being selected, so the number of elements in the sample space \(S\) is equal to the number of combinations of 10 stocks taken 3 at a time, namely,

\[
C(10, 3) = \frac{10!}{3! \cdot 7!} = 120
\]

So, \(n(S) = 120\).

Define \(F\) as the event “at least one of the three stocks has a yield above 4.25%.” Event \(F\) is equivalent to “exactly one with yield above 4.25%” or “exactly two with
yields above 4.25%” or “exactly three with yields above 4.25%.” Since these are mutually exclusive, we have

\[ P(F) = P(\text{exactly 1 above 4.25%}) + P(\text{exactly 2 above 4.25%}) + P(\text{exactly 3 above 4.25%}) \]

Selecting exactly one stock with a yield above 4.25% means we simultaneously select exactly two stocks with yields below 4.25%. Since there are 4 stocks in the list with yields above 4.25% and 6 stocks with yields below 4.25%, we can do this in \( \binom{4}{1} \cdot \binom{6}{2} = 60 \) ways. Likewise, we can select exactly two stocks with yields above 4.25% in \( \binom{4}{2} \cdot \binom{6}{1} = 36 \) ways, and we can select exactly three stocks with yields above 4.25% in \( \binom{4}{3} \cdot \binom{6}{0} = 4 \) ways. So, the desired probability is

\[
P(F) = \frac{\binom{4}{1} \cdot \binom{6}{2}}{\binom{10}{3}} + \frac{\binom{4}{2} \cdot \binom{6}{1}}{\binom{10}{3}} + \frac{\binom{4}{3} \cdot \binom{6}{0}}{\binom{10}{3}}
\]

\[
= \frac{60}{120} + \frac{36}{120} + \frac{4}{120} = \frac{100}{120} = 0.833
\]

**NOTE** We could have solved Example 14(b) alternatively by using the complement of \( F \). Then \( \overline{F} \) is the event “none of the three stocks have yield above 4.25%.” The event \( \overline{F} \) can occur in \( \binom{4}{0} \cdot \binom{6}{3} = 20 \) ways, and

\[
P(\overline{F}) = \frac{\binom{4}{0} \cdot \binom{6}{3}}{\binom{10}{3}} = \frac{20}{120} = 0.167
\]

So,

\[ P(F) = 1 - P(\overline{F}) = 1 - 0.167 = 0.833 \]

**NOW WORK PROBLEM 41.**

**EXAMPLE 15 Quality Control**

A laboratory contains 10 electron microscopes, of which 2 are defective. If all microscopes are equally likely to be chosen and if 4 are chosen, what is the expected number of defective microscopes?

**SOLUTION** The sample of 4 microscopes can contain 0, 1, or 2 defective microscopes. The probability \( p_0 \) that none in the sample is defective is

\[
p_0 = \frac{\binom{2}{0} \cdot \binom{8}{4}}{\binom{10}{4}} = \frac{1}{3}
\]

Similarly, the probabilities \( p_1 \) and \( p_2 \) for 1 and 2 defective microscopes are

\[
p_1 = \frac{\binom{2}{1} \cdot \binom{8}{3}}{\binom{10}{4}} = \frac{8}{15} \quad \text{and} \quad p_2 = \frac{\binom{2}{2} \cdot \binom{8}{2}}{\binom{10}{4}} = \frac{2}{15}
\]

Since we are interested in determining the expected number of defective microscopes, we assign a payoff of 0 to the outcome “0 defectives are selected,” a payoff of 1 to the outcome “1 defective is chosen,” and a payoff of 2 to the outcome “2 defectives are chosen.” The expected value \( E \) is then

\[
E = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 = \frac{8}{15} + \frac{4}{15} = \frac{4}{5}
\]

Of course, we cannot have \( \frac{4}{5} \) of a defective microscope. However, we can interpret this to mean that in the long run, such a sample will average just under 1 defective microscope. ■
Observe that \(\frac{4}{5}\) is a reasonable answer for the expected number of defective microscopes since \(\frac{1}{5}\) of the microscopes in the laboratory are defective and a random sample consisting of 4 of these microscopes is selected.

**The Pascal Triangle**

Sometimes the notation \(\binom{n}{r}\), read as "from \(n\) choose \(r\)," is used in place of the combination notation \(C(n, r)\).

Suppose that we arrange the values of \(\binom{n}{r}\) in a triangular display, as shown next and in Figure 24.

This display is called the **Pascal triangle**, named after Blaise Pascal (1623–1662), a French mathematician.

The Pascal triangle has 1’s down the sides. To get any other entry, add the two nearest entries in the row above it. The shaded triangles in Figure 24 illustrate this feature of the Pascal triangle. Based on this feature, the row corresponding to \(n = 6\) is found as follows:

\[
\begin{align*}
\binom{5}{0} &= 1 \\
\binom{5}{1} &= 5 \\
\binom{5}{2} &= 10 \\
\binom{5}{3} &= 10 \\
\binom{5}{4} &= 5 \\
\binom{5}{5} &= 1
\end{align*}
\]

Although the Pascal triangle provides an interesting and organized display of the values of \(\binom{n}{r}\), in practice it is not all that helpful. For example, if you wanted to know the value of \(\binom{12}{5}\), you would need to produce 13 rows of the triangle before seeing the answer. It is much faster to use the fact that

\[
\binom{n}{r} = C(n, r) = \frac{n!}{(n - r)!r!}
\]
The numbers \( C(n, r) = \binom{n}{r} \) are also called \textbf{binomial coefficients}. To learn why, read ‘The Binomial Theorem’ in Section A.5 of Appendix A.

**EXERCISE 8.5**

Answers Begin on Page AN−41.

### Concepts and Vocabulary

1. A \_\_\_\_\_ is an arrangement, without regard to order, of \( r \) objects chosen from \( n \) distinct objects without repetition, where \( r \leq n \).

2. The number of combinations of 5 objects taken 2 at a time is \( C(5, 2) = \_\_\_\_\_\_\_\_ \).

3. \textit{True or False} If \( r \leq n \), then \( P(n, r) = n!C(n, r) \).

4. \textit{True or False} Some permutations involve \( n \) objects that are not distinct.

### Skill Building

In Problems 5–12, find the value of each expression.

- **5.** \( C(6, 4) \)
- **6.** \( C(5, 4) \)
- **7.** \( C(7, 2) \)
- **8.** \( C(8, 7) \)
- **9.** \( C(5, 1) \)
- **10.** \( C(8, 1) \)
- **11.** \( C(8, 6) \)
- **12.** \( C(8, 4) \)

- **26.** \textit{Forming Words} How many different 11-letter words (real or imaginary) can be formed from the letters in the word \textsc{MATHEMATICS}?

- **27.** \textit{Arranging Lights} How many different ways can 3 red, 4 yellow, and 5 blue bulbs be arranged in a string of Christmas tree lights with 12 sockets?

- **28.** \textit{Arranging Trees} In how many ways can 3 apple trees, 4 peach trees, and 2 plum trees be arranged along a fence line if one does not distinguish between trees of the same kind?

- **29.** \textit{Forming a Committee} A student dance committee is to be formed consisting of 2 boys and 3 girls. If the membership is to be chosen from 4 boys and 8 girls, how many different committees are possible?

- **30.** \textit{Forming a Committee} The student relations committee of a college consists of 2 administrators, 3 faculty members, and 5 students. Four administrators, 8 faculty members, and 20 students are eligible to serve. How many different committees are possible?

- **31.** \textit{Forming Teams} In how many ways can 12 children be placed on 3 distinct teams of 3, 5, and 4 members?

- **32.** \textit{Forming Committees} A group of 9 people is going to be formed into committees of 4, 3, and 2 people. How many committees can be formed if
  - (a) A person can serve on any number of committees?
  - (b) No person can serve on more than one committee?

- **33.** \textit{Powerball} Powerball is a lottery played in 29 states, Washington, D.C., and the U.S. Virgin Islands. A ticket consists of five distinct numbers selected from 1 to 55 and a sixth “Powerball” number selected from 1 to 42. The order of the first five numbers is unimportant. The Powerball can be the same as one of the first five numbers because it is drawn from
a separate number pool from the first five numbers. How many distinct Powerball tickets are possible?

**Source:** www.powerball.com

### 34. Mega Millions

Mega Millions is a lottery played in 12 states. A ticket consists of five distinct numbers between 1 and 56 and one “Mega Ball” number between 1 and 46. The order of the first five numbers is unimportant. The Mega Ball number can be the same as one of the first five numbers because it is drawn from a separate number pool from the first five numbers. How many distinct Mega Millions tickets are possible?

**Source:** www.megamillions.com

### 35. Contract Negotiations

A college has 90 full-time faculty members. Of these, 25 are in the Math and Science Division, 23 are in the Allied Health Division, 15 are in the Liberal Arts Division, 8 are in the Business Division, and 19 are in the Computer and Technology Division. For contract negotiations, a 7-member team will be formed by selecting one member from each division, plus 2 “at large” members who may be selected from any of the remaining faculty. How many negotiating teams are possible?

### 36. Company Drawing

A company employs 80 people. At the annual company picnic, the company randomly draws the names of 5 employees for prizes. The prize list consists of the following: two fully paid trips for two to Jamaica, three $1500 cash prizes. The remaining 75 employees get $100 each. Assuming no employee receives more than one prize, how many different ways can the prizes be distributed among the employees?

### 37. Auditing

To audit the 58 accounts payable of a small firm, the auditor randomly selects a sample of 8 account balances for review. How many samples are possible?

### 38. Auditing

Refer to Problem 37. Suppose that 5 of the 58 accounts contain errors.

(a) How many samples (of the 8 account balances) are possible that contain exactly 1 error?

(b) Exactly 2 errors?

(c) Exactly 3 errors?

(d) Exactly 4 errors?

(e) Exactly 5 errors?

(f) How many samples are possible that contain at least one error?

(g) How many samples are possible that contain no errors?

### 39. Forming Committees

The U.S. Senate has 100 members. Suppose it is desired to place each senator on exactly 1 of 7 possible committees. The first committee has 22 members, the second has 13, the third has 10, the fourth has 5, the fifth has 16, and the sixth and seventh have 17 apiece. In how many ways can these committees be formed?

### 40. Hockey

There are 15 teams in the Eastern Conference of the National Hockey League for the 2009–2010 season, divided into 3 divisions of 5 teams each. Eight of these teams will participate in the playoffs. The division winner from each division goes to the playoffs; in addition, the next 5 ranked teams in the conference, based on regular season records, will participate in the playoffs. How many different collections of eight teams from the Eastern Conference could go to the playoffs?

**Source:** National Hockey League.

### 41. Dogs of the Dow

Refer to Example 14 on page 473. The table below shows the top 10 dividend yielding stocks along with their prices that were listed on the “Dog of the Dow” Web site on December 31, 2009.

<table>
<thead>
<tr>
<th>Company</th>
<th>Price on 12/31/2009</th>
<th>%Yield on 12/31/2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>28.03</td>
<td>5.85</td>
</tr>
<tr>
<td>Verizon</td>
<td>33.13</td>
<td>5.73</td>
</tr>
<tr>
<td>DuPont</td>
<td>33.67</td>
<td>4.87</td>
</tr>
<tr>
<td>Kraft</td>
<td>27.18</td>
<td>4.27</td>
</tr>
<tr>
<td>Merck</td>
<td>36.54</td>
<td>4.16</td>
</tr>
<tr>
<td>Chevron</td>
<td>76.99</td>
<td>3.53</td>
</tr>
<tr>
<td>McDonald’s</td>
<td>62.44</td>
<td>3.52</td>
</tr>
<tr>
<td>Pfizer</td>
<td>18.19</td>
<td>3.52</td>
</tr>
<tr>
<td>Home Depot</td>
<td>28.93</td>
<td>3.11</td>
</tr>
<tr>
<td>Boeing</td>
<td>54.13</td>
<td>3.1</td>
</tr>
</tbody>
</table>

**Source:** www.dogsofthedow.com

(a) If two of these stocks are randomly selected, what is the probability both have prices above $50?

(b) If two of these stocks are randomly selected, what is the probability neither will have prices above $50?

(c) If two of these stocks are randomly selected, what is the probability at least one will have a price above $50?
42. **Dogs of the Dow** Refer to Problem 41.
   (a) If three of these stocks are randomly selected, what is the probability all three will have prices below $50?
   (b) If three of these stocks are randomly selected, what is the probability exactly two will have prices below $50?
   (c) If three of these stocks are randomly selected, what is the probability at least two will have prices below $50?

43. **Puppies of the Dow** Refer to Problem 41. The “Puppies of the Dow” are the five lowest-priced stocks in the Dogs of the Dow list.
   (a) List the five Puppies of the Dow.
   (b) If two of these “puppies” are randomly selected, what is the probability both will have yields above 4%?
   (c) If two of these “puppies” are randomly selected, what is the probability exactly one will have a yield above 4%?
   (d) If two of these “puppies” are randomly selected, what is the probability neither will have a yield above 4%?
   (e) If two of these “puppies” are randomly selected, what is the probability at least one will have a yield above 4%?

44. **Puppies of the Dow** Refer to Problems 41 and 43.
   (a) If five stocks from the Dogs of the Dow list are randomly selected, what is the probability all five will be Puppies of the Dow?
   (b) If five stocks from the Dogs of the Dow list are randomly selected, what is the probability exactly three will be Puppies of the Dow?
   (c) If five stocks from the Dogs of the Dow list are randomly selected, what is the probability at least two will be Puppies of the Dow?

45. **Auditing** For an internal audit of the 35 accounts payable of a small firm, the auditor randomly selects a sample of 5 account balances for review. Suppose 3 of the 35 accounts contain errors.
   (a) What is the probability the auditor will not select any accounts with errors?
   (b) What is the probability the auditor will select exactly one account with errors?
   (c) What is the probability the auditor will select at least two accounts with errors?

46. **Top Global Brands** Each year, *BusinessWeek* publishes an annual ranking of the best global brands by brand value. The table below identifies the top 10 global brands for 2009.

<table>
<thead>
<tr>
<th>2009 Rank</th>
<th>Name</th>
<th>Country</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Coca-Cola</td>
<td>U.S.</td>
</tr>
<tr>
<td>2</td>
<td>IBM</td>
<td>U.S.</td>
</tr>
<tr>
<td>3</td>
<td>Microsoft</td>
<td>U.S.</td>
</tr>
<tr>
<td>4</td>
<td>GE</td>
<td>U.S.</td>
</tr>
<tr>
<td>5</td>
<td>Nokia</td>
<td>Finland</td>
</tr>
<tr>
<td>6</td>
<td>McDonald’s</td>
<td>U.S.</td>
</tr>
<tr>
<td>7</td>
<td>Google</td>
<td>U.S.</td>
</tr>
<tr>
<td>8</td>
<td>Toyota</td>
<td>Japan</td>
</tr>
<tr>
<td>9</td>
<td>Intel</td>
<td>U.S.</td>
</tr>
<tr>
<td>10</td>
<td>Disney</td>
<td>U.S.</td>
</tr>
</tbody>
</table>

**Source:** *BusinessWeek*

(a) If three of these top 10 brands are selected at random, what is the probability all three will be U.S. brands?
(b) If four of these top 10 brands are selected at random, what is the probability all four will be U.S. brands?
(c) If five of these top 10 brands are selected at random, what is the probability all five will be U.S. brands?

47. **Cell Phone SIM Codes** A cell phone provider assigns a code to each of its SIM (subscriber identity module) cards. The codes start with one letter followed by five numbers and end with two additional letters. (Assume repetition is allowed, and order is important.)
   (a) How many different cell phone codes are there?
   (b) What is the probability a code selected at random ends in AA?
   (c) What is the probability a code selected at random begins with A and ends with A?
   (d) What is the probability a code selected at random has no repeated letter and no repeated number?
   (e) What is the probability a code selected at random has no repeated letter and all five numbers are the same?

48. **Pin Numbers** A student loan administrator distributes pin numbers to its debtors. Each pin consists of two letters followed by three numbers. (Assume repetition is allowed, and order is important.)
   (a) How many different pin numbers are there?
   (b) What is the probability a pin number selected at random ends in 000?
   (c) What is the probability a pin number selected at random begins with AA?
(d) What is the probability a pin number selected at random begins with A and ends with 00?
(e) What is the probability a pin number selected at random has a repeated letter and different numbers?

49. Defective Refrigerator Through a mix-up on the production line, 6 defective refrigerators were shipped out with 44 good ones. If 5 are selected at random, what is the probability that all 5 are defective? What is the probability that at least 2 of them are defective?

50. Defective Transformers In a shipment of 50 transformers, 10 are known to be defective. If 30 transformers are picked at random, what is the probability that all 30 are nondefective? Assume that all transformers look alike and have an equal probability of being chosen.

51. Favorable Extensions The employees of a company have four-digit extensions. Extensions ending in 00 are considered favorable because they are easy to remember. Suppose extensions are given out randomly and digits can be repeated.
(a) What is the probability of having a favorable extension?
(b) What is the probability of having a favorable extension if the first digit cannot be 0?

52. Winning a Pool Each week eight persons contribute $10.00 to a pool. Every Friday one name is drawn out of a hat containing the eight names and the winner receives the $80.00.
(a) What is the probability that the same person wins three weeks in a row?
(b) What is the probability that a particular person does not win in 5 weeks?
(c) What is the probability that 5 different people win in each of the next 5 weeks?

53. Forming Committees There are 59 Democrats and 41 Republicans in the U.S. Senate. A committee of seven senators is to be formed by selecting members of the Senate randomly.
(a) What is the probability the committee is composed of all Democrats?
(b) What is the probability the committee is composed of all Republicans?
(c) What is the probability the committee is composed of 4 Democrats and 3 Republicans?

54. Quality Control A shipment of 120 computer cases that contains 5 defective cases was sent to an assembly plant. The quality control manager at the assembly plant randomly selects 5 cases and inspects them. What is the probability exactly one case is defective?

55. Quality Control The purchasing director of a bank has just received a shipment of 100 printers. She has been told that the shipment includes 5 defective printers. She randomly selects two printers and tests them. If both printers work the purchasing director accepts the shipment. What is the probability the shipment is rejected?

56. United States Senate as of January 2010 Use the information in the table to answer the questions.
8.6 The Binomial Probability Model

OBJECTIVES

1. Find binomial probabilities (p. 482)
2. Solve applied problems involving the binomial probability model (p. 484)
3. Find the expected value in a Bernoulli trial (p. 487)
4. Model: error-correcting codes for the digital transmission of data (p. 489)

Bernoulli Trials

In this section we study experiments that use a probability model called the binomial probability model. The model was first studied by J. Bernoulli about 1700, and for this reason the model is sometimes referred to as a Bernoulli trial.

The binomial probability model is a sequence of trials, each of which consists of the repetition of a single experiment. We assume the outcome of one experiment does not affect the outcome of any other one; that is, we assume the trials to be independent. Furthermore, we assume that there are only two possible outcomes for each trial and label them A, for success, and F, for failure.

We denote the probability of success by \( p = P(A); p = P(A) \) remains the same from trial to trial. In addition, since there are only two outcomes in each trial, the probability of failure, denoted by \( q \), must be \( 1 - p \), so

\[
q = 1 - p = P(F)
\]

Notice that \( p + q = 1 \).

Any random experiment for which the binomial probability model is appropriate is called a Bernoulli trial.

Definition

Bernoulli Trial

Random experiments are called Bernoulli trials if

(a) The same experiment is repeated two or more times.
(b) There are only two possible outcomes, success and failure, on each trial.
(c) The repeated trials are independent.
(d) The probability of each outcome remains the same for each trial.

Many real-world situations have the characteristics of the binomial probability model. Here are some examples.

- In repeatedly running a subject through a T-maze, we may label a turn to the left by A and a turn to the right by F. The assumption of independence on each trial is equivalent to presuming the subject has no memory.
- In opinion polls one person's response is independent of any other person's response, and we may designate the answer “Yes” by an A and any other answer (“No” or “Don't know”) by an F.
- In testing TVs, we have a sequence of independent trials (each test of a particular TV is a trial), and we label a nondefective TV with an A and a defective one with an F.
In determining whether 9 out of 12 persons will recover from a tropical disease, we assume that each of the 12 persons has the same chance of recovery from the disease and that their recoveries are independent (they are not treated by the same doctor or in the same hospital). We may designate “recovery” by \( A \) and “nonrecovery” by \( F \).

**EXAMPLE 1 Tossing a Coin Three Times**

Discuss the experiment of tossing a coin three times.

**SOLUTION**

We define a success as heads (\( H \)) and a failure as tails (\( T \)). The coin may be fair or loaded, so we let \( p \) be the probability of heads and \( q \) be the probability of tails.

The sample space for this experiment is

\[
\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}
\]

As before, \( HHT \) means that the first two tosses are heads and the third is tails. The tree diagram in Figure 25 lists all the possible outcomes and their respective probabilities.

![Tree diagram](image)

In a binomial probability problem we are usually interested in the probability of an exact number of successes (or in the probability of at least a certain number of successes).

For example, suppose we are interested in the probability of obtaining exactly two heads in the experiment of tossing a coin three times. The event “exactly two heads” consists of the outcomes

\[
HHT, HTH, THH
\]

The outcome \( HHT \) has probability \( ppq = p^2q \) since the two heads each have probability \( p \) and the tail has probability \( q \). In the same way, the other two outcomes, \( HTH \) and \( THH \), in which there are two heads and one tail, also have probabilities \( p^2q \). Therefore, the probability that exactly two heads appear is equal to the sum of the probabilities of the three outcomes and so is given by \( 3p^2q \).

The outcome \( HHH \), in which all three tosses are heads, has a probability \( ppp = p^3 \).

In a similar way we can compute the remaining probabilities:

\[
P(\text{Exactly 1 head}) = 3pq^2 \quad \text{and} \quad P(\text{No heads}) = q^3
\]

8.6 The Binomial Probability Model 481

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Find Binomial Probabilities

When considering repeated trials in excess of three, it would be extremely tedious to solve the problem using a tree diagram. This is why we seek a general formula.

Suppose the probability of a success in a Bernoulli trial is \( p \) and suppose we wish to find the probability of exactly \( k \) successes in \( n \) repeated trials. One possible outcome is

\[
\text{AAA} \cdots A \cdot \text{FF} \cdots F
\]

where \( k \) successes come first, followed by \( n - k \) failures. The probability of this outcome is

\[
p^k q^{n-k}
\]

The \( k \) successes could also be obtained by rearranging the letters \( A \) and \( F \) in Display (1) above. Then the number of such sequences must equal the number of ways of choosing \( k \) successes in \( n \) trials—namely, \( C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \). If we multiply this number by the probability of obtaining any one such sequence, we arrive at the following general result:

**Theorem**

**Formula for \( b(n, k; p) \)**

In a Bernoulli trial the probability of exactly \( k \) successes in \( n \) trials is given by

\[
b(n, k; p) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k}
\]

where \( p \) is the probability of success and \( q = 1 - p \) is the probability of failure.

The symbol \( b(n, k; p) \) represents the probability of exactly \( k \) successes in \( n \) trials and is called a **binomial probability**.

**NOW WORK PROBLEM 7.**

**EXAMPLE 2** Example of a Bernoulli Trial; Finding a Binomial Probability

A common example of a Bernoulli trial is the experiment of tossing a fair coin.

1. There are exactly two possible mutually exclusive outcomes on each trial or toss (heads or tails).
2. The probability of a particular outcome (say, heads) remains constant from trial to trial (toss to toss).
3. The outcome on any trial (toss) is independent of the outcome on any other trial (toss).

Find the probability of obtaining exactly one tail in six tosses of a fair coin.

* Recall that the notation \( \binom{n}{k} \), read as “from \( n \) choose \( k \),” can be used in place of the combination notation \( C(n, k) \).
Let \( T \) denote the outcome “Tail shows” (success) and let \( H \) denote the outcome “Head shows” (failure). Using Formula (2), in which \( k = 1 \) (one success), \( n = 6 \) (the number of trials), and \( p = \frac{1}{2} = P(T) \) (the probability of success using a fair coin), we obtain

\[
P(\text{Exactly 1 success}) = b\left(6, 1; \frac{1}{2}\right) = \binom{6}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{6-1} = \frac{6}{64} = 0.0938
\]

NOW WORK PROBLEM 21.

COMMENT: A graphing utility can be used to compute binomial probabilities. Figure 26 shows \( b(6, 1; 0.5) \) using a TI-84 Plus calculator. Check your manual. Notice that for this calculator \( b(6, 1; 0.5) \) is entered as \( b(6, 0.5, 1) \).

FIGURE 26

\[
\binom{6}{1} \cdot 0.5 \cdot 0.5 = 0.09375
\]

USING TECHNOLOGY

EXAMPLE 3 Using Excel

Use Excel to solve Example 2. Make a table to find the probabilities of getting exactly 0, 1, 2, 3, 4, 5, or 6 tails.

SOLUTION

STEP 1 Use the Excel function \( \text{BINOMDIST}() \). The function is found under the category \( \text{Statistical} \).

STEP 2 The syntax for the function is

\[
\text{BINOMDIST} \left( \text{# of successes}, \text{# of trials}, \text{probability of success}, \text{cumulative} \right)
\]

\( \text{Cumulative} \) is a logical variable. Set \( \text{cumulative} \) equal to FALSE to find the probability.

STEP 3 Set up the Excel spreadsheet.
**STEP 4** Enter the function in F2. Keep the number of trials and the probability of success constant by using $\cdot$

![Function Arguments](image)

**STEP 5** The probability of getting zero tails when tossing a coin 6 times is given below.

<table>
<thead>
<tr>
<th>Successes</th>
<th>Trials</th>
<th>Probability of Success</th>
<th>Binomial Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>0.5</td>
<td>0.015625</td>
</tr>
</tbody>
</table>

**STEP 6** To find the probability of getting exactly 1, 2, 3, 4, 5, or 6 tails, highlight row 2 and click and drag to row 8.

<table>
<thead>
<tr>
<th>Successes</th>
<th>Trials</th>
<th>Probability of Success</th>
<th>Binomial Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>0.5</td>
<td>0.015625</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>0.5</td>
<td>0.09375</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>0.5</td>
<td>0.234375</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>0.5</td>
<td>0.3125</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>0.5</td>
<td>0.234375</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>0.5</td>
<td>0.09375</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>0.5</td>
<td>0.015625</td>
</tr>
</tbody>
</table>

NOW WORK PROBLEM 21 USING EXCEL.

**EXAMPLE 4** Baseball

A pitcher gives up a hit on the average of once every five pitches. If nine pitches are thrown, what is the probability that
(a) Exactly three pitches result in hits?
(b) No pitch results in a hit?
(c) Eight or more pitches result in hits?
(d) No more than seven pitches result in hits?
SOLUTION

In this example \( p = P(\text{Success}) = P(\text{Allowing a hit}) = \frac{1}{5} = 0.2 \). The number of trials is the number of pitches, so \( n = 9 \). Finally, \( k \) is the number of pitches that result in hits.

(a) If exactly three pitches result in hits, then \( k = 3 \). Using \( n = 9, p = 0.2, \) and \( q = 1 - p = 0.8 \), we find

\[
P(\text{Exactly 3 hits}) = \binom{9}{3} (0.2)^3 (0.8)^6 = 0.1762
\]

(b) If none of the nine pitches resulted in a hit, \( k = 0 \).

\[
P(\text{Exactly 0 hits}) = \binom{9}{0} (0.2)^0 (0.8)^9 = 0.1342
\]

(c) Nine pitches are thrown, so the event “eight or more pitches result in hits” is equivalent to the events: “exactly 8 result in hits” or “exactly 9 result in hits.” Since these events are mutually exclusive, we have

\[
P(\text{At least 8 hits}) = P(\text{Exactly 8 hits}) + P(\text{Exactly 9 hits})
\]

\[
= \binom{9}{8} (0.2)^8 (0.8)^1 + \binom{9}{9} (0.2)^9 (0.8)^0 = 0.0000189
\]

(d) “No more than seven pitches result in hits” means 0, 1, 2, 3, 4, 5, 6, or 7 pitches result in a hit. We could add \( \binom{9}{0} (0.2)^0 (0.8)^9 \), \( \binom{9}{1} (0.2)^1 (0.8)^8 \), and so on, but it is easier to use the formula \( P(E) = 1 - P(\overline{E}) \). The complement of “No more than 7” is “at least 8.” We use the result of part (c) to find:

\[
P(\text{No more than 7 hits}) = 1 - P(\text{At least 8 hits})
\]

\[
= 1 - [\binom{9}{8} (0.2)^8 (0.8)^1 + \binom{9}{9} (0.2)^9 (0.8)^0] = 0.999981
\]

NOW WORK PROBLEM 55.

COMMENT: A graphing utility can be used to compute cumulative probabilities. Figure 27 shows the probability of 0 through 7 successes in 9 trials where the probability of success is 0.2 using a TI-84 Plus. This is the same result obtained in Example 4(d).

FIGURE 27

\[
\text{binomcdf}(9, 0.2, 7)
\]

\[
.999981056
\]

EXAMPLE 5 Quality Control

A machine produces lightbulbs to meet certain specifications, and 80% of the bulbs produced meet these specifications. A sample of six bulbs is taken from the machine’s production and placed in a box. What is the probability that at least three of them fail to meet the specifications?
In this example the number of trials is \( n = 6 \). We are looking for the probability of the event \( E: \) at least 3 fail to meet specifications.

Since the experiment consists of choosing 6 bulbs, the event \( E \) is the union of the mutually exclusive events: “Exactly 3 fail,” “Exactly 4 fail,” “Exactly 5 fail,” and “Exactly 6 fail.” We use Formula (2) with \( n = 6 \) and \( k = 3, 4, 5, \) and 6. Since the probability of a bulb failing to meet specifications is 0.20, we have

\[
P(\text{Exactly 3 fail}) = b(6; 3; 0.20) = 0.0819 \\
P(\text{Exactly 4 fail}) = b(6; 4; 0.20) = 0.0154 \\
P(\text{Exactly 5 fail}) = b(6; 5; 0.20) = 0.0015 \\
P(\text{Exactly 6 fail}) = b(6; 6; 0.20) = 0.0001
\]

Then,

\[
P(\text{At least 3 fail}) = P(E) = 0.0819 + 0.0154 + 0.0015 + 0.0001 = 0.0989
\]

Another way of getting this answer is to compute the probability of the complementary event \( E: \) fewer than 3 fail.

Then,

\[
P(\bar{E}) = P(\text{Exactly 2 fail}) + P(\text{Exactly 1 fail}) + P(\text{Exactly 0 fail}) \\
= b(6; 2; 0.20) + b(6; 1; 0.20) + b(6; 0; 0.20) \\
= 0.2458 + 0.3932 + 0.2621 = 0.9011
\]

Then,

\[
P(\text{At least 3 fail}) = 1 - P(\bar{E}) = 1 - 0.9011 = 0.0989
\]

as before.

**NOW WORK PROBLEM 37.**
There is a likelihood of 0.254 (just over 25%) that he will pass if he is just guessing.  

(b) If the claim is correct, the probability of success is 0.9. The probability of at least 6 successes in this instance is

$$b(6; 0.9) + b(7; 0.9) + b(8; 0.9) + b(9; 0.9) = 0.0446 + 0.1722 + 0.3874 + 0.3874 = 0.9916$$

If the claim is correct, the probability of success exceeds 99%.

Notice that the test in Example 6 is fair to the claimant since it practically assures him the position if his claim is true. However, the company may not like the test because 25% of the time a person who guesses will pass the test.

**NOW WORK PROBLEM 61.**

**EXAMPLE 7 Testing a Serum or Vaccine**

Suppose that the normal rate of infection of a certain disease in cattle is 25%. To test a newly discovered serum, healthy animals are injected with the serum. How can we evaluate the result of the experiment?

**SOLUTION**

In a probability experiment, an event with a probability less than 0.05 is considered unusual. So we can evaluate the effects of the serum by determining how many treated animals contract the disease. Then we calculate the probability that number of cattle would have been infected if they had not been vaccinated. If the probability is less than 0.05 we conclude the serum is effective.

In this example, \( n \) is the number of cattle vaccinated; \( k \) is the number of cattle that are infected; and \( p = 0.25 \) is the probability an untreated cow is infected.

For example, if we vaccinate 10 cattle and none of them are infected, we compute

\[ b(10, 0; 0.25) = 0.056. \]

This is a small probability, indicating that the serum may have had an effect, but it is not conclusive.

However, if we enlarge the sample and vaccinate 20 animals, the probability that less than 2 become infected is

\[ b(20, 0; 0.25) + b(20, 1; 0.25) = 0.0032 + 0.0211 = 0.0243. \]

Since this is less than 0.05, we conclude the serum is effective.

**3 Find the Expected Value in a Bernoulli Trial**

In 100 tosses of a coin, what is the expected number of heads? If a student guesses at random on a true–false exam with 50 questions, what is the expected grade? These are specific instances of the following more general question:

In \( n \) trials of a Bernoulli process, what is the expected number of successes?

In a Bernoulli trial, let \( p \) denote the probability of success on any individual trial.

If \( n = 1 \) (one trial), then the expected number of successes is

\[ E = 1 \cdot p + 0 \cdot (1 - p) = p \]

If \( n = 2 \) (two trials), then either 0, 1, or 2 successes can occur. The expected number \( E \) of successes is

\[ E = 2 \cdot p^2 + 1 \cdot 2p(1 - p) + 0 \cdot (1 - p)^2 = 2p \]
If \( n = 3 \) (three trials), then either 0, 1, 2, or 3 successes can occur. The expected number \( E \) of successes is

\[
E = 3 \cdot p^3 + 2 \cdot 3p^2(1 - p) + 1 \cdot 3p(1 - p)^2 + 0 \cdot (1 - p)^3
\]

\[
= 3p^3 + 6p^2 - 6p^3 + 3p - 6p^2 + 3p^3
\]

\[
= 3p
\]

This pattern leads to the following result:

**Theorem Expected Value for Bernoulli Trials**

In a Bernoulli process with \( n \) trials the expected number of successes is

\[
E = np
\]

where \( p \) is the probability of success on any single trial.

The intuitive idea behind the result is fairly simple. Thinking of probabilities as percentages, if success results \( p \) percent of the time, then out of \( n \) attempts, \( p \) percent of them, namely, \( np \), should be successful.

**EXAMPLE 8 Expected Value in a Bernoulli Trial**

In flipping a fair coin five times, what is the expected number of tails?

**Solution**

The six events 0 tails, 1 tail, 2 tails, 3 tails, 4 tails, 5 tails are events that constitute a partition of the sample space. The respective probability of each of these events is

\[
\binom{5}{0} \left(\frac{1}{2}\right)^5 \binom{5}{1} \left(\frac{1}{2}\right)^5 \binom{5}{2} \left(\frac{1}{2}\right)^5 \binom{5}{3} \left(\frac{1}{2}\right)^5 \binom{5}{4} \left(\frac{1}{2}\right)^5 \binom{5}{5} \left(\frac{1}{2}\right)^5
\]

If we assign the payoffs 0, 1, 2, 3, 4, 5 respectively to each event, then the expected number of tails is

\[
E = 0 \cdot \binom{5}{0} \left(\frac{1}{2}\right)^5 + 1 \cdot \binom{5}{1} \left(\frac{1}{2}\right)^5 + 2 \cdot \binom{5}{2} \left(\frac{1}{2}\right)^5
\]

\[
+ 3 \cdot \binom{5}{3} \left(\frac{1}{2}\right)^5 + 4 \cdot \binom{5}{4} \left(\frac{1}{2}\right)^5 + 5 \cdot \binom{5}{5} \left(\frac{1}{2}\right)^5 = \frac{5}{2}
\]

Using the result \( E = np \) is much easier. For \( n = 5 \) and \( p = \frac{1}{2} \), we obtain

\[
E = 5 \cdot \left(\frac{1}{2}\right) = \frac{5}{2}.
\]

**NOW WORK PROBLEM 65.**
EXAMPLE 9  Expected Value in a Bernoulli Trial

A multiple-choice test contains 100 questions, each with four choices. If a person guesses, what is the expected number of correct answers?

SOLUTION  This is an example of a Bernoulli trial. The probability for success (a correct answer) when guessing is \( p = \frac{1}{4} \). Since there are \( n = 100 \) questions, the expected number of correct answers is

\[
E = np = (100)\left(\frac{1}{4}\right) = 25
\]

NOW WORK PROBLEM 71.

Model:  Error-Correcting Codes for the Digital Transmission of Data

Electronically transmitted data, whether from computer to computer or from a satellite to a ground station, are normally in the form of strings of 0s and 1s—that is, in binary form. Bursts of noise or faults in relays, for example, may at times garble the transmission and produce errors so that the message received is not the same as the one originally sent. For example,

\[
\begin{array}{c}
001 \\
\text{Message sent}
\end{array} \quad \rightarrow \quad \begin{array}{c}
101 \\
\text{Noisy channel}
\end{array} \quad \begin{array}{c}
101001 \\
\text{Message received}
\end{array}
\]

is a transmission where the message received is in error since the initial 0 has been changed to a 1.

A naive way of trying to protect against such error would be to repeat the message. So instead of transmitting 001, we would send 001001. Then, were the same error to creep in as it did before, the received message would be

\[
101001
\]

The receiver would certainly know an error had occurred since the last half of the message is not a duplicate of the first half. But the receiver would have no way of recovering the original message since there is no way to know where the error happened. That is, the receiver would not be able to distinguish between the two messages

\[
001001 \quad \text{and} \quad 101101
\]

There are more sophisticated ways of coding binary data with redundancy that not only allow the detection of errors but simultaneously locate and correct them so that the original message can be recovered. These are referred to as error-correcting codes and are commonly used in computer-implemented transmissions.

One such code is the (7, 4) Hamming code named after Richard Hamming, a former researcher at AT&T Laboratories. It is a code of length seven, meaning that an individual message is a string consisting of seven items, each of which is either a 0 or 1. (The 4 refers to the fact that the first four elements in the string can be freely chosen by the sender, while the remaining three are determined by a fixed rule and constitute the redundancy that gives the code its error-correction capability.) The (7, 4) Hamming code is capable of locating and correcting a single error. That is, if during transmission a 1 has been changed to a 0 or vice versa in one of the seven locations, the Hamming code is capable
of detecting and correcting this. While we will not explain how or why the Hamming code works, we will analyze the benefit obtained by its use.

By an error we mean that an individual 1 has been changed to a 0 or that a 0 has been changed to a 1. We assume that the probability $q$ of an error happening remains constant during transmission. Then $p = 1 - q$ is the probability that an individual symbol remains unchanged. In practice, values of $q$ are close to 0 and values of $p$ close to 1 since we would usually be using a reliable channel. See Figure 28.

![Figure 28](image)

We also assume that errors occur randomly and independently. This means we can think of the transmission of a binary string of length seven as a Bernoulli trial.

**EXAMPLE 10 Analyzing a Message of Length 7**

A message consisting of 7 symbols is transmitted. The probability $p$ a symbol remains unchanged by the transmission is $p = 0.98$.

**(a)** What is the probability the message is received without any alteration?

**(b)** If a Hamming code is employed, the message will be received without alteration even if one symbol is changed. What is the probability the message is received correctly using a Hamming code?

**SOLUTION**

**(a)** Since $p$ is the probability of no alteration of a single symbol and the message consists of 7 symbols, the probability the message is received without any changes is

$$b(7, 7; p) = p^7 = (0.98)^7 = 0.8681$$

The message is received without error almost 87% of the time.

**(b)** Using a Hamming code, the message is received correctly even if one symbol is changed. The probability the message is received without any change now is

$$b(7, 7; p) + b(7, 6; p) = p^7 + 7p^6q = (0.98)^7 + 7(0.98)^6(0.02) = 0.9921$$

Using the Hamming code, the probability the message is received without error has improved to over 99%.

There are codes in use that correct more than a single error. One such code, known by the initials of its originators as a BCH code, is a code of length 15 (messages are binary strings of length 15) that corrects up to two errors. Sending a message of length 15 with no error-correcting code results in a probability of $b(15, 15; p) = p^{15}$ of the correct message being received. Using the BCH code that can correct two or fewer errors, the probability that a message will be correctly received becomes

$$b(15, 15; p) + b(15, 14; p) + b(15, 13; p)$$

No errors 1 error 2 errors
EXAMPLE 11  Analyzing a Message of Length 15

A message consisting of 15 symbols is transmitted. The probability $p$ a symbol remains unchanged by the transmission is $p = 0.98$.

(a) What is the probability the message is received without any alteration?

(b) If a BCH code is employed, the message will be received without alteration even if one or two symbols are changed. What is the probability the message is received correctly using a BCH code?

SOLUTION  (a) Since $p$ is the probability of no alteration of a single symbol and the message consists of 15 symbols, the probability the message is received without any changes is

$$b(15, 15; p) = p^{15} = (0.98)^{15} = 0.7386$$

The message is received without error almost 74% of the time.

(b) Using a BCH code, the message is received correctly even if one or two symbols are changed. The probability the message is received without any change now is

$$b(15, 15; p) + b(15, 14; p) + b(15, 13; p) = p^{15} + 15p^{14}q + 105p^{13}q^2$$

$$= (0.98)^{15} + 15(0.98)^{14}(0.02) + 105(0.98)^{13}(0.02)^2$$

$$= 0.9970$$

Using the BCH code, the probability the message is received without error has improved to over 99%.

Codes that can correct a high number of errors are clearly very desirable. Yet a basic result in the theory of codes states that as the error-correcting capability of a code increases, so, of necessity, must its length. But lengthier codes require more time for transmission and are clumsier to use. Here, speed and correctness are at odds.

EXERCISE 8.6  Answers Begin on Page AN–42.

Concepts and Vocabulary

1. **True or False** In a Bernoulli trial, the same experiment is repeated two or more times, but the probability of a particular outcome varies from trial to trial.
2. **True or False** In a Bernoulli trial, the same experiment is repeated two or more times and the repeated trials are independent.
3. In a Bernoulli trial, the probability of exactly $k$ successes in $n$ trials is $b(n, k; p) =$________, where $p$ is the probability of success and $q = 1 - p$ is the probability of failure.
4. In a binomial probability model, the probability of exactly 3 successes in 7 trials with the probability of success being 0.3 is written as $b(______, ______; ______)$.
5. **True or False** In a Bernoulli trial, the probability of at least 5 successes in 6 trials is $b(6, 5; p) + b(6, 6; p)$, where $p$ is the probability of success and $q = 1 - p$ is the probability of failure.
6. In a Bernoulli process with $n$ trials, the expected number of successes is $E =$____, where $p$ is the probability of success.
In Problems 27–32, a fair coin is tossed 8 times.

21. Find the probability of obtaining exactly 6 successes in 10 trials when the probability of success is 0.3.
22. Find the probability of obtaining exactly 5 successes in 9 trials when the probability of success is 0.2.
23. Find the probability of obtaining exactly 9 successes in 12 trials when the probability of success is 0.8.
24. Find the probability of obtaining exactly 8 successes in 15 trials when the probability of success is 0.75.
25. Find the probability of obtaining at least 5 successes in 8 trials when the probability of success is 0.30.
26. Find the probability of obtaining at most 3 successes in 7 trials when the probability of success is 0.20.

In Problems 27–32, a fair coin is tossed 8 times.
27. What is the probability of obtaining exactly 1 head?
28. What is the probability of obtaining exactly 2 heads?
29. What is the probability of obtaining at least 5 tails?
30. What is the probability of obtaining at most 2 tails?
31. What is the probability of obtaining exactly 2 heads if it is known that at least 1 head appeared?
32. What is the probability of obtaining exactly 3 heads if it is known that at least 1 head appeared?

Applications

37. **Quality Control** Suppose that 5% of the items produced by a factory are defective. If 8 items are chosen at random, what is the probability
   (a) Exactly 1 is defective?
   (b) Exactly 2 are defective?
   (c) At least 1 is defective?
   (d) Fewer than 3 are defective?
38. **Opinion Poll** Suppose that 60% of the voters intend to vote for a conservative candidate. What is the probability a survey polling 8 people reveals that 3 or fewer intend to vote for a conservative candidate?
39. **Family Structure** What is the probability in a family with exactly 6 children:
   (a) 3 are boys?
   (b) At least 5 are boys?
   (c) At least 2, but less than 4, are girls?
40. **Family Structure** What is the probability in a family of 7 children:
   (a) 4 are girls?
   (b) At least 2 are girls?
   (c) At least 2 and not more than 4 are girls?
41. **Guessing on a True–False Exam** In a 20-item true–false examination, a student guesses on each question.
   (a) What is the probability the student will obtain all correct answers?
   (b) If 12 correct answers constitute a passing grade, what is the probability the student will pass?
   (c) What are the odds in favor of passing?
42. **True–False Tests**
   (a) In a 15-item true–false examination, what is the probability that a student who guesses on each question will get at least 10 correct answers?
   (b) If another student has 0.8 probability of correctly answering each question, what is the probability that this student will answer at least 12 questions correctly?
43. **Cheating on Taxes**  
*Money* magazine reported that 12% of people admit they think it is acceptable to cheat on their taxes. Suppose 15 people are randomly selected.

(a) What is the probability that at least five admit they think it is acceptable to cheat on taxes?

(b) What is the probability that fewer than three admit they think it is acceptable to cheat on taxes?

(c) What is the probability that none admit they think it is acceptable to cheat on taxes?


44. **Individual Tax Audits**  
In 2009, the IRS audited about 1% of all individual income tax returns. Suppose that 10 individual income tax returns from 2009 are randomly selected.

(a) What is the probability that none was audited?

(b) What is the probability that exactly one was audited?

(c) What is the probability that more than one was audited?

*Source:* Internal Revenue Service

45. **Filing Tax Returns**  
The IRS estimates that 10% of all individual income tax returns filed each year are submitted on the last day of filing. Suppose that eight individual income tax returns are randomly selected.

(a) What is the probability that exactly two were submitted on the last day of filing?

(b) What is the probability that exactly five were submitted on the last day of filing?

(c) What is the probability that none were submitted on the last day of filing?


46. **IRS Prosecutions**  
For tax evasion cases that actually reached prosecution in 2009, the IRS reported a conviction rate of 87.2%. Suppose 20 tax evasion cases prosecuted by the IRS in 2009 are randomly selected.

(a) What is the probability that all 20 resulted in convictions?

(b) What is the probability that exactly 19 resulted in convictions?

(c) What is the probability that at least 18 resulted in convictions?

*Source:* Internal Revenue Service

47. **Worker Satisfaction**  
A study by the Society for Human Resource Management, an international association of human resource professionals, found that 80% of workers liked their jobs. Suppose 10 workers are randomly selected.

(a) What is the probability that all ten like their jobs?

(b) What is the probability that exactly nine like their jobs?

(c) What is the probability that no more than seven like their jobs?

*Source:* Society for Human Resource Management, June 27, 2005

48. **Worker Satisfaction**  
A Sperion® Workplace Snapshot survey conducted by Harris Interactive® found that 32% of workers listen to music while working, a practice believed to increase job satisfaction and productivity. Suppose 6 workers are selected at random.

(a) What is the probability that exactly three listen to music while working?

(b) What is the probability that more than four listen to music while working?

(c) What is the probability that two or fewer listen to music while working?


49. **Clipping Coupons**  
According to the Promotion Marketing Association, 77% of all Americans use coupons, saving more than $3 billion each year on purchases. Suppose 18 Americans are randomly selected.

(a) What is the probability that exactly 14 use coupons?

(b) What is the probability that at least 14 use coupons?

(c) What is the probability that at least 16 use coupons?


50. **Volunteerism**  
According to the Bureau of Labor Statistics, 26.8% of the U.S. population did volunteer work during the twelve months prior to September 2009. Suppose 14 people from the U.S. population are randomly selected.

(a) What is the probability that exactly six had volunteered?

(b) What is the probability that fewer than six had volunteered?

(c) What is the probability that more than six had volunteered?


51. **Insurance Fraud**  
The National Insurance Crime Bureau estimates that 15% of auto theft claims involve fraud. Suppose 12 auto theft claims are randomly selected.

(a) Determine the probability none of the claims involve fraud.

(b) Determine the probability six claims involve fraud.

(c) Determine the probability at least one of the claims involve fraud.

*Source:* www.statefarm.com citing the National Insurance Crime Bureau
52. Auto Loan Approval  Bev is a loan officer at a privately owned bank. From years of experience, Bev knows that 84% of auto loan applications are approvable. Bev has recently received 15 auto loan applications that she must examine.

(a) What is the probability that all 15 applications are approvable?

(b) What is the probability that at least one application must be rejected?

(c) What is the probability that fewer than 10 of the applications are approvable?

53. Air Passenger No-Shows  The airline industry reports that 12% of ticketed passengers fail to show up on time for the scheduled flight. A scheduled flight of a small jet with 44 seats is sold out.

(a) Determine the probability two passengers will be no-shows.

(b) Determine the probability none of the passengers will be no-shows.

(c) Determine the probability at least two of the passengers will be no-shows.

Source: De Lollis, Barbara, "Airlines give fliers fewer chances to do the bump," USA Today, Dec. 19, 2005 (www.usatoday.com)

54. Airline Bumps  To counter the cost of passenger no-shows, airlines frequently overbook flights. If the number of passengers that show up for a flight is greater than the number of seats available, then some passengers must be bumped from that flight (usually voluntarily through incentives). A USA Today analysis of government statistics showed that airlines bump about 12 of every 10,000 passengers. Suppose a shuttle from an airport parking lot is carrying 18 independent air passengers.

(a) Determine the probability that none of them will be bumped from their flight.

(b) Determine the probability that exactly one of them will be bumped from their flight.

(c) Determine the probability that more than one of them will be bumped from their flights.

Source: De Lollis, Barbara, "Airlines give fliers fewer chances to do the bump," USA Today, Dec. 19, 2005 (www.usatoday.com)

55. Batting Averages  A baseball player has a 0.250 batting average.

(a) What is the probability that the player will have at least 2 hits in 4 times at bat?

(b) What is the probability of at least 1 hit in 4 times at bat?

56. Target Shooting  If the probability of hitting a target is \( \frac{2}{3} \) and 10 shots are fired independently, what is the probability of the target being hit at least twice?

57. Opinion Poll  Mr. Austin and Ms. Moran are running for public office. A survey conducted just before the day of election indicates that 60% of the voters prefer Mr. Austin and 40% prefer Ms. Moran. If 8 people are chosen at random and asked their preference, find the probability that all 8 people will express a preference for Ms. Moran.

58. Screening Employees  To screen prospective employees, a company gives a 10-question multiple-choice test. Each question has 4 possible answers, of which 1 is correct. The chance of answering the questions correctly by just guessing is or 25%. Find the probability of answering, by chance:

(a) Exactly 3 questions correctly.

(b) No questions correctly.

(c) At least 8 questions correctly.

(d) No more than 7 questions correctly.

59. Heart Attack  Approximately 23% of North American unexpected deaths are due to heart attacks. What is the probability that 4 of the next 10 unexpected deaths reported in a certain community will be due to heart attacks?

60. Support for the President  A Fox News/Opinion Dynamics Poll conducted with 900 registered voters in the United States on April 20–21, 2010, found that 46% approved of the job Barack Obama was doing as president. Assuming that this poll reflects national opinion, if 30 voters were randomly selected, what is the probability that at least 10, but less than 15, would approve of the job Barack Obama is doing as president?

Source: Fox News Opinion/Dynamics Poll

61. Product Testing  A supposed coffee connoisseur claims she can distinguish between a cup of instant coffee and a cup of drip coffee 80% of the time. You give her 6 cups of coffee and tell her that you will grant her claim if she correctly identifies at least 5 of the 6 cups.

(a) What are her chances of having her claim granted if she is in fact only guessing?

(b) What are her chances of having her claim rejected when in fact she really does have the ability she claims?

62. Opinion Poll  Opinion polls based on small samples often yield misleading results. Suppose 65% of the people in a city are opposed to a bond issue and the others favor it. If 7 people are asked for their opinion, what is the probability that a majority of them will favor the bond issue?
63. The Aging Population  The 2008 United States Census showed that 12.3% of the United States population was 65 years of age or older. Suppose that in the year 2008, 10 people were randomly selected from the United States population.
   (a) What is the probability that exactly four of these people would be 65 years of age or older?
   (b) What is the probability that none of these people would be 65 years of age or older?
   (c) What is the probability that at most five of these people would be 65 years of age or older?
   Source: United States Census Bureau

64. Age Distribution  The 2008 United States Census showed that 87.7% of the U.S. population was younger than 65 years old. Suppose that in the year 2008, 10 people were randomly selected from the U.S. population.
   (a) What is the probability that exactly 8 of these people would be younger than 65?
   (b) What is the probability that all of these people would be younger than 65?
   (c) What is the probability that at most five of these people would be 65 years of age or older?
   Source: United States Census Bureau, 2008

65. Tossing a Die  Find the number of times the face 5 is expected to occur in a sequence of 2000 throws of a fair die.

66. Tossing a Coin  What is the expected number of tails that will turn up if a fair coin is tossed 582 times?

67. Quality Control  A certain kind of lightbulb has been found to have a 0.02 probability of being defective. A shop owner receives 500 lightbulbs of this kind. How many of these bulbs are expected to be defective?

68. Pass/Fail  A student enrolled in a math course has a 0.9 probability of passing the course. In a class of 20 students, how many would you expect to fail the math course?

69. Healthcare Costs  In a December 2009 Gallup poll, 14% of adults in the United States indicated that healthcare costs were the most important financial problem facing their family. For a random sample of 20 adults, what is the expected number that would say healthcare costs were the most important financial problem facing their family?
   Source: The Gallup Organization

70. Emergency Savings  A report by CNN in February 2007 stated that only 40% of adult Americans have separate savings for an emergency. For a random sample of 120 adult Americans, what is the expected number that will have separate savings for an emergency?
   Source: www.cnn.com

71. Drug Reaction  A doctor has found that the probability that a patient who is given a certain drug will have unfavorable reactions to the drug is 0.002. If a group of 500 patients is going to be given the drug, how many of them does the doctor expect to have unfavorable reactions?

72. True–False Test  A true–false test consisting of 30 questions is scored by subtracting the number of wrong answers from the number of right ones. Find the expected number of correct answers of a student who just guesses on each question. What will the expected test score be?

73. Hamming Code  There is a Hamming code of length 15 that corrects a single error. Assuming \( p = 0.98 \), find the probability that a message transmitted using this code will be correctly received.

74. Golay Code  There is a binary code of length 23 (called the Golay code) that can correct up to 3 errors. If the probability that an individual character is transmitted correctly is 0.97, find the probability that a message transmitted using this code will be correctly received.

75. BCH Code  There is a binary code of length 31 [called the binary (31, 31, \( p \)] BCH code] used for some pagers that can correct up to two errors. If the probability that an individual character is transmitted correctly is 0.97, find the probability that a message transmitted using this code will be correctly received.
   Source: www.eccpage.com

Sometimes experiments are simulated using a random number function* instead of actually performing the experiment.
In Problems 76–79, use a graphing utility to simulate each experiment.

76. Tossing a Fair Coin  Consider the experiment of tossing a coin 4 times and counting the number of heads occurring in these 4 tosses. Simulate the experiment using a random number function on your calculator, considering a toss to be tails (T) if the result is less than 0.5, and considering a toss to be heads (H) if the result is greater than or equal to 0.5. Record the number of heads in 4 tosses. [Note: Most calculators repeat the action of the last entry if you simply press the ENTER, or EXE, key again.] Repeat the experiment 10 times, obtaining a sequence of 10 numbers. Using these 10 numbers you can estimate the probability of \( k \) heads, \( P(k) \), for each \( k = 0, 1, 2, 3, 4 \), by the ratio
   \[
   \frac{\text{Number of times } k \text{ appears in your sequence}}{10}
   \]
Enter your estimates in the table on the next page. Calculate the actual probabilities using the binomial probability formula, and enter these numbers in the table. How close are your numbers to the actual values?

*Most graphing utilities have a random number function (usually RAND or RND) for generating numbers between 0 and 1. Check your user’s manual to see how to use this function on your graphing calculator.
77. Tossing a Loaded Coin  Consider the experiment of tossing a loaded coin 4 times and counting the number of heads occurring in these 4 tosses. Simulate the experiment using a random number function on your calculator, considering a toss to be tails (T) if the result is less than 0.80, and considering a toss to be heads (H) if the result is greater than or equal to 0.80. Record the number of heads in 4 tosses. Repeat the experiment 10 times, obtaining a sequence of 10 numbers. Using these 10 numbers you can estimate the probability of $k$ heads, $P(k)$, for each $k = 0, 1, 2, 3, 4$, by the ratio

$$
\frac{\text{Number of times } k \text{ appears in your sequence}}{10}
$$

Enter your estimates in a table. Calculate the actual probabilities using the binomial probability formula, and enter these numbers in your table. How close are your numbers to the actual values?

<table>
<thead>
<tr>
<th>$k$</th>
<th>Your Estimate of $P(k)$</th>
<th>Actual Value of $P(k)$</th>
</tr>
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<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
<td></td>
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</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

78. Tossing a Fair Coin  Consider the experiment of tossing a fair coin 8 times and counting the number of heads occurring in these 8 tosses. Simulate the experiment using a random number function on your calculator, considering a toss to be tails (T) if the result is less than 0.50, and considering a toss to be heads (H) if the result is greater than or equal to 0.50. Record the number of heads in 8 tosses. Repeat the experiment 10 times, obtaining a sequence of 10 numbers. Using these 10 numbers you can estimate the probability of 3 heads, $P(3)$, by the ratio

$$
\frac{\text{Number of times 3 appears in your sequence}}{10}
$$

Calculate the actual probability using the binomial probability formula. How close is your estimate to the actual value?

79. Tossing a Loaded Coin  Consider the experiment of tossing a loaded coin 8 times and counting the number of heads occurring in these 8 tosses. Simulate the experiment using a random number function on your calculator, considering a toss to be tails (T) if the result is less than 0.80, and considering a toss to be heads (H) if the result is greater than or equal to 0.80. Record the number of heads in 8 tosses. Repeat the experiment 10 times, obtaining a sequence of 10 numbers. Using these 10 numbers you can estimate the probability of 3 heads, $P(3)$, by the ratio

$$
\frac{\text{Number of times 3 appears in your sequence}}{10}
$$

Calculate the actual probability using the binomial probability formula. How close is your estimate to the actual value?

CHAPTER 8 REVIEW  OBJECTIVES

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<th>Examples</th>
<th>You should be able to</th>
<th>Review Exercises</th>
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<td>1–4, 13–18, 58a, 58b, 59c, 60c, 66</td>
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<td></td>
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<td>2 Find probabilities using the Product Rule (p. 425)</td>
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<td>3 Find probabilities using a tree diagram (p. 426)</td>
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<tr>
<td>8</td>
<td></td>
<td>4 Find conditional probabilities from a data table (p. 428)</td>
<td>61a, 61b</td>
</tr>
</tbody>
</table>
### THINGS TO KNOW

**Conditional Probability** (p. 423)

\[ P(E|F) = \frac{P(E \cap F)}{P(F)}, \quad P(F) \neq 0 \]

**Product Rule** (p. 425)

\[ P(E \cap F) = P(F) \cdot P(E|F) \]

**E is independent of F** (p. 434)

\[ P(E|F) = P(E) \]

**Criterion for Independent Events** (p. 435)

E, F are independent if and only if \( P(E \cap F) = P(E) \cdot P(F) \)

**Probability of an Event E in a Partitioned Sample Space** (p. 449)

\[
P(E) = P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + \cdots + P(A_n) \cdot P(E|A_n)
\]

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<table>
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<td>4, 5, 6, 7</td>
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<tr>
<td></td>
<td>8, 9</td>
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</tbody>
</table>
498 Chapter 8 Additional Probability Topics

Bayes’ Theorem (p. 450)

\[ P(A_j|E) = \frac{P(A_j) \cdot P(E|A_j)}{P(E)} \]

Factorial (p. 458) 

\[ 0! = 1 \quad n! = n(n - 1) \cdots (3)(2)(1) \]

The product of the first \( n \) positive integers.

Permutation (p. 459)

An ordered arrangement of \( r \) objects chosen from \( n \) objects.

Distinct, with Repetition (p. 459) 

\[ n^r \]

The \( n \) objects are distinct (different) and repetition is allowed in the selection of \( r \) of them.

Distinct, without Repetition (p. 461)

\[ P(n, r) = n(n-1) \cdots [n - (r - 1)] \]

The \( n \) objects are distinct (different) and repetition is not allowed in the selection of \( r \) of them, where \( r \leq n \).

Combination (pp. 466–467) 

\[ C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \]

An arrangement, without regard to order, of \( r \) objects selected from \( n \) distinct objects without repetition, where \( r \leq n \).

\( n \) Objects, Not Distinct (p. 470)

\[ \frac{n!}{n_1!n_2! \cdots n_k!} \]

The number of permutations of \( n \) objects of which \( n_1 \) are of one kind, \( n_2 \) are of a second kind, \ldots, and \( n_k \) are of a \( k \)th kind, where \( n = n_1 + n_2 + \cdots + n_k \).

Binomial Probability Formula (p. 482)

\[ b(n, k; p) = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p \]

Expected Value for Bernoulli Trials (p. 488)

\[ E = np \]

REVIEW EXERCISES Answers to odd-numbered problems begin on page AN–43.

Blue Problem numbers indicate the author’s suggestions for a practice test.

In Problems 1–12, use the tree diagram below to find the indicated probability:

1. \( P(E|A) \) 
2. \( P(F|A) \) 
3. \( P(E|B) \) 
4. \( P(F|B) \) 
5. \( P(A \cap E) \) 
6. \( P(A \cap F) \) 
7. \( P(B \cap E) \) 
8. \( P(B \cap F) \) 
9. \( P(A|E) \) 
10. \( P(A|F) \) 
11. \( P(B|E) \) 
12. \( P(B|F) \)

In Problems 13–32, use the tree diagram below to find the indicated probability:

1. \( P(E|A) \) 
2. \( P(F|A) \) 
3. \( P(E|B) \) 
4. \( P(F|B) \) 
5. \( P(A \cap E) \) 
6. \( P(A \cap F) \) 
7. \( P(B \cap E) \) 
8. \( P(B \cap F) \) 
9. \( P(A|E) \) 
10. \( P(A|F) \) 
11. \( P(B|E) \) 
12. \( P(B|F) \)
In Problems 33–38, evaluate each expression.

13. \( P(E|A) \)  
14. \( P(F|A) \)  
15. \( P(E|B) \)  
16. \( P(F|B) \)  
17. \( P(E|C) \)  
18. \( P(F|C) \)  
19. \( P(A \cap E) \)  
20. \( P(A \cap F) \)  
21. \( P(B \cap E) \)  
22. \( P(B \cap F) \)  
23. \( P(C \cap E) \)  
24. \( P(C \cap F) \)  
25. \( P(E) \)  
26. \( P(F) \)  
27. \( P(A|E) \)  
28. \( P(A|F) \)  
29. \( P(B|E) \)  
30. \( P(B|F) \)  
31. \( P(C|E) \)  
32. \( P(C|F) \)  

\[ 33. \ 0! \]  
\[ 34. \ (8 - 3)! \]  
\[ 35. \ \frac{7!}{4!} \]  
\[ 36. \ \frac{10!}{2!8!} \]  
\[ 37. \ \frac{12!}{11!} \]  
\[ 38. \ \frac{6!}{(6 - 3)!3!} \]

39. In how many different ways can a committee of 3 people be formed from a group of 5 people?  
40. In how many different ways can 4 people line up?  
41. In how many different ways can 3 books be placed on a shelf?  
42. In how many different ways can 4 people line up?  
43. You are to set up a code of 2-digit words using the digits 1, 2, 3, 4, 5, 6 without using any digit more than once in the same word.  
   (a) What is the maximum number of words in such a language?  
   (b) If the words 124, 142, etc., designate the same word, how many different words are possible?  
44. You are to set up a code of 2-digit words using the digits 1, 2, 3, 4 without using any digit more than once.  
   (a) What is the maximum number of words in such a language?  
   (b) If all words of the form \( ab \) and \( ba \) are the same, how many words are possible?  
45. Forming Committees There are 7 boys and 6 girls willing to serve on a committee. How many 7-member committees are possible if a committee is to contain:  
   (a) 3 boys and 4 girls?  
   (b) At least one member of each sex?  
46. Choosing Double-Dip Cones Juan’s Ice Cream Parlor offers 31 different flavors to choose from and specializes in double-dip cones.  
   (a) How many different cones are there to choose from if you may select the same flavor for each dip?  
   (b) How many different cones are there to choose from if you cannot repeat any flavor? Assume that a cone with vanilla on top of chocolate is different from a cone with chocolate on top of vanilla.  
   (c) How many different cones are there if you consider any cone having chocolate on top and vanilla on the bottom the same as having vanilla on top and chocolate on the bottom?  
47. Arranging Books A person has 4 history, 5 English, and 6 mathematics books. How many ways can the books be arranged on a shelf if books on the same subject must be together?  
48. Choosing Names A newborn child can be given 1, 2, or 3 names. In how many ways can a child be named if we can choose from 100 names?  
49. Forming Committees In how many ways can a committee of 8 boys and 5 girls be formed if there are 10 boys and 11 girls eligible to serve on the committee?  
50. Football Teams A football squad has 7 linemen, 11 linebackers, and 9 safeties. How many different teams composed of 5 linemen, 3 linebackers, and 3 safeties can be formed?  
51. There are 5 rotten plums in a crate of 25 plums. How many samples of 4 of the 25 plums contain  
   (a) Only good plums?  
   (b) Three good plums and 1 rotten plum?  
   (c) One or more rotten plums?  
52. An admissions test given by a university contains 10 true–false questions. Eight or more of the questions must be answered correctly in order to be admitted.  
   (a) How many different ways can the answer sheet be filled out?  
   (b) How many different ways can the answer sheet be filled out so that 8 or more questions are answered correctly?  
53. How many 6-letter words (real or imaginary) can be made from the word FINITE?  
54. How many 7-letter words (real or imaginary) can be made from the word MESSAGE?  
55. Arranging Books Jessica has 10 books to arrange on a shelf. How many different arrangements are possible if she has 3 identical copies of Harry Potter and the Sorcerer's Stone and 2 identical copies of Finite Mathematics?  
56. Arranging Pennants Mike has 9 pennants to arrange on a pole. How many different arrangements are possible if 4 pennants are blue, 2 are yellow, and 3 are green?  
57. Rolling a Loaded Die A die is loaded so that when it is rolled an outcome of 6 is 3 times more likely to occur than any other number. The die is rolled twice. Define  
   \( E \): A 3 appears on the first roll.  
   \( F \): A 6 appears on the second roll.  
Show that the events \( E \) and \( F \) are independent.
58. The records of Midwestern University show that in one semester, 38% of the students failed mathematics, 27% of the students failed physics, and 9% of the students failed mathematics and physics. A student is selected at random.
   (a) If a student failed physics, what is the probability that he or she failed mathematics?
   (b) If a student failed mathematics, what is the probability that he or she failed physics?
   (c) What is the probability that he or she failed mathematics or physics?
59. In a certain population of people, 25% are blue-eyed and 75% are brown-eyed. Also, 10% of the blue-eyed people are left-handed and 5% of the brown-eyed people are left-handed.
   (a) What is the probability that a person chosen at random is blue-eyed and left-handed?
   (b) What is the probability that a person chosen at random is left-handed?
   (c) What is the probability that a person is blue-eyed, given that the person is left-handed?
60. **College Majors** At a local college 55% of the students are female and 45% are male. Also 40% of the female students are education majors, and 15% of the males are education majors.
   (a) What is the probability a student selected at random is male and an education major?
   (b) What is the probability a student selected is an education major?
   (c) What is the probability a student is female given the person is an education major?
61. **Score Distribution** Two forms of a standardized math exam were given to 100 students. The following are the results.

<table>
<thead>
<tr>
<th>Score</th>
<th>Form A</th>
<th>Form B</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Over 80%</td>
<td>8</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>Under 80%</td>
<td>32</td>
<td>48</td>
<td>80</td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td>60</td>
<td>100</td>
</tr>
</tbody>
</table>

   (a) What is the probability that a student who scored over 80% took form A?
   (b) What is the probability that a student who took form A scored over 80%?
   (c) Show that the events “scored over 80%” and “took form A” are independent.
   (d) Are the events “scored over 80%” and “took form B” independent?
62. **ACT Scores** The following data compare ACT scores of students with their performance in the classroom [based on a maximum 4.0 grade point average (GPA)].

<table>
<thead>
<tr>
<th>GPA</th>
<th>Below 21</th>
<th>22–27</th>
<th>Above 28</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6–4.0</td>
<td>8</td>
<td>56</td>
<td>104</td>
<td>168</td>
</tr>
<tr>
<td>3.0–3.5</td>
<td>47</td>
<td>70</td>
<td>30</td>
<td>147</td>
</tr>
<tr>
<td>Below 3</td>
<td>47</td>
<td>34</td>
<td>4</td>
<td>85</td>
</tr>
<tr>
<td>Total</td>
<td>102</td>
<td>160</td>
<td>138</td>
<td>400</td>
</tr>
</tbody>
</table>

   A graduating student is selected at random. Find the probability that
   (a) The student scored above 28.
   (b) The student’s GPA is 3.6–4.0.
   (c) The student scored above 28 with GPA of 3.6–4.0.
   (d) The student’s ACT score was in the 22–27 range.
   (e) The student had a 3.0–3.5 GPA.
   (f) Show that “ACT above 28” and “GPA below 3” are not independent.
63. $E$ and $F$ are independent events. Find $P(E|F)$ if $P(F) = 0.4$ and $P(E \cap F) = 0.2$.
64. $E$ and $F$ are independent events. Find $P(E)$ if $P(E \cup F) = 0.6$ and $P(F) = 0.1$.
65. **Shooting Free Throws** A basketball player hits 70% of his free throws. Assuming independence on successive throws, what is the probability of
   (a) Missing the first throw and then getting 3 in a row?
   (b) Making 10 free throws in a row?
66. **Majoring in Business** At the College at Old Westbury, 20% of the students are business majors and the rest major in something else. Although 70% of the business majors take Finite Mathematics, only 10% of the other majors take Finite Mathematics. A student is chosen at random. She is taking Finite Mathematics. What is the probability she is a business major?
67. Three envelopes are addressed for 3 secret letters written in invisible ink. A secretary randomly places each of the letters in an envelope and mails them. What is the probability that at least 1 person receives the correct letter?
68. Jones lives at $O$ (see the figure). He owns 5 gas stations located 4 blocks away (dots). Each afternoon he checks on one of his gas stations. He starts at $O$. At each intersection he flips a fair coin. If it shows heads, he will head north (N); otherwise, he will head east (E). What is
the probability that he will end up at gas station $G$ before coming to one of the other stations?

69. Defective Machine The calibration of a filling machine broke, causing it to underfill 10 jars of jam. The jars were accidentally mixed in with 62 properly filled jars. All the jars look the same. Four jars are chosen from the batch and weighed. What is the probability
(a) All 4 jars are underweight?
(b) Exactly 2 jars are underweight?
(c) At most 1 jar is underweight?

70. Broken Calculators Three broken calculators were inadvertently packed in a case of 12 calculators. Two were chosen from the case.
(a) What is the probability both are broken?
(b) What is the probability neither is broken?
(c) What is the probability at least 1 is broken?

71. Quality Control In a factory three machines, $A_1$, $A_2$, and $A_3$, produce 55%, 30%, and 15% of total production, respectively. The percentage of defective output of these machines is 1%, 2%, and 3%, respectively. An item is chosen at random and it is defective.
(a) What is the probability that it came from machine $A_1$?
(b) From $A_2$?
(c) From $A_3$?

72. Test for Cancer A lung cancer test has been found to have the following reliability. The test detects 85% of the people who have cancer and does not detect 15% of these people. Among the noncancerous group it detects 92% of the people not having cancer, whereas 8% of this group are detected erroneously as having lung cancer. Statistics show that about 1.8% of the population has lung cancer. Suppose an individual is given the test for lung cancer and it detects the disease. What is the probability that the person actually has lung cancer?

73. Advertising Management believes that 1 out of 5 people watching a television advertisement about their new product will purchase the product. Five people who watched the advertisement are picked at random.
(a) What is the probability none of these people will purchase the product?
(b) What is the probability exactly 3 will purchase the product?

74. Baseball Suppose that the probability of a player hitting a home run is $\frac{1}{20}$. In 5 tries what is the probability that the player hits at least 1 home run?

75. Tossing Coins What is the expected number of heads when a coin is tossed 7 times?

76. Recovery Rate The recovery rate from a flu is 0.9. If 4 people have this flu, what is the probability that
(a) All will recover?  
(b) Exactly 2 will recover? 
(c) At least 2 will recover? 
Assume independence.

Chapter 8 Project

ASKING SENSITIVE QUESTIONS USING CONDITIONAL PROBABILITY

In your journalism class, a group project places you in charge of determining what proportion of the students at your school are overweight. You decide to conduct a survey of a sample of the student body, and (of course) you will need to ask the survey respondents the question “Are you overweight?” There is a problem in being so direct, however. Even though the survey is anonymous, some respondents might not want to answer this question truthfully. They might be afraid that you could somehow figure out who they were by their responses, or they just might not care to share that information with you. You could call a question such as “Are you overweight?” a sensitive question. Nevertheless, you want to estimate the proportion of overweight students at your school. How can you get an idea of that proportion without driving some of your respondents to lie? It turns out that conditional probability provides a way to handle this problem.

Consider conducting the survey in the following manner: Before answering the sensitive question, the respondent flips a coin. We assume that it is a fair coin; that is, heads appears with probability $\frac{1}{2}$. If the flip is tails, the respondent is instructed to answer “Yes” to the sensitive question; if the flip is heads, the respondent is instructed to answer the question truthfully. Since no one except the respondent has any idea whether the
flip was a head or a tail, no one except the respondent knows whether a “Yes” response means “the coin came up tails” or “I am overweight.” In this way, the respondent might be more inclined to answer the question truthfully if required to do so.

Define the events:

\[ E: \text{The respondent answer “Yes”} \]
\[ H: \text{The flip is a head} \]
\[ T: \text{The flip is a tail} \]

After you tabulate the results to your survey, you will know \( P(E) \), the proportion of your respondents that answered “Yes.”

1. Why are you interested in finding \( P(E|H) \)?
2. Find an equation for \( P(E) \) that involves \( P(E|H) \) and \( P(E|T) \).
3. Explain why then solve the equation found in part 2 for \( P(E|H) \).
4. If 82 of 100 respondents answered “Yes,” what is your best estimate for the proportion of overweight students at your school?

Suppose that you want to get a bit more information: You want respondents to check one of the three boxes labeled “underweight,” “less than 20 pounds overweight,” or “more than 20 pounds overweight.” Once again, use a coin-flipping strategy. If the flip is tails, the respondent is instructed to check the “more than 20 pounds overweight” box; if the flip is heads, the respondent is instructed to answer the question truthfully.

Define the events:

\[ A: \text{The respondent answers “underweight.”} \]
\[ B: \text{The respondent answers “less than 20 pounds overweight.”} \]
\[ C: \text{The respondent answers “more than 20 pounds overweight.”} \]
\[ H: \text{The flip is heads.} \]
\[ T: \text{The flip is tails.} \]

5. Express as an equation.
6. Express as an equation.
7. Express as an equation.

Suppose you survey 300 students and get the following results:

- Number checking “Underweight” 30
- Number checking “Less than 20 pounds overweight” 50
- Number checking “More than 20 pounds overweight” 220

8. What is your best estimate for the proportion of students at your school who are underweight, less than 20 pounds overweight, and more than 20 pounds overweight?

9. Write a one-page report that summarizes your findings and the methodology you used.

---

**Mathematical Questions from Professional Exams**

1. **Actuary Exam—Part II** If \( P \) and \( Q \) are events having positive probability in the same sample space \( S \) such that \( P \cap Q = \emptyset \), then all of the following pairs are independent EXCEPT

   (a) \( \emptyset \) and \( P \)
   (b) \( P \) and \( Q \)
   (c) \( P \) and \( S \)
   (d) \( P \) and \( P \cap Q \)
   (e) \( \emptyset \) and the complement of \( P \)

2. **Actuary Exam—Part II** A box contains 12 varieties of candy and exactly 2 pieces of each variety. If 12 pieces of candy are selected at random, what is the probability that a given variety is represented?

   (a) \( \frac{12}{(12!)^2} \)
   (b) \( \frac{12}{24!} \)
   (c) \( \frac{12}{24} \)
   (d) \( \frac{11}{46} \)
   (e) \( \frac{35}{46} \)

3. **Actuary Exam—Part II** What is the probability that a 3-card hand drawn at random and without replacement from a regular deck consists entirely of black cards?

   (a) \( \frac{1}{17} \)
   (b) \( \frac{2}{17} \)
   (c) \( \frac{1}{8} \)
   (d) \( \frac{3}{17} \)
   (e) \( \frac{4}{17} \)

4. **Actuary Exam—Part II** Events \( S \) and \( T \) are independent with \( \Pr(S) < \Pr(T) \), \( \Pr(S \cap T) = \frac{6}{25} \), and \( \Pr(S|T) + \Pr(T|S) = 1 \). What is \( \Pr(S) \)?

   (a) \( \frac{1}{25} \)
   (b) \( \frac{1}{5} \)
   (c) \( \frac{5}{25} \)
   (d) \( \frac{2}{5} \)
   (e) \( \frac{3}{5} \)
5. **Actuary Exam—Part II** What is the least number of independent times that an unbiased die must be thrown to make the probability that all throws do not give the same result greater than .999?  
(a) 3  
(b) 4  
(c) 5  
(d) 6  
(e) 7

6. **Actuary Exam—Part II** In a group of 20,000 men and 10,000 women, 6% of the men and 3% of the women have a certain affliction. What is the probability that an afflicted member of the group is a man?  
(a) \(\frac{3}{5}\)  
(b) \(\frac{2}{3}\)  
(c) \(\frac{3}{4}\)  
(d) \(\frac{4}{5}\)  
(e) \(\frac{8}{9}\)

7. **Actuary Exam—Part II** An unbiased die is thrown 2 independent times. Given that the first throw resulted in an even number, what is the probability that the sum obtained is 8?  
(a) \(\frac{3}{36}\)  
(b) \(\frac{1}{6}\)  
(c) \(\frac{4}{21}\)  
(d) \(\frac{7}{36}\)  
(e) \(\frac{1}{3}\)

8. **Actuary Exam—Part II** If the events S and T have equal probability and are independent with \(\Pr(S \cap T) = p > 0\), then \(\Pr(S) = \)  
(a) \(\sqrt{p}\)  
(b) \(p^2\)  
(c) \(\frac{P}{2}\)  
(d) \(p\)  
(e) \(2p\)

9. **Actuary Exam—Part II** The probability that both S and T occur, the probability that S occurs and T does not, and the probability that T occurs and S does not are all equal to \(p\). What is the probability that either S or T occurs?  
(a) \(p\)  
(b) \(2p\)  
(c) \(3p\)  
(d) \(3p^2\)  
(e) \(p^3\)

10. **Actuary Exam—Part II** What is the probability that a bridge hand contains 1 card of each denomination (i.e., 1 ace, 1 king, 1 queen, . . . , 1 three, 1 two)?  
(a) \(\frac{\binom{13}{1}}{\binom{13}{15}}\)  
(b) \(\frac{\binom{4}{13}}{\binom{52}{13}}\)  
(c) \(\frac{\binom{52}{4}}{\binom{13}{13}}\)  
(d) \(\frac{\binom{13}{13}}{\binom{52}{13}}\)  
(e) \(\frac{13^4}{\binom{52}{13}}\)

11. **Actuary Exam—Part II** What is the probability that 10 independent tosses of an unbiased coin result in no fewer than 1 head and no more than 9 heads?  
(a) \(\left(\frac{1}{2}\right)^9\)  
(b) \(1 - 11\left(\frac{1}{2}\right)^9\)  
(c) \(1 - 11\left(\frac{1}{2}\right)^{10}\)  
(d) \(1 - \left(\frac{1}{2}\right)^9\)  
(e) \(1 - \left(\frac{1}{2}\right)^{10}\)
Final exams are over and a weekend trip to New York City next month in June is the plan. Weather can be a problem wherever one travels, but New York in June seems a safe choice. But is it? The highest temperature on record is 101°F and the lowest a brisk 44°F. Wow, what a difference! Can we rely on average temperatures to feel more secure? How do we use information about average temperatures to assess the likelihood the trip will not be affected by extreme weather? This chapter on statistics will give you the background to do an analysis, and the Chapter 9 Project will guide you.

Have a great trip!
OBJECTIVES

1. Identify continuous and discrete variables (p. 506)
2. Obtain a simple random sample (p. 507)
3. Identify sources of bias in a sample (p. 507)

We have all heard of statistics, but do we really know what statistics is?

**Definition**

Statistics is the science of collecting, arranging, analyzing, and interpreting information to draw conclusions or answer questions.

By making observations, statisticians collect data in the form of measurements or counts. A measurable characteristic is called a variable. Variables fall into two general groups. If a variable measures an attribute or a trait of the individual being studied, it is a qualitative variable. Eye color and favorite type of music are examples of qualitative variables. Other variables provide numerical measures of the individual. These are called quantitative variables. If a quantitative variable can assume any real value between certain limits, it is called a continuous variable. A quantitative variable is called a discrete variable if it can assume only a finite set of values or as many values as there are whole numbers. Examples of continuous variables are weight, height, length, and time. Examples of discrete variables are the number of votes a candidate gets in an election and the number of new cars sold by a dealership.

1. **Identify Continuous and Discrete Variables**

   It is important to be able to identify whether a variable is continuous or discrete because it determines the kind of statistical analysis that can be used.

**EXAMPLE 1** Identifying Continuous and Discrete Variables

State the variable in the following experiments and determine whether it is continuous or discrete.

(a) Record the number of months in 2007 that a salesperson signs more than 10 new accounts.

(b) Measure the length of time a smoker can blow into a tube before taking a breath.

(c) Count the number of defective parts produced by a certain machine in an hour.

**SOLUTION**

(a) The variable is the number of months. It is discrete because it can have only the values 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.

(b) The variable is time. It is continuous because it can take on any value within certain limits, say from 0 to 90 seconds.

(c) The variable is the number of defective parts. It is discrete; it is a whole number.

**NOW WORK PROBLEMS 5 AND 7.**
9.1 Introduction to Statistics: Data and Sampling

The set of observations made of the variable in an experiment is called data. In collecting data it is often impossible or impractical to observe an entire group, called the population. So instead of examining the entire population, often a small segment, called a sample, is chosen to be observed. It would be difficult, for example, to question all cigarette smokers in order to study the effects of smoking. Therefore, a sample of smokers is usually selected.

The method of selecting the sample is important if we want the results to be reliable. It is essential that the individuals in the sample have characteristics similar to those of the population they are intended to represent.

2 Obtain a Simple Random Sample

There are several ways to select a representative sample. The simplest and most reliable method of sampling is called simple random sampling. In a simple random sample, every member of the population has an equal probability of being selected into the sample.

**EXAMPLE 2 Obtaining a Simple Random Sample**

Janet is the president of the Marketing Society. She wants to know the opinion of the members concerning a proposed guest speaker. There are 150 members in the club, and rather than ask each one’s opinion, Janet decides to ask a simple random sample of 10 members. How should Janet choose the sample?

**SOLUTION**

Janet needs to choose 10 members of the Marketing Society, and she wants every member of the society to have an equal opportunity to be chosen. She writes each member’s name on a separate sheet of paper and puts the papers in a box. She mixes the papers and selects 10 sheets. She asks the 10 members whose names she chose their opinions.

**NOW WORK PROBLEM 17.**

A more efficient way to select a simple random sample would be to assign a number to each member of the population. Then a random number generator or a list of random numbers can be used to identify the members that will make up the simple random sample.

**COMMENT:** Graphing utilities have random generators built into them. Check your user’s manual to learn how to generate random numbers on your graphing utility.

3 Identify Sources of Bias in a Sample

If the sample is not chosen in a way that every member of the population has an equal probability of being selected, a biased sample could result. Bias occurs when a segment of the population is either overrepresented or underrepresented in the sample. The data collected from a biased sample often do not reflect the population accurately. For example, if we want to study the relationship between smoking cigarettes and lung cancer, we cannot choose a sample of smokers who all live in the same location. The individuals chosen might have characteristics peculiar to their environment that give a false impression with respect to all smokers.
EXAMPLE 3 Identify Sources of Bias in a Sample

Janet decides that it is too difficult to obtain a simple random sample of the members of the Marketing Society, so she asks 10 freshmen from her Introduction to Marketing class their opinions about the proposed speaker. Identify a possible source of bias in this sample.

SOLUTION Janet polled only freshmen. As a result, sophomore, junior, and senior members of the Marketing Society were underrepresented in the sample. If the speaker had appeal to a particular level of student, the results would not represent the opinion of the population.

NOW WORK PROBLEM 23.

When a sample is representative of a population, important conclusions about the population can often be inferred from analysis of the data collected. The phase of statistics dealing with conditions under which such inference is valid is called inductive statistics or inferential statistics. Since such inference cannot be absolutely certain, the language of probability is often used in stating conclusions. When a meteorologist makes a weather forecast, weather data collected over a large region are studied; based on the study, the forecast is given in terms of probabilities. A typical forecast might be, “There is a 20% chance of rain tomorrow.”

EXERCISE 9.1 Answers Begin on Page AN–44.

Concepts and Vocabulary

1. A measurable characteristic is called a(n) _____.
2. In a(n) _____ _____ _____ every member of the population has an equal probability of being selected.
3. True or False A discrete variable can assume any real value between certain limits.
4. If, in a sample, a segment of the population is overrepresented or underrepresented, the sample is _____.

Skill Building

In Problems 5–16, identify the variable in each experiment and determine whether it is continuous or discrete.

5. A statistician observes the number of heads that occur when a coin is tossed 1000 times.
6. A statistician counts the number of red pieces of candy in each of 100 one-ounce bags of M&Ms.
7. The Environmental Protection Agency (EPA) determines the average gas mileage for each of 500 randomly chosen Honda Accords.
8. A General Electric (GE) quality control team measures how long it takes each of 500 lightbulbs to burn out.
9. A statistician measures the time a person who arrives at a neighborhood McDonald’s between noon and 1:00 P.M. waits in line.
10. A pharmaceutical firm measures the time it takes for a drug to take effect.
11. A marketing analyst surveys 500 people and asks the number of airplane flights they took in the last 12 months.
12. A sales manager counts the number of calls made in a day before a sale is made.
13. An urban planner counts the number of people who cross a particular intersection between 1:00 P.M. and 3:00 P.M.
14. A carpet manufacturer counts the number of missed stitches in a 100-yard roll of carpeting.
15. A quality control team measures how long a phone battery lasts.
16. A psychologist observes how many times a subject blinks when being told a scary story.

In Problems 17–22, list some possible ways to choose random samples for each study.

17. A study to determine public opinion about a certain television program.
18. A study to determine favorite brands of cereal.
19. A study of the opinions of people toward Medicare.
20. A study to determine opinions about an election of a U.S. president.
21. A study of the number of savings accounts per family in the United States.
22. A national study of the monthly budget for a family of four.
9.2 Representing Qualitative Data Graphically: Bar Graphs; Pie Charts

Discussion and Writing

23. The following is an example of a biased sample: In a study of political party preferences, poorly dressed interviewers obtained a significantly greater proportion of answers favoring Democratic party candidates in their samples than did their well-dressed and wealthier-looking counterparts. Give two more examples of biased samples.

24. In a study of the number of savings accounts per family, a sample of accounts totaling less than $10,000 was taken and, from the owners of these accounts, information about the total number of accounts owned by all family members was obtained. Criticize this sample.

25. It is customary for news reporters to sample the opinions of a few people to find out how the population at large feels about the events of the day. A reporter questions people on a downtown street corner. Is there anything wrong with such an approach?

26. In 1936 the Literary Digest conducted a poll to predict the presidential election. Based on its poll, it predicted the election of Landon over Roosevelt. In the actual election, Roosevelt won. The sample was taken by drawing the mailing list from telephone directories and lists of car owners. What was wrong with the sample?

9.2 Representing Qualitative Data Graphically: Bar Graphs; Pie Charts

**OBJECTIVES**

1. Construct a bar graph from data (p. 509)
2. Construct a pie chart from data (p. 510)
3. Analyze a graph (p. 513)

In this section we discuss qualitative data. The data as collected are called raw data. Raw data are often organized to make them more useful. The organization of data involves the presentation of the collected measurements or counts in a form suitable for determining logical conclusions. Usually tables or graphs are used to represent the collected data. There are two popular methods for graphically displaying qualitative data: **bar graphs** and **pie charts**. Both are used when the data can be separated into categories. A bar graph will show the number or the percent of data that is in each category, while a pie chart will show only the percent of data in each category.

**EXAMPLE 1 Constructing a Bar Graph from Data**

For the fiscal year 2009 (October 2008–September 2009), the federal government spent a total of $3,517.6 trillion. The breakdown of expenditures (in billions of dollars) is given in Table 1. Construct a bar graph of the data.

<table>
<thead>
<tr>
<th>Category</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social Security, Medicare, and other retirement</td>
<td>$1168.7</td>
</tr>
<tr>
<td>National defense, veterans, and foreign affairs</td>
<td>$794.0</td>
</tr>
<tr>
<td>Net interest (interest on the public debt)</td>
<td>$186.9</td>
</tr>
<tr>
<td>Physical, human, and community development</td>
<td>$553.3</td>
</tr>
<tr>
<td>Social programs</td>
<td>$741.2</td>
</tr>
<tr>
<td>Law enforcement and general government</td>
<td>$73.5</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td><strong>$3517.6</strong></td>
</tr>
</tbody>
</table>

*Source: U.S. Treasury Department.*

**SOLUTION** A horizontal axis is used to indicate each category of spending and a vertical axis is used to represent the amount spent in each category. For each category of spending we draw rectangles of equal width whose heights represent the amount spent in that category. The rectangles will not touch each other. See Figure 1.
FIGURE 1

NOW WORK PROBLEM 5(a)–(c).

EXAMPLE 2 Constructing a Pie Chart from Data

Use the data given in Table 1 to construct a pie chart.

SOLUTION To construct a pie chart, a circle is divided into sectors, one sector for each category of data. The size of each sector is proportional to the total amount spent. Since Social Security, Medicare, and other retirement is $1168.7 billion and total spending is $3.5176 trillion = $3517.6 billion, the percent of data in this category is

\[
\frac{1168.7}{3517.6} \approx 0.33 = 33\%.
\]

Therefore, Social Security, Medicare, and other retirement will make up 33% of the pie chart. Since a circle has 360°, the degree measure of the sector for this category of spending is \(0.33(360^\circ) \approx 120^\circ\). Following this procedure for the remaining categories of spending, we obtain Table 2.

<table>
<thead>
<tr>
<th>Category</th>
<th>Spending (in billions)</th>
<th>Percent of Total Spending</th>
<th>Degree Measure of Sector*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social Security, Medicare, and other</td>
<td>$1168.7</td>
<td>33%</td>
<td>120°</td>
</tr>
<tr>
<td>retirement</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>National defense, veterans, and foreign</td>
<td>$794.0</td>
<td>23%</td>
<td>81°</td>
</tr>
<tr>
<td>affairs</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Net interest (interest on the public</td>
<td>$186.9</td>
<td>5%</td>
<td>19°</td>
</tr>
<tr>
<td>debt)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Physical, human, and community development</td>
<td>$553.3</td>
<td>16%</td>
<td>57°</td>
</tr>
<tr>
<td>Social programs</td>
<td>$741.2</td>
<td>21%</td>
<td>76°</td>
</tr>
<tr>
<td>Law enforcement and general government</td>
<td>$73.5</td>
<td>2%</td>
<td>8°</td>
</tr>
<tr>
<td>TOTAL</td>
<td>$3517.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*The data in column 4 do not add up to 360° due to rounding.
9.2 Representing Qualitative Data Graphically: Bar Graphs; Pie Charts

To construct a pie chart by hand, use a protractor to approximate the angles for each sector. See Figure 2.

**FIGURE 2**

Government Spending 2009

- **Law enforcement and general government**: 2%
- **Social programs**: 21%
- **Social Security, Medicare, and other retirement**: 33%
- **Physical, human, and community development**: 16%
- **National defense, veterans, and foreign affairs**: 23%
- **Net interest (interest on the public debt)**: 5%

**NOW WORK PROBLEM 7.**

**USING TECHNOLOGY**

**EXAMPLE 3** Using Excel to Construct a Pie Chart

Use Excel to construct a pie chart for the data in Table 1.

**SOLUTION**

**STEP 1** Enter the data into Excel, highlight it, and under Insert select the Chart Wizard.
**STEP 2**  For the chart type, select Pie.

**STEP 3**  Click Next. The basic pie chart will appear. Click Next again. This screen provides the options for the pie chart. Click on Legend. Turn the legend off by clicking in the box Show Legend. Then highlight the Data Label tab.

**STEP 4**  Click on Category name and Percentage.

**NOW WORK PROBLEM 7(b) USING EXCEL.**
9.2 Representing Qualitative Data Graphically: Bar Graphs; Pie Charts

Analyze a Graph

One reason for graphing data by drawing bar graphs or pie charts is to quickly determine certain information about the data. The decision of which graph to use sometimes depends on preference but other times depends on the amount of data available. Bar graphs can be used whenever the data can be divided into categories, regardless if all categories are considered. However, pie charts can be created only if all possible categories of the variable under consideration are represented. Pie charts are most useful when comparing parts to the whole, whereas bar graphs are more useful for comparing the categories to each other but not to the whole.

EXAMPLE 4 Analyzing a Pie Chart

The pie chart in Figure 3 represents the sources of revenue for the United States federal government in its fiscal year 2009. Answer the questions below using the pie chart. Total revenue for the fiscal year 2009 was $2.103 trillion.

(a) What is the largest source of revenue for the federal government? What is the amount?
(b) What is the smallest source of revenue for the federal government? What is the amount?
(c) How much revenue does the government collect from Social Security, Medicare, etc.?

SOLUTION

(a) The largest source of revenue for the federal government is personal income taxes. The government collects

\[
(0.44)(2.103 \text{ trillion}) = 0.925 \text{ trillion} = 925 \text{ billion}
\]

from this source.

(b) The smallest source of revenue for the federal government is other sources. The government collects

\[
(0.02)(2.103 \text{ trillion}) = 0.042 \text{ trillion} = 42 \text{ billion}
\]

from this source.

Source: U.S. Treasury Department.
EXERCISE 9.2 Answers Begin on Page AN-44.

Concepts and Vocabulary

1. Two popular ways to display data that can be separated into categories are in _____ and _____.
2. True or False In a bar graph the width of the rectangle depends on the size of the category.
3. True or False In a pie chart a circle is divided into equal sectors, one for each category represented by the data.
4. True or False A bar graph can be used any time there are categorical data, but a pie chart can be used only if all the categories under consideration are represented.

Skill Building

5. Family Income The data below represent median family income (in dollars) by region of the country in 2007.

<table>
<thead>
<tr>
<th>Region</th>
<th>Median Family Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Northeast</td>
<td>$51,624</td>
</tr>
<tr>
<td>Midwest</td>
<td>$48,014</td>
</tr>
<tr>
<td>South</td>
<td>$45,464</td>
</tr>
<tr>
<td>West</td>
<td>$56,291</td>
</tr>
</tbody>
</table>

Source: U.S. Bureau of the Census.

(a) Draw a bar graph of the data.
(b) Which region has the highest median income?
(c) Which region has the lowest median income?
(d) Discuss the use of a pie chart to represent this situation.

6. Family Income The data below represent the median family income (in dollars) by region of the country in 2005.

<table>
<thead>
<tr>
<th>Region</th>
<th>Median Family Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Northeast</td>
<td>$62,133</td>
</tr>
<tr>
<td>Midwest</td>
<td>$57,453</td>
</tr>
<tr>
<td>South</td>
<td>$51,352</td>
</tr>
<tr>
<td>West</td>
<td>$56,291</td>
</tr>
</tbody>
</table>

Source: U.S. Bureau of the Census.

(a) Draw a bar graph of the data.
(b) Which region has the highest median income?
(c) Which region has the lowest median income?
(d) Discuss the use of a pie chart to represent this situation.


<table>
<thead>
<tr>
<th>Region</th>
<th>Number of Families</th>
</tr>
</thead>
<tbody>
<tr>
<td>Northeast</td>
<td>54,123</td>
</tr>
<tr>
<td>Midwest</td>
<td>53,885</td>
</tr>
<tr>
<td>South</td>
<td>91,951</td>
</tr>
<tr>
<td>West</td>
<td>57,945</td>
</tr>
</tbody>
</table>

Source: U.S. Bureau of the Census.

(a) Draw a bar chart of the data.
(b) Which chart seems to summarize the data better? Why?
(c) Which region has the most families?
(d) Which region has the fewest families?

8. Family Size The data below represent the number (in thousands) of families by size of the family in 2008.

<table>
<thead>
<tr>
<th>Family Size</th>
<th>Number of Families</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 people</td>
<td>35,797</td>
</tr>
<tr>
<td>3 people</td>
<td>17,431</td>
</tr>
<tr>
<td>4 people</td>
<td>14,972</td>
</tr>
<tr>
<td>5 people</td>
<td>6,952</td>
</tr>
<tr>
<td>6 people</td>
<td>2,382</td>
</tr>
<tr>
<td>7 or more</td>
<td>1,340</td>
</tr>
</tbody>
</table>

Source: U.S. Bureau of the Census.
9.2 Representing Qualitative Data Graphically: Bar Graphs; Pie Charts

(a) Draw a bar graph of the data.
(b) Draw a pie chart of the data.
(c) Which chart seems to summarize the data better? Why?
(d) What is the most common size of a family in the United States?
(e) What percent of families have 5 or more members?
(f) If you were a marketing consultant planning to market a product to families, what size families would you target? Why?

9. Household Income The data that follow represent the median income (in dollars) of families by type of household in 2007.

<table>
<thead>
<tr>
<th>Family Household</th>
<th>Median Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Married-couple families</td>
<td>$72,589</td>
</tr>
<tr>
<td>Female householder—no spouse</td>
<td>$30,296</td>
</tr>
<tr>
<td>Male householder—no spouse</td>
<td>$44,358</td>
</tr>
</tbody>
</table>

Source: U.S. Bureau of the Census.

(a) Draw a bar graph of the data.
(b) Which household type has the highest median income?
(c) Which household type has the lowest median income?
(d) Why do you think that there is such a large discrepancy?

10. Household Income The data that follow represent the median income (in dollars) of families by type of household in 2005.

<table>
<thead>
<tr>
<th>Family Household</th>
<th>Median Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Married-couple families</td>
<td>$66,067</td>
</tr>
<tr>
<td>Female householder—no spouse</td>
<td>$30,650</td>
</tr>
<tr>
<td>Male householder—no spouse</td>
<td>$46,756</td>
</tr>
</tbody>
</table>


(a) Draw a bar graph of the data.
(b) Which household type has the highest median income?
(c) Which household type has the lowest median income?
(d) Why do you think that there is such a large discrepancy?
(e) Comparing the data here with that in Problem 9, which group had the largest increase in median income from 2005 to 2007?

11. Causes of Death The data in the following table represent the causes of death for Americans in 2006.

<table>
<thead>
<tr>
<th>Cause of Death</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heart disease</td>
<td>631,636</td>
</tr>
<tr>
<td>Cancer</td>
<td>559,888</td>
</tr>
<tr>
<td>Cerebrovascular diseases</td>
<td>137,119</td>
</tr>
<tr>
<td>Respiratory diseases</td>
<td>124,583</td>
</tr>
<tr>
<td>Accidents</td>
<td>121,599</td>
</tr>
<tr>
<td>Diabetes mellitus</td>
<td>72,449</td>
</tr>
<tr>
<td>Alzheimer’s disease</td>
<td>72,432</td>
</tr>
<tr>
<td>Influenza and pneumonia</td>
<td>56,326</td>
</tr>
<tr>
<td>Kidney failure</td>
<td>45,344</td>
</tr>
<tr>
<td>Septicemia</td>
<td>34,234</td>
</tr>
<tr>
<td>Suicide</td>
<td>33,300</td>
</tr>
<tr>
<td>All other causes</td>
<td>537,354</td>
</tr>
</tbody>
</table>


(a) Draw a bar graph of the data.
(b) Is it appropriate to draw a pie chart? Justify your answer.
(c) What was the leading cause of death among Americans in 2006?
(d) What is the largest expense?
(e) What percent of expenditures is spent on maintenance and repair?

12. Automobile Costs The following data represent the total amount spent (in billions of dollars) in the United States on automobiles and their maintenance in 2008.

<table>
<thead>
<tr>
<th>Category</th>
<th>Expenditures</th>
</tr>
</thead>
<tbody>
<tr>
<td>New motor vehicles</td>
<td>$184.5</td>
</tr>
<tr>
<td>Net purchases of used motor vehicles</td>
<td>$105.4</td>
</tr>
<tr>
<td>Motor vehicle parts and accessories</td>
<td>$52.4</td>
</tr>
<tr>
<td>Motor vehicle fuels, lubricants, and fluids</td>
<td>$386.4</td>
</tr>
<tr>
<td>Motor vehicle maintenance and repair</td>
<td>$158.5</td>
</tr>
<tr>
<td>Other motor vehicle services</td>
<td>$63</td>
</tr>
</tbody>
</table>

Source: U.S. Bureau of Economic Analysis.

(a) Draw a bar graph of the data.
(b) Draw a pie chart of the data.
(c) Which graph seems to summarize the data better?
(d) What is the largest expense?
(e) What portion of the expenditures is spent on maintenance and repair?
13. **On-Time Performance**

The bar graph below represents the overall percentage of reported flight operations arriving on time in May 2010.

![Bar graph showing on-time performance of various airlines.](image)

**Source:** Bureau of Transportation Statistics.

(a) Which airline has the highest percentage of on-time flights?
(b) Which airline has the lowest percentage of on-time flights?
(c) Estimate the percentage of United Airlines’ flights that are on time.

14. **Income Required for a Loan**

The bar graph that follows shows the minimum annual income required for a $150,000 loan using fixed-loan interest rates available on July 11, 2010. Taxes and insurance are assumed to be $300 monthly.

![Bar graph showing income required for loans.](image)

**Source:** www.interest.com.

(a) Which loan requires the most annual income?  
(b) Which loan requires the least annual income?  
(c) Estimate the minimum annual income required for the loan types found in part (a) and part (b).

15. **Consumer Price Index**

The Consumer Price Index (CPI) is an index that measures inflation. It is calculated by obtaining the prices of a market basket of goods each month. The market basket, along with the percentages of each product (relative importance), for May 2010 is given in the following pie chart:

![Pie chart showing consumer price index.](image)
9.3 Organizing and Displaying Quantitative Data

**Source:** Bureau of Labor Statistics.

(a) What is the largest component of the CPI?

(b) What is the smallest component of the CPI?

(c) Senior citizens spend about 17.2% of their cash incomes on medical care. Why do you think they feel the CPI weight for medical care is too low?


16. **Asset Allocation**

According to financial planners, an individual’s investment mix should change over a person’s lifetime. The longer an individual’s time horizon, the more the individual should invest in stocks. Jim has 40 years to retirement, has a preference for growth, and can withstand significant fluctuation in market value, so his financial planner suggests that his retirement portfolio be diversified according to the asset mix provided in the pie chart.

**Source:** Fidelity Investments

(a) How much should Jim invest in stocks?

(b) How much should Jim invest in bonds?

(c) The return on bonds over long periods of time is less than that of stocks. Explain why you think the financial planner recommended what she did for bonds.

**9.3 Organizing and Displaying Quantitative Data**

**PREPARING FOR THIS SECTION**

Before getting started, review the following:

- Rectangular Coordinates (Section 1.1, pp. 2–3)

**NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 528.**

**OBJECTIVES**

1. Form a frequency table (p. 518)
2. Construct a line chart (p. 519)
3. Group data into class intervals (p. 519)
4. Build a histogram (p. 522)
5. Draw a frequency polygon (p. 526)
6. Draw an ogive (p. 527)
7. Identify the shape of a distribution (p. 528)

Often studies result in data represented by a large collection of numbers. If the data are to be interpreted, they must be organized. In this section we discuss the organization of quantitative data. One way to organize such data is by using a *frequency table*. 
EXAMPLE 1 Forming a Frequency Table

Table 3 lists the weights of a random sample of 71 children selected from a group of 10,000.

(a) List the data from smallest to highest in a table.

(b) Form a frequency table of the data.

### TABLE 3 WEIGHTS OF 71 STUDENTS, IN POUNDS

<table>
<thead>
<tr>
<th>69</th>
<th>71</th>
<th>71</th>
<th>55</th>
<th>52</th>
<th>55</th>
<th>58</th>
<th>58</th>
<th>62</th>
<th>67</th>
<th>94</th>
</tr>
</thead>
<tbody>
<tr>
<td>82</td>
<td>94</td>
<td>95</td>
<td>89</td>
<td>89</td>
<td>104</td>
<td>93</td>
<td>93</td>
<td>58</td>
<td>62</td>
<td>67</td>
</tr>
<tr>
<td>94</td>
<td>85</td>
<td>92</td>
<td>75</td>
<td>75</td>
<td>79</td>
<td>79</td>
<td>82</td>
<td>94</td>
<td>105</td>
<td>115</td>
</tr>
<tr>
<td>105</td>
<td>109</td>
<td>94</td>
<td>92</td>
<td>89</td>
<td>85</td>
<td>85</td>
<td>89</td>
<td>95</td>
<td>92</td>
<td>105</td>
</tr>
<tr>
<td>72</td>
<td>72</td>
<td>79</td>
<td>79</td>
<td>85</td>
<td>72</td>
<td>79</td>
<td>119</td>
<td>89</td>
<td>72</td>
<td>72</td>
</tr>
<tr>
<td>79</td>
<td>79</td>
<td>69</td>
<td>93</td>
<td>85</td>
<td>93</td>
<td>79</td>
<td>85</td>
<td>69</td>
<td>79</td>
<td></td>
</tr>
</tbody>
</table>

**SOLUTION** (a) Certain information available from the sample becomes more evident once the data are ordered according to some scheme. If the 71 measurements are written from smallest to highest, we obtain Table 4.

### TABLE 4

<table>
<thead>
<tr>
<th>52</th>
<th>55</th>
<th>55</th>
<th>58</th>
<th>58</th>
<th>58</th>
<th>58</th>
<th>62</th>
<th>62</th>
<th>67</th>
<th>67</th>
</tr>
</thead>
<tbody>
<tr>
<td>69</td>
<td>69</td>
<td>69</td>
<td>69</td>
<td>71</td>
<td>71</td>
<td>71</td>
<td>72</td>
<td>72</td>
<td>72</td>
<td></td>
</tr>
<tr>
<td>75</td>
<td>75</td>
<td>75</td>
<td>79</td>
<td>79</td>
<td>79</td>
<td>79</td>
<td>79</td>
<td>79</td>
<td>79</td>
<td></td>
</tr>
<tr>
<td>82</td>
<td>85</td>
<td>85</td>
<td>85</td>
<td>85</td>
<td>85</td>
<td>85</td>
<td>89</td>
<td>89</td>
<td>89</td>
<td></td>
</tr>
<tr>
<td>89</td>
<td>92</td>
<td>92</td>
<td>92</td>
<td>93</td>
<td>93</td>
<td>93</td>
<td>94</td>
<td>94</td>
<td>94</td>
<td></td>
</tr>
<tr>
<td>94</td>
<td>95</td>
<td>95</td>
<td>104</td>
<td>104</td>
<td>105</td>
<td>105</td>
<td>105</td>
<td>109</td>
<td>115</td>
<td>119</td>
</tr>
</tbody>
</table>

(b) Once Table 4 has been constructed, it becomes easy to present the data in a **frequency table**. This is done as follows: Each weight that occurs is listed once. Tally marks are used to record each occurrence of a weight. Then the **frequency** \( f \) with which each weight occurs is recorded. See Table 5.
9.3 Organizing and Displaying Quantitative Data

TABLE 5

<table>
<thead>
<tr>
<th>Weight</th>
<th>Tally</th>
<th>Frequency, ( f )</th>
<th>Weight</th>
<th>Tally</th>
<th>Frequency, ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>/</td>
<td>1</td>
<td>85</td>
<td>( \backslash \backslash \backslash )</td>
<td>7</td>
</tr>
<tr>
<td>55</td>
<td>( \backslash \backslash )</td>
<td>2</td>
<td>89</td>
<td>( \backslash \backslash )</td>
<td>5</td>
</tr>
<tr>
<td>58</td>
<td>( \backslash \backslash \backslash \backslash )</td>
<td>4</td>
<td>92</td>
<td>( \backslash \backslash )</td>
<td>3</td>
</tr>
<tr>
<td>62</td>
<td>( \backslash \backslash \backslash )</td>
<td>3</td>
<td>93</td>
<td>( \backslash \backslash \backslash )</td>
<td>4</td>
</tr>
<tr>
<td>67</td>
<td>( \backslash \backslash )</td>
<td>2</td>
<td>94</td>
<td>( \backslash \backslash \backslash )</td>
<td>5</td>
</tr>
<tr>
<td>69</td>
<td>( \backslash \backslash \backslash \backslash )</td>
<td>4</td>
<td>95</td>
<td>( \backslash \backslash )</td>
<td>2</td>
</tr>
<tr>
<td>71</td>
<td>( \backslash \backslash \backslash )</td>
<td>3</td>
<td>104</td>
<td>( \backslash \backslash )</td>
<td>2</td>
</tr>
<tr>
<td>72</td>
<td>( \backslash \backslash \backslash )</td>
<td>5</td>
<td>105</td>
<td>( \backslash \backslash \backslash )</td>
<td>3</td>
</tr>
<tr>
<td>75</td>
<td>( \backslash \backslash \backslash )</td>
<td>3</td>
<td>109</td>
<td>( \backslash \backslash )</td>
<td>1</td>
</tr>
<tr>
<td>79</td>
<td>( \backslash \backslash \backslash \backslash )</td>
<td>8</td>
<td>115</td>
<td>( \backslash \backslash )</td>
<td>1</td>
</tr>
<tr>
<td>82</td>
<td>( \backslash \backslash \backslash )</td>
<td>2</td>
<td>119</td>
<td>( \backslash \backslash )</td>
<td>1</td>
</tr>
</tbody>
</table>

NOW WORK PROBLEM 5(a).

EXAMPLE 2 Constructing a Line Chart

Use Table 5 to construct a line chart for the data given in Table 3.

SOLUTION A line chart is obtained using rectangular coordinates. The vertical axis (y-axis) denotes the frequency \( f \) and the horizontal axis (x-axis) denotes the weight data. Points are plotted according to the information in Table 5 and a vertical line is drawn from each point to the horizontal axis. See Figure 4.

NOW WORK PROBLEM 5(b).

Group Data into Class Intervals

When data are collected and few repeated entries are obtained, the data are usually easier to organize by grouping them and constructing a histogram.

Table 6 lists the monthly electric bills, in dollars, of a sample of 71 residential customers, beginning with the smallest bill.
The first step in grouping data is to calculate the range.

**Definition**

The range of a set of numbers is the difference between the largest and the smallest number in the set. That is

\[
\text{Range} = \text{Largest value} - \text{Smallest value}
\]

For the data in Table 6 the range is

\[
\text{Range} = 119.38 - 52.30 = 67.08
\]

To group these data, we divide the range into intervals of equal size, called class intervals. Table 7 shows the data using 14 class intervals, each of size $5$, beginning with the interval 50–54.99.

**Table 7**

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 50–54.99</td>
<td>/</td>
<td>1</td>
</tr>
<tr>
<td>2 55–59.99</td>
<td>//</td>
<td>6</td>
</tr>
<tr>
<td>3 60–64.99</td>
<td>///</td>
<td>3</td>
</tr>
<tr>
<td>4 65–69.99</td>
<td>//</td>
<td>6</td>
</tr>
<tr>
<td>5 70–74.99</td>
<td>///</td>
<td>8</td>
</tr>
<tr>
<td>6 75–79.99</td>
<td>///</td>
<td>11</td>
</tr>
<tr>
<td>7 80–84.99</td>
<td>//</td>
<td>2</td>
</tr>
<tr>
<td>8 85–89.99</td>
<td>///</td>
<td>12</td>
</tr>
<tr>
<td>9 90–94.99</td>
<td>///</td>
<td>12</td>
</tr>
<tr>
<td>10 95–99.99</td>
<td>//</td>
<td>2</td>
</tr>
<tr>
<td>11 100–104.99</td>
<td>//</td>
<td>2</td>
</tr>
<tr>
<td>12 105–109.99</td>
<td>///</td>
<td>4</td>
</tr>
<tr>
<td>13 110–114.99</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>14 115–119.99</td>
<td>//</td>
<td>2</td>
</tr>
</tbody>
</table>
9.3 Organizing and Displaying Quantitative Data

Example 3 below shows how to group the data of Table 6 into class intervals of size $10.

**EXAMPLE 3  Grouping Data into Class Intervals**

For the monthly electric bills listed in Table 6, group the data into class intervals each of size $10.

**SOLUTION** First we determine the first class interval. Since the smallest bill is $52.30 and we want a class interval of size $10, we choose $50 to $59.99 as the first interval. The intervals will be from $50.00 to $59.99, from $60.00 to $69.99, up to the interval $110 to $119.99. We can stop here since the largest bill, $119.38, is in this interval. Table 8 shows the result of using class intervals of size $10.

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 50–59.99</td>
<td><img src="image" alt="Tally" /></td>
<td>7</td>
</tr>
<tr>
<td>2 60–69.99</td>
<td><img src="image" alt="Tally" /></td>
<td>9</td>
</tr>
<tr>
<td>3 70–79.99</td>
<td><img src="image" alt="Tally" /></td>
<td>19</td>
</tr>
<tr>
<td>4 80–89.99</td>
<td><img src="image" alt="Tally" /></td>
<td>14</td>
</tr>
<tr>
<td>5 90–99.99</td>
<td><img src="image" alt="Tally" /></td>
<td>14</td>
</tr>
<tr>
<td>6 100–109.99</td>
<td><img src="image" alt="Tally" /></td>
<td>6</td>
</tr>
<tr>
<td>7 110–119.99</td>
<td><img src="image" alt="Tally" /></td>
<td>2</td>
</tr>
</tbody>
</table>

Now work Problem 5(c).

We use Tables 7 and 8 to introduce some vocabulary.

The class intervals shown in Tables 7 and 8 each begin at 50 and end at 119.99, so as to include all the data from Table 6. The first number in a class interval is called the lower class limit; the second number is called the upper class limit. We choose these limits so that each item in Table 6 can be assigned to one and only one class interval. The midpoint of a class interval is found by adding two consecutive lower class limits and dividing the sum by 2. The class width is the difference between consecutive lower class limits.

When the data are represented in the form of Table 7 (or Table 8), they are said to be grouped data. Notice that once raw data are converted to grouped data, it is impossible to retrieve or recover the original data. The best we can do is to choose the midpoint of each class interval as a representative for each class. In Table 8, for example, the actual electric bills of $104.31, $104.71, $105.21, $105.37, $105.71, and $109.34 are viewed as being represented by the midpoint of the class interval from 100 to 109.99, namely, $100 + 110 \over 2 = 105$.

Next we present the grouped data of Table 8 in a graph called a histogram.
EXAMPLE 4  Building a Histogram

Build a histogram for the grouped data of Table 8.

**SOLUTION**

To build a histogram for the data in Table 8, we construct a set of adjoining rectangles having as base the size of the class interval and as height the frequency of occurrence of data in that particular interval. The center of the base is the midpoint of each class interval. Figure 5 shows the histogram for the data in Table 8.

**FIGURE 5**

![Histogram](image)

NOW WORK PROBLEM 5(d).

---

**EXAMPLE 5  Using Excel**

Use Excel to work Examples 3 and 4.

**SOLUTION**

**STEP 1** Enter the data given in Table 6 in column B. [This is the tedious part. Usually the data will be entered electronically.]

In column D, enter the maximum value of the class intervals of the frequency table. These are called the **Bin values** in Excel.

<table>
<thead>
<tr>
<th></th>
<th>Electric Bills</th>
<th>Bin</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>52.3</td>
<td>43.99</td>
</tr>
<tr>
<td>2</td>
<td>56.61</td>
<td>53.99</td>
</tr>
<tr>
<td>3</td>
<td>55.71</td>
<td>63.99</td>
</tr>
<tr>
<td>4</td>
<td>58.01</td>
<td>73.99</td>
</tr>
<tr>
<td>5</td>
<td>58.41</td>
<td>83.99</td>
</tr>
<tr>
<td>6</td>
<td>58.51</td>
<td>93.99</td>
</tr>
<tr>
<td>7</td>
<td>58.91</td>
<td>103.99</td>
</tr>
<tr>
<td>8</td>
<td>62.33</td>
<td>113.99</td>
</tr>
<tr>
<td>9</td>
<td>62.5</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>62.71</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>67.13</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>67.23</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>68.51</td>
<td></td>
</tr>
</tbody>
</table>
9.3 Organizing and Displaying Quantitative Data

**STEP 2** Click on Tools and then Data Analysis. (If Data Analysis isn’t available, click on Add-ins.)

![Data Analysis dialog box](image1)

**STEP 3** Highlight Histogram and click on OK.

![Histogram selection](image2)
**STEP 4**  
**Input Range** is the cells containing the data.  
**Bin Range** is the cells containing the class intervals.  
**Output Range** is the upper left cell where the frequency table will go.  
**New Worksheet Ply** is the name of the histogram.  
**Cumulative Percentage** gives the percentage of the data less than or equal to the bin value.  
**Chart Output** gives the histogram.

To enter the data, click and drag on the appropriate cells.

The frequency table and histogram are produced after clicking on OK.
**STEP 5** To eliminate the spaces between the bars, double click on one of the bars to get the screen below.

Select the Options tab and make the gap zero.
The final histogram is given next.
NOW WORK PROBLEMS 5(a), (b), (c), AND (d) USING EXCEL.

5 Draw a Frequency Polygon

Look again at Figure 5. If we connect the midpoints of the tops of the rectangles in Figure 5 with straight line segments, we obtain a line graph called a **frequency polygon**. Figure 6 shows the frequency polygon for the histogram obtained in Example 4.

NOW WORK PROBLEM 5(e).

Sometimes it is useful to determine how many data points fall below a certain value. This is called **calculating cumulative frequencies**, and it is done by adding the frequencies from all class intervals less than or equal to the class interval being considered.

**EXAMPLE 6 Creating a Cumulative Frequency Table**

Create a cumulative frequency table for the data in Table 8.

**SOLUTION**

We compute the cumulative frequency by determining the frequency of measurements at or below a given point. For the data in Table 8 we start in the lowest class interval ($50–$59.99) and note that there are 7 bills in this interval. So we put 7 in the column labeled $cf$ (cumulative frequency) of Table 9 in the row for $50–$59.99. Next we add the number of bills in the second class interval ($60–$69.99), 9, to the cumulative total from the first interval $7 + 9 = 16$, and place 16 in column $cf$ of the second interval. We then continue with the third interval, adding 19 to 16, and place the sum, 35, in the $cf$ column
of the third interval, and so on. The last entry in the $cf$ column should equal the total number of bills in the sample. The numbers in the $cf$ column are called the **cumulative frequencies**.

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>$f$</th>
<th>$cf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50–59.99</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>60–69.99</td>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>70–79.99</td>
<td>19</td>
<td>35</td>
</tr>
<tr>
<td>80–89.99</td>
<td>14</td>
<td>49</td>
</tr>
<tr>
<td>90–99.99</td>
<td>14</td>
<td>63</td>
</tr>
<tr>
<td>100–109.99</td>
<td>6</td>
<td>69</td>
</tr>
<tr>
<td>110–119.99</td>
<td>2</td>
<td>71</td>
</tr>
</tbody>
</table>

**NOW WORK PROBLEM 5(f).**

### 6 Draw an Ogive

The graph in which the horizontal axis represents class intervals and the vertical axis represents cumulative frequencies is called an **ogive** (pronounced “Oh jive”).

**EXAMPLE 7 Draw an Ogive**

Use the cumulative frequency table in Table 9 to draw an ogive.

**SOLUTION**

Draw two axes and put the class intervals on the horizontal axis and the cumulative frequencies on the vertical axis. Plot 0 at the lower class limit of the smallest interval and then plot each cumulative frequency at the upper class limit of each class interval. Finally connect the points with straight line segments. See Figure 7.

**FIGURE 7**

NOW WORK PROBLEM 5(g).
Identify the Shape of a Distribution

One advantage of using histograms to display data is that they show the shape of a distribution. While many shapes are possible, two shapes are of particular interest. When the data in the frequency table are evenly spaced around a central point, the distribution is called **symmetric**.

Symmetric distributions are either bell shaped or uniform. Common examples of symmetric distributions include IQ scores (bell shaped) and the outcomes when a fair die is repeatedly tossed (uniform). Histograms depicting symmetric distributions are shown in Figures 8(a) and (b).

(a) Bell shaped  (b) Uniform  (c) Skewed right  (d) Skewed left

**Skewed** distributions have most outcomes clustered at one end of the distribution with fewer values at the other end. When the sparse data are to the right of the clustered data, the distribution is called **skewed right**. A histogram of the distribution will have a tail on the right. See Figure 8(c). A good example of a skewed right distribution is annual income. Most individuals have incomes clustered around $46,000, but a few individuals, such as some CEOs and entertainers, have much greater incomes.

When the sparse data are to the left of the clustered data, the distribution is called **skewed left** and a histogram of the distribution will have a longer tail on the left, as in Figure 8(d). An example of a skewed left distribution is age of death in the United States.

**EXERCISE 9.3** Answers Begin on Page AN–45.

`Are You Prepared?` Problem  Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. List the quadrant in which each point is located. (pp. 2–3)
   (a)  (b) (1, 5)

Concepts and Vocabulary

2. True or False  If data are skewed right, then the histogram has a long tail at the right end of the graph.

3. When grouping data, the range is divided into ______ of equal size.

4. The graph of a cumulative frequency distribution is called an ______.

Skill Building

5. The following scores were made on a 60-item test:

<table>
<thead>
<tr>
<th>25</th>
<th>30</th>
<th>34</th>
<th>37</th>
<th>41</th>
<th>42</th>
<th>46</th>
<th>49</th>
<th>53</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>31</td>
<td>34</td>
<td>37</td>
<td>41</td>
<td>42</td>
<td>46</td>
<td>50</td>
<td>53</td>
</tr>
<tr>
<td>28</td>
<td>31</td>
<td>35</td>
<td>37</td>
<td>41</td>
<td>43</td>
<td>47</td>
<td>51</td>
<td>54</td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>36</td>
<td>38</td>
<td>41</td>
<td>44</td>
<td>48</td>
<td>52</td>
<td>54</td>
</tr>
<tr>
<td>30</td>
<td>33</td>
<td>36</td>
<td>39</td>
<td>41</td>
<td>44</td>
<td>48</td>
<td>52</td>
<td>55</td>
</tr>
<tr>
<td>30</td>
<td>33</td>
<td>37</td>
<td>40</td>
<td>42</td>
<td>45</td>
<td>48</td>
<td>52</td>
<td></td>
</tr>
</tbody>
</table>

(a) Set up a frequency table for the above data.
(b) Draw a line chart for the data.
(c) Group the data into class intervals of size 2, beginning with the interval 24–25.9.
(d) Build the histogram for the data.
(e) Draw the frequency polygon for this histogram.
(f) Find the cumulative frequencies.
(g) Draw the ogive.
9.3 Organizing and Displaying Quantitative Data

In Problems 9 and 10, identify the shape of the distribution as uniform, bell shaped, skewed right, or skewed left.

6. Use the test scores given in Problem 5.
   (a) Group the data into class intervals of size 5, beginning with the interval 24–28.9.
   (b) Build the histogram for the data.
   (c) Draw the frequency polygon for the histogram.
   (d) Find the cumulative frequencies.
   (e) Draw the ogive.

7. Use Table 5 (p. 519) in the text.
   (a) Group the data into class intervals of size 5 beginning with the interval 50–54.9.
   (b) Build the histogram for the data.
   (c) Draw the frequency polygon for this histogram.
   (d) Find the cumulative frequencies.
   (e) Draw the ogive.

8. Use Table 5 (p. 519) in the text.
   (a) Group the data into class intervals of size 10, beginning with the interval 50–59.9. How many class intervals now exist?
   (b) Build the histogram for the data.
   (c) Draw the frequency polygon for this histogram.
   (d) Find the cumulative frequencies.
   (e) Draw the ogive.
   (f) Discuss the advantages/disadvantages of using a class interval size of 10 compared to a smaller interval size such as 5.

Applications

11. Licensed Drivers in Florida The histogram that follows gives the number of licensed drivers between the ages of 20 and 84 in the state of Florida in 2008.

   ![Histogram of licensed drivers in Florida]

   **Source:** Federal Highway Administration, Highway Statistics 2008.
   (a) Determine the number of class intervals.
   (b) What is the lower class limit of the first class interval? What is the upper class limit of the first class interval?
   (c) Determine the class width.
   (d) Approximately how many licensed drivers are 70 to 84 years old?
   (e) Which class interval has the most licensed drivers?
   (f) Which class interval has the fewest licensed drivers?
   (g) Describe the shape of the distribution.
   (h) Draw a frequency polygon for the given data.

12. IQ Scores The histogram below represents the IQ scores of students enrolled in College Algebra at a local university.

   ![Histogram of IQ scores]

   (a) Determine the number of class intervals.
   (b) What is the lower class limit of the first class interval? What is the upper class limit of the first class interval?
   (c) Determine the class width.
   (d) How many students have an IQ between 100 and 104?
   (e) How many students have an IQ above 110?
   (f) How many students are enrolled in College Algebra?
   (g) Describe the shape of the distribution.
   (h) Draw a frequency polygon for the given data.
13. **Licensed Drivers in Tennessee** The frequency table below provides the number of licensed drivers between the ages of 20 and 84 in the state of Tennessee in 2008.

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of Licensed Drivers</th>
</tr>
</thead>
<tbody>
<tr>
<td>20–24</td>
<td>348,587</td>
</tr>
<tr>
<td>25–29</td>
<td>372,032</td>
</tr>
<tr>
<td>30–34</td>
<td>365,111</td>
</tr>
<tr>
<td>35–39</td>
<td>404,931</td>
</tr>
<tr>
<td>40–44</td>
<td>406,831</td>
</tr>
<tr>
<td>45–49</td>
<td>435,706</td>
</tr>
<tr>
<td>50–54</td>
<td>422,420</td>
</tr>
<tr>
<td>55–59</td>
<td>380,898</td>
</tr>
<tr>
<td>60–64</td>
<td>331,010</td>
</tr>
<tr>
<td>65–69</td>
<td>250,679</td>
</tr>
<tr>
<td>70–74</td>
<td>182,974</td>
</tr>
<tr>
<td>75–79</td>
<td>133,888</td>
</tr>
<tr>
<td>80–84</td>
<td>91,318</td>
</tr>
</tbody>
</table>

**Source:** Federal Highway Administration, Highway Statistics 2008.

(a) Determine the number of class intervals.
(b) What is the lower class limit of the first class interval? What is the upper class limit of the first class interval?
(c) Determine the class width.
(d) Build the histogram for the data.
(e) Draw a frequency polygon for the data.
(f) Which age group has the most licensed drivers?
(g) Which age group has the fewest licensed drivers?

14. **Licensed Drivers in Hawaii** The frequency table below provides the number of licensed drivers between the ages of 20 and 84 in the state of Hawaii in 2008.

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of Licensed Drivers</th>
</tr>
</thead>
<tbody>
<tr>
<td>20–24</td>
<td>73,199</td>
</tr>
<tr>
<td>25–29</td>
<td>85,097</td>
</tr>
<tr>
<td>30–34</td>
<td>78,820</td>
</tr>
<tr>
<td>35–39</td>
<td>83,318</td>
</tr>
<tr>
<td>40–44</td>
<td>83,966</td>
</tr>
<tr>
<td>45–49</td>
<td>89,520</td>
</tr>
<tr>
<td>50–54</td>
<td>88,939</td>
</tr>
</tbody>
</table>

**Source:** Federal Highway Administration, Highway Statistics 2008.

(a) Determine the number of class intervals.
(b) What is the lower class limit of the first class interval? What is the upper class limit of the first class interval?
(c) Determine the class width.
(d) Build the histogram for the data.
(e) Draw a frequency polygon for the data.
(f) Which age group has the most licensed drivers?
(g) Which age group has the fewest licensed drivers?

15. **Undergraduate Tuition** The data below represent the cost of undergraduate tuition and fees at all four-year public and private nonprofit colleges in the United States for 2006–2007.

<table>
<thead>
<tr>
<th>Tuition (Dollars)</th>
<th>Number of 4-Year Colleges</th>
<th>Tuition (Dollars)</th>
<th>Number of 4-Year Colleges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0–1,999</td>
<td>19</td>
<td>$20,000–21,999</td>
<td>117</td>
</tr>
<tr>
<td>$2,000–3,999</td>
<td>132</td>
<td>$22,000–23,999</td>
<td>100</td>
</tr>
<tr>
<td>$4,000–5,999</td>
<td>308</td>
<td>$24,000–25,999</td>
<td>83</td>
</tr>
<tr>
<td>$6,000–7,999</td>
<td>180</td>
<td>$26,000–27,999</td>
<td>53</td>
</tr>
<tr>
<td>$8,000–9,999</td>
<td>106</td>
<td>$28,000–29,999</td>
<td>45</td>
</tr>
<tr>
<td>$10,000–11,999</td>
<td>99</td>
<td>$30,000–31,999</td>
<td>36</td>
</tr>
<tr>
<td>$12,000–13,999</td>
<td>89</td>
<td>$32,000–33,999</td>
<td>46</td>
</tr>
<tr>
<td>$14,000–15,999</td>
<td>98</td>
<td>$34,000–35,999</td>
<td>32</td>
</tr>
<tr>
<td>$16,000–17,999</td>
<td>125</td>
<td>$36,000–37,999</td>
<td>6</td>
</tr>
<tr>
<td>$18,000–19,999</td>
<td>116</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Source:** Chronicle of Higher Education, Tuition and Fees, 2006–2007.

(a) Determine the number of class intervals.
(b) What is the lower class limit of the first class interval? What is the upper class limit of the first class interval?
(c) Determine the class width.
18. **Birth Rates** The following data give the 2007 birth rates per 1000 women for women between the ages of 15 and 44 years old.

<table>
<thead>
<tr>
<th>Age</th>
<th>Birth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>15–19</td>
<td>42.5</td>
</tr>
<tr>
<td>20–24</td>
<td>106.4</td>
</tr>
<tr>
<td>25–29</td>
<td>117.5</td>
</tr>
<tr>
<td>30–34</td>
<td>99.9</td>
</tr>
<tr>
<td>35–39</td>
<td>47.5</td>
</tr>
<tr>
<td>40–44</td>
<td>9.5</td>
</tr>
</tbody>
</table>


(a) Determine the number of class intervals.
(b) What is the lower class limit of the first class interval?
   What is the upper class limit of the first class interval?
(c) Determine the class width.
(d) Build the histogram for the data.
(e) Draw a frequency polygon of the data.
(f) What range of tuition occurs most frequently?

16. **Hourly Earnings** The following table gives the average hourly earnings of production workers in manufacturing in 2008 for 20 states.

<table>
<thead>
<tr>
<th>State</th>
<th>Hourly Earnings</th>
</tr>
</thead>
<tbody>
<tr>
<td>AK</td>
<td>17.31</td>
</tr>
<tr>
<td>CA</td>
<td>16.79</td>
</tr>
<tr>
<td>HI</td>
<td>18.93</td>
</tr>
<tr>
<td>KY</td>
<td>17.38</td>
</tr>
<tr>
<td>MI</td>
<td>22.11</td>
</tr>
<tr>
<td>PA</td>
<td>15.77</td>
</tr>
<tr>
<td>VT</td>
<td>16.51</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State</th>
<th>Hourly Earnings</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL</td>
<td>15.68</td>
</tr>
<tr>
<td>CT</td>
<td>21.42</td>
</tr>
<tr>
<td>IL</td>
<td>16.44</td>
</tr>
<tr>
<td>MA</td>
<td>20.33</td>
</tr>
<tr>
<td>NH</td>
<td>17.31</td>
</tr>
<tr>
<td>RI</td>
<td>13.94</td>
</tr>
<tr>
<td>WI</td>
<td>17.94</td>
</tr>
</tbody>
</table>

**Source:** Northeast-Midwest Institute.

(a) Group the data into 5 class intervals of equal width beginning with the interval 13.00–14.99.
(b) Build the histogram for these data.
(c) Draw the frequency polygon for this histogram.

17. **Birth Rates** The following data give the 2007 birth rates per 1000 people for 20 states.

<table>
<thead>
<tr>
<th>State</th>
<th>Birth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL</td>
<td>14</td>
</tr>
<tr>
<td>CA</td>
<td>15.5</td>
</tr>
<tr>
<td>FL</td>
<td>13.1</td>
</tr>
<tr>
<td>KS</td>
<td>15.1</td>
</tr>
<tr>
<td>NV</td>
<td>16.1</td>
</tr>
<tr>
<td>PA</td>
<td>12.1</td>
</tr>
<tr>
<td>VA</td>
<td>14.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State</th>
<th>Birth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AZ</td>
<td>16.2</td>
</tr>
<tr>
<td>CT</td>
<td>11.9</td>
</tr>
<tr>
<td>GA</td>
<td>15.9</td>
</tr>
<tr>
<td>LA</td>
<td>15.4</td>
</tr>
<tr>
<td>NJ</td>
<td>13.4</td>
</tr>
<tr>
<td>SC</td>
<td>14.3</td>
</tr>
<tr>
<td>WI</td>
<td>13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State</th>
<th>Birth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AK</td>
<td>16.2</td>
</tr>
<tr>
<td>DE</td>
<td>14.1</td>
</tr>
<tr>
<td>IL</td>
<td>14.1</td>
</tr>
<tr>
<td>MI</td>
<td>12.4</td>
</tr>
<tr>
<td>OR</td>
<td>13.2</td>
</tr>
<tr>
<td>TN</td>
<td>14.1</td>
</tr>
</tbody>
</table>

**Source:** National Vital Statistics Reports, Vol. 57, No. 12, March 2009.

(a) Group the data into 8 class intervals of equal width beginning with the interval 11.9–12.4.
(b) Build the histogram for these data.
(c) Draw the frequency polygon for this histogram.

19. **HIV Death Rates** The following data give the 2007 death rates (number of deaths per 100,000 people) from HIV-related illnesses for 20 states.

<table>
<thead>
<tr>
<th>State</th>
<th>Death Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL</td>
<td>4.0</td>
</tr>
<tr>
<td>CT</td>
<td>4.0</td>
</tr>
<tr>
<td>FL</td>
<td>8.4</td>
</tr>
<tr>
<td>IL</td>
<td>2.4</td>
</tr>
<tr>
<td>IA</td>
<td>3.3</td>
</tr>
<tr>
<td>KY</td>
<td>1.3</td>
</tr>
<tr>
<td>MD</td>
<td>7.8</td>
</tr>
<tr>
<td>MI</td>
<td>1.9</td>
</tr>
<tr>
<td>NV</td>
<td>3.1</td>
</tr>
<tr>
<td>NJ</td>
<td>5.7</td>
</tr>
<tr>
<td>NC</td>
<td>4.2</td>
</tr>
<tr>
<td>OR</td>
<td>1.4</td>
</tr>
<tr>
<td>VA</td>
<td>3.0</td>
</tr>
<tr>
<td>WA</td>
<td>1.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State</th>
<th>Death Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AZ</td>
<td>1.7</td>
</tr>
<tr>
<td>HI</td>
<td>1.6</td>
</tr>
<tr>
<td>IA</td>
<td>3.3</td>
</tr>
<tr>
<td>KY</td>
<td>1.3</td>
</tr>
<tr>
<td>MO</td>
<td>2.2</td>
</tr>
<tr>
<td>NY</td>
<td>7.0</td>
</tr>
<tr>
<td>VA</td>
<td>3.0</td>
</tr>
<tr>
<td>WI</td>
<td>0.9</td>
</tr>
</tbody>
</table>


(a) Group the data into 5 class intervals of equal width beginning with the interval 0–1.9.
(b) Build the histogram for these data.
(c) Draw the frequency polygon for this histogram.
20. **Cancer Death Rates** The following data give the 2007 death rates (number of deaths per 100,000 people) attributed to cancer for 20 states.

<table>
<thead>
<tr>
<th>State</th>
<th>Death Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL</td>
<td>216.6</td>
</tr>
<tr>
<td>AZ</td>
<td>159.9</td>
</tr>
<tr>
<td>CA</td>
<td>150.5</td>
</tr>
<tr>
<td>CT</td>
<td>194.9</td>
</tr>
<tr>
<td>FL</td>
<td>219.6</td>
</tr>
<tr>
<td>HI</td>
<td>172.5</td>
</tr>
<tr>
<td>IL</td>
<td>187.6</td>
</tr>
<tr>
<td>IA</td>
<td>213.4</td>
</tr>
<tr>
<td>KY</td>
<td>228.5</td>
</tr>
<tr>
<td>MD</td>
<td>181.2</td>
</tr>
<tr>
<td>MI</td>
<td>199.4</td>
</tr>
<tr>
<td>MO</td>
<td>210.6</td>
</tr>
<tr>
<td>NV</td>
<td>168.8</td>
</tr>
<tr>
<td>NJ</td>
<td>196.8</td>
</tr>
<tr>
<td>NY</td>
<td>183.9</td>
</tr>
<tr>
<td>NC</td>
<td>192.9</td>
</tr>
<tr>
<td>OR</td>
<td>197.3</td>
</tr>
<tr>
<td>VA</td>
<td>181.6</td>
</tr>
<tr>
<td>WA</td>
<td>178.8</td>
</tr>
<tr>
<td>WI</td>
<td>195.7</td>
</tr>
</tbody>
</table>


(a) Group the data into 4 class intervals of equal width beginning with the interval 150–169.9.

(b) Build the histogram for these data.

(c) Draw the frequency polygon for this histogram.

**9.4 Measures of Central Tendency**

The idea of taking an average is familiar to practically everyone. Often we hear people talk about average salary, average height, average grade, and so on. The idea of average is so commonly used it should not surprise you to learn that several kinds of averages are used in statistics.

Averages are called *measures of central tendency* because they estimate the “center” of the data collected. The three most common measures of central tendency are the (arithmetic) mean, median, and mode. Of these, the mean is the one most often used.

**Definition**

**Arithmetic Mean**

The **arithmetic mean**, or **mean**, of a set of \( n \) real numbers \( x_1, x_2, \ldots, x_n \) is defined as the number

\[
\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}
\]

(1)

where \( n \) is the number of items in the set.
Find the Mean of a Set of Data

**EXAMPLE 1** Finding the Mean of a Set of Data

Professor Murphy tested 8 students in her Finite Mathematics class. The scores on the test were 100, 94, 85, 79, 70, 69, 65, and 62. What is the mean score?

**SOLUTION** To compute the mean $\bar{x}$, add up the 8 scores and divide by 8.

$$\bar{x} = \frac{100 + 94 + 85 + 79 + 70 + 69 + 65 + 62}{8} = \frac{624}{8} = 78$$

Now work Problem 7(a).

An interesting fact about the mean is that the sum of deviations of each item from the mean is zero. In Example 1 the deviation of each score from the mean $\bar{x} = 78$ is $(100 - 78), (94 - 78), (85 - 78), (79 - 78), (70 - 78), (69 - 78), (65 - 78),$ and $(62 - 78)$. Table 10 lists each score, the mean, and the deviation from the mean. If we add the deviations from the mean, we obtain a sum of zero.

<table>
<thead>
<tr>
<th>Score</th>
<th>Mean</th>
<th>Deviation from Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>62</td>
<td>78</td>
<td>-16</td>
</tr>
<tr>
<td>65</td>
<td>78</td>
<td>-13</td>
</tr>
<tr>
<td>69</td>
<td>78</td>
<td>-9</td>
</tr>
<tr>
<td>70</td>
<td>78</td>
<td>-8</td>
</tr>
<tr>
<td>79</td>
<td>78</td>
<td>-8</td>
</tr>
<tr>
<td>85</td>
<td>78</td>
<td>1</td>
</tr>
<tr>
<td>94</td>
<td>78</td>
<td>7</td>
</tr>
<tr>
<td>100</td>
<td>78</td>
<td>16</td>
</tr>
</tbody>
</table>

Sum of Deviations: 0

For any set of data the following result is true:

**Theorem** For any set of data, the sum of the deviations from the mean is zero.
Find the Mean for Grouped Data

EXAMPLE 2 Finding the Mean for Grouped Data

Find the mean for the grouped data given in Table 8 (repeated in the first two columns of Table 11 below).

SOLUTION Follow the steps below.

STEP 1 Find the midpoint \( m_i \) of each of the class intervals and enter the result in column 3 of Table 11. For example, the midpoint of the class interval 80–89.99 is

\[
\frac{80 + 90}{2} = 85
\]

STEP 2 Multiply the entry \( m_i \) in column 3 by the frequency \( f_i \) for that class interval and enter the product in column 4, which is labeled \( f_i m_i \).

STEP 3 Add the entries in column 4; sum of \( f_i m_i = 5775 \).

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>( f_i )</th>
<th>( m_i )</th>
<th>( f_i m_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50–59.99</td>
<td>7</td>
<td>55</td>
<td>385</td>
</tr>
<tr>
<td>60–69.99</td>
<td>9</td>
<td>65</td>
<td>585</td>
</tr>
<tr>
<td>70–79.99</td>
<td>19</td>
<td>75</td>
<td>1425</td>
</tr>
<tr>
<td>80–89.99</td>
<td>14</td>
<td>85</td>
<td>1190</td>
</tr>
<tr>
<td>90–99.99</td>
<td>14</td>
<td>95</td>
<td>1330</td>
</tr>
<tr>
<td>100–109.99</td>
<td>6</td>
<td>105</td>
<td>630</td>
</tr>
<tr>
<td>110–119.99</td>
<td>2</td>
<td>115</td>
<td>230</td>
</tr>
<tr>
<td>( n = 71 )</td>
<td></td>
<td></td>
<td>5775 = Sum of ( f_i m_i )</td>
</tr>
</tbody>
</table>

STEP 4 The mean \( \bar{x} \) is then computed by dividing the sum of \( f_i m_i \) by the number \( n \) of entries. That is,

\[
\bar{x} = \frac{5775}{71} = 81.34
\]

Definition Arithmetic Mean for Grouped Data

For grouped data, the mean \( \bar{x} \) is given as

\[
\bar{x} = \frac{\sum f_i m_i}{n} \tag{2}
\]

where

- \( \sum \) denotes the sum of the entries \( f_i m_i \)
- \( f_i \) = number of entries in the \( i \)th class interval
- \( m_i \) = midpoint of \( i \)th class interval
- \( n \) = number of items
When data are grouped, the original data are lost due to grouping. As a result, the mean obtained by using Formula (2) is only an approximation to the actual mean. The reason for this is that using Formula (2) amounts to computing the weighted average midpoint of the class intervals (weighted by the frequency of items in that interval).

**COMMENT:** Graphing utilities can be used to find measures of central tendency for both grouped and ungrouped data. Figure 9 illustrates the calculation of the mean for the grouped data in Example 2 on a TI-84 Plus. The midpoint for each interval was entered into L₁ and the frequency into L₂. Then one-variable statistics were computed using L₁ and L₂. The mean is $\bar{x} = 81.34$ (rounded to two decimal places). Consult your user’s manual for the commands needed to compute measures of central tendency on your graphing utility.

**EXAMPLE 3 Finding the Median of a Set of Data**

(a) The set of data 2, 2, 3, 4, 5, 7, 7, 7, 11 has median 5.

(b) The set of data 2, 2, 3, 3, 4, 5, 7, 7, 7, 11 has median 4.5 since

\[
\frac{4 + 5}{2} = 4.5
\]

NOW WORK PROBLEM 7(b).

**4 Find the Median for Grouped Data**

Finding the median for grouped data requires more work. As with the mean, the median for grouped data only approximates the actual median that would have been obtained prior to grouping the data.
EXAMPLE 4  Finding the Median for Grouped Data

Find the grouped data in Table 12 (based on Table 8, p. 521.)

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 50–59.99</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>2 60–69.99</td>
<td>//</td>
<td>9</td>
</tr>
<tr>
<td>3 70–79.99</td>
<td>/////</td>
<td>19</td>
</tr>
<tr>
<td>4 80–89.99</td>
<td>/////</td>
<td>14</td>
</tr>
<tr>
<td>5 90–99.99</td>
<td>/////</td>
<td>14</td>
</tr>
<tr>
<td>6 100–109.99</td>
<td>//</td>
<td>6</td>
</tr>
<tr>
<td>7 110–119.99</td>
<td>//</td>
<td>2</td>
</tr>
</tbody>
</table>

SOLUTION  Follow the steps below.

STEP 1  Find the interval containing the median.

The median is the middle value when the 71 items are in ascending order. So the median in this sample is the 36th entry. We start counting the tallies, beginning with the class interval 50–59.99, until we come as close as we can to 36. This brings us to the class interval 70–79.99, since through this interval there are 35 data items. The median will lie in the next class interval, 80–89.99.

STEP 2  In the interval containing the median, count the number \( p \) of items remaining to reach the median.

Since we have accounted for 35 data items, there is \( 36 - 35 = 1 \) item left, so \( p = 1 \).

STEP 3  If \( f \) is the frequency for the interval containing the median and \( i \) is the class width, then the interpolation factor is

\[
\text{interpolation factor} = \frac{p}{f} \cdot i
\]

For the grouped data in Table 12, we have \( p = 1, f = 14, \) and \( i = 10 \), so

\[
\text{interpolation factor} = \frac{p}{f} \cdot i = \frac{1}{14} \cdot 10 = 0.71 \quad \text{for} \quad p = 1, f = 14, \ i = 10
\]

STEP 4  The median \( M \) is

\[
M = \left[ \text{lower limit of interval containing the median} \right] + \text{[interpolation factor]}
\]

For the grouped data in Table 12, the median \( M \) is

\[
M = 80 + 0.71 = 80.71
\]
### Definition: Median for Grouped Data

The median $M$ for grouped data is approximated by

$$M = \left( \text{lower limit of interval containing the median} \right) + \left( \frac{P \cdot i}{f} \right)$$

where

- $p = \text{number required to reach the median from the lower limit of the interval containing the median}$
- $f = \text{frequency in this interval}$
- $i = \text{class width}$

**NOW WORK PROBLEM 19(b).**

As with the mean, the median of a set of grouped data is an approximation to the actual median. For Example 4, if we go back to the original data listed in Table 6, we obtain the true median $M = 82.32$.

The median of a set of data or grouped data is sometimes called the **fiftieth percentile** and is denoted by $Q_{50}$ to indicate that 50% of the data are less than or equal to it. Similarly, we can define $Q_{25}$, or the first quartile, as the point that separates the lowest 25% of the data from the upper 75%, and $Q_{75}$, or the third quartile, as the point that separates the lower 75% of the data from the upper 25% of the data.

### USING TECHNOLOGY

**EXAMPLE 5** Using Excel to Find the Mean and the Median

Use Excel to find the mean and the median of the data given in Table 6. (Refer to Example 5 Using Excel on page 522).

**SOLUTION**

1. **STEP 1** Click on the Insert function on the tool bar.

   Excel has the statistical functions `Average()` and `Median()`.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Electric Bills</td>
<td>Average</td>
<td>Median</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>52.3</td>
<td>82.10225</td>
<td>82.32</td>
<td></td>
</tr>
</tbody>
</table>

   **Click on the Insert function on the tool bar.**
STEP 2 Select Statistical in the Category drop-down menu. Then highlight Average and click OK. After STEPS 3 and 4, find the median by highlighting Median and clicking OK.

STEP 3 There are two ways to insert the cells that are used to find the average and the median.

(a) Highlight the cells.

or

(b) Type the location of the first cell followed by a colon and the location of the last cell.

\[
\begin{array}{|c|c|}
\hline
\text{Average} & \text{Median} \\ 
\hline
=AVERAGE(B2:B72) & =MEDIAN(B2:B72) \\
\hline
\end{array}
\]

STEP 4 The average and median are given below.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Electric Bills</td>
<td>Average</td>
<td>Median</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>52.3</td>
<td>82.10225</td>
<td>82.32</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

NOW WORK PROBLEMS 7(a) AND 7(b) USING EXCEL.

5 Identify the Mode of a Set of Data

A third commonly used measure of central tendency is called the mode.

**Definition** Mode

The mode of a set of real numbers is the value that occurs with the greatest frequency.

The mode does not necessarily exist, and if it does, it is not always unique.
1. Three common measures of central tendency are _____, _____, and _____.

2. True or False The mean found from grouped data only approximates the true mean.
In Problems 7–14, find (a) the mean, (b) the median, and (c) the mode (if it exists) of each set of data.

7. 21, 25, 43, 36
8. 16, 18, 24, 30
9. 55, 55, 80, 92, 70
10. 90, 80, 82, 82, 70
11. 65, 82, 82, 95, 70
12. 62, 71, 83, 90, 75
13. 48, 65, 80, 92, 80, 75
14. 95, 90, 91, 82, 80, 80

Applications

15. Baseball Players During 2010 spring training, the ages of the 40 men on the New York Yankees roster were:

Source: Major League Baseball.

(a) Find the mean age of the players.
(b) Find the median age of the players.
(c) Identify the modal age of the players if one exists.

16. Baseball Players During 2010 spring training, the ages of the 40 men on the Atlanta Braves roster were:

Source: Major League Baseball.

(a) Find the mean age of the players.
(b) Find the median age of the players.
(c) Identify the modal age of the players if one exists.

17. Investments If an investor purchased 50 shares of IBM stock at $85 per share, 90 shares at $105 per share, 120 shares at $110 per share, and another 75 shares at $130 per share, what is the mean cost per share?

18. Revenue If a farmer sells 120 bushels of corn at $4 per bushel, 80 bushels at $4.10 per bushel, 150 bushels at $3.90 per bushel, and 120 bushels at $4.20 per bushel, what is the mean income per bushel?

19. Mother’s Age The following data give the number of births in the United States in 2007 by the age of the mother for women aged 15 to 45. (The numbers of births are given in thousands.)

<table>
<thead>
<tr>
<th>Age of Mother</th>
<th>No. of Births</th>
</tr>
</thead>
<tbody>
<tr>
<td>15–19</td>
<td>445</td>
</tr>
<tr>
<td>20–24</td>
<td>1082</td>
</tr>
<tr>
<td>25–29</td>
<td>1208</td>
</tr>
<tr>
<td>30–34</td>
<td>962</td>
</tr>
<tr>
<td>35–39</td>
<td>499</td>
</tr>
<tr>
<td>40–44</td>
<td>105</td>
</tr>
</tbody>
</table>


(a) Find the mean age of a woman who gave birth in 2007.
(b) Find the median age of a woman who gave birth in 2007.

20. Multiple Births The following data give the number of multiple births (two or more children) in the United States in the year 2007 of mothers aged 15–45.

<table>
<thead>
<tr>
<th>Age of Mother</th>
<th>No. of Births</th>
</tr>
</thead>
<tbody>
<tr>
<td>15–19</td>
<td>7,201</td>
</tr>
<tr>
<td>20–24</td>
<td>25,115</td>
</tr>
<tr>
<td>25–29</td>
<td>37,270</td>
</tr>
<tr>
<td>30–34</td>
<td>40,444</td>
</tr>
<tr>
<td>35–39</td>
<td>25,860</td>
</tr>
<tr>
<td>40–44</td>
<td>6,202</td>
</tr>
</tbody>
</table>


(a) Find the mean age of a mother whose pregnancy resulted in multiple births in 2007.
(b) Find the median age of a mother whose pregnancy resulted in multiple births in 2007.
21. **Licensed Drivers in Tennessee** Refer to the data provided in Problem 13, Exercise 9.3. Find
   (a) The mean age of a licensed driver in Tennessee.
   (b) The median age of a licensed driver.

22. **Licensed Drivers in Hawaii** Refer to the data provided in Problem 14, Exercise 9.3. Find
   (a) The mean age of a licensed driver in Hawaii.
   (b) The median age of a licensed driver.

23. **Undergraduate Tuition 2006–2007** Refer to the data provided in Problem 15, Exercise 9.3. Find the mean tuition at a four-year college in 2006–2007.

24. **Birth Rates** Refer to the data provided in Problem 18, Exercise 9.3. Find the mean age.

25. **Faculty Salary** The annual salaries of five faculty members in the mathematics department at a large university are $34,000, $35,000, $36,000, $36,500, and $65,000.

26. **Administrative Professionals** The distribution of the annual earnings of 3072 administrative professionals in 2008 by the International Association of Administrative Professionals is summarized in the table below. Find (a) the mean salary and (b) the median salary.

<table>
<thead>
<tr>
<th>Annual Earnings, $</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>15,000–24,999</td>
<td>224</td>
</tr>
<tr>
<td>25,000–34,999</td>
<td>864</td>
</tr>
<tr>
<td>35,000–44,999</td>
<td>1120</td>
</tr>
<tr>
<td>45,000–54,999</td>
<td>640</td>
</tr>
<tr>
<td>55,000–64,999</td>
<td>224</td>
</tr>
</tbody>
</table>

### 9.5 Measures of Dispersion

**OBJECTIVES**

1. Find the standard deviation of sample data (p. 543)
2. Find the standard deviation for grouped data (p. 546)
3. Use the Empirical Rule to describe a bell-shaped distribution (p. 547)
4. Use Chebychev’s theorem (p. 549)

**EXAMPLE 1** Comparing Mean and Median for a Set of Sample Data

Find the mean and median for each of the following sets of sample data:

- \( S_1: 4, 6, 8, 10, 12, 14, 16 \)
- \( S_2: 6, 7, 9, 10, 11, 13, 14 \)

**SOLUTION** For \( S_1 \), the mean \( \bar{x}_1 \) and median \( M_1 \) are

\[
\bar{x}_1 = \frac{4 + 6 + 8 + 10 + 12 + 14 + 16}{7} = \frac{70}{7} = 10 \quad M_1 = 10
\]
For \( S_2 \), the mean \( \bar{x}_2 \) and median \( M_2 \) are

\[
\bar{x}_2 = \frac{6 + 7 + 9 + 10 + 11 + 13 + 14}{7} = \frac{70}{7} = 10 \quad M_2 = 10
\]

Notice that each set of scores has the same mean and the same median. Now look at Figure 10. Do you see that the data in \( S_1 \) seem to be more spread out from the mean 10 than those in \( S_2 \)?

We seek a way to measure the extent to which scores are spread out. Such measures are called \textit{measures of dispersion}.

**Range**

The simplest measure of dispersion is the \textit{range}, which we have already defined as the difference between the largest value and the smallest value. For the sets \( S_1 \) and \( S_2 \) in Example 1, we compute the range of \( S_1 \) to be \( R_1 = 16 - 4 = 12 \) and the range of \( S_2 \) to be \( R_2 = 14 - 6 = 8 \). We conclude that the range is easy to find but that it depends on only two data items so it is affected by extreme values in the data set, and it tells us nothing about how the rest of the data are spread out.

**Variance**

Another measure of dispersion, the \textit{variance}, uses all the data values. The variance is a measure of deviation from the mean. Recall that the sum of the deviations from the mean adds up to zero, so just using deviations from the mean will not be of much use. Since some of the deviations are positive and some are negative, by squaring the deviations and finding their arithmetic mean, we obtain a positive number called the \textit{variance}. Like the population mean and sample mean, we distinguish between the \textit{population variance} and the \textit{sample variance}.

**Definition**

**Population Variance**

If the data used are the entire population \( x_1, x_2, x_3, \ldots x_N \), the \textit{population variance} \( \sigma^2 \), the Greek letter sigma squared, is defined as

\[
\sigma^2 = \frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}
\]

where

\[
\mu = \frac{x_1 + x_2 + x_3 + \cdots + x_N}{N}
\]

is the population mean and \( N \) is the number of items in the population.
Definition

**Sample Variance**
If the data used are a sample $x_1, x_2, x_3, \ldots x_n$ of the population, the *sample variance* $s^2$ is defined as

\[
s^2 = \frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n - 1}
\]

(2)

where

\[
\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}
\]

is the sample mean and $n$ is the number of items in the sample.

Notice that we divide by $n - 1$ in the formula for the sample variance [Formula (2)]. The reason for this is that statisticians have found that when using the sample variance to estimate the population variance, a better agreement is obtained by dividing by $n - 1$.

**EXAMPLE 2  Finding the Sample Variance for a Set of Sample Data**

Calculate the sample variance for sets $S_1$ and $S_2$ of Example 1.

**SOLUTION** For $S_1$, $\overline{x} = 10$ and $n = 7$, so that

\[
s^2_1 = \frac{(4 - 10)^2 + (6 - 10)^2 + (8 - 10)^2 + (10 - 10)^2 + (12 - 10)^2 + (14 - 10)^2 + (16 - 10)^2}{7 - 1} = 18.67
\]

For $S_2$, $\overline{x} = 10$ and $n = 7$, so that

\[
s^2_2 = \frac{(6 - 10)^2 + (7 - 10)^2 + (9 - 10)^2 + (10 - 10)^2 + (11 - 10)^2 + (13 - 10)^2 + (14 - 10)^2}{7 - 1} = 8.67
\]

Since the sample variance for set $S_1$ is larger than the sample variance for set $S_2$, we conclude that the data in set $S_1$ are more widely dispersed than the data in set $S_2$, confirming what we saw in Figure 10.

**1  Find the Standard Deviation of Sample Data**

In computing the variance, we square the deviations from the mean. This means, for example, that if our data represent dollars, then the variance has the units “dollars squared.” To remedy this, we use the square root of the variance, called the *standard deviation*. 
544 Chapter 9 Statistics

**Definition**

**Standard Deviation of Sample Data**

The standard deviation of a set \( x_1, x_2, \ldots, x_n \) of \( n \) data items taken from the population is defined as

\[
s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}
\]

where \( \bar{x} \) is the sample mean and \( \sum \) means to add up all the deviations squared.

For the sample data in Example 1 the standard deviation for \( S_1 \) is

\[
s_1 = \sqrt{\frac{36 + 16 + 4 + 0 + 4 + 16 + 36}{7 - 1}} = \sqrt{\frac{112}{6}} = \sqrt{18.67} = 4.32
\]

and the standard deviation for \( S_2 \) is

\[
s_2 = \sqrt{\frac{16 + 9 + 1 + 0 + 1 + 9 + 16}{7 - 1}} = \sqrt{\frac{57}{6}} = \sqrt{9.5} = 2.94
\]

Again, the fact that the standard deviation of the set \( S_2 \) is less than the standard deviation of the set \( S_1 \) indicates that the data of \( S_2 \) are more clustered around the mean than those of \( S_1 \).

**EXAMPLE 3** Finding the Standard Deviation of Sample Data

Find the standard deviation for the sample data

\[100, 90, 90, 85, 80, 75, 75, 70, 70, 65, 65, 60, 40, 40, 40\]

**SOLUTION**

The sample mean is

\[
\bar{x} = \frac{100 + 2\cdot 90 + 85 + 80 + 3\cdot 75 + 2\cdot 70 + 2\cdot 65 + 60 + 3\cdot 40}{16} = 70
\]

The deviations from the mean and their squares are computed in Table 13 on page 545. Based on Formula (3), the standard deviation is

\[
s = \sqrt{\frac{4950}{15}} = 18.17
\]

NOW WORK PROBLEM 7.

*The standard deviation \( \sigma \) for population data is defined as*

\[
\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}} = \sqrt{\frac{\sum (x_i - \mu)^2}{N}}
\]

where \( \mu \) is the population mean and \( x_1, x_2, x_3, \ldots, x_N \) is the population.
EXAMPLE 4  
Finding the Standard Deviation of Sample Data

Find the standard deviation of the sample data
80, 80, 80, 80, 75, 75, 75, 75, 70, 70, 65, 65, 60, 60, 55, 55

SOLUTION  Here the mean is $\bar{x} = 70$ for the 16 scores. Table 14 below gives the deviations from the mean and their squares. The standard deviation is

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n-1}} = \sqrt{\frac{1200}{15}} = 8.94$$

These two examples show that although the samples have the same mean, 70, and the same sample size, 16, the data in Example 3 deviate further from the mean than do the data in Example 4.

In general, a relatively small standard deviation indicates that the measures tend to cluster close to the mean, and a relatively large standard deviation shows that the measures are widely scattered from the mean.

**TABLE 13**

<table>
<thead>
<tr>
<th>Scores, $x$</th>
<th>Deviation from the Mean, $x - \bar{x}$</th>
<th>Deviation Squared, $(x - \bar{x})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>-30</td>
<td>900</td>
</tr>
<tr>
<td>40</td>
<td>-30</td>
<td>900</td>
</tr>
<tr>
<td>40</td>
<td>-30</td>
<td>900</td>
</tr>
<tr>
<td>60</td>
<td>-10</td>
<td>100</td>
</tr>
<tr>
<td>65</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>65</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>75</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>75</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>75</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>80</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>85</td>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>90</td>
<td>20</td>
<td>400</td>
</tr>
<tr>
<td>90</td>
<td>20</td>
<td>400</td>
</tr>
<tr>
<td>100</td>
<td>30</td>
<td>900</td>
</tr>
<tr>
<td>Mean $\bar{x} = 70$</td>
<td>Sum = 0</td>
<td>Sum = 4950</td>
</tr>
<tr>
<td>$n = 16$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 14**

<table>
<thead>
<tr>
<th>Scores, $x$</th>
<th>Deviation from the Mean, $x - \bar{x}$</th>
<th>Deviation Squared, $(x - \bar{x})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>-15</td>
<td>225</td>
</tr>
<tr>
<td>55</td>
<td>-15</td>
<td>225</td>
</tr>
<tr>
<td>60</td>
<td>-10</td>
<td>100</td>
</tr>
<tr>
<td>60</td>
<td>-10</td>
<td>100</td>
</tr>
<tr>
<td>65</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>65</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>0</td>
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<tr>
<td>70</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>75</td>
<td>5</td>
<td>25</td>
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<tr>
<td>75</td>
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<td>25</td>
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<td>75</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>75</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>80</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>80</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>80</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>80</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>Mean $\bar{x} = 70$</td>
<td>Sum = 0</td>
<td>Sum = 1200</td>
</tr>
<tr>
<td>$n = 16$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Find the Standard Deviation for Grouped Data

To find the standard deviation for grouped data, we use the formula

\[ s = \sqrt{\frac{(m_1 - \bar{x})^2 \cdot f_1 + (m_2 - \bar{x})^2 \cdot f_2 + \cdots + (m_k - \bar{x})^2 \cdot f_k}{n - 1}} \]

where \( m_1, m_2, \ldots, m_k \) are the class midpoints; \( f_1, f_2, \ldots, f_k \) are the respective frequencies of each class interval; \( n \) is the number of data items, that is, \( n = f_1 + f_2 + \cdots + f_k \); and \( \bar{x} \) is the sample mean.

**EXAMPLE 5** Finding Standard Deviation for Grouped Data

Find the standard deviation for the grouped data given in Table 8, page 521, and repeated below in Table 15.

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 50–59.99</td>
<td>7</td>
</tr>
<tr>
<td>2 60–69.99</td>
<td>9</td>
</tr>
<tr>
<td>3 70–79.99</td>
<td>19</td>
</tr>
<tr>
<td>4 80–89.99</td>
<td>14</td>
</tr>
<tr>
<td>5 90–99.99</td>
<td>14</td>
</tr>
<tr>
<td>6 100–109.99</td>
<td>6</td>
</tr>
<tr>
<td>7 110–119.99</td>
<td>2</td>
</tr>
</tbody>
</table>

**SOLUTION** We have already found (see Example 2, p. 534) that the mean for this grouped data is \( \bar{x} = 81.34 \).

The class midpoints are 55, 65, 75, 85, 95, 105, and 115. The deviations of the mean from the class midpoints, their squares, and the products of the squares by the respective frequencies are listed in Table 16.

<table>
<thead>
<tr>
<th>Class Midpoint</th>
<th>( f_i )</th>
<th>( m_i - \bar{x} )</th>
<th>( (m_i - \bar{x})^2 )</th>
<th>( (m_i - \bar{x})^2 \cdot f_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>7</td>
<td>-26.34</td>
<td>693.80</td>
<td>4856.6</td>
</tr>
<tr>
<td>65</td>
<td>9</td>
<td>-16.34</td>
<td>267.00</td>
<td>2403.0</td>
</tr>
<tr>
<td>75</td>
<td>19</td>
<td>-6.34</td>
<td>40.20</td>
<td>763.8</td>
</tr>
<tr>
<td>85</td>
<td>14</td>
<td>3.66</td>
<td>13.40</td>
<td>187.6</td>
</tr>
<tr>
<td>95</td>
<td>14</td>
<td>13.66</td>
<td>186.60</td>
<td>2612.4</td>
</tr>
<tr>
<td>105</td>
<td>6</td>
<td>23.66</td>
<td>559.80</td>
<td>3358.8</td>
</tr>
<tr>
<td>115</td>
<td>2</td>
<td>33.66</td>
<td>1133.00</td>
<td>2266.0</td>
</tr>
<tr>
<td>Sum</td>
<td>71</td>
<td></td>
<td></td>
<td>16,448.2</td>
</tr>
</tbody>
</table>
9.5 Measures of Dispersion

Using Formula (5), the standard deviation is

\[ s = \sqrt{\frac{16,448.2}{70}} = 15.33 \]

**NOW WORK PROBLEM 13.**

A little computation shows that the sum of the deviations of the approximate mean from the class midpoints is not exactly zero. This is due to the fact that we are using an approximation to the mean. Remember, we cannot compute the exact mean for grouped data.

**USING TECHNOLOGY**

**EXAMPLE 6 Using Excel to Find Standard Deviation**

Use Excel to find the standard deviation of the grouped data given in Table 15.

**SOLUTION**

**COMMENT:** Finding standard deviations by hand is extremely tedious. We have just seen that Excel calculates the standard deviation quickly and accurately. Graphing utilities are also designed to compute the standard deviation of a set of data points. Figure 11 illustrates the standard deviation of the grouped data from Table 15 on a TI-84 Plus. Here the class midpoints are put in L1 and the class frequencies are put in L2, as in Figure 11(a). The standard deviation is given by \( s_x \), seen in Figure 11(b). Note that the population standard deviation is also calculated and is given by \( \sigma X \). Consult your user’s manual to find the method of calculating standard deviations on your graphing utility.

**FIGURE 11**

(a) (b)

**3 Use the Empirical Rule to Describe a Bell-Shaped Distribution**

If the data being analyzed have a distribution that is bell shaped, the **Empirical Rule** describes the percentage of outcomes that are within \( k \) standard deviations of the mean.
Theorem

The Empirical Rule

If a distribution is bell shaped, then

- Approximately 68% of the data are within one standard deviation of the mean.
- Approximately 95% of the data are within two standard deviations of the mean.
- Approximately 99.7% of the data are within three standard deviations of the mean.

The Empirical Rule can be used for both sample data and population data provided that the distribution is roughly bell shaped. See Figure 12.

EXAMPLE 7 Using the Empirical Rule

The histogram of the data in Table 8 is approximately bell shaped. Refer to Figure 5 (p. 522).

(a) According to the Empirical Rule, what percentage of electric bills should be within two standard deviations of the mean?

(b) Using the Empirical Rule, determine the percentage of electric bills that are between $50.68 and $112.00.

(c) Using the raw data in Table 6 (p. 520), calculate the exact percentage of electric bills that are between $50.68 and $112.00.

SOLUTION

(a) According to the Empirical Rule, approximately 95% of the electric bills should be within two standard deviations of the mean.

(b) Since $50.68 is two standard deviations below the mean and $112.00 is two standard deviations above the mean, the Empirical Rule states that approximately 95% of the bills are between $50.68 and $112.00.

(c) Using Table 6 (p. 520), we find that 69 (97%) of the customers surveyed had electric bills between $50.68 and $112.00. This is close to the percentage predicted by the Empirical Rule.

Now work Problem 27.
9.5 Measures of Dispersion

4 Use Chebychev’s Theorem

Suppose we are analyzing an experiment with numerical outcomes and that the experiment has population mean $\mu$ and population standard deviation $\sigma$. We wish to estimate the probability that a randomly chosen outcome lies within $k$ units of the mean.

**Chebychev’s Theorem**

For any distribution of numbers with population mean $\mu$ and population standard deviation $\sigma$, the probability that a randomly chosen outcome lies between $\mu - k\sigma$ and $\mu + k\sigma$ is at least $1 - \frac{\sigma^2}{k^2}$.

**EXAMPLE 8 Using Chebychev’s Theorem**

Suppose that an experiment with numerical outcomes has population mean 4 and population standard deviation 1. Use Chebychev’s theorem to estimate the probability that an outcome lies between 2 and 6.

**SOLUTION** Here, $\mu = 4, \sigma = 1$. Since we wish to estimate the probability that an outcome lies between 2 and 6, the value of $k$ is $k = 6 - \mu = 6 - 4 = 2$ (or $k = \mu - 2 = 4 - 2 = 2$). Then by Chebychev’s theorem, the desired probability is at least

$$1 - \frac{\sigma^2}{k^2} = 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = 0.75$$

That is, we expect at least 75% of the outcomes of this experiment to lie between 2 and 6. ■

**NOW WORK PROBLEM 31.**

**EXAMPLE 9 Using Chebychev’s Theorem**

An office supply company sells boxes containing 100 paper clips. Because of the packaging procedure, not every box contains exactly 100 clips. From previous data it is known that the average number of clips in a box is indeed 100 and the standard deviation is 2.8. If the company ships 10,000 boxes, estimate the number of boxes having between 94 and 106 clips, inclusive.

**SOLUTION** Our experiment involves counting the number of clips in the box. For this experiment we have $\mu = 100$ and $\sigma = 2.8$. Therefore, by Chebychev’s theorem the fraction of boxes having between $100 - 6 = 94$ and $100 + 6 = 106$ clips ($k = 6$) should be at least

$$1 - \frac{(2.8)^2}{6^2} = 1 - \frac{7.84}{36} = 1 - 0.22 = 0.78$$

That is, we expect at least 78% of 10,000 boxes, or about 7800 boxes, to have between 94 and 106 clips. ■

*Named after the nineteenth-century Russian mathematician P. L. Chebychev.*
The importance of Chebychev's theorem stems from the fact that it applies to any data—only the population mean and population standard deviation must be known. However, the estimate is a crude one. Other results (such as the normal distribution given later) produce more accurate estimates about the probability of falling within \( k \) units of the mean.

**EXERCISE 9.5**

Answers Begin on Page AN–49.

**Concepts and Vocabulary**

1. The range, variance, and standard deviation measure the ______ of the distribution.
2. **True or False** A disadvantage of using the range to measure dispersion is that it only uses the two extreme data points.
3. The Empirical Rule states that if the distribution of the data is bell shaped, about ______ percent of the outcomes will be within one standard deviation of the mean.
4. When calculating the sample standard deviation for \( n \) items, we divide by ______.
5. **True or False** The standard deviation is preferred to the variance because the units are squared.
6. **True or False** Chebychev’s theorem is used to estimate the probability that an outcome is within \( k \) standard deviations of the mean.

**Skill Building**

In Problems 7–12, compute the standard deviation for each set of sample data.

7. 4, 5, 9, 9, 10, 14, 25
8. 6, 8, 10, 11, 12, 18
10. 55, 65, 80, 80, 90
11. 85, 75, 62, 78, 100
12. 92, 82, 75, 75, 82

In Problems 13 and 14, calculate the mean and the standard deviation of the sample data below.

<table>
<thead>
<tr>
<th>Class</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>10–16</td>
<td>1</td>
</tr>
<tr>
<td>17–23</td>
<td>3</td>
</tr>
<tr>
<td>24–30</td>
<td>10</td>
</tr>
<tr>
<td>31–37</td>
<td>12</td>
</tr>
<tr>
<td>38–44</td>
<td>5</td>
</tr>
<tr>
<td>45–51</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Class</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–3</td>
<td>2</td>
</tr>
<tr>
<td>4–7</td>
<td>5</td>
</tr>
<tr>
<td>8–11</td>
<td>8</td>
</tr>
<tr>
<td>12–15</td>
<td>6</td>
</tr>
<tr>
<td>16–19</td>
<td>3</td>
</tr>
</tbody>
</table>

**Applications**

15. **Lightbulb Life** A sample of 6 lightbulbs was chosen and their lifetimes measured. The bulbs lasted 968, 893, 769, 845, 922, and 915 hours. Calculate the mean and the standard deviation of the lifetimes of the lightbulbs.

16. **Aptitude Scores** A sample of 25 applicants for admission to Midwestern University had the following scores on the quantitative part of an aptitude test:

<table>
<thead>
<tr>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>591</td>
</tr>
<tr>
<td>490</td>
</tr>
<tr>
<td>606</td>
</tr>
<tr>
<td>502</td>
</tr>
<tr>
<td>550</td>
</tr>
<tr>
<td>570</td>
</tr>
<tr>
<td>415</td>
</tr>
<tr>
<td>614</td>
</tr>
<tr>
<td>506</td>
</tr>
<tr>
<td>551</td>
</tr>
</tbody>
</table>

Find the mean and standard deviation of these scores.

17. **Baseball Players** During 2010 spring training, the ages of the 40 men on the New York Yankees roster were

<table>
<thead>
<tr>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
</tr>
<tr>
<td>29</td>
</tr>
<tr>
<td>25</td>
</tr>
</tbody>
</table>

Source: Major League Baseball.

(a) Find the range of the players’ ages.

(b) Find the standard deviation, assuming sample data.

(c) Find the standard deviation, assuming population data.

(d) Decide whether the data are sample or population. Give reasons.
18. **Baseball Players**  During 2010 spring training, the ages of the 40 men on the Atlanta Braves roster were

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>2</td>
</tr>
<tr>
<td>23</td>
<td>3</td>
</tr>
<tr>
<td>24</td>
<td>4</td>
</tr>
<tr>
<td>34</td>
<td>5</td>
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<td>24</td>
<td>6</td>
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<tr>
<td>35</td>
<td>7</td>
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<td>27</td>
<td>9</td>
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<td>30</td>
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<td>37</td>
<td>11</td>
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<td>38</td>
<td>15</td>
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<td>28</td>
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<td>17</td>
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<td>27</td>
<td>18</td>
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<td>25</td>
<td>39</td>
</tr>
<tr>
<td>24</td>
<td>40</td>
</tr>
</tbody>
</table>

Source: Major League Baseball.

(a) Find the range of the players’ ages.

(b) Find the standard deviation, assuming sample data.

(c) Find the standard deviation, assuming population data.

(d) Decide whether the data are sample or population. Give reasons.

19. **Mother’s Age**  The following data give the number of births in the United States in 2007 by the age of the mother for women under the age of 45. (The numbers of births are given in thousands.)

<table>
<thead>
<tr>
<th>Age of Mother</th>
<th>No. of Births</th>
</tr>
</thead>
<tbody>
<tr>
<td>10–14</td>
<td>6</td>
</tr>
<tr>
<td>15–19</td>
<td>445</td>
</tr>
<tr>
<td>20–24</td>
<td>1082</td>
</tr>
<tr>
<td>25–29</td>
<td>1208</td>
</tr>
<tr>
<td>30–34</td>
<td>962</td>
</tr>
<tr>
<td>35–39</td>
<td>499</td>
</tr>
<tr>
<td>40–44</td>
<td>105</td>
</tr>
</tbody>
</table>


(a) Are these sample data or population data? Justify your reasoning.

(b) Find the standard deviation of the mothers’ ages.

20. **Charges Accounts**  A department store takes a sample of its customer charge accounts and finds the following:

<table>
<thead>
<tr>
<th>Outstanding Balance ($)</th>
<th>Number of Accounts</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–49</td>
<td>15</td>
</tr>
<tr>
<td>50–99</td>
<td>41</td>
</tr>
<tr>
<td>100–149</td>
<td>80</td>
</tr>
<tr>
<td>150–199</td>
<td>60</td>
</tr>
<tr>
<td>200–249</td>
<td>8</td>
</tr>
</tbody>
</table>

Find the mean and the standard deviation of the outstanding balances.

21. **Earthquakes**  The data below list the number of earthquakes recorded worldwide during 2009 that measured below 8 on the Richter scale.

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Earthquakes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–0.9</td>
<td>21</td>
</tr>
<tr>
<td>1.0–1.9</td>
<td>26</td>
</tr>
<tr>
<td>2.0–2.9</td>
<td>3009</td>
</tr>
<tr>
<td>3.0–3.9</td>
<td>2899</td>
</tr>
<tr>
<td>4.0–4.9</td>
<td>6908</td>
</tr>
<tr>
<td>5.0–5.9</td>
<td>1776</td>
</tr>
<tr>
<td>6.0–6.9</td>
<td>142</td>
</tr>
<tr>
<td>7.0–7.9</td>
<td>16</td>
</tr>
</tbody>
</table>

Source: National Earthquake Information Center.

(a) Are these sample data or population data?

(b) Find the mean magnitude of the earthquakes worldwide in 2009.

(c) Find the standard deviation of the magnitude of the earthquakes recorded in 2009.

22. **Earthquakes**  The data below list the number of earthquakes recorded in the United States in 2009 that measured below 8 on the Richter scale.

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Earthquakes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–0.9</td>
<td>17</td>
</tr>
<tr>
<td>1.0–1.9</td>
<td>26</td>
</tr>
<tr>
<td>2.0–2.9</td>
<td>2374</td>
</tr>
<tr>
<td>3.0–3.9</td>
<td>1491</td>
</tr>
<tr>
<td>4.0–4.9</td>
<td>293</td>
</tr>
<tr>
<td>5.0–5.9</td>
<td>55</td>
</tr>
<tr>
<td>6.0–6.9</td>
<td>4</td>
</tr>
<tr>
<td>7.0–7.9</td>
<td>0</td>
</tr>
</tbody>
</table>

Source: National Earthquake Information Center.

(a) Are these sample data or population data?

(b) Find the mean magnitude of the earthquakes in the United States in 2009.

(c) Find the standard deviation of the magnitude of the earthquakes recorded in the United States in 2009.

23. **Licensed Drivers in Tennessee**  Refer to the data in Problem 13, Exercise 9.3.

(a) Find the standard deviation assuming sample data.

(b) Find the standard deviation assuming population data.

(c) Decide whether the data are sample or population. Give reasons.
24. Licensed Drivers in Hawaii  Refer to the data provided in Problem 14, Exercise 9.3.
   (a) Find the standard deviation assuming sample data.
   (b) Find the standard deviation assuming population data.
   (c) Decide whether the data are sample or population.
      Give reasons.
25. Undergraduate Tuition 2006–2007  Refer to the data provided in Problem 15, Exercise 9.3.
   (a) Are these data from a population or from a sample? Justify your answer.
   (b) Find the standard deviation of tuition.
26. Birth Rates  Refer to the data provided in Problem 18, Exercise 9.3.
   (a) Are these data from a population or from a sample? Justify your answer.
   (b) Find the standard deviation.
27. The Empirical Rule  One measure of intelligence is the Stanford-Binet Intelligence Quotient (IQ). IQ scores have a bell-shaped distribution with a mean of 100 and a standard deviation of 15.
   (a) What percentage of persons have an IQ score between 70 and 130?
   (b) What percentage of persons have an IQ score less than 70 or greater than 130?
   (c) What percentage of persons have an IQ score greater than 130?
   (a) What percentage of SAT scores are between 399 and 631?
   (b) What percentage of SAT scores are less than 399 or greater than 631?
   (c) What percentage of SAT scores are greater than 747?
29. The Empirical Rule  The weight, in grams, of the pair of kidneys in adult males between the ages of 40 and 49 have a bell-shaped distribution with a mean of 325 grams and a standard deviation of 30 grams.
   (a) About 95% of kidneys will be between what weights?
   (b) What percentage of kidneys weigh between 235 grams and 415 grams?
   (c) What percentage of kidneys weigh less than 235 grams or more than 415 grams?
   (d) What percentage of kidneys weigh between 295 grams and 385 grams?
30. The Empirical Rule  The distribution of the length of bolts has a bell shape with a mean of 4 inches and a standard deviation of 0.007 inch.
   (a) About 68% of bolts manufactured will be between what lengths?
   (b) What percentage of bolts will be between 3.986 inches and 4.014 inches?
   (c) If the company discards any bolts less than 3.986 inches or greater than 4.014 inches, what percentage of bolts manufactured will be discarded?
   (d) What percentage of bolts manufactured will be between 4.007 inches and 4.021 inches?
31. Chebychev’s Theorem  Suppose that an experiment with numerical outcomes has mean 25 and standard deviation 3. Use Chebychev’s theorem to tell what percentage of outcomes lie
   (a) Between 19 and 31.
   (b) Between 20 and 30.
   (c) Between 16 and 34.
   (d) Less than 19 or more than 31.
   (e) Less than 16 or more than 34.
32. Cost of Meat  A survey reveals that the mean price for a pound of beef is $3.20 with a standard deviation of $0.40. Use Chebychev’s theorem to determine the probability that a randomly selected pound of beef costs
   (a) Between $2.80 and $3.60.
   (b) Between $2.50 and $3.90.
   (c) Between $2.20 and $4.20.
   (d) Between $2.40 and $4.00.
   (e) Less than $2.40 or more than $4.00.
33. Quality Control  A watch company determines that each box of 500 watches has an average of 6 defective watches with standard deviation 2. Suppose that 1000 boxes are produced. Use Chebychev’s theorem to estimate the number of boxes having between 0 and 12 defective watches.
34. Sales  The average sale at a department store is $51.25, with a standard deviation of $8.50. Find the smallest interval such that by Chebychev’s theorem at least 90% of the store’s sales fall within it.

<table>
<thead>
<tr>
<th>Year</th>
<th>Births</th>
</tr>
</thead>
<tbody>
<tr>
<td>2006</td>
<td>4,265,555</td>
</tr>
<tr>
<td>2005</td>
<td>4,138,349</td>
</tr>
<tr>
<td>2004</td>
<td>4,112,052</td>
</tr>
<tr>
<td>2003</td>
<td>4,089,950</td>
</tr>
<tr>
<td>2002</td>
<td>4,021,726</td>
</tr>
<tr>
<td>2001</td>
<td>4,025,933</td>
</tr>
</tbody>
</table>

   (a) Are these population or sample data? Justify your reasoning.
   (b) Calculate the mean number of births over the six-year period.
   (c) Calculate the standard deviation of births over the six-year period.
   (d) Are the mean and the standard deviations you calculated exact or are they approximations? Explain your reasoning.
(c) Do the number of births differ greatly over the six-year period? Justify your reasoning.

36. Fishing  The number of salmon caught in each of two rivers over the past 15 years is as follows:

<table>
<thead>
<tr>
<th>River I Number Caught</th>
<th>Years</th>
<th>River II Number Caught</th>
<th>Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>500–1499</td>
<td>4</td>
<td>750–1249</td>
<td>2</td>
</tr>
<tr>
<td>1500–2499</td>
<td>8</td>
<td>1350–1799</td>
<td>3</td>
</tr>
<tr>
<td>2500–3499</td>
<td>2</td>
<td>1800–2249</td>
<td>4</td>
</tr>
<tr>
<td>3500–4499</td>
<td>1</td>
<td>2250–2699</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2700–3149</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Are these population or sample data?
(b) Find the mean and the standard deviation of the number of fish caught in each river.
(c) Are the mean and the standard deviation calculated in part (b) approximate or exact? Explain.
(d) Using the statistics you calculated, discuss which river should be preferred for fishing.

9.6 The Normal Distribution

PREPARING FOR THIS SECTION
Before getting started, review the following:
- The Binomial Probability Model (Section 8.6, pp. 480–491)

NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 562.

OBJECTIVES
1. Find a Z-score (p. 556)
2. Use the standard normal curve (p. 557)
3. Approximate a binomial distribution by the standard normal distribution (p. 560)

Frequency polygons or frequency distributions can assume almost any shape or form, depending on the data. However, the data obtained from many experiments often follow a common pattern. For example, heights of adults, weights of adults, and test scores all lead to data that have the same kind of frequency distribution. This distribution is referred to as the normal distribution or the Gaussian distribution. Because it occurs so often in practical situations, it is generally regarded as the most important distribution, and much statistical theory is based on it. The graph of the normal distribution, called the normal curve, is the bell-shaped curve shown in Figure 13.

We list some properties of the normal distribution next.
Properties of the Normal Distribution

1. Graphs of normally distributed data are bell shaped and are symmetric with respect to a vertical line at the mean $\mu$. See Figure 14.
2. The mean, median, and mode of a normal distribution are equal.
3. The area enclosed by the normal curve and the $x$-axis is always equal to 1 square unit. The shaded region in Figure 14 has an area of 1 square unit.
4. The probability that an outcome of a normally distributed experiment is between $a$ and $b$ equals the area under the associated normal curve from $x = a$ to $x = b$. See the shaded region in Figure 15.
5. The standard deviation of a normal distribution plays a major role in describing the area under the normal curve. As shown in Figure 16, the standard deviation is related to the area under the normal curve as follows:
   - (a) About 68.27% of the total area under the curve is within 1 standard deviation of the mean ($\mu - \sigma$ to $\mu + \sigma$).
   - (b) About 95.45% of the total area under the curve is within 2 standard deviations of the mean ($\mu - 2\sigma$ to $\mu + 2\sigma$).
   - (c) About 99.73% of the total area under the curve is within 3 standard deviations of the mean ($\mu - 3\sigma$ to $\mu + 3\sigma$).

Notice that Property 5 is essentially the Empirical Rule we discussed in Section 9.5. It is also worth noting that, in theory, the normal curve will never touch the $x$-axis but will extend to infinity in either direction.

EXAMPLE 1 Using the Normal Curve to Analyze IQ Scores

At Jefferson High School the average IQ score of the 1200 students is 100, with a standard deviation of 15. The IQ scores have a normal distribution.

(a) How many students have an IQ between 85 and 115?
(b) How many students have an IQ between 70 and 130?
(c) How many students have an IQ between 55 and 145?
(d) How many students have an IQ under 55 or over 145?
(e) How many students have an IQ over 145?
SOLUTION

Figure 17 shows a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$.

(a) The IQ scores have a normal distribution and the mean is 100. Since the standard deviation $\sigma$ is 15, then $1\sigma$ on either side of the mean is from 85 to 115. By Property 5(a) we know that 68.27% of 1200, or $(0.6827)(1200) = 819$ students have IQs between 85 and 115.

(b) The scores from 70 to 130 extend on either side of the mean. By Property 5(b) we know that 95.45% of 1200, or $(0.9545)(1200) = 1145$ students have IQs between 70 and 130.

(c) The scores from 55 to 145 extend on either side of the mean. By Property 5(c) we know that 99.73% of 1200, or $(0.9973)(1200) = 1197$ students have IQs between 55 and 145.

(d) There are three students $(1200 - 1197)$ who have scores that are not between 55 and 145.

(e) One or two students have IQs above 145.

NOW WORK PROBLEM 29.

A normal distribution is completely determined by the mean $\mu$ and the standard deviation $\sigma$. Normal distributions of data with different means or different standard deviations give rise to different normal curves.

Figure 18 indicates how the normal curve changes when the standard deviation changes. Each normal curve has the same mean 0.

As the standard deviation increases, the normal curve spreads out [Figure 18(c)], indicating a greater likelihood for the outcomes to be far from the mean. A compressed curve [Figure 18(a)] indicates that the outcomes are more likely to be close to the mean.
Standard Normal Curve

It would be a hopeless task to attempt to tabulate areas under a normal curve for every conceivable value of \( \mu \) and \( \sigma \). Fortunately, we are able to transform all the observations to one table—the table corresponding to the so-called **standard normal curve**, which is the normal curve for which \( \mu = 0 \) and \( \sigma = 1 \). This is accomplished by introducing new data, called **Z-scores**. A Z-score transforms any normal value with a mean \( \mu \) and a standard deviation \( \sigma \) to a normal value with mean 0 and standard deviation 1.

**Definition**

If data are normally distributed, then the **Z-score** of a value \( x \) is defined by

\[
Z = \frac{x - \mu}{\sigma}
\]

where

- \( x \) = original data point
- \( \mu \) = mean of the original data
- \( \sigma \) = standard deviation of the original data

The transformed data obtained using Equation (1) will always have a zero mean and a standard deviation of one. Such data are said to be expressed in **standard units** or **standard scores**.

A Z-score can be interpreted as the number of standard deviations that the original score is away from its mean. So, by expressing data in terms of standard units, it becomes possible to make a comparison of distributions.

---

**Find a Z-score**

**EXAMPLE 2 Finding Z-Scores**

On a test, 80 is the mean and 7 is the standard deviation. What is the Z-score of a score of (a) 88? (b) 62?

Interpret your results. Graph the normal curve.

**SOLUTION**

(a) Here, 88 is the original score. Using Equation (1) with \( x = 88, \mu = 80, \sigma = 7 \), we get

\[
Z = \frac{x - \mu}{\sigma} = \frac{88 - 80}{7} = \frac{8}{7} = 1.1429
\]

(b) Here, 62 is the original score. Using Equation (1) with \( x = 62, \mu = 80, \sigma = 7 \), we get

\[
Z = \frac{62 - 80}{7} = \frac{-18}{7} = -2.5714
\]

The Z-score of 1.1429 tells us that the original score of 88 is 1.1429 standard deviations above the mean. The Z-score of -2.5714 tells us that the original score of 62 is 2.5714 standard deviations below the mean. A negative Z-score always means that the score is below the mean. See Figure 19 for a graph of the normal curve.
9.6 The Normal Distribution

The normal curve in Figure 20 with mean $\mu = 0$ and standard deviation $\sigma = 1$ is the standard normal curve. For this curve the areas between $Z = -1$ and 1, $Z = -2$ and 2, $Z = -3$ and 3 are equal, respectively, to 68.27%, 95.45%, and 99.73% of the total area under the curve, which is 1. To find the areas cut off between other points, we proceed as follows:

The area under portions of the standard normal curve can be calculated by hand by using the standard normal curve table, which is printed on the inside back cover of this text. The table gives the area under the curve between the mean $\mu = 0$ and select positive $Z$-scores. Areas between $\mu = 0$ and negative values of $Z$ are obtained using symmetry. Since a normal distribution is symmetric about its mean, the area under the standard normal curve to the right of $\mu = 0$ equals the area under the curve to the left of $\mu = 0$; each area equals $\frac{1}{2}$.

Given a $Z$-score rounded to two decimal points, the area between 0 and absolute value of $|Z|$ is found by locating the standard score truncated to one decimal place in the left column of the table and then by moving across the row to the entry under the second decimal place. This value represents the area between 0 and $Z$.

**EXAMPLE 3 Using the Standard Normal Curve**

(a) Find the area included between 0 and 0.6 on a standard normal curve. Refer to the shaded area in Figure 21(a).

(b) Find the area included between 0.6 and 1.86 on a standard normal curve. Refer to the shaded area in Figure 21(b).

**SOLUTION** Use the standard normal curve table, which is printed on the inside back cover.
To find the area between 0 and 0.6, we find \( Z = 0.6 \) in the table. Corresponding to \( Z = 0.6 \) is the value 0.2257, which is the area under the curve between the mean 0 and 0.6. In other words, 22.57\% of the area under the standard normal curve will lie between 0 and 0.6.

We begin by checking the table to find the area of the curve cut off between the mean and a point equivalent to a standard score of 0.6. This value is 0.2257, as we found in part (a). Next, we continue down the table in the left-hand column until we come to a standard score of 1.8. By looking across the row to the column below 0.06, we find that 0.4686 of the area is included between the mean and 1.86. The area under the curve between the points 0.6 and 1.86 is the difference between the two areas, 0.4686 \( - \) 0.2257, which is 0.2429. We can then state that approximately 24.29\% of the area under the curve falls between 0.6 and 1.86, or that the probability of a score falling between these two values is about 0.2429.

NOW WORK PROBLEM 15.

EXAMPLE 4 Using the Standard Normal Curve

We want to determine the area under the standard normal curve that falls between a standard score of \(-0.39\) and one of 1.86.

**SOLUTION**

See Figure 22.

There are no values for negative standard scores in the standard normal curve table. Because of the symmetry of normal curves, standard scores equal in absolute value, give equal areas when taken from the mean. From the table we find that a standard score for 0.39 cuts off an area of 0.1517 between it and the mean. A standard score of 1.86 includes 0.4686 of the area of the curve between it and the mean. The area included between \(-0.39\) and 1.86 is then equal to the sum of these two areas, 0.1517 \( + \) 0.4686, which is 0.6203. Approximately 62.03\% of the area is between \(-0.39\) and 1.86. In other words, the probability of a score falling between these two values is about 0.6203.

EXAMPLE 5 Using the Standard Normal Curve

The scores on a test are normally distributed with a mean of 78 and a standard deviation of 7. What is the probability that a test chosen at random has a score between 80 and 90?

**SOLUTION**

To find the probability of obtaining a score between 80 and 90 on the test, we need to find the area under a normal curve from \( x_1 = 80 \) to \( x_2 = 90 \).
9.6 The Normal Distribution

We begin by finding the Z-scores $Z_1$ of $x_1 = 80$ and $Z_2$ of $x_2 = 90$.

$$Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{80 - 78}{7} = 0.29$$

$$Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{90 - 78}{7} = 1.71$$

From the standard normal curve table, the area $A_1$ from the mean to $Z_1$ and the area $A_2$ from the mean to $Z_2$ are $A_1 = 0.1141$ and $A_2 = 0.4564$.

See Figure 23. The area between $Z_1$ and $Z_2$ is the difference between $A_2$ and $A_1$.

$$A_2 - A_1 = 0.4564 - 0.1141 = 0.3423$$

So, the probability of obtaining a test score between 80 and 90 is 0.3423, or 34.23%.

**COMMENT:** A graphing utility can be used to find the area under a portion of a normal curve. The graphing utility uses the actual distribution mean and standard deviation and computes the area beneath the normal curve over the interval of interest all in one step. See Figure 24. The information needed for computing the solution is $x_1 = 80, x_2 = 90, \mu = 78$, and $\sigma = 7$. The result, normalcdf(80, 90, 78, 7) = 0.344 indicates that for a normal curve with mean 78 and standard deviation 7, 34.4% of the area beneath the curve is between $x_1 = 80$, and $x_2 = 90$, so the probability of obtaining a test score between 80 and 90 on the test is 0.344 or 34.4%. The two answers vary due to rounding.

An advantage of transforming data points to $Z$-scores is that it allows us to compare scores that are on different scales. That is, using $Z$-scores we can compare a variable $x_1$ from a normal distribution with mean $\mu_1$ and standard deviation $\sigma_1$ to a variable $x_2$ from a different normal distribution that has a mean $\mu_2$ and standard deviation $\sigma_2$. This is illustrated in the next example.

**EXAMPLE 6 Comparing Exam Scores**

A student receives a grade of 82 on a final examination in biology for which the mean is 73 and the standard deviation is 9. In his final examination in sociology, for which the mean grade is 81 and the standard deviation is 15, he receives an 89. In which examination is his relative standing higher?
In their present forms these distributions are not comparable since they have different means and, more important, different standard deviations. In order to compare the data, we transform the data to standard scores. For the biology test data the $Z$-score for the student’s examination score of 82 is

$$Z = \frac{82 - 73}{9} = \frac{9}{9} = 1$$

For the sociology test data, the $Z$-score for the student’s examination score of 89 is

$$Z = \frac{89 - 81}{15} = \frac{8}{15} = 0.533$$

This means the student’s score in the biology exam is 1 standard deviation above the mean, while his score in the sociology exam is 0.533 standard deviation above the mean. So the student’s relative standing is higher in biology.

NOW WORK PROBLEM 37.

3 Approximate a Binomial Distribution by the Standard Normal Distribution
We start with an example.

**EXAMPLE 7 Finding the Frequency Distribution for a Binomial Probability**

Consider an experiment in which a fair coin is tossed 10 times. Find the frequency distribution for the probability of tossing a head. Graph the frequency distribution.

The probability for obtaining exactly $k$ heads is given by a binomial distribution $b\left(10, k; \frac{1}{2}\right)$. The distribution is given in Table 17. If we graph this frequency distribution, we obtain the line chart shown in Figure 25. When we connect the tops of the lines of the line chart, we obtain a curve that approximates a normal curve, as shown.

This particular distribution for $n = 10$ and $p = \frac{1}{2}$ is a result of the choice of $n$ and $p$. As a matter of fact, the line chart for any binomial probability $b(n, k; p)$ will give an approximation to a normal curve provided $n$ is large or $p$ is close to $\frac{1}{2}$.
Probabilities associated with binomial experiments are readily obtainable from the formula $b(n, k; p)$ when $n$ is small. If $n$ is large, we can compute the binomial probabilities by an approximating procedure using a normal curve. It turns out that the normal distribution provides a very good approximation to the binomial distribution when the product $np(1 - p) \geq 10$.

The mean $\mu$ for the binomial distribution is given by $\mu = np$ (see Expected Value for Bernoulli Trials on page 488). It can be shown that the standard deviation is $\sigma = \sqrt{npq}$, where $q = 1 - p$.

**Example 8 Quality Control**

A company manufactures 60,000 pencils each day. Quality control studies have shown that, on the average, 4% of the pencils are defective. A random sample of 500 pencils is selected from each day’s production and tested. What is the probability that in the sample there are

(a) At least 12 and no more than 24 defective pencils?
(b) 32 or more defective pencils?

**Solution**

(a) Since $n = 500$ is very large, it is appropriate to use a normal curve approximation for the binomial distribution. With $n = 500$, $p = 0.04$, and $q = 1 - p = 0.96$, we have

$\mu = np = 500(0.04) = 20 \quad \sigma = \sqrt{npq} = \sqrt{500(0.04)(0.96)} = 4.38$

To find the approximate probability that the number of defective pencils in a sample is at least 12 and no more than 24, we find the area under a normal curve from $x = 11.5$ to $x = 24.5$.

We subtract 0.5 from the lower limit and add 0.5 to the upper limit because we are using a continuous distribution to approximate discrete variables. Whenever we approximate a binomial probability using the normal distribution we enlarge the interval by 0.5 on each end. This is called *correcting for continuity*. See Figure 26.

Areas $A_1$ and $A_2$ are found by converting to $Z$-scores and using the standard normal curve table.

$x = 11.5: \quad Z_1 = \frac{x - \mu}{\sigma} = \frac{11.5 - 20}{4.38} = -1.94 \quad A_1 = 0.4738$

$x = 24.5: \quad Z_2 = \frac{x - \mu}{\sigma} = \frac{24.5 - 20}{4.38} = 1.03 \quad A_2 = 0.3485$

Total area $= A_1 + A_2 = 0.4738 + 0.3485 = 0.8223$

The approximate probability of the number of defective pencils in the sample being at least 12 and no more than 24 is 0.8223.
(b) Remember, to correct for continuity, we subtract 0.5 from the lower limit of 32 and use 31.5. We want to find the area $A_2$ to the right of 31.5, as indicated in Figure 27. We know that the area to the right of the mean is 0.5, and if we subtract the area $A_1$ from 0.5, we will obtain $A_2$.

First, we find the area $A_1$:

$$Z = \frac{x - \mu}{\sigma} = \frac{31.5 - 20}{4.38} = 2.63 \quad A_1 = 0.4957$$

Then

$$A_2 = 0.5 - A_1 = 0.5 - 0.4957 = 0.0043$$

The approximate probability of finding 32 or more defective pencils in the sample is 0.0043.

**FIGURE 27**

**NOW WORK PROBLEM 43.**

COMMENT: Graphing utilities can calculate exact values for the binomial distribution even if the number of Bernoulli trials, $n$, is large. This makes the normal approximation less important than it had been in the past. Check the user’s manual for your graphing utility to learn how to compute the cumulative binomial distribution. Then redo Example 8 with exact values, and compare the exact solutions to the approximations obtained in Example 8.

The results of Example 8, using a TI-84 Plus graphing calculator, are shown in Figures 28 and 29. When calculating the probability that the number of defective pencils is between 12 and 24, we find the difference between the probability that the number of defective pencils is less than or equal to 24 and the probability that the number of defective pencils is less than or equal to 24 and the probability that the number of defective pencils is less than or equal to 11. See Figure 28.

To find the probability that there are at least 32 defective pencils, we use one minus the probability of the complement. That is, we calculate one minus the probability there are 31 or fewer defective pencils, as illustrated in Figure 29.

The solutions obtained with the graphing utility differ from the normal approximations in Example 8 because the approximation uses the continuous variables of a normal distribution in place of the discrete variables of the binomial distribution.

**EXERCISE 9.6  Answers Begin on Page AN–49.**

**‘Are You Prepared?’ Problems**  Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. **True or False** In a binomial probability model, each trial has only two possible outcomes. (p. 480)

2. Find the probability of obtaining 1, 2, or 3 successes in 5 trials of a binomial experiment if the probability of success is 0.2. (p. 482)

3. **True or False** In a binomial probability model, with $n$ trials, where the probability of success is $p$, the expected number $E$ of successes is $E = np.$ (p. 488)

**Concepts and Vocabulary**

4. The graph of a normal probability distribution is symmetric about the _____.

5. To standardize a normal random variable, we find a _____.

6. A standard normal distribution always has a mean of _____ and a standard deviation of _____.

7. If the area under the standard normal curve between 0 and Z is 0.4, what is the area under the curve between $-Z$ and 0?

8. **True or False** A $Z$-score can be interpreted as the number of standard deviations the original score is from its mean.
9.6 The Normal Distribution

Skill Building

In Problems 9–12, for each normal curve determine \( \mu \) and \( \sigma \) by inspection.

9. \[ \begin{array}{c|c|c}
6 & 7 & 8 \\
\hline
9 & 10 & \text{34.135\%}
\end{array} \]

10. \[ \begin{array}{c|c|c}
80 & 90 & 100 \\
\hline
110 & 120 & \text{34.135\%}
\end{array} \]

11. \[ \begin{array}{c|c|c}
16 & 17 & 18 \\
\hline
19 & 20 & \text{68.27\%}
\end{array} \]

12. \[ \begin{array}{c|c|c}
1.6 & 1.8 & 2.0 \\
\hline
2.2 & 2.4 & \text{68.27\%}
\end{array} \]

13. Given a normal distribution with a mean of 13.1 and a standard deviation of 9.3, find the \( Z \)-score equivalent of the following scores in this distribution:
   (a) \( x = 7 \)  
   (b) \( x = 9 \)  
   (c) \( x = 13 \)  
   (d) \( x = 29 \)  
   (e) \( x = 37 \)  
   (f) \( x = 41 \)

14. Given a normal distribution with a mean of 15.2 and a standard deviation of 5.1, find the \( Z \)-score equivalent of the following scores in this distribution:
   (a) \( x = 8 \)  
   (b) \( x = 9 \)  
   (c) \( x = 16 \)  
   (d) \( x = 22 \)  
   (e) \( x = 23 \)  
   (f) \( x = 25 \)

15. Given the following \( Z \)-scores on a standard normal distribution, find the area under the standard normal curve between each score and the mean.
   (a) \( Z = 0.89 \)  
   (b) \( Z = 1.10 \)  
   (c) \( Z = 3.06 \)  
   (d) \( Z = -1.22 \)  
   (e) \( Z = 2.30 \)  
   (f) \( Z = -0.75 \)

16. Given the following \( Z \)-scores on a standard normal distribution, find the area under the standard normal curve between each score and the mean.
   (a) \( Z = 1.85 \)  
   (b) \( Z = 1.15 \)  
   (c) \( Z = -1.60 \)  
   (d) \( Z = 2.50 \)  
   (e) \( Z = -2.31 \)  
   (f) \( Z = 0.25 \)

17. \[ \begin{array}{c|c|c|c|c|c}
-0.5 & - & 1 & 2 & \text{not shaded} & \text{shaded}
\end{array} \]

18. \[ \begin{array}{c|c|c|c|c|c}
-1.2 & 1.5 & -0.5 & 0.5 & \text{not shaded} & \text{shaded}
\end{array} \]

19. \[ \begin{array}{c|c|c|c|c|c}
1 & 2 & - & 1 & \text{not shaded} & \text{shaded}
\end{array} \]

20. \[ \begin{array}{c|c|c|c|c|c}
-0.5 & 0.5 & - & 0.5 & \text{not shaded} & \text{shaded}
\end{array} \]

In Problems 21–26, suppose a binomial experiment consists of 750 trials and the probability of success for each trial is 0.4. Then approximate the probability of obtaining the number of successes indicated by using a normal curve.

21. \( x = 285 \) – 315  
22. \( x = 280 \) – 320  
23. \( x = 300 \) or more  
24. \( x = 300 \) or less  
25. \( x = 325 \) or more  
26. \( x = 275 \) or less

Applications

27. Assigning Grades An instructor assigns grades in an examination according to the following procedure:
   - A if the score exceeds \( \mu + 1.6\sigma \)
   - B if the score is between \( \mu + 0.6\sigma \) and \( \mu + 1.6\sigma \)
   - C if the score is between \( \mu - 0.3\sigma \) and \( \mu + 0.6\sigma \)
   - D if the score is between \( \mu - 1.4\sigma \) and \( \mu - 0.3\sigma \)
   - F if the score is below \( \mu - 1.4\sigma \)

   What percent of the class receives each grade, assuming that the scores are normally distributed?

28. Assigning Grades Professor Morgan uses a normal distribution to assign grades in his Finite Mathematics class. He assigns an A to students scoring more than 2 standard deviations above the mean and an F to students scoring more than two standard deviations below the mean. He assigns a B to students who score between 1.2 and 2 standard deviations above the mean and a D to students who score between 1.6 and 2 standard deviations below the mean. All other students get a C. What percent of the class receives each grade assuming the scores are normally distributed?

29. Women’s Heights The average height of 2000 women in a random sample is 64 inches. The standard deviation is 2 inches. The heights have a normal distribution.
   (a) How many women in the sample are between 62 and 66 inches tall?
   (b) How many women in the sample are between 60 and 68 inches tall?
   (c) How many women in the sample are between 58 and 70 inches tall?
   (d) How many women in the sample are more than 70 inches tall?
   (e) How many women in the sample are shorter than 58 inches?

30. Weight of Corn Flakes in a Box Corn flakes come in a box that says it holds a mean weight of 16 ounces of cereal. The standard deviation is 0.1 ounce. Suppose that the manufacturer packages 600,000 boxes with weights that have a normal distribution.
   (a) How many boxes weigh between 15.9 and 16.1 ounces?
   (b) How many boxes weigh between 15.8 and 16.2 ounces?
   (c) How many boxes weigh between 15.7 and 16.3 ounces?
   (d) How many boxes weigh under 15.7 or over 16.3 ounces?
   (e) How many boxes weigh under 15.7 ounces?
31. **Student Weights**  The weight of 100 college students closely follows a normal distribution with a mean of 130 pounds and a standard deviation of 5.2 pounds.

(a) How many of these students would you expect to weigh at least 142 pounds?
(b) What range of weights would you expect to include the middle 70% of the students in this group?

32. **Time to Do Taxes**  The Internal Revenue Service claims it takes an average of 3.7 hours to complete a 1040 tax form. Assuming the time to complete the form is normally distributed with a standard deviation of 30 minutes:

(a) What percent of people would you expect to complete the form in less than 5 hours?
(b) What time interval would you expect to include the middle 50% of the tax filers?

33. **Life Expectancy of Clothing**  If the average life of a certain make of clothing is 40 months with a standard deviation of 7 months, what percentage of these clothes can be expected to last from 28 months to 42 months? Assume that clothing lifetime follows a normal distribution.

34. **Life Expectancy of Shoes**  Records show that the average life expectancy of a pair of shoes is 2.2 years with a standard deviation of 1.7 years. A manufacturer guarantees that shoes lasting less than a year are replaced free. For every 1000 pairs sold, how many pairs should the manufacturer expect to replace free? Assume a normal distribution.

35. **Movie Theater Attendance**  The attendance over a weekly period of time at a movie theater is normally distributed with a mean of 10,000 and a standard deviation of 1000 persons. Find

(a) The number in the lowest 70% of the attendance figures.
(b) The percent of attendance figures that falls between 8500 and 11,000 persons.
(c) The percent of attendance figures that differs from the mean by 1500 persons or more.

36. **Test Scores**  Scores on an aptitude test are normally distributed with a mean of 980 and a standard deviation of 110. If 10,000 students take the test,

(a) How many would you expect to score between 900 and 1200?
(b) How many would you expect to score above 1400?
(c) How many would you expect to score below 750?

37. **Comparing Test Scores**  Colleen, Mary, and Kathleen are vying for a position as editor. Colleen, who is tested with group I, gets a score of 76 on her test; Mary, who is tested with group II, gets a score of 89; and Kathleen, who is tested with group III, gets a score of 21. If the average score for group I is 82, for group II is 93, and for group III is 24, and if the standard deviation for each group is 7, 2, and 9, respectively, which person has the highest relative standing?

38. In Mathematics 135 the average final grade is 75.0 and the standard deviation is 10.0. The professor’s grade distribution shows that 15 students with grades from 68.0 to 82.0 received Cs. Assuming the grades follow a normal distribution, how many students are in Mathematics 135?

39. (a) Draw the line chart and frequency curve for the probability of a head in an experiment in which a biased coin is tossed 15 times and the probability that a head occurs is 0.3. [Hint: Find $b(15, k; 0.30)$ for $k = 0, 1, \ldots, 15$.]  
(b) Discuss the shape of the curve.
(c) What are the mean and the standard deviation of the distribution?

40. Follow the same directions as in Problem 39 for an experiment in which a biased coin is tossed 25 times and the probability that a head appears is $\frac{3}{4}$.

In Problems 41–44, use a normal approximation to the binomial distribution.

41. **Lifetime Batting Averages**  A baseball player has a lifetime batting average of 0.250. In a season, this player comes to bat 300 times.

(a) What is the probability that at least 80 and no more than 90 hits occur?
(b) What is the probability that 85 or more hits occur?

42. **Hitting a Target**  A skeet shooter has a long-established probability of hitting a target of 0.75. In a particular session, 200 attempts are made.

(a) What is the probability that at least 135 and no more than 160 are successful?
(b) What is the probability that 160 or more are successful?

43. **Quality Control**  A company manufactures 100,000 packages of jelly beans each week. On average, 3% of the packages do not seal properly. A random sample of 500 packages is selected at the end of the week. What is the probability that in this sample at least 10 are not properly sealed?

44. **Quality Control**  A company manufactures 200,000 pairs of pantyhose per week. On average 4% of the pairs are defective. A random sample of 300 pairs are selected each week. What is the probability that more than 5 pairs in the sample are defective?

45. Graph the standard normal curve using a graphing utility. For what value of $x$ does the function assume its maximum? The equation is given by 
$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

46. Graph the normal curve with $\mu = 10$ and $\sigma = 2$ using a graphing utility. For what value of $x$ does the function assume its maximum? The equation is given by 
$$y = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}(x-10)^2}$$

47. **Quality Control**  Refer to Problem 43. Use a graphing utility to compute the exact probability that the sample contains at least 10 bags of improperly sealed jelly beans.

48. **Quality Control**  Refer to Problem 44. Use a graphing utility to compute the exact probability that the sample contains at least 5 pairs that are defective.
Discussion and Writing

49. **Quality Control** Refer to Problem 43. Suppose that each week a random sample of 500 packages of jelly beans is selected and every week there are at least 25 improperly sealed bags of jelly beans. What would you conclude about the statistics given in Problem 43?

50. **Quality Control** Refer to Problem 44. Suppose that each week a random sample of 300 pairs of pantyhose is selected and every week there are at least 20 defective pairs in the sample. What would you conclude about the statistics given in Problem 44?

'Are You Prepared?' Answers

1. True
2. \( P(1, 2, \text{ or } 3 \text{ successes}) = 0.6656 \)
3. True

**CHAPTER 9 REVIEW**

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<td></td>
<td>2</td>
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<tr>
<td></td>
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<td>1, 5</td>
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<tr>
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<td>2</td>
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</tr>
<tr>
<td></td>
<td>3, 5</td>
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<td>27b–32b</td>
</tr>
<tr>
<td></td>
<td>4</td>
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<td>27c–32c</td>
</tr>
<tr>
<td>9.5</td>
<td>3, 4</td>
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<tr>
<td></td>
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</tr>
<tr>
<td></td>
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<tr>
<td></td>
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</tr>
<tr>
<td></td>
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</tr>
</tbody>
</table>
566 Chapter 9 Statistics

IMPORTANT FORMULAS

Mean for Sample Data (p. 532 and p. 539)
\[ \bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} \]

Mean for Population Data (p. 539)
\[ \mu = \frac{x_1 + x_2 + \cdots + x_N}{N} \]

Mean for Grouped Data (p. 534)
\[ \bar{x} = \frac{\sum f_i m_i}{n} \]

Standard Deviation of Sample Data (p. 544)
\[ s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n - 1}} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n - 1}} \]

Standard Deviation for Population Data (p. 544)
\[ \sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}} = \sqrt{\frac{\sum (x_i - \mu)^2}{N}} \]

Standard Deviation for Grouped Data (p. 546)
\[ s = \sqrt{\frac{(m_1 - \bar{x})^2 f_1 + (m_2 - \bar{x})^2 f_2 + \cdots + (m_k - \bar{x})^2 f_k}{n - 1}} \]
\[ = \sqrt{\frac{\sum [(m_i - \bar{x})^2 f_i]}{n - 1}} \]

Z-Score (p. 556)
\[ Z = \frac{x - \mu}{\sigma} \]

REVIEW EXERCISES Answers to odd-numbered problems begin on page AN–50.

Blue problem numbers represent the author’s suggestions for a Practice Test.

In Problems 1–6, determine the variable in the experiment and state whether it is continuous or discrete.

1. A municipal hospital epidemiologist measures the circumference of each newborn baby’s head.
2. An immigration officer asks each alien how long he or she expects to stay in the country.
3. A pollster counts how many people are in favor of a new law.
4. A statistician counts the number of hits a major league baseball player gets in a season.
5. A quality control manager lists the number of defective products manufactured each day.
6. A human resources clerk records the days an employee calls in sick.

7. Opinion Poll The student activities president wants to poll students for their opinions regarding a band to play on campus. Describe how she can choose a simple random sample of 100 students to poll.

8. Retirement Plans An employee benefits officer wants to poll employees of a bank regarding their retirement plans. How can he choose a simple random sample of 50 employees?

9. Opinion Poll An urban planner wants to poll the citizens of East Norwich regarding their opinions about a proposed road construction project that would interfere with rush hour traffic for the next 18 months. He asks 200 people as they leave the neighborhood supermarket between 10:00 A.M. and 3:00 P.M. on Thursday. What is wrong with the urban planner’s sample?

10. Theater Attendance A researcher wants to determine how often the citizens of Muttontown attend the theater. She interviews 50 people at the local senior citizens’ center. What is wrong with this researcher’s sample?

11. The following data show how office workers in Chicago get to work:

<table>
<thead>
<tr>
<th>Means of Transportation</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ride alone</td>
<td>64</td>
</tr>
<tr>
<td>Carpool</td>
<td>5</td>
</tr>
<tr>
<td>Ride bus</td>
<td>30</td>
</tr>
<tr>
<td>Other</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Construct a bar graph of the data.
(b) Construct a pie chart of the data.
(c) Which of the graphs seems more informative to you? Why?
12. **Getting to Work** According to the 2000 census, Americans traveled to work in the following ways. The numbers are given in thousands.

<table>
<thead>
<tr>
<th>Means of Transportation</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drove</td>
<td>112,904</td>
</tr>
<tr>
<td>Used public transportation</td>
<td>6,030</td>
</tr>
<tr>
<td>Walked</td>
<td>3,721</td>
</tr>
<tr>
<td>Other</td>
<td>1,540</td>
</tr>
</tbody>
</table>

(a) Construct a bar graph of the data.
(b) Construct a pie chart of the data.
(c) Which of the graphs seems more informative to you? Why?

13. To study their attitudes toward a new product, 1000 people were interviewed. Their response is given in the following table:

<table>
<thead>
<tr>
<th>Attitude</th>
<th>No. of Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do not like</td>
<td>420</td>
</tr>
<tr>
<td>Like</td>
<td>360</td>
</tr>
<tr>
<td>Like very much</td>
<td>220</td>
</tr>
</tbody>
</table>

(a) Construct a bar graph of the data.
(b) Construct a pie chart of the data.
(c) Which of the graphs seems more informative to you? Why?

14. **Vacation Travel** An Internet travel agency surveyed 1000 of its clients to determine their mode of transportation for their last vacation. The agency found that 635 people drove, 280 flew, and 85 cruised to their destination.

(a) Construct a bar graph of the data.
(b) Construct a pie chart of the data.
(c) Which of the graphs seems more informative to you? Why?

15. **College Enrollment** The pie chart represents the race or ethnicity of students enrolled in four-year colleges in the United States in the fall of 2007. A total of 9,160,973 students were enrolled.

- **Source:** National Science Foundation, Division of Science Resources Statistics.
  - (a) What group made up the smallest percentage of four-year college enrollment?
  - (b) If Asian Americans represent roughly 4% of the total population in the United States, were they underrepresented or overrepresented in four-year colleges in 2007?
  - (c) Approximately how many Hispanics were enrolled in four-year colleges in 2007?

16. **Home Runs** In 2001 Barry Bonds hit 73 home runs, breaking Mark McGwire’s major league record. The bar graph shows where he hit each home run ball.

(a) In what direction were the fewest of his home runs hit?
(b) What percent of the home runs were hit to left center field?
(c) Where were almost 25% of the home runs hit?
(d) If you were a fielding coach for the opposing team and Barry Bonds was at bat, what advice would you give to your outfielders?

17. **Educational Attainment** The bar graph depicts the highest level of education attained by people in the United States in 2009.

- **Source:** U.S. Census Bureau.
  - (a) What is the highest level of education that most Americans have obtained?
  - (b) How many Americans have at least a bachelor’s degree?
  - (c) How many people in the United States do not have a high school diploma?
  - (d) How many people have gone to college but do not have a bachelor’s degree?
18. **College Enrollment**  The pie chart represents the race or ethnicity of students enrolled in two-year colleges in the United States in the fall of 2007. A total of 6,638,938 students were enrolled.

![Pie chart showing enrollment by race/ethnicity.](image-url)

**Source:** National Science Foundation, Division of Science Resources Statistics.

(a) What group represents the largest portion of the student population?
(b) Approximately how many American Indians were enrolled in a two-year college in 2007?
(c) Approximately how many African Americans were enrolled in two-year colleges in 2007?

19. **Test Scores**  The following scores were made on a math exam:

<table>
<thead>
<tr>
<th>Score</th>
<th>80</th>
<th>90</th>
<th>82</th>
<th>21</th>
<th>100</th>
<th>55</th>
<th>80</th>
<th>62</th>
<th>78</th>
<th>52</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>21</td>
<td>73</td>
<td>80</td>
<td>44</td>
<td>72</td>
<td>63</td>
<td>91</td>
<td>85</td>
<td>33</td>
<td>66</td>
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<td></td>
<td>78</td>
<td>42</td>
<td>87</td>
<td>90</td>
<td>90</td>
<td>70</td>
<td>48</td>
<td>75</td>
<td>83</td>
<td>77</td>
</tr>
<tr>
<td></td>
<td>63</td>
<td>85</td>
<td>69</td>
<td>80</td>
<td>41</td>
<td>87</td>
<td>66</td>
<td>52</td>
<td>71</td>
<td>60</td>
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<td></td>
<td>74</td>
<td>70</td>
<td>73</td>
<td>95</td>
<td>89</td>
<td>41</td>
<td>92</td>
<td>68</td>
<td>100</td>
<td>72</td>
</tr>
</tbody>
</table>

(a) Set up a frequency table for the above data. What is the range?
(b) Draw a line chart for the data.
(c) Build a histogram for the data using a class interval of size 10 beginning with the interval 20.0–29.9.
(d) Draw a frequency polygon on the histogram.
(e) Describe the shape of the distribution.
(f) Find the cumulative frequencies.
(g) Draw the ogive.

20. **Test Scores**  Use the data from Problem 19:

(a) Group the data into class intervals of size 5, starting with the interval 20.0–24.9.
(b) Build a histogram for the data using a class interval of size 5.
(c) Draw a frequency polygon on the histogram.
(d) Find the cumulative frequencies.
(e) Draw the ogive.

21. **Track Championship**  At the high school track championships, 42 boys participated in the 1600-meter race. Their times (rounded to the nearest second) were

<table>
<thead>
<tr>
<th>4 min 30 sec</th>
<th>4 min 12 sec</th>
<th>4 min 46 sec</th>
<th>4 min 15 sec</th>
<th>4 min 46 sec</th>
<th>4 min 10 sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 30</td>
<td>4 22</td>
<td>4 39</td>
<td>4 56</td>
<td>4 50</td>
<td>5 01</td>
</tr>
<tr>
<td>5 08</td>
<td>5 12</td>
<td>5 20</td>
<td>5 20</td>
<td>5 31</td>
<td>5 18</td>
</tr>
<tr>
<td>5 06</td>
<td>4 40</td>
<td>4 52</td>
<td>4 36</td>
<td>5 02</td>
<td>5 06</td>
</tr>
<tr>
<td>5 12</td>
<td>5 31</td>
<td>5 37</td>
<td>5 40</td>
<td>5 55</td>
<td>5 43</td>
</tr>
<tr>
<td>5 48</td>
<td>5 40</td>
<td>6 01</td>
<td>6 32</td>
<td>6 12</td>
<td>6 10</td>
</tr>
<tr>
<td>6 40</td>
<td>6 02</td>
<td>7 05</td>
<td>7 15</td>
<td>6 30</td>
<td>5 20</td>
</tr>
</tbody>
</table>

(a) Set up a frequency table for the above data. What is the range?
(b) Draw a line chart for the data.
(c) Group the data into class intervals of size 30 seconds, starting with the interval 4 minutes 0 seconds–4 minutes 29 seconds.
(d) Build the histogram for the data.
(e) Draw a frequency polygon on the histogram.
(f) Describe the shape of the distribution.
(g) Find the cumulative frequencies.
(h) Draw the ogive.
22. **Track Championship**  Use the data in Problem 21.
(a) Group the data into class intervals of size 1 minute, starting with the interval 4 minutes 0 seconds–4 minutes 59 seconds.
(b) Build the histogram for the data.
(c) Draw a frequency polygon on the histogram.
(d) Find the cumulative frequencies.
(e) Draw the ogive.

23. **Baseball Players**  During 2010 spring training, the ages of the 40 men on the New York Yankees roster were

<table>
<thead>
<tr>
<th>Age (Years)</th>
<th>Number of Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>20–24</td>
<td>10</td>
</tr>
<tr>
<td>25–29</td>
<td>11</td>
</tr>
<tr>
<td>30–34</td>
<td>8</td>
</tr>
<tr>
<td>35–39</td>
<td>5</td>
</tr>
<tr>
<td>40–44</td>
<td>6</td>
</tr>
</tbody>
</table>

**Source:** Major League Baseball.
(a) Make a frequency table of the players’ ages.
(b) Construct a line chart of the data.
(c) Group the data into class intervals of size 5, beginning with the interval 20.0–24.9. How many class intervals are there?
(d) Build the histogram for the data.
(e) Draw the frequency polygon on the histogram.
(f) Describe the shape of the distribution.
(g) Find the cumulative frequencies.
(h) Draw the ogive.

24. **Baseball Players**  During 2010 spring training, the ages of the 40 men on the Atlanta Braves roster were

<table>
<thead>
<tr>
<th>Age (Years)</th>
<th>Number of Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>20–24</td>
<td>9</td>
</tr>
<tr>
<td>25–29</td>
<td>12</td>
</tr>
<tr>
<td>30–34</td>
<td>10</td>
</tr>
<tr>
<td>35–39</td>
<td>6</td>
</tr>
<tr>
<td>40–44</td>
<td>3</td>
</tr>
</tbody>
</table>

**Source:** Major League Baseball.
(a) Make a frequency table of the players’ ages.
(b) Construct a line chart of the data.
(c) Group the data into class intervals of size 5, beginning with the interval 20.0–24.9. How many class intervals are there?
(d) Build the histogram for the data.
(e) Draw the frequency polygon on the histogram.
(f) Describe the shape of the distribution.
(g) Find the cumulative frequencies.
(h) Draw the ogive.

25. **Tax Returns**  The following table gives the number of individual tax returns (in millions) with an adjusted gross income under $50,000 for the 2007 tax year.

<table>
<thead>
<tr>
<th>Adjusted Gross Income, $</th>
<th>Number of Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–9,999</td>
<td>3.75</td>
</tr>
<tr>
<td>10,000–19,999</td>
<td>13.2</td>
</tr>
<tr>
<td>20,000–29,999</td>
<td>15.5</td>
</tr>
<tr>
<td>30,000–39,999</td>
<td>12.5</td>
</tr>
<tr>
<td>40,000–49,999</td>
<td>9.2</td>
</tr>
</tbody>
</table>

**Source:** IRS, Statistics of Income Division.
(a) Graph the data using a histogram.
(b) Draw a frequency polygon on the histogram.
(c) Compare the data to the previous problem. Comment on any similarities or differences.

26. **Tax Returns**  The following table gives the number of individual tax returns (in millions) with an adjusted gross income under $50,000 for the 2006 tax year.

<table>
<thead>
<tr>
<th>Adjusted Gross Income, $</th>
<th>Number of Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–9,999</td>
<td>3.8</td>
</tr>
<tr>
<td>10,000–19,999</td>
<td>13.2</td>
</tr>
<tr>
<td>20,000–29,999</td>
<td>15.4</td>
</tr>
<tr>
<td>30,000–39,999</td>
<td>12.1</td>
</tr>
<tr>
<td>40,000–49,999</td>
<td>8.7</td>
</tr>
</tbody>
</table>

**Source:** IRS, Statistics of Income Division.
(a) Graph the data using a histogram.
(b) Draw a frequency polygon on the histogram.
(c) Compare the data to the previous problem. Comment on any similarities or differences.
In Problems 27–32, find (a) the mean, (b) the median, (c) the mode, if it exists, (d) the range, and (e) the standard deviation of each sample of data.

27. 12, 10, 8, 2, 0, 4, 10, 5, 4, 8, 0
28. 195, 5, 2, 2, 2, 1, 0
30. −5, 2, 3, 8, −6, 9, 11, 10, −2, 8, −4, 0
31. 5, 7, 7, 9, 10, 11, 1, 6, 2, 12
32. 12, 10, 8, 10, 6, 10, 14

33. In seven different rounds of golf, Joe scores 74, 72, 76, 81, 77, 76, and 73.
   (a) Do you think these are sample data or population data? Explain your answer.
   (b) What is Joe's mean golf score?
   (c) Find the standard deviation of Joe's golf scores.

34. The 10 fish caught on Tuesday weighed 16.2 pounds, 15 pounds, 12.3 pounds, 20 pounds, 8 pounds, 6.5 pounds, 8 pounds, 10.8 pounds, 12 pounds, and 9 pounds.
   (a) Do you think these are sample data or population data? Explain your answer.
   (b) What is the mean weight of the fish?
   (c) Find the standard deviation of the fishes' weights.

35. Population Distribution The U.S. Census Bureau gives information about the age distribution of persons living in the United States. The following data give the estimated number (in thousands) of females for various ages in 2009.

<table>
<thead>
<tr>
<th>Age</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–4</td>
<td>10,412</td>
</tr>
<tr>
<td>5–9</td>
<td>10,073</td>
</tr>
<tr>
<td>10–14</td>
<td>9,751</td>
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<tr>
<td>15–19</td>
<td>10,486</td>
</tr>
<tr>
<td>20–24</td>
<td>10,446</td>
</tr>
<tr>
<td>25–29</td>
<td>10,562</td>
</tr>
<tr>
<td>30–34</td>
<td>9,780</td>
</tr>
<tr>
<td>35–39</td>
<td>10,185</td>
</tr>
<tr>
<td>40–44</td>
<td>10,487</td>
</tr>
<tr>
<td>45–49</td>
<td>11,535</td>
</tr>
<tr>
<td>50–54</td>
<td>11,083</td>
</tr>
<tr>
<td>55–59</td>
<td>9,770</td>
</tr>
<tr>
<td>60–64</td>
<td>8,234</td>
</tr>
<tr>
<td>65–69</td>
<td>6,273</td>
</tr>
<tr>
<td>70–74</td>
<td>4,925</td>
</tr>
<tr>
<td>75–79</td>
<td>4,176</td>
</tr>
<tr>
<td>80–84</td>
<td>3,524</td>
</tr>
<tr>
<td>85–89</td>
<td>2,395</td>
</tr>
<tr>
<td>90–94</td>
<td>1,077</td>
</tr>
<tr>
<td>95–99</td>
<td>319</td>
</tr>
<tr>
<td>100–104</td>
<td>55</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau.

(a) Approximate the mean age of a female in 2009.
(b) Approximate the median age of a female in 2009.
(c) Approximate the standard deviation of the age of the females.

36. Population Distribution The U.S. Census Bureau gives information about the age distribution of persons living in the United States. The following data give the estimated number (in thousands) of males for various ages in 2009.

<table>
<thead>
<tr>
<th>Age</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–4</td>
<td>10,887</td>
</tr>
<tr>
<td>5–9</td>
<td>10,535</td>
</tr>
<tr>
<td>10–14</td>
<td>10,222</td>
</tr>
<tr>
<td>15–19</td>
<td>11,051</td>
</tr>
<tr>
<td>20–24</td>
<td>11,093</td>
</tr>
<tr>
<td>25–29</td>
<td>11,115</td>
</tr>
<tr>
<td>30–34</td>
<td>10,107</td>
</tr>
<tr>
<td>35–39</td>
<td>10,353</td>
</tr>
<tr>
<td>40–44</td>
<td>10,504</td>
</tr>
<tr>
<td>45–49</td>
<td>11,295</td>
</tr>
<tr>
<td>50–54</td>
<td>10,677</td>
</tr>
<tr>
<td>55–59</td>
<td>9,204</td>
</tr>
<tr>
<td>60–64</td>
<td>7,576</td>
</tr>
<tr>
<td>65–69</td>
<td>5,511</td>
</tr>
<tr>
<td>70–74</td>
<td>4,082</td>
</tr>
<tr>
<td>75–79</td>
<td>3,149</td>
</tr>
<tr>
<td>80–84</td>
<td>2,298</td>
</tr>
<tr>
<td>85–89</td>
<td>1,266</td>
</tr>
<tr>
<td>90–94</td>
<td>424</td>
</tr>
<tr>
<td>95–99</td>
<td>82</td>
</tr>
<tr>
<td>100–104</td>
<td>87</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau.

(a) Approximate the mean age of a male in 2009.
(b) Approximate the median age of a male in 2009.
(c) Approximate the standard deviation of the age of the males.

37. The Empirical Rule A random sample of 200 lightbulbs has a mean life of 600 hours and a standard deviation of 53 hours.

(a) A histogram of the data indicates the sample data follow a bell-shaped distribution. According to the Empirical Rule, 99.7% of lightbulbs have lifetimes between _____ and _____ hours.
(b) Assuming the data are bell shaped, determine the percentage of lightbulbs that will have a life between 494 and 706 hours.
(c) Assuming the data are bell shaped, what percentage of lightbulbs will last between 547 and 706 hours?
(d) If the company that manufactures the lightbulbs guarantees to replace any bulb that does not last at least 441 hours, what percentage of lightbulbs can the firm expect to have to replace, according to the Empirical Rule?
38. **The Empirical Rule**  In a random sample of 250 toner cartridges, the mean number of pages a toner cartridge can print is 4302 and the standard deviation is 340.

(a) Suppose a histogram of the data indicates that the sample data follow a bell-shaped distribution. According to the Empirical Rule, 99.7% of toner cartridges will print between _____ and _____ pages.

(b) Assuming that the distribution of the data is bell shaped, determine the percentage of toner cartridges whose print total is between 3622 and 4982 pages.

(c) If the company that manufactures the toner cartridges guarantees to replace any cartridge that does not print at least 3622 pages, what percent of cartridges can the firm expect to be responsible for replacing, according to the Empirical Rule?

In Problems 39–42, use Chebychev’s theorem to find the probability.

39. **Jelly Jars**  A machine fills jars with 12 ounces of jam. The machine is not exact and jars have a standard deviation of 0.05 ounce. In a lot of 1000 jars, how many jars would you expect to have between 11.9 and 12.1 ounces of jam?

40. **Machining**  A die cast stamps out metal washers with a mean outer diameter of 2.5 cm and a variance of 0.01 cm². In a shipment of 10,000 washers, how many would you expect to have an outer diameter between 2.3 and 2.7 cm?

41. **Sacks of Potatoes**  Potatoes are packed in 10-pound bags. Each bag has a mean weight of 10 pounds with a standard deviation of 0.25 pound. What is the probability that a bag picked at random weighs less than 9.5 pounds or more than 10.5 pounds?

42. **Shrimp by the Pound**  Shrimp are priced by size; the larger the shrimp, the more expensive the cost. Jumbo shrimp have 9 shrimp to the pound with a standard deviation of 0.75 shrimp. What is the probability that a pound of jumbo shrimp contains fewer than 8 or more than 10 shrimp?

In Problems 43–48, compute the Z-score for the data point x, using the given population mean and standard deviation.

43. \( \mu = 10, \sigma = 3, x = 8 \)

44. \( \mu = 14, \sigma = 3, x = 20 \)

45. \( \mu = 1, \sigma = 5, x = 8 \)

46. \( \mu = 140, \sigma = 15, x = 125 \)

47. \( \mu = 55, \sigma = 3, x = 60 \)

48. \( \mu = 32, \sigma = 13, x = 10 \)

In Problems 49–52, calculate the area under the normal curve.

49. Between \( Z = -2.75 \) and \( Z = -1.35 \)

50. Between \( Z = 1.2 \) and \( Z = 1.75 \)

51. Between \( Z = -0.75 \) and \( Z = 2.1 \)

52. Between \( Z = -1.5 \) and \( Z = 0.34 \)

53. A normal distribution has a mean of 25 and a standard deviation of 5.

(a) What proportion of the scores will fall between 20 and 30?

(b) What proportion of the scores will lie above 35?

54. A set of 600 scores is normally distributed. How many scores would you expect to find

(a) Between \( \pm 1\sigma \) of the mean?

(b) Between \( 1\sigma \) and \( 3\sigma \) above the mean?

(c) Between \( \pm \frac{2}{3}\sigma \) of the mean?

55. **Average Life of a Dog**  The average life expectancy of a dog is 14 years, with a standard deviation of about 1.25 years. Assuming that the life spans of dogs are normally distributed, approximately how many dogs will die before reaching the age of 10 years, 4 months?

56. **Life of a Battery**  The average life span of a phone battery is 20 months with a standard deviation of 1.5 months. Assuming the battery life is normally distributed, what is the probability that a battery chosen at random will last more than 24 months?
57. **Comparing Test Scores**  
Bob got an 89 on the final exam in mathematics and a 79 on the sociology exam. In the mathematics class the average grade was 79 with a standard deviation of 5, and in the sociology class the average grade was 72 with a standard deviation of 3.5. Assuming that the grades in both subjects were normally distributed, in which class did Bob rank higher?

58. **Comparing Test Scores**  
Marty took 3 tests in science this semester. His grades were 58, 79, and 83. If the mean score and standard deviation of the tests were 50 and 7, 73 and 4, and 75 and 8, respectively, on which test did Marty have the highest standing?

59. Suppose it is known that the number of items produced each week in a factory has a mean of 40. If the variance of a week’s production is known to equal 25, then what can be said about the probability that this week’s production will be between 30 and 50?

60. From past experience a teacher knows that the test scores of students taking an examination have a mean of 75 and a variance of 25. What can be said about the probability that a student chosen at random will score between 65 and 85?

In Problems 61–64 use the normal approximation to the binomial distribution.

61. **Blood Tests**  
A blood test for an antigen gives a positive result with a probability of 0.7. In testing 200 blood samples, what is the probability of obtaining more than 160 positive results?

62. **Color-blindness**  
The probability a male is color-blind is 0.30. In a sample of 1000 males, what is the probability that fewer than 200 are color-blind?

63. **Toll Gates**  
An automatic gate malfunctions, failing to open, with a probability 0.05. What is the probability that in 500 operations, the gate fails to work between 20 and 30 times?

64. **Licensing Tests**  
A professional licensing test has a 60% pass rate. What is the probability that, in a group of 200 randomly selected test takers, between 110 and 125 pass the test?

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**Chapter 9 Project**

**“AVERAGE” WEATHER**

The daily weather forecast that appears in a local newspaper or on television often contains information regarding the average high and low temperatures, air quality, and other information. This information helps a reader or viewer to make a comparison between historical and actual temperatures, rainfall, humidity level, pollen, air pollutants, and so on. These figures are the result of the application of statistics to a large collection of data. In this example, we compare the actual weather to the reported averages.

Part of what makes travel comfortable and enjoyable is good weather. Suppose that you want to visit New York City, which is notorious for extreme weather. To avoid the cold of winter and the extreme heat of summer, you decide to visit in June. According to *USA Today*, the average high temperature is 80°F, the average low temperature is 63°F, and the average rainfall is 3.6 inches. The highest temperature recorded was 101°F and the lowest temperature recorded was 44°F. However, you are curious about the deviation from these averages and discover the data shown in Table 1 for June 2009.
We shall investigate how June 2009 compared to an average month.

1. Calculate the mean high temperature, the mean low temperature, and the total rainfall.
2. The mean temperature for the month is defined as the mean of the daily temperatures (the midpoint of the high and low temperatures). What is the mean temperature for this month?
3. Show that we can calculate the mean monthly temperature by calculating the mean of the mean high temperature and the mean low temperature.
4. What are the median high and the median low temperatures for the month? How do they compare with the corresponding mean temperatures? Interpret the differences.
5. Is there a modal high temperature? If so, what is it? Is there a modal low temperature? If so, what is it?
6. Compare the mean, median, and modal high temperatures. Do the same for the low temperatures. What do you conclude?
7. Calculate the standard deviation of the high and low temperatures. A normal distribution is symmetric around its mean, so the mean, the median, and the mode are all equal. What kind of errors might one expect if one were to use the normal distribution to estimate the probability of obtaining a low temperature between 60°F and 65°F on a day in June based on the 2009 data? [Hint: Try a geometric argument. That is, use a graph.]
8. Make a frequency table for the daily high temperatures and one for the daily low temperatures. Then group each set of data into class intervals of equal degree width. Use 51°–55°F as the first interval.
9. Build a histogram for each set of grouped data and draw a frequency polygon on each histogram.
10. Examine the histograms and determine the modes of the high and low temperature frequency distributions. If the earlier calculations indicated that the month was close to the average, what do the modes tell you?
11. Find the approximate mean high temperature for June 2009 and the approximate standard deviation using the grouped data. Are they good approximations of the actual mean high temperature and its standard deviation? Explain your reasoning.
12. Repeat Problem 11 for the grouped low temperatures.

### TABLE 1 DATA FOR JUNE 2009

<table>
<thead>
<tr>
<th>Day</th>
<th>High of (°F)</th>
<th>Low of (°F)</th>
<th>Precip. (in)</th>
<th>Day</th>
<th>High of (°F)</th>
<th>Low of (°F)</th>
<th>Precip. (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>71</td>
<td>53</td>
<td>0</td>
<td>16</td>
<td>70</td>
<td>55</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>60</td>
<td>0.1</td>
<td>17</td>
<td>68</td>
<td>60</td>
<td>0.38</td>
</tr>
<tr>
<td>3</td>
<td>72</td>
<td>54</td>
<td>1</td>
<td>18</td>
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<td>61</td>
<td>1.92</td>
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<td>68</td>
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<td>0.06</td>
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<td>77</td>
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</tr>
<tr>
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<td>71</td>
<td>59</td>
<td>0.7</td>
<td>24</td>
<td>76</td>
<td>64</td>
<td>0.13</td>
</tr>
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<td>67</td>
<td>59</td>
<td>0.09</td>
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<td>79</td>
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<td>63</td>
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<td>67</td>
<td>0.01</td>
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<td>81</td>
<td>64</td>
<td>0.02</td>
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<tr>
<td>13</td>
<td>73</td>
<td>61</td>
<td>0.15</td>
<td>28</td>
<td>82</td>
<td>65</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>73</td>
<td>57</td>
<td>0.24</td>
<td>29</td>
<td>81</td>
<td>67</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>72</td>
<td>56</td>
<td>0.26</td>
<td>30</td>
<td>84</td>
<td>65</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Source: National Weather Service.
Mathematical Questions from Professional Exams

1. **Actuary Exam—Part II**  Under the hypothesis that a pair of dice are fair, the probability is approximately 0.95 that the number of 7s appearing in 180 throws of the dice will lie within $30 \pm K$. What is the value of $K$?
   (a) 2  (b) 4  (c) 6  (d) 8  (e) 10

2. **Actuary Exam—Part II**  If $X$ is normally distributed with mean $\mu$ and variance $\mu^2$ and if $P(-4 < X < 8) = 0.9974$, then $\mu = $
   (a) 1  (b) 2  (c) 4  (d) 6  (e) 8

3. **Actuary Exam—Part II**  A manufacturer makes golf balls whose weights average 1.62 ounces, with a standard deviation of 0.05 ounce. What is the probability that the weight of a group of 100 balls will lie in the interval $162 \pm 0.5$ ounces?
   (a) 0.18  (b) 0.34  (c) 0.68  (d) 0.84  (e) 0.96
Markov Chains; Games

10

Education, Job Security, and Salary

According to the Bureau of Labor Statistics, for individuals 25 years of age or older in 2009, unemployment rates declined as the level of education increased. According to the report, for individuals with only an elementary school diploma, the unemployment rate was 14.6%. For high school graduates, the unemployment rate was 9.7% and for those with a college degree, the rate was 5.7%. The report also determined that salaries increased as the level of education increased. The median weekly earnings for those with only an elementary school diploma were $454 ($23,608 annually). With a high school diploma, median weekly earnings rose to $626 ($32,552 annually), and for those with a bachelor’s degree, earnings were $1025 ($53,000 annually). As it turns out, the likelihood of a child achieving a certain level of education depends on the level of education of the parents. The Chapter Project at the end of this chapter provides data on this phenomenon and uses Markov chains to investigate conclusions that can be drawn from the data.

A Look Back, A Look Forward

In Chapter 3, we studied matrices and in Chapter 7 we studied probability. Both of these topics were rich in applications. In this chapter we study applications that involve both matrices and probability.

Chapter 10 is divided into two parts: Markov chains (Sections 10.1–10.3) and games (Sections 10.4–10.6). These parts are completely independent of one another, so one may be covered and the other skipped without any difficulty.

A Markov chain is a probability model that consists of repeated trials of an experiment in which some change or transition is occurring, such as consumers changing their preferred choice of a soft drink. Information about how this change takes place is then used to predict future market share.

Games provide a mathematical way to express the strategies available to people or corporations. Ways to arrive at a “best” strategy are discussed.
Introduction

In Chapter 8 Bernoulli trials were introduced. In discussing Bernoulli trials, the assumption was made that the outcome of each trial is independent of the outcome of any previous trial.

A Markov chain is another type of probability model in which there is some connection between one trial and the next. Loosely speaking, a Markov chain is one in which what happens next is determined by what happened immediately before. At any trial, the experiment is in one of a finite number of states, with the next trial of the experiment consisting of movement to a possibly different state. The probability of moving to a certain state depends only on the state currently occupied and does not vary with time. Here are some situations that may be thought of as Markov chains:

- There are yearly population shifts between a city and its surrounding suburbs. At any time, a person is either in the city or in the suburbs. So there are two states here with movement between them. As we track the population over time, we can ask: What percentage of the population is, say, in the city?
- Several detergents compete in the market. Each year there is some shift of customer loyalty from one brand to another. At any point in time the state of the experiment would correspond to the brand of detergent a customer uses.
- A psychology experiment consists of placing a mouse in a maze composed of rooms. The mouse moves from room to room, and we think of this movement as moving from state to state.

We use this psychology experiment to introduce some terminology.

Figure 1 shows a maze consisting of four connecting rooms. The rooms are numbered 1, 2, 3, 4 for convenience, and each room contains pulsating lights of a different color. The experiment consists of releasing a mouse in a particular room and observing its behavior.
We assume an observation is made whenever a movement occurs or after a fixed time interval, whichever comes first. Since the movement of the mouse is not predictable, probabilistic terms are used to describe it.

We refer to the rooms as states and use \( p_{ij} \) to represent the probability of moving from state \( i \) to state \( j \) in one observation interval. For example, the probability \( p_{12} \) that the mouse moves from state 1 to state 2 might be \( p_{12} = \frac{1}{2} \), while the probability of moving from state 1 to state 3 might be \( p_{13} = \frac{1}{4} \). Note that \( p_{14} = 0 \) since there is no direct passage between rooms 1 and 4. Then \( p_{11} = \frac{1}{4} \) would be the probability that the mouse remains in room 1 for one observation interval.

A convenient way of writing the probabilities \( p_{ij} \) is to display them in a matrix. The resulting matrix

\[
P = \begin{bmatrix}
p_{11} & p_{12} & p_{13} & p_{14} \\
p_{21} & p_{22} & p_{23} & p_{24} \\
p_{31} & p_{32} & p_{33} & p_{34} \\
p_{41} & p_{42} & p_{43} & p_{44}
\end{bmatrix}
\]

is called the transition matrix of the experiment.

In our example there are four states and \( P \) is a \( 4 \times 4 \) matrix. Were we to assign values to the remaining probabilities, a possible choice for the transition matrix \( P \) might be

\[
P = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 1 & 1 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\
\frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \\
1 & 0 & 1 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 1 & 1 & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4}
\end{bmatrix}
\]

where the rows and columns are indexed by the four states (rooms).

We can display the entries in \( P \) using a tree diagram. See Figure 2. The entries in \( P \) are conditional probabilities representing the probabilities of where the mouse will go next given that we know where it is now. This essential idea of movement from state to state with attached probabilities forms the basis of a Markov chain.
Definition

Markov Chain

A Markov chain is a sequence of experiments, each of which results in one of a finite number of states that we label 1, 2, …, m. The probability of being in a particular state depends only on the state previously occupied.

If \( p_{ij} \) is the probability of moving from state \( i \) to state \( j \), then the transition matrix \( P = [p_{ij}] \) of a Markov chain is the \( m \times m \) matrix

\[
P = \begin{bmatrix}
1 & 2 & \cdots & m \\
p_{11} & p_{12} & \cdots & p_{1m} \\
p_{21} & p_{22} & \cdots & p_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{m1} & p_{m2} & \cdots & p_{mm}
\end{bmatrix}
\]

Notice that the transition matrix \( P \) is a square matrix with entries that are always between 0 and 1, inclusive, since they represent probabilities. Also, the sum of the entries in every row is 1 since, as in the mouse and maze example, once the mouse is in a given room, it either stays there or moves to one of the other rooms.

The transition matrix contains the information necessary to predict what happens next, given that we know what happened before. It remains to specify what the situation was at the start of the experiment. For example, we need to specify what room the mouse was placed in at the start. A row vector is used for this purpose.

Definition

Initial Probability Distribution

In a Markov chain with \( m \) states, the initial probability distribution is a \( 1 \times m \) row vector \( v^{(0)} \), whose \( i \)th entry is the probability the experiment was in state \( i \) at the start.

For example, if the mouse was equally likely to be placed in any one of the rooms at the start, then we would have \( v^{(0)} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \). If it was decided to always place the mouse initially in room 1, then we would have \( v^{(0)} = [1 \ 0 \ 0 \ 0] \).

Example 1

Finding the Probability Distribution of a Markov Chain

Look at the maze in Figure 3. Suppose we assign the following transition probabilities:

From room 1 to room

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix}
\]
Here, the mouse starts in room 1, and \( \frac{1}{3} \) of the time it remains there during the time interval of observation, \( \frac{1}{3} \) of the time it enters room 2, and \( \frac{1}{3} \) of the time it enters room 3. Since it cannot go to room 4 directly from room 1, the probability assignment is 0. Similarly, the transition probabilities in moving from room 2, room 3, and room 4 are assigned as follows:

(a) Find the transition matrix \( P \).

(b) If the initial placement of the mouse is in room 4, find the initial probability distribution.

(c) What is the probability distribution after two observations assuming the initial placement of the mouse is in room 4?

**SOLUTION**

(a) The transition matrix \( P \) is

\[
P = \begin{bmatrix}
1 & 2 & 3 & 4 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]

(b) Since the initial placement of the mouse is in room 4, the initial probability distribution is

\[
\nu^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}
\]

(c) To find the probabilities after two observations, use a tree diagram.

The tree begins at 4, since the initial placement of the mouse is in room 4. Then, using row 4 of the transition matrix, we branch from 4, to the rooms 2, 3, and 4. (We skip room 1, since \( p_{14} = 0 \).) From these rooms branch out using rows 2, 3, and 4 of the transition matrix. See Figure 4 on page 580.
From this tree diagram we deduce, for example, that the mouse will be in state 1 after two observations with probability $\frac{1}{9} + \frac{1}{9} = \frac{2}{9}$. That is,

The probability of moving from 4 to 1 in two trials $= \frac{2}{9}$

Similarly,

The probability of moving from 4 to 2 in two trials $= \frac{1}{3} + \frac{1}{3} = \frac{2}{9}$

The probability of moving from 4 to 3 in two trials $= \frac{1}{3} + \frac{1}{3} = \frac{2}{9}$

The probability of moving from 4 to 4 in two trials $= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1}{3}$

The probability distribution $\nu^{(2)}$ after two observations is

$$
\nu^{(2)} = \begin{bmatrix} \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{3} \end{bmatrix}
$$

**NOW WORK PROBLEM 9.**

Using the notation introduced in Example 1(c), the row vector $\nu^{(k)}$ is used to record the probabilities of being in the various states after $k$ trials. The $i$th entry of $\nu^{(k)}$ is the probability of being in state $i$ after $k$ trials.

Instead of following the procedure of Example 1 to find $\nu^{(2)}$, it is possible to compute $\nu^{(k)}$ directly from the transition matrix $P$ and the initial probability distribution $\nu^{(0)}$.

**Theorem**

**Probability Distribution after $k$ Trials**

In a Markov chain the probability distribution $\nu^{(k)}$ after $k$ trials is

$$
\nu^{(k)} = \nu^{(k-1)}P
$$

where $P$ is the transition matrix.
10.1 Markov Chains and Transition Matrices

Equation (1) shows that

\[ \nu^{(k)} = \nu^{(0)} P \quad k = 1 \]
\[ \nu^{(2)} = \nu^{(1)} P \quad k = 2 \]
\[ \nu^{(3)} = \nu^{(2)} P \quad k = 3 \]

and so on. The succeeding distribution can always be derived from the previous one by multiplying by the transition matrix, \( P \).

**EXAMPLE 2 Finding the Probability Distribution After \( k \) Trials**

Using the information given in Example 1, use Equation (1) to find the probability distribution after two observations.

**SOLUTION** The initial probability distribution is \( \nu^{(0)} = [0 \ 0 \ 0 \ 1] \), so by Equation (1),

\[
\nu^{(1)} = \nu^{(0)} P = [0 \ 0 \ 0 \ 1] \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}
\]

Using Equation (1) again gives

\[
\nu^{(2)} = \nu^{(1)} P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix} = \begin{bmatrix} 2/9 & 2/9 & 2/9 & 1/3 \end{bmatrix}
\]

This way of obtaining \( \nu^{(2)} \) agrees with the results found in Example 1(c).

**NOW WORK PROBLEM 11 USING EQUATION (1).**

A few computations using Equation (1) lead to another result.

\[
\nu^{(1)} = \nu^{(0)} P \\
\nu^{(2)} = \nu^{(1)} P = [\nu^{(0)} P] P = \nu^{(0)} P^2 \\
\nu^{(3)} = \nu^{(2)} P = [\nu^{(0)} P^2] P = \nu^{(0)} P^3 \\
\nu^{(4)} = \nu^{(3)} P = [\nu^{(0)} P^3] P = \nu^{(0)} P^4 \\
\nu^{(5)} = \nu^{(4)} P = [\nu^{(0)} P^4] P = \nu^{(0)} P^5
\]
Continuing in this way, we have the following result.

**Theorem**

**Probability Distribution after \( k \) Trials**

In a Markov chain the probability distribution \( \nu^{(k)} \) after \( k \) trials is

\[
\nu^{(k)} = \nu^{(0)} P^k
\]

where \( P^k \) is the \( k \)th power of the transition matrix \( P \) and \( \nu^{(0)} \) is the initial probability distribution.

The solution to Example 2(c) could also have been arrived at by squaring the transition matrix \( P \) and computing \( \nu^{(2)} = \nu^{(0)} P^2 \):

\[
\nu^{(2)} = \nu^{(0)} P^2 = \begin{bmatrix} 0.85 & 0.15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & 9 & 9 & 9 \\ 2 & 1 & 2 & 2 \\ 9 & 3 & 9 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 9 & 9 & 3 & 3 \end{bmatrix}
\]

**NOW WORK PROBLEM 11 USING EQUATION (2).**

2 Solve Applied Problems Involving a Markov Chain

**EXAMPLE 3** Population Movement

Suppose that the city of Oak Lawn is experiencing a movement of its population to the suburbs. At present, 85% of the total population lives in the city and 15% lives in the suburbs. But each year 7% of the city people move to the suburbs, while only 1% of the suburb people move back to the city. Assuming that the total population (city and suburbs together) remains constant, what percent of the total will remain in the city after 5 years?

**SOLUTION**

This problem can be expressed as a sequence of experiments in which each experiment measures the proportion of people in the city and the proportion of people in the suburbs.

In the \((n + 1)\)st year these proportions will depend for their value only on the proportions in the \(n\)th year and not on the proportions found in earlier years. This experiment can be modeled by a Markov chain.

The initial probability distribution for this Markov chain is

\[
\nu^{(0)} = \begin{bmatrix} 0.85 & 0.15 \end{bmatrix}
\]
That is, initially, 85% of the people reside in the city and 15% in the suburbs.

The transition matrix \( P \) is

\[
P = \begin{bmatrix}
0.93 & 0.07 \\
0.01 & 0.99
\end{bmatrix}
\]

That is, each year 7% of the city people move to the suburbs (so that 93% remain in the city) and 1% of the suburb people move to the city (so that 99% remain in the suburbs).

To find the probability distribution after 5 years, we need to compute \( v^{(5)} \). We’ll use Equation (1) five times.

\[
v^{(1)} = v^{(0)}P = \begin{bmatrix} 0.85 & 0.15 \end{bmatrix} \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix} = \begin{bmatrix} 0.792 & 0.208 \end{bmatrix}
\]

\[
v^{(2)} = v^{(1)}P = \begin{bmatrix} 0.792 & 0.208 \end{bmatrix} \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix} = \begin{bmatrix} 0.73864 & 0.26136 \end{bmatrix}
\]

\[
v^{(3)} = v^{(2)}P = \begin{bmatrix} 0.73864 & 0.26136 \end{bmatrix} \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix} = \begin{bmatrix} 0.68955 & 0.31045 \end{bmatrix}
\]

\[
v^{(4)} = v^{(3)}P = \begin{bmatrix} 0.68955 & 0.31045 \end{bmatrix} \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix} = \begin{bmatrix} 0.64439 & 0.35561 \end{bmatrix}
\]

\[
v^{(5)} = v^{(4)}P = \begin{bmatrix} 0.64439 & 0.35561 \end{bmatrix} \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix} = \begin{bmatrix} 0.60284 & 0.39716 \end{bmatrix}
\]

After 5 years, 60.28% of the residents live in the city and 39.72% live in the suburbs. ■

NOW WORK PROBLEM 17.

Example 3 leads us to inquire whether the situation in Oak Lawn ever stabilizes. That is, after a certain number of years is an equilibrium reached? Also, does the equilibrium, if attained, depend on what the initial distribution of the population was, or is it independent of the initial state? We deal with these questions in the next section.

### USING TECHNOLOGY

#### EXAMPLE 4 Population Movement

Rework Example 3 using a graphing utility.

The initial probability vector \( v^{(0)} \) and the transition matrix \( P \) are

\[
v^{(0)} = \begin{bmatrix} 0.85 & 0.15 \end{bmatrix} \quad P = \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix}
\]

Enter these as \( A \) and \( B \), respectively, in your graphing utility. Since the graphing utility can easily raise a matrix to a power, we use Equation (2) to find the probability distribution after 5 years. Figure 5 shows the result using a TI-84 Plus. We see that

\[
v^{(5)} = v^{(0)}P^5 = \begin{bmatrix} 0.6028 & 0.3972 \end{bmatrix} \quad A = v^{(0)}, \quad B = P
\]

as before. ■
EXERCISE 10.1 Answers Begin on Page AN–55.

‘Are You Prepared?’ Problems  Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. A sample space consists of four distinct objects. If the elements are equally likely to occur, what probability is assigned to each one? (p. 378–385)
2. In constructing a probability model, the sum of the probabilities assigned to each element of the sample space will add up to _________. (p. 378–385)
3. What is the dimension of the matrix \[\begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}\]? (p. 104–116)

4. Find the product:
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 3 & 0 \\
1 & 0 & 1 & 3 \\
0 & 1 & 1 & 3
\end{bmatrix}
\]

5. True or False  The matrix \[\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\] is a transition matrix for a Markov chain.
6. True or False  If \(P\) is the transition matrix of a Markov chain, each row of \(P\) has entries that add up to 1.

7. In a Markov chain with \(m\) states the initial probability distribution is a row vector of dimension _________.
8. In a Markov chain with transition matrix \(P\), the probability distribution \(\nu^{(k)}\) after \(k\) observations satisfies the two equations:
\[\nu^{(k)} = ___P\] and \[\nu^{(k)} = ___P^k\]

Skill Building

9. Consider a Markov chain with transition matrix

\[
\begin{bmatrix}
1 & 2 \\
3 & 3
\end{bmatrix}
\]

(a) What does the entry \(\frac{1}{4}\) in this matrix represent?

(b) Assuming that the system is initially in state 1, find the probability distribution one observation later. What is it two observations later?

(c) Assuming that the system is initially in state 2, find the probability distribution one observation later. What is it two observations later?

10. Consider a Markov chain with transition matrix

\[
\begin{bmatrix}
0.3 & 0.7 \\
0.4 & 0.6
\end{bmatrix}
\]

(a) What does the entry 0.7 in the matrix represent?

(b) Assuming that the system is initially in state 1, find the probability distribution two observations later.

(c) Assuming that the system is initially in state 2, find the probability distribution two observations later.

11. Consider the transition matrix of Problem 9. If the initial probability distribution is \(\begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}\), what is the probability distribution after two observations?

12. Consider the transition matrix of Problem 10. If the initial probability distribution is \(\begin{bmatrix} 0.5 & 0.5 \end{bmatrix}\), what is the probability distribution after two observations?

13. Consider a Markov chain with transition matrix

\[
\begin{bmatrix}
0.7 & 0.2 & 0.1 \\
0.6 & 0.2 & 0.2 \\
0.4 & 0.1 & 0.5
\end{bmatrix}
\]

If the initial distribution is \(\begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix}\), what is the probability distribution in the next observation?

14. Consider a Markov chain with transition matrix

\[
\begin{bmatrix}
0.1 & 0.1 & 0.8 \\
0.5 & 0.5 & 0 \\
0.7 & 0 & 0.3
\end{bmatrix}
\]

If the initial distribution is \(\begin{bmatrix} 0.3 & 0.3 & 0.4 \end{bmatrix}\), what is the probability distribution in the next observation?

15. Find the values of \(a, b,\) and \(c\) that will make the following matrix a transition matrix for a Markov chain:

\[
\begin{bmatrix}
0.2 & a & 0.4 \\
b & 0.6 & 0.3 \\
0 & c & 0
\end{bmatrix}
\]
16. Find the values of \( a, b, \) and \( c \) that will make the following matrix a transition matrix for a Markov chain:

\[
P = \begin{bmatrix}
1 & 1 & 2 \\
\frac{1}{3} & 2 & a \\
0 & b & \frac{2}{5} \\
\frac{1}{2} & c & \frac{1}{2}
\end{bmatrix}
\]

17. In Example 3, page 582, if the initial probability distribution for Oak Lawn and the suburbs is \( v^{(0)} = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \), what is the population distribution after 5 years?

18. In the maze given in Figure 1 (page 576), if the initial probability distribution is find the probability distribution after two observations.

19. **Mass Transit** A new rapid transit system has just been installed. It is anticipated that each week 90% of the commuters who used the rapid transit will continue to do so. Of those who traveled by car, 5% will begin to use the rapid transit instead.

   (a) Explain why the above is a Markov chain.
   
   (b) Set up the \( 2 \times 2 \) transition matrix \( P \) with columns and rows labeled \( R \) (rapid transit) and \( C \) (car) to display these transitions.
   
   (c) Compute \( P^2 \) and \( P^3 \).

20. **Voting Patterns** The voting pattern for a certain group of cities is such that 60% of the Democratic (\( D \)) mayors were succeeded by Democrats and 40% by Republicans (\( R \)). Also 30% of the Republican mayors were succeeded by Democrats and 70% by Republicans.

   (a) Explain why the above is a Markov chain.
   
   (b) Set up the \( 2 \times 2 \) transition matrix \( P \) with columns and rows labeled \( D \) and \( R \) to display these transitions.
   
   (c) Compute \( P^2 \) and \( P^3 \).

21. **Maze Experiment** Consider the maze with 9 rooms shown in the figure. The system consists of the maze and a mouse. We assume that the following learning pattern exists: If the mouse is in room 1, 2, 3, 4, or 5, it either remains in that room or it moves to any room that the maze permits, each having the same probability; if it is in room 8, it moves directly to room 9; if it is in room 6, 7, or 9, it remains in that room.

22. **Walk or Drive?** A professor either walks or drives to a university. He never drives 2 days in a row, but if he walks 1 day, he is just as likely to walk the next day as to drive his car. Show that this forms a Markov chain and give the transition matrix.

23. **Market Penetration** A company is promoting Pinot Grigio wine. The result of this is that 75% of the people drinking Pinot Grigio over any given period of 1 month continue to drink it the next month; of those people drinking other white wines in the period of 1 month, 35% change over to Pinot Grigio the next month. We would like to know what fraction of wine drinkers will drink Pinot Grigio after 2 months if 50% drink another white wine now.

24. **Insurance** Suppose that, during the year 2010, 45% of the drivers in a certain metropolitan area had Travelers automobile insurance, 30% had General American insurance, and 25% were insured by other companies. Suppose also that a year later (i) of those who had been insured by Travelers in 2010, 92% continued to be insured by Travelers, but 8% had switched their insurance to General American; (ii) of those who had been insured by General American in 2010, 90% continued to be insured by General American, but 4% had switched to Travelers and 6% had switched to other companies; (iii) of those who had been insured by other companies in 2010, 82% continued but 10% had switched to Travelers, and 8% had switched to General American. Using these data, answer the following questions:

   (a) What percentage of drivers in the metropolitan area will be insured by Travelers and General American in 2011?
   
   (b) If these trends continued for one more year, what percentage of the drivers will be insured by Travelers and General American in 2012?

25. Consider the transition matrix

\[
P = \begin{bmatrix}
0.5 & 0.2 & 0.1 & 0.2 \\
0.3 & 0.3 & 0.2 & 0.2 \\
0.1 & 0.5 & 0.1 & 0.3 \\
0.25 & 0.25 & 0.25 & 0.25
\end{bmatrix}
\]

If the initial probability distribution is \( v^{(0)} = \begin{bmatrix} 0.4 & 0.3 & 0.1 & 0.2 \end{bmatrix} \), what is the probability distribution after 10 observations?

26. Consider the transition matrix

\[
P = \begin{bmatrix}
0.7 & 0.15 & 0.05 & 0.1 \\
0.15 & 0.7 & 0.05 & 0.1 \\
0.1 & 0.05 & 0.7 & 0.15 \\
0.15 & 0.05 & 0.1 & 0.7
\end{bmatrix}
\]
If the initial probability distribution is 
\( v^{(0)} = [0.15, 0.15, 0.55, 0.15] \),
what is the probability distribution after eight observations?

27. **Psychology Experiment** Consider the maze with six rooms shown in the figure. The system consists of the maze and a mouse. We assume the following learning pattern exists: If the mouse is in room 1, 2, 3, or 4, it either remains in that room or it moves to any room that the maze permits, each having the same probability; if it is in room 5, it moves directly to room 6; if it is in room 6, it remains in that room.

(a) Construct the transition matrix \( P \).

(b) If the mouse is initially in room 2, what is the initial probability distribution?

(c) What is the probability distribution for the mouse after 10 trials?

(d) What is the most likely room for the mouse to be in after 10 trials?

28. **Psychology Experiment** Consider the maze with five rooms shown in the figure. The system consists of the maze and a mouse. We assume the following learning pattern exists: If the mouse is in room 1, 2, or 4, it either remains in that room or it moves to any room that the maze permits, each having the same probability; if it is in room 3, it moves directly to room 5; if it is in room 5, it remains in that room.

(a) Construct the transition matrix \( P \).

(b) If the mouse is initially in room 2, what is the initial probability distribution?

(c) What is the probability distribution for the mouse after 10 trials?

(d) What is the most likely room for the mouse to be in after 10 trials?

29. A **probability vector** is a row matrix whose entries are probabilities that add up to 1. Let

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]

be a transition matrix and

\[ u = [u_1 \quad u_2] \]

be a probability vector. Prove that \( uA \) is a probability vector.

30. Let

\[ P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \]

be a transition matrix. Prove that \( v^{(k)} = v^{(k-1)}P \).

#### Discussion and Writing

31. Explain why the matrix below cannot be the transition matrix for a Markov chain.

\[ \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix} \]

32. Explain why the matrix below cannot be the transition matrix for a Markov chain.

\[ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

33. If \( A \) is a transition matrix, what about \( A^2 \) and \( A^3 \)? Are they also transition matrices? What do you conjecture about \( A^0 \)?

34. If you are asked to find the probability distribution of a Markov chain after six observations, would you use Equation (1) or Equation (2)? Give reasons for your choice.

#### ‘Are You Prepared?’ Answers

1. \( \frac{1}{4} \) 2. 1 3. \( 1 \times 4 \) 4. \( \begin{bmatrix} 2/9 & 2/9 & 2/9 & 1/3 \end{bmatrix} \)
One important question about Markov chains is: What happens in the long run? Does the distribution of the states tend to stabilize over time? In this section we investigate the conditions under which a Markov chain produces an equilibrium, or steady-state, situation. We also give a procedure for finding this equilibrium distribution when it exists. As it turns out, Markov chains that are regular provide the answer.

**1 Identify Regular Markov Chains**

**Definition**

A Markov chain is said to be regular if, for some power of its transition matrix \( P \), all of the entries are positive.

**Example 1 Identifying Regular Markov Chains**

The transition matrix \( P \) of a Markov chain is given as

\[
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
1 & 0
\end{bmatrix}
\]

Since the square of \( P \), namely,

\[
P^2 = \left(\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
1 & 0
\end{bmatrix}\right)^2 = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{3}{4} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

has all positive entries, the Markov chain is regular.
**EXAMPLE 2** Identifying Markov Chains That Are Not Regular

The transition matrix of a Markov chain is given as

\[
P = \begin{bmatrix}
1 & 0 \\
3 & 1 \\
4 & 4
\end{bmatrix}
\]

Since every power of \( P \) will always have \( p_{12} = 0 \) (do you see why?), the Markov chain is not regular.

**NOW WORK PROBLEMS 5 AND 13.**

The next example sets the stage for determining the long-run behavior of a Markov chain.

**EXAMPLE 3** Consumer Loyalty

A wine company is promoting its Syrah wine. The result of this is that 75% of the people drinking Syrah wine over any given period of 1 month continue to drink it the next month; of those people drinking other kinds of red wine in the period of 1 month, 35% change over to the Syrah wine the next month.

The transition matrix \( P \) of this experiment is

\[
P = \begin{bmatrix}
E_1 & E_2 \\
0.75 & 0.25 \\
0.35 & 0.65
\end{bmatrix}
\]

(a) Find the probability of passing from \( E_1 \) to either \( E_1 \) or \( E_2 \) in two trials.

(b) Find the probability of passing from \( E_2 \) to either \( E_1 \) or \( E_2 \) in two trials.

**SOLUTION** A tree diagram depicting two trials of this experiment is given in Figure 6.

(a) The probability \( p_{11}^{(2)} \) of proceeding from \( E_1 \) to \( E_1 \) in two trials is

\[
p_{11}^{(2)} = (0.75)(0.75) + (0.25)(0.35) = 0.65
\]
The probability $p_{12}^{(2)}$ from $E_1$ to $E_2$ in two trials is

$$p_{12}^{(2)} = (0.75)(0.25) + (0.25)(0.65) = 0.35$$

(b) The probability $p_{21}^{(2)}$ from $E_2$ to $E_1$ in two trials is

$$p_{21}^{(2)} = (0.35)(0.75) + (0.65)(0.35) = 0.49$$

The probability $p_{22}^{(2)}$ from $E_2$ to $E_2$ in two trials is

$$p_{22}^{(2)} = (0.35)(0.25) + (0.65)(0.65) = 0.51$$

If we square the transition matrix $P$, we obtain

$$P^2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{bmatrix}$$

$$= \begin{bmatrix} (0.75)(0.75) + (0.25)(0.35) & (0.75)(0.25) + (0.25)(0.65) \\ (0.35)(0.75) + (0.65)(0.35) & (0.35)(0.25) + (0.65)(0.65) \end{bmatrix}$$

$$= \begin{bmatrix} 0.65 & 0.35 \\ 0.49 & 0.51 \end{bmatrix}$$

and we notice that

$$P^2 = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{bmatrix}$$

That is, the square of the transition matrix gives the probabilities for moving from one state to another state in two trials.

This is true in general, and higher powers of the transition matrix carry similar interpretations.

**Theorem**

**Probability of Passing from State $i$ to State $j$ in $n$ Trials**

If $P$ is the transition matrix of a Markov chain, then the $(i, j)$th entry of $P^n$ gives the probability of passing from state $i$ to state $j$ in $n$ trials.

In other words, we can investigate the long-run behavior of a Markov chain by raising its transition matrix to a high enough power. We use the transition matrix $P$ of Example 3 and compute $P^2$ through $P^{12}$.*

$$P = \begin{bmatrix} 0.7500 & 0.2500 \\ 0.3500 & 0.6500 \end{bmatrix}$$
$$P^5 = \begin{bmatrix} 0.5876 & 0.4124 \\ 0.5774 & 0.4226 \end{bmatrix}$$
$$P^9 = \begin{bmatrix} 0.5832 & 0.4168 \\ 0.5832 & 0.4166 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.6500 & 0.3500 \\ 0.4900 & 0.5100 \end{bmatrix}$$
$$P^6 = \begin{bmatrix} 0.5850 & 0.4150 \\ 0.5809 & 0.4191 \end{bmatrix}$$
$$P^{10} = \begin{bmatrix} 0.5834 & 0.4166 \\ 0.5833 & 0.4167 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.6100 & 0.3900 \\ 0.5460 & 0.4540 \end{bmatrix}$$
$$P^7 = \begin{bmatrix} 0.5840 & 0.4160 \\ 0.5824 & 0.4176 \end{bmatrix}$$
$$P^{11} = \begin{bmatrix} 0.5834 & 0.4166 \\ 0.5833 & 0.4167 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.5940 & 0.4060 \\ 0.5684 & 0.4316 \end{bmatrix}$$
$$P^8 = \begin{bmatrix} 0.5836 & 0.4164 \\ 0.5830 & 0.4170 \end{bmatrix}$$
$$P^{12} = \begin{bmatrix} 0.5833 & 0.4167 \\ 0.5833 & 0.4167 \end{bmatrix}$$

* A graphing utility or a spreadsheet such as Excel makes this easy.
Notice that the higher powers $P^n$ seem to be converging and stabilizing, becoming equal to a fixed matrix. Also notice that the rows of the matrix $P^{12}$ are equal. This means the probability is 0.5833 that a person will drink Syrah wine regardless of what he or she drank previously! In other words, the situation has stabilized, with 58.33% drinking Syrah and 41.67% drinking other red wines.

A probability vector is a row matrix whose entries are probabilities that add up to 1. The probability vector $\mathbf{t} = [0.5833 \ 0.4167]$ is called the fixed probability vector $\mathbf{t}$ of the Markov chain whose transition matrix is $P$.

### 2 Find the Fixed Probability Vector of a Regular Markov Chain

The results stated below hold only if the Markov chain is regular.

**Theorem Long-Term Behavior of a Regular Markov Chain**

Let $P$ be the transition matrix of a regular Markov chain. Then the following properties hold.

1. The matrices $P^n$ approach a fixed matrix $T$ as $n$ gets large. We write $P^n \to T$ (as $n$ gets large).
2. The rows of $T$ are all identical and equal to a probability row vector $\mathbf{t}$.
3. $\mathbf{t}$ is the unique probability vector that satisfies $\mathbf{t}P = \mathbf{t}$.
4. If $\mathbf{v}(0)$ is any initial distribution, then $\mathbf{v}(n) \to \mathbf{t}$ as $n$ gets large.

Properties 1 and 2 have already been demonstrated. To see Property 3, return to the Equations (1) and, from $P^{12}$, let

$$\mathbf{t} = [0.5833 \ 0.4167]$$

Compute $\mathbf{t}P$,

$$\mathbf{t}P = [0.5833 \ 0.4167] \begin{bmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{bmatrix} = [0.5833 \ 0.4167] = \mathbf{t}$$

The row vector $\mathbf{t}$ obeys the equation $\mathbf{t}P = \mathbf{t}$.

This means it is not necessary to compute higher and higher powers $P^n$ of the transition matrix to find $T$ and, in the process, $\mathbf{t}$. Property 3 allows us to find $\mathbf{t}$ by solving a system of equations.

### EXAMPLE 4 Finding the Fixed Probability Vector of a Regular Markov Chain

Find the fixed probability vector $\mathbf{t}$ of the Markov chain whose transition matrix $P$ is

$$P = \begin{bmatrix} 0.93 & 0.07 \\ 0.01 & 0.99 \end{bmatrix}$$

(This is the transition matrix from Example 3, page 582, which represents population movement.)
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**Example 5: Finding the Fixed Probability Vector of a Regular Markov Chain**

Find the fixed probability vector \( \mathbf{t} \) of the transition matrix

\[
P = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]

**Solution**

First, \( P \) is regular, since \( P^2 \) has all positive entries (you should verify this). Let \( \mathbf{t} = [t_1 \ t_2 \ t_3] \) be the fixed vector. Then

\[
\begin{align*}
\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} P &= \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} \\
\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix} &= \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}
\end{align*}
\]

Solve the system of equations:

\[
\begin{align*}
\frac{1}{2}t_1 + 0t_2 + \frac{1}{2}t_3 &= t_1 \\
\frac{1}{2}t_1 + 1t_2 + \frac{1}{2}t_3 &= t_2 \\
\frac{1}{3}t_1 + \frac{1}{3}t_2 + \frac{1}{3}t_3 &= t_3
\end{align*}
\]

Simplifying, we get:

\[
\begin{align*}
\frac{1}{2}t_1 &= t_1 \\
\frac{1}{2}t_2 &= t_2 \\
\frac{1}{3}t_1 + \frac{1}{3}t_2 + \frac{1}{3}t_3 &= t_3
\end{align*}
\]

From the first two equations, we have:

\[
\begin{align*}
t_1 &= \frac{2}{3}t_1 \\
t_2 &= \frac{2}{3}t_2
\end{align*}
\]

Substituting into the third equation:

\[
\frac{1}{3}t_1 + \frac{1}{3}t_2 + t_3 = t_3
\]

\[
\frac{1}{3}t_1 + \frac{1}{3}t_2 = 0
\]

Solving for \( t_1 \) and \( t_2 \):

\[
\begin{align*}
t_1 &= \frac{1}{2}t_1 \\
t_2 &= \frac{1}{2}t_2
\end{align*}
\]

Thus, the system of equations reduces to:

\[
\begin{align*}
t_1 &= \frac{1}{2}t_1 \\
t_2 &= \frac{1}{2}t_2
\end{align*}
\]

and

\[
\begin{align*}
t_1 &= \frac{1}{2}t_1 \\
t_2 &= \frac{1}{2}t_2
\end{align*}
\]

Solving for \( t_1 \) and \( t_2 \) separately:

\[
\begin{align*}
t_1 &= \frac{2}{3}t_1 \\
t_2 &= \frac{2}{3}t_2
\end{align*}
\]

Thus, the fixed probability vector \( \mathbf{t} \) is

\[
\mathbf{t} = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}
\]

We interpret \( \mathbf{t} \) as follows: In the long run, \( \frac{2}{3} \) or 66.7% of the population will live in the city, while \( \frac{1}{3} \) or 33.3% will live in the suburbs.

**Now work Problem 15.**
The matrix equation on the right yields three additional equations.

\[
\begin{align*}
\frac{1}{2}t_1 + \frac{1}{2}t_2 + \frac{1}{3}t_3 &= t_1 \\
\frac{1}{2}t_2 + \frac{1}{3}t_3 &= t_2 \\
\frac{1}{2}t_1 + \frac{1}{3}t_3 &= t_3
\end{align*}
\]

Upon simplifying, we have

\[
\begin{align*}
-\frac{1}{2}t_1 + \frac{1}{2}t_2 + \frac{1}{3}t_3 &= 0 \\
-\frac{1}{2}t_2 + \frac{1}{3}t_3 &= 0 \\
\frac{1}{2}t_1 - \frac{2}{3}t_3 &= 0
\end{align*}
\]

Simplify the three equations by removing the fractions and insert the equation \( t_1 + t_2 + t_3 = 1 \). Then we have the following system of four equations containing three variables:

\[
\begin{align*}
t_1 + t_2 + t_3 &= 1 \quad (1) \\
3t_1 - 3t_2 - 2t_3 &= 0 \quad (2) \\
-3t_2 + 2t_3 &= 0 \quad (3) \\
3t_1 - 4t_3 &= 0 \quad (4)
\end{align*}
\]

Note that the sum of Equations (3) and (4) yields Equation (2), so that we really have a system of three equations containing three variables. Solve using substitution. From Equation (3), \( t_2 = \frac{2}{3}t_3 \) and from Equation (4), \( t_1 = \frac{4}{3}t_3 \). Substituting into Equation (1),

\[
\begin{align*}
t_1 + t_2 + t_3 &= 1 \quad (1) \\
\frac{4}{3}t_3 + \frac{2}{3}t_3 + t_3 &= 1 \\
3t_3 &= 1 \\
t_3 &= \frac{1}{3}
\end{align*}
\]

So, \( t_1 = \frac{4}{3}t_3 = \frac{4}{9}, t_2 = \frac{2}{3}t_3 = \frac{2}{9}, \) and \( t_3 = \frac{1}{3} \). The fixed probability vector \( t \) of the transition matrix \( P \) is \( t = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{1}{3} \end{bmatrix} \).

**Now work Problem 19.**

One feature of a regular Markov chain is that the long-run probability distribution \( t \) can be found simply by solving a system of equations. A more remarkable feature is that the same equilibrium \( t \) is reached no matter what the initial probability distribution is.

We prove this for a \( 2 \times 2 \) transition matrix.
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Proof Suppose \( \nu^{(0)} = [a \ b] \) is any initial probability distribution of a regular Markov chain with a \( 2 \times 2 \) transition matrix \( P \). Let \( t = [t_1 \ t_2] \) be the fixed vector. By Property 1,

\[
P^n \rightarrow T = \begin{bmatrix} t_1 & t_2 \\ t_1 & t_2 \end{bmatrix} \quad \text{as } n \text{ gets large}
\]

Multiply by \( \nu^{(0)} \). Then

\[
\nu^{(0)} P^n \rightarrow \nu^{(0)} T \quad \text{as } n \text{ gets large}
\]

But \( \nu^{(n)} = \nu^{(0)} P^n \), so

\[
\nu^{(n)} \rightarrow \nu^{(0)} T \quad \text{as } n \text{ gets large}
\]

Now

\[
\nu^{(0)} T = [a \ b] \begin{bmatrix} t_1 & t_2 \\ t_1 & t_2 \end{bmatrix} = \begin{bmatrix} (a + b)t_1 & (a + b)t_2 \end{bmatrix}
\]

Since \( \nu^{(0)} \) is a probability vector, it follows that \( a + b = 1 \). As a result, \( \nu^{(0)} T = [t_1 \ t_2] \) and

\[
\nu^{(n)} \rightarrow \nu^{(0)} T = [t_1 \ t_2] = t
\]

That is, the same \( t \) is reached regardless of the entries used in \( \nu^{(0)} \). \( \blacksquare \)

3 Solve Applied Problems Involving Regular Markov Chains

**EXAMPLE 6 Consumer Loyalty**

A certain community has three grocery stores. Within this community there exists a shift of customers from one grocery store to another. A study was made on January 1, and it was found that \( \frac{1}{4} \) of the population shopped at store I, \( \frac{1}{3} \) at store II, and \( \frac{5}{12} \) at store III. Each month store I retains 90% of its customers and loses 10% of them to store II. Store II retains 5% of its customers and loses 85% of them to store I and 10% of them to store III. Store III retains 40% of its customers and loses 50% of them to store I and 10% to store II. Assume the population of the community remains constant.

(a) What is the transition matrix \( P \)?

(b) What is the initial probability distribution \( \nu^{(0)} \)?

(c) What proportion of customers will each store retain by February 1?

(d) What proportion of customers will each store retain by March 1?

(e) Assuming the same pattern continues, what will be the long-run distribution of customers among the three stores?
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SOLUTION  (a) The transition matrix \( P \) is

\[
P = \begin{pmatrix}
0.90 & 0.10 & 0 \\
0.85 & 0.05 & 0.10 \\
0.50 & 0.10 & 0.40 \\
\end{pmatrix}
\]

(b) The initial probability distribution is the distribution as of January 1, so

\[
\nu^{(0)} = \begin{pmatrix}
\frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\
\end{pmatrix}
\]

(c) By February 1, one month later, the probability distribution is

\[
\nu^{(1)} = \nu^{(0)}P = \begin{pmatrix}
\frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\
\end{pmatrix} \begin{pmatrix}
0.90 & 0.10 & 0.00 \\
0.85 & 0.05 & 0.10 \\
0.50 & 0.10 & 0.40 \\
\end{pmatrix} = \begin{pmatrix}
0.7167 & 0.0833 & 0.2000 \\
\end{pmatrix}
\]

(d) To find the probability distribution after 2 months (March 1), compute \( \nu^{(2)} \):

\[
\nu^{(2)} = \nu^{(0)}P^2 = \begin{pmatrix}
\frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\
\end{pmatrix} \begin{pmatrix}
0.895 & 0.095 & 0.010 \\
0.8575 & 0.0975 & 0.045 \\
0.735 & 0.095 & 0.170 \\
\end{pmatrix} = \begin{pmatrix}
0.8158 & 0.0958 & 0.0883 \\
\end{pmatrix}
\]

(e) Since \( P^2 \) has all positive entries, \( P \) is regular. To find the long-run distribution, we seek the fixed probability vector \( t \) of the regular transition matrix \( P \). Let \( t = [t_1, t_2, t_3] \). Then \( tP = t \) and \( t_1 + t_2 + t_3 = 1 \), so that

\[
t_1 + t_2 + t_3 = 1 \quad \text{and} \quad [t_1, t_2, t_3] \begin{pmatrix}
0.90 & 0.10 & 0 \\
0.85 & 0.05 & 0.10 \\
0.50 & 0.10 & 0.40 \\
\end{pmatrix} = [t_1, t_2, t_3]
\]

The solution (which you should verify) is

\[
[t_1, t_2, t_3] = [0.8889, 0.0952, 0.0159]
\]

In the long run, store I will have about 88.9% of all customers, store II will have 9.5%, and store III will have 1.6%.

NOW WORK PROBLEM 21.

EXAMPLE 7  Spread of Rumor

A U.S. senator has decided whether to vote yes or no on an important bill pending in Congress and conveys this decision to an aide. The aide then passes this news on to another individual, who passes it on to a friend, and so on, each time to a new individual. Assume \( p \) is the probability that any one person passes on the information opposite to the way he or she heard it. Then \( 1 - p \) is the probability a person passes on the information the same way he or she heard it. With what probability will the \( n \)th person receive the information as a yes vote?
This can be viewed as a Markov chain model. Although it is not intuitively obvious, we shall find that the answer is essentially independent of $p$, provided $p$ does not equal 0 or 1.

To obtain the transition matrix $P$, we observe that two states are possible: a yes is heard or a no is heard. The transition from a yes to a no or from a no to a yes occurs with probability $p$. The transition from yes to yes or from no to no occurs with probability $1 - p$. The transition matrix $P$ is

\[
P = \begin{bmatrix}
1 - p & p \\
p & 1 - p
\end{bmatrix}
\]

The probability that the $n$th person will receive the information in one state or the other is given by successive powers of the matrix $P$, that is, by the matrices $P^n$. In fact, the answer in this case rapidly approaches $t_1 = \frac{1}{2}$, $t_2 = \frac{1}{2}$, after any considerable number of people are involved. This can be shown by verifying that the fixed probability vector is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. Hence, no matter what the senator’s initial decision is, eventually (after enough information exchange) half the people hear that the senator is going to vote yes and half hear that the senator is going to vote no.

An interesting interpretation of this result applies to successive roll calls in Congress on the same issue. We let $p$ represent the probability that a member changes his or her mind on an issue from one roll call to the next. Then the above example shows that if enough roll calls are forced, the Congress will eventually be near an even split on the issue. This could perhaps serve as a model for explaining a minority party’s use of the parliamentary device of delaying actions.

**USING TECHNOLOGY**

**EXAMPLE 8 Using Excel**

Use Excel to do Example 6. Create a table for the probability distribution vector for 24 months, starting with January.

**SOLUTION**

To find the probability distribution for February, use the Excel command for matrix multiplication, mmult().

**STEP 1** Set up the Excel spreadsheet.
STEP 2  Multiply the vector times the transition matrix. Keep the transition matrix constant by using $.

(a) Highlight where the product is to go: cells B5, C5, and D5.
(b) Enter = mmult() into the formula bar.
(c) Highlight the probability distribution vector and enter a comma.
(d) Highlight the transition matrix. Put dollar signs before the row numbers.
(e) Press Ctrl-Shift-Enter.

STEP 3  Repeat the process for March.

STEP 4  The results are given below.

STEP 5  To find the long-run probability distribution $t$, highlight rows 5 and 6, and click and drag. The result is given next.
In about 18 months, store I will have 88.9% of the customers; store II, 9.5%; and store III, 1.6%. Notice that the distribution remains fixed from this point on.

NOW WORK PROBLEMS 19 AND 21 USING EXCEL.


‘Are You Prepared?’ Problems Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Solve the system of equations:
   \[
   \begin{align*}
   x_1 + x_2 &= 1 \quad (1) \\
   0.07x_1 - 0.01x_2 &= 0 \quad (2)
   \end{align*}
   \]
   (pp. 54 and 56)

2. Solve the system of equations:
   \[
   \begin{align*}
   x_1 + x_2 + x_3 &= 1 \quad (1) \\
   3x_1 - 3x_2 - 2x_3 &= 0 \quad (2) \\
   -3x_2 + 2x_3 &= 0 \quad (3) \\
   3x_1 - 4x_3 &= 0 \quad (4)
   \end{align*}
   \]
   (p. 60)

Concepts and Vocabulary

3. True or False The transition matrix \( P \) of a regular Markov chain has positive entries.

4. True or False If the transition matrix \( P \) of a regular Markov chain is raised to a high enough power, the rows of that matrix are all the same.
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Skill Building

In Problems 5–14, determine whether the Markov chain whose transition matrix \( P \) is given is regular.

5. \( P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \)
6. \( P = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \)
7. \( P = \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \)
8. \( P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

9. \( P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \)
10. \( P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{2} \end{bmatrix} \)
11. \( P = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{3} \\ 0 \end{bmatrix} \)
12. \( P = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} \)
13. \( P = \begin{bmatrix} \frac{1}{4} & \frac{4}{4} \\ \frac{1}{4} & \frac{4}{4} \\ \frac{1}{4} & \frac{4}{4} \end{bmatrix} \)
14. \( P = \begin{bmatrix} \frac{1}{1} & \frac{1}{1} & \frac{1}{1} \\ \frac{1}{4} & \frac{4}{4} \\ \frac{1}{4} & \frac{4}{4} \end{bmatrix} \)

In Problems 15–20, find the fixed probability vector \( \mathbf{t} \) of each regular Markov chain.

15. \( P = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} \)
16. \( P = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{4} \\ \frac{1}{3} \end{bmatrix} \)
17. \( P = \begin{bmatrix} \frac{1}{4} \\ \frac{2}{3} \\ \frac{1}{2} \end{bmatrix} \)
18. \( P = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{2}{5} \end{bmatrix} \)
19. \( P = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{4} \\ \frac{1}{2} \end{bmatrix} \)
20. \( P = \begin{bmatrix} \frac{1}{4} \\ \frac{4}{2} \\ \frac{1}{4} \end{bmatrix} \)

Applications

21. Consumer Loyalty  A grocer stocks her store with three types of detergents, A, B, C. When brand A is sold out, the probability is 0.7 that she stocks up with brand A and 0.15 each with brands B and C. When she sells out brand B, the probability is 0.8 that she will stock up again with brand B and 0.1 each with brands A and C. Finally, when she sells out brand C, the probability is 0.6 that she will stock up with brand C and 0.2 each with brands A and B.

(a) Find the transition matrix.

(b) In the long run, what is the stock of detergents?

22. Consumer Loyalty  A housekeeper buys three kinds of cereal: A, B, C. She never buys the same cereal in successive weeks. If she buys cereal A, then the next week she buys cereal B. However, if she buys either B or C, then the next week she is three times as likely to buy A as the other brand.

(a) Find the transition matrix.

(b) In the long run, how often does she buy each of the three brands?
23. **Voting Loyalty**  In England, of the sons of members of the Conservative party, 70% vote Conservative and the rest vote Labour. Of the sons of Labourites, 50% vote Labour, 40% vote Conservative, and 10% vote Socialist. Of the sons of Socialists, 40% vote Socialist, 40% vote Labour, and 20% vote Conservative.

(a) What is the probability that the grandson of a Labourite will vote Socialist?

(b) In the long run how many will vote Conservative?

24. **Family Traits**  The probabilities that a blond mother will have a blond, brunette, or redheaded daughter are 0.6, 0.2, and 0.2, respectively. The probabilities that a brunette mother will have a blond, brunette, or redheaded daughter are 0.1, 0.7, and 0.2, respectively. And the probabilities that a redheaded mother will have a blond, brunette, or redheaded daughter are 0.4, 0.2, and 0.4, respectively.

(a) What is the probability that a blond woman is the grandmother of a brunette?

(b) If the population of women is now 50% brunettes, 30% blonds, and the rest redheads, what will the distribution be in the long run?

25. **Sales Strategy**  A representative of a book publishing company has to cover three universities, U₁, U₂, and U₃. She never sells at the same university in successive months. If she sells at university U₁, then the next month she sells at U₂. However, if she sells at either U₂ or U₃, then the next month she is three times as likely to sell at U₁ as at the other university.

(a) Show that this situation forms a Markov chain and find the transition matrix that models it.

(b) If in her first month on the job she has an initial probability vector \( v^{(0)} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \), what will her probability distribution be next month?

(c) In the long run how often does she sell at each university?

26. **Family Traits**  Assume that the probability of a fat father having a fat son is 0.7 and that of a thin father having a thin son is 0.4.

(a) Show that this situation forms a Markov chain and find the transition matrix that models it.

(b) If currently 60% of men are fat and 40% are thin, what percent of the males will be fat after two generations?

(c) In the long run, what will be the distribution of fat and thin males?

---

**Discussion and Writing**

33. If \( \begin{bmatrix} 1/3 & 0 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix} \) is the fixed probability vector of a matrix \( P \), can \( P \) be regular? Give reasons.
10.3 Absorbing Markov Chains

PREPARING FOR THIS SECTION  Before getting started, review the following:

- The Inverse of a Matrix (Section 3.3, pp. 135–148)

NOW WORK THE ‘ARE YOU PREPARED’ PROBLEMS ON PAGE 608.

OBJECTIVES

1. Identify absorbing Markov chains (p. 600)
2. Subdivide the transition matrix of an absorbing Markov chain (p. 601)
3. Analyze the gambler’s ruin problem (p. 602)
4. Model: the rise and fall of stock prices (p. 606)

1. Identify Absorbing Markov Chains

We have seen examples of transition matrices of Markov chains in which there are states that are impossible to leave. Such states are called absorbing. For example, in the transition matrix below, state 2 is absorbing since, once it is reached, it is impossible to leave.

<table>
<thead>
<tr>
<th></th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>1/4</td>
<td>3/4</td>
</tr>
<tr>
<td>State 2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Definition

Absorbing State; Absorbing Markov Chain

In a Markov chain let $p_{ii}$ denote the probability of remaining in state $E_i$. If $p_{ii} = 1$ then $E_i$ is called an absorbing state.

A Markov chain is said to be an absorbing Markov chain if and only if it contains at least one absorbing state and it is possible to go from any nonabsorbing state to an absorbing state in one or more trials.

In other words, an absorbing state will capture the process and will not allow any state to pass from it.

‘Are You Prepared?’ Answers

1. $x_1 = \frac{1}{8}, x_2 = \frac{7}{8}$
2. $x_1 = \frac{4}{9}, x_2 = \frac{2}{9}, x_3 = \frac{1}{3}$
EXAMPLE 1 Identifying Absorbing Markov Chains

Consider two Markov chains with $P_1$ and $P_2$ as their transition matrices:

$$P_1 = \begin{bmatrix}
E_1 & E_2 & E_3 & E_4 \\
0.4 & 0.2 & 0.4 & 0 \\
0 & 1 & 0 & 0 \\
0.1 & 0 & 0.5 & 0.4 \\
0.1 & 0 & 0.3 & 0.6 \\
\end{bmatrix} \quad P_2 = \begin{bmatrix}
E_1 & E_2 & E_3 \\
1 & 0 & 0 \\
0 & 1 & 3 \\
0.1 & 0 & 3 \\
0 & 1 & 0 \\
\end{bmatrix}$$

Determine whether either or both are absorbing chains.

**SOLUTION**

The chain having $P_1$ as its transition matrix is absorbing since $E_3$ is an absorbing state and it is possible to pass from each of the other states to $E_2$. Specifically, it is possible to pass from $E_1$ directly to $E_2$, and from either $E_3$ or $E_4$ to $E_2$ and then to $E_2$.

On the other hand, the matrix $P_2$ is an example of a nonabsorbing Markov chain since it is impossible to go from the nonabsorbing state $E_2$ to the absorbing state $E_1$.

**NOW WORK PROBLEM 5.**

2 **Subdivide the Transition Matrix of an Absorbing Markov Chain**

When working with an absorbing Markov chain, it is convenient to rearrange the states so that the absorbing states come first and the nonabsorbing states follow.

Once this rearrangement is made, the transition matrix can be subdivided into four submatrices as follows:

$$\begin{bmatrix}
\text{Absorbing} & \text{Nonabsorbing} \\
I & S \\
0 & Q \\
\end{bmatrix}$$

Here, $I$ is an identity matrix, $0$ denotes the zero matrix, and the matrices $S$ and $Q$ are the two submatrices corresponding to the absorbing and nonabsorbing states.
EXAMPLE 2 Subdividing the Transition Matrix of an Absorbing Markov Chain

Subdivide the transition matrix given below and find the submatrices \( S \) and \( Q \):

\[
\begin{bmatrix}
E_1 & E_2 & E_3 & E_4 & E_5 \\
E_1 & 0 & 0.5 & 0 & 0 & 0.5 \\
E_2 & 0 & 0 & 0.9 & 0 & 0.1 \\
E_3 & 0 & 0 & 0 & 0.7 & 0.3 \\
E_4 & 0 & 0 & 0 & 0 & 1 \\
E_5 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

**SOLUTION** First, note that \( E_4 \) and \( E_5 \) are absorbing states and that this is an absorbing Markov chain. Move the absorbing states \( E_4 \) and \( E_5 \) so they appear first.

\[
\begin{bmatrix}
E_4 & E_5 & E_1 & E_2 & E_3 \\
E_4 & 1 & 0 & 0 & 0 & 0 \\
E_5 & 0 & 1 & 0 & 0 & 0 \\
E_1 & 0 & 0.5 & 0 & 0.5 & 0 \\
E_2 & 0 & 0.1 & 0 & 0 & 0.9 \\
E_3 & 0.7 & 0.3 & 0 & 0 & 0
\end{bmatrix}
\]

The subdivided matrix is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0.1 & 0 & 0 & 0.9 \\
0.7 & 0.3 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
S & Q
\end{bmatrix}
\]

from which

\[
S = \begin{bmatrix}
0 & 0.5 \\
0 & 0.1 \\
0.7 & 0.3
\end{bmatrix}
\quad Q = \begin{bmatrix}
0 & 0.5 & 0 \\
0 & 0 & 0.9 \\
0 & 0 & 0
\end{bmatrix}
\]

**NOW WORK PROBLEM 19.**

3 Analyze the Gambler’s Ruin Problem

**Gambler’s Ruin Problem** Consider the following game involving two players, an example of a gambler’s ruin problem. Player I has $3 and player II has $2. They flip a fair coin; if it is a head, player I pays player II $1, and if it is a tail, player II pays player I $1. The total amount of money in the game is, of course, $5. We would like to know how long the game will last; that is, how long it will take for one of the players to go broke or win all the money. (This game can easily be generalized by assuming that player I has \( M \) dollars and player II has \( N \) dollars. Also, the coin need not be fair.)

In this experiment how much money a player has after any given flip of the coin depends only on how much he had after the previous flip and will not depend (directly) on how much he had in the preceding stages of the game. This experiment can be represented by a Markov chain.
The coin being flipped is fair so that a probability of \(\frac{1}{2}\) is assigned to each outcome. The states are the amounts of money each player has at each stage of the game. Note that each player can increase or decrease the amount of money he has by only $1 at a time. The transition matrix \(P\) is then

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Here, \(p_{21} = \frac{1}{2}\) is the probability of starting with $1 and losing it. Also, \(p_{11} = 1\) is the probability of having $0 given that a player has started with $0. This is a sure event since, once a player is in state 0, he stays there forever (he is broke). Similarly, \(p_{55} = 1\) represents the probability of having $5, given that a player started with $5, which is again a sure event (the player has won all the money). In other words, 0 and 5 are absorbing states.

The following questions are of interest:

(a) Given that one gambler is in a nonabsorbing state, what is the expected number of times he will hold between $1 and $4 inclusive before the termination of the game? That is, on the average, how many times will the process be in nonabsorbing states?

(b) What is the expected length of the process (game)?

(c) What is the probability that an absorbing state is reached (that is, that one gambler will eventually be wiped out)?

To answer these questions, first subdivide the transition matrix \(P\).

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Now \(P\) is in the form

\[
P = \begin{bmatrix}
I & \mathbf{0} \\
I & P
\end{bmatrix}
\]

where
In general, if \( P \) is the transition matrix of an absorbing Markov chain and if \( P \) has \( r \) absorbing states, then \( P \) can be subdivided and written in the form

\[
P = \begin{bmatrix}
I_r & 0 \\
S & Q
\end{bmatrix}
\]

where \( I_r \) is the \( r \times r \) identity matrix, \( 0 \) is the zero matrix of dimension \( r \times s \), \( S \) is of dimension \( s \times r \), and \( Q \) is of dimension \( s \times s \).

The role \( Q \) plays is given in the following result:

**Theorem**

**Fundamental Matrix of an Absorbing Markov Chain**

For an absorbing Markov chain that has a transition matrix \( P \) of the form

\[
P = \begin{bmatrix}
I_r & 0 \\
S & Q
\end{bmatrix}
\]

where \( S \) is of dimension \( s \times r \) and \( Q \) is of dimension \( s \times s \), define the matrix \( T \), called the fundamental matrix of the Markov chain, to be

\[
T = (I_r - Q)^{-1}
\]

The entries of \( T \) give the expected number of times the process is in each non-absorbing state provided the process began in a nonabsorbing state.

Return now to the gambler’s ruin problem and the matrices in Display (1), where \( S \) and \( Q \) were found. The fundamental matrix \( T \) is

\[
T = (I_4 - Q)^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 0
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
0.4 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.4 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.4 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.4
\end{bmatrix}
\]

The entries give the expected number of times the process is in each non-absorbing state provided the process began in a nonabsorbing state.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
1.6 & 1.2 & 0.8 & 0.4 \\
1.2 & 2.4 & 1.6 & 0.8 \\
0.8 & 1.6 & 2.4 & 1.2 \\
0.4 & 0.8 & 1.2 & 1.6
\end{bmatrix}
\]
The fundamental matrix $T$ provides the answer to question (a). For example, the entry 0.8 in row 3, column 1 indicates that 0.8 is the expected number of times the player will have $1 if he started with $3. In the fundamental matrix $T$ the column headings indicate present money, while the row headings indicate money started with.

To answer question (b), we need the following result:

**Theorem Expected Number of Trials before Absorption**

The expected number of trials before absorption for each nonabsorbing state is found by adding the entries in the corresponding row of the fundamental matrix $T$.

So to answer question (b), again look at the fundamental matrix $T$ in Equation (3). The expected number of games before absorption (when one of the players wins or loses all the money) can be found by adding the entries in each row of $T$. For example, if a player starts with $3, the expected number of games before absorption is

$$0.8 + 1.6 + 2.4 + 1.2 = 6.0$$

That is, on the average, we expect this game to end on the seventh play.

If a player starts with $1, the expected number of games before absorption is

$$1.6 + 1.2 + 0.8 + 0.4 = 4.0$$

To answer question (c), we need the following result:

**Theorem Probability of Being Absorbed**

The $(i,j)$th entry in the matrix product $T \cdot S$ gives the probability that, starting in nonabsorbing state $i$, we reach the absorbing state $j$.

So, to answer question (c), find the product of the matrices $T$ and $S$:

$$T \cdot S = \begin{bmatrix}
1.6 & 1.2 & 0.8 & 0.4 \\
1.2 & 2.4 & 1.6 & 0.8 \\
0.8 & 1.6 & 2.4 & 1.2 \\
0.4 & 0.8 & 1.2 & 1.6
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & 0 \\
0 & 0 \\
0 & \frac{1}{2}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
$0$ & $5$ \\
$1$ & $0.8$ & $0.2$ \\
$2$ & $0.6$ & $0.4$ \\
$3$ & $0.4$ & $0.6$ \\
$4$ & $0.2$ & $0.8$
\end{bmatrix}$$

The entry in row 3, column 2, indicates the probability is 0.6 that the player starting with $3 will win all the money. The entry in row 2, column 1, indicates the probability is 0.6 that a player starting with $2 will lose all his money.

The above results are applicable in general to any transition matrix $P$ of an absorbing Markov chain. Table 1 gives probabilities for ruin and expected length for other kinds of betting situations for which the bet is 1 unit.
Suppose, for example, that player I starts with $90 and player II with $10, with player I having a probability of 0.45 of winning (the game being unfavorable to player I). If at each trial, the stake is $1, Table 1 shows that the probability is 0.866 that player I is ruined. If the same game is played with player I having $9 to start and player II having $1 to start, the probability that player I is ruined drops to 0.210.

**NOW WORK PROBLEM 25.**

### 4 Model: The Rise and Fall of Stock Prices*

This model involves using Markov chains to analyze the movement of stock prices. The stocks (shares) were those whose daily progress was recorded by the London *Financial Times*. At any point in time a given stock was classified as being in one of three states: increase (I), decrease (D), or no change (N), depending on whether its price had risen, fallen, or remained the same compared to the previous trading day. The intent of the model was to study the long-range movement of stocks through these states.

Data reporting the past history of stock prices over a period of 1097 trading days were used to statistically fit a transition matrix to the model. The transition matrix used was

\[
P = \begin{bmatrix}
I & D & N \\
0.586 & 0.074 & 0.340 \\
0.070 & 0.638 & 0.292 \\
0.079 & 0.064 & 0.857
\end{bmatrix}
\]

According to this model, a share that increased one day would have probability 0.586 of also increasing the next day and a probability of 0.340 of registering no change the next day. Note that the tendency to remain in the same state is strongest for the no-change state since \( p_{33} \) is the largest of the diagonal elements.

The fixed probability vector \( t \) for the transition matrix can be computed to be

\[
t = [0.157 \quad 0.154 \quad 0.689]
\]

In the long run a stock will have increased its price 15.7% of the time, decreased its price 15.4% of the time, and remained unchanged 68.9% of the time.

How long will a stock increase before its price begins to fall? Or, in general, how long will a stock be in a given state before it moves to another one? These questions are highly reminiscent of those encountered while studying absorbing chains, yet here the transition matrix \( P \) is not absorbing. As a result, techniques used earlier do not apply.

Suppose we wanted to know on the average how long a stock would spend in states \( I \) or \( N \) before arriving in state \( D \). We could then assume state \( D \) is an absorbing one and replace the current probability \( P_{22} \) with 1 and make all other elements in the second row of \( P \) zero. The new matrix \( P' \) we have created is

\[
P' = \begin{bmatrix}
I & D & N \\
0.586 & 0.074 & 0.340 \\
0 & 1 & 0 \\
0.079 & 0.064 & 0.857
\end{bmatrix}
\]

which is the matrix of an absorbing chain.

Subdividing this matrix, we obtain

\[
Q = \begin{bmatrix}
I & N \\
0.586 & 0.340 \\
0.079 & 0.857
\end{bmatrix}
\]

The fundamental matrix \( T \) of this absorbing Markov chain is

\[
T = (I - Q)^{-1} = \begin{bmatrix}
0.414 & -0.340 \\
-0.079 & 0.143
\end{bmatrix}^{-1} = \begin{bmatrix}
4.42 & 10.51 \\
2.44 & 12.80
\end{bmatrix}
\]

The entries of \( T = (I - Q)^{-1} \) are interpreted as follows. The \((i, j)\)th entry gives the average time spent in state \( j \) having started in state \( i \) before reaching the absorbing state \( D \).

So, a stock currently increasing would spend an average of 4.42 days increasing and 10.51 days not changing before declining. Hence, a total of 14.93 days on the average would elapse before an increasing stock first began to decrease. Likewise, if a stock is currently exhibiting no change, the second row of the above matrix shows that an average of 2.44 + 12.80 = 15.24 days would go by before it began to decrease.

Suppose a stock is increasing and then either levels off or declines. How long will it take before it starts rising again? In general, given we are in state \( i \) and leave, what is the average time that elapses before we return again to state \( i \)? The answer, which we state without proof, is given next.
Theorem

Average Time between Visits

The average amount of time elapsed between visits to state $i$ (called mean recurrence time) is given by the reciprocal of the $i$th component of the fixed probability vector $t$.

For our example,

$$t = [0.157 \ 0.154 \ 0.689]$$

So we can then compute

<table>
<thead>
<tr>
<th>State</th>
<th>Mean Recurrence Time (Days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>6.37</td>
</tr>
<tr>
<td>$D$</td>
<td>6.49</td>
</tr>
<tr>
<td>$N$</td>
<td>1.45</td>
</tr>
</tbody>
</table>

In this model, then, days on which a share’s price fails to change follow each other fairly closely.

EXERCISE 10.3 Answers Begin on Page AN–56.

‘Are You Prepared?’ ProblemsAnswers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find the inverse of the matrix

$$
\begin{bmatrix}
2 & -3 \\
-1 & 5
\end{bmatrix}
$$

(p. 137)

2. Use a graphing utility to find the inverse of the matrix

$$
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 1 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 1
\end{bmatrix}
$$

(p. 139)

Concepts and Vocabulary

3. True or False The transition matrix $P$ of an absorbing Markov chain has at least one entry equal to 1.

4. The transition matrix $P$ of an absorbing Markov chain can be subdivided into _______ submatrices.
10.3 Absorbing Markov Chains

Skill Building

In Problems 5–16, determine if the Markov chain, whose transition matrix \( P \) is given, is absorbing. If it is, identify the absorbing state(s).

5. \( P = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \)

6. \( P = \begin{bmatrix} 1 & 2 \\ \frac{3}{3} & \frac{3}{3} \\ 0 & 1 \end{bmatrix} \)

7. \( P = \begin{bmatrix} 1 & 1 \\ \frac{2}{2} & \frac{2}{2} \end{bmatrix} \)

8. \( P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \)

9. \( P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \)

10. \( P = \begin{bmatrix} 1 & 1 & 1 \\ \frac{4}{4} & \frac{2}{4} & \frac{2}{4} \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{3}{3} & \frac{3}{3} \end{bmatrix} \)

11. \( P = \begin{bmatrix} 1 & 1 \\ \frac{1}{1} & \frac{1}{1} \\ 0 & 1 & 0 \\ \frac{2}{2} & \frac{2}{2} \end{bmatrix} \)

12. \( P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ \frac{5}{5} & \frac{5}{5} \end{bmatrix} \)

13. \( P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 \end{bmatrix} \)

14. \( P = \begin{bmatrix} 1 & 1 \\ \frac{4}{4} & \frac{4}{4} \\ 1 & 0 & 0 \\ \frac{2}{2} & \frac{2}{2} \end{bmatrix} \)

15. \( P = \begin{bmatrix} 1 & 1 & 1 \\ \frac{4}{4} & \frac{4}{4} & \frac{4}{4} \\ 0 & 1 & 0 & 0 \\ \frac{1}{1} & \frac{1}{1} & \frac{1}{1} \end{bmatrix} \)

16. \( P = \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{1} & \frac{1}{1} & \frac{1}{1} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \)

In Problems 17–22, subdivide the transition matrix of each absorbing Markov chain.

17. \( P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \)

18. \( P = \begin{bmatrix} 1 & 1 & 1 \\ \frac{4}{4} & \frac{2}{4} & \frac{2}{4} \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{3}{3} & \frac{3}{3} \end{bmatrix} \)

19. \( P = \begin{bmatrix} 1 & 1 & 1 \\ \frac{4}{4} & \frac{4}{4} & \frac{4}{4} \\ 0 & 1 & 0 & 0 \\ \frac{1}{1} & \frac{1}{1} & \frac{1}{1} \end{bmatrix} \)

20. \( P = \begin{bmatrix} 1 & 1 & 1 \\ \frac{3}{3} & \frac{3}{3} & \frac{3}{3} \\ \frac{2}{2} & \frac{2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

21. \( P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \)

22. \( P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \)

23. Suppose that for a certain absorbing Markov chain the fundamental matrix \( T \) is found to be

\[
\begin{bmatrix}
S1 & S2 & S3 \\
1.5 & 1.0 & 0.5 \\
S2 & 1.0 & 2.0 & 1.0 \\
S3 & 0.5 & 1.0 & 1.5
\end{bmatrix}
\]

(a) What is the expected number of times a person will have $3 given that she started with $1? With $2?

(b) If a player starts with $3, how many games can she expect to play before the game ends?

24. Use the matrix \( T \) from Problem 23 to answer the following questions:

(a) What is the expected number of times a person will have $2 given that she started with $1? With $2?

(b) If a player starts with $1, how many games can she expect to play before absorption?
25. **Gambler’s Ruin Problem** A person repeatedly bets $1 each day. If he wins, he wins $1 (he receives his bet of $1 plus winnings of $1). He stops playing when he goes broke, or when he accumulates $3. His probability of winning is 0.4 and of losing is 0.6. What is the probability of eventually accumulating $3 if he starts with $1? With $2?

26. The following data were obtained from the admissions office of a 2-year junior college. Of the first-year class (F), 75% became sophomores (S) the next year and 25% dropped out (D). Of those who were sophomores during a particular year, 90% graduated (G) by the following year and 10% dropped out.

(a) Set up a Markov chain with states D, F, G, and S that describes the situation.
(b) How many states are absorbing?
(c) Determine the matrix $T$.
(d) Determine the probability that an entering first-year student will eventually graduate.

27. **Gambler’s Ruin Problem** Colleen wants to purchase a $4000 used car, but has only $1000 available. Not wishing to finance the purchase, she makes a series of wagers in which the winnings equal whatever is bet. The probability of winning is 0.4 and the probability of losing is 0.6. In a daring strategy Colleen decides to bet all her money or at least enough to obtain $4000 until she loses everything or has $4000. That is, if she has $1000, she bets $1000; if she has $2000, she bets $2000; if she has $3000, she bets $1000.

(a) What is the expected number of wagers placed before the game ends?
(b) What is the probability that Colleen is wiped out?
(c) What is the probability that Colleen wins the amount needed to purchase the car?

28. (a) Answer the questions in Problem 27 if the probability of winning is 0.5.
(b) Answer the questions in Problem 27 if the probability of winning is 0.6.

29. **Military Strategy** Three armored cars, A, B, and C, are engaged in a three-way battle. Armored car A has probability $\frac{1}{3}$ of destroying its target, B has probability $\frac{1}{2}$ of destroying its target, and C has probability $\frac{1}{6}$ of destroying its target. The armored cars fire at the same time and each fires at the strongest opponent not yet destroyed. Using as states the surviving cars at any round, set up a Markov chain and answer the following questions:
(a) How many states are in this chain?
(b) How many are absorbing?
(c) Find the expected number of rounds fired.
(d) Find the probability that A survives.

30. **Stock Price Model** In the model discussed on pages 606–608 a stock currently showing no change would, on the average, stay how long in that state?

31. **Stock Price Model** Refer to the model discussed on pages 606–608. By making state I an absorbing state, compute the average number of days elapsed before a stock currently decreasing would start to increase again.

32. Suppose that for a certain absorbing Markov chain the transition matrix $P$ is given by

$$
P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0.05 & 0.95 \\
0.75 & 0.25 & 0 & 0
\end{bmatrix}
$$

Determine the matrices $I$, $S$, and $Q$. Find the fundamental matrix $T$. Determine the product $T \cdot S$.

33. Consider the maze with five rooms shown in the figure. The system consists of the maze and a mouse. We assume the following learning pattern exists: If the mouse is in room 2, 3, or 4, it moves with equal probability to any room the maze permits; if it is in room 1, it gets caught in the trap and remains there; if it is in room 5, it remains in that room with the cheese forever.

(a) Construct the transition matrix $P$.
(b) How many rooms represent an absorbing state for this Markov chain?
(c) If a mouse begins in room 3, what is the probability that it will find the cheese in 5 steps? Determine the powers of the matrix $P$.
(d) Subdivide the transition matrix.
(e) Determine the matrices $I$, $S$, and $Q$.
(f) Find the fundamental matrix $T$.
(g) Determine the product $T \cdot S$.
(h) How many steps, on the average, will the mouse make in the maze before it ends up in the trap?


10.4 Two-Person Games

PREPARING FOR THIS SECTION  Before getting started, review the following:

• Matrix Algebra (Section 3.1, pp. 104–116)

NOW WORK THE ‘ARE YOU PREPARED’ PROBLEM ON PAGE 614.

OBJECTIVES  1 Identify a strictly determined game (p. 613)

Game theory, as a branch of mathematics, is concerned with the analysis of human behavior in conflicts of interest. In other words, game theory gives mathematical expression to the strategies of opposing players and offers techniques for choosing the best possible strategy. In most parlor games it is relatively easy to define winning and losing and, on this basis, to quantify the best strategy for each player. However, game theory is not merely a tool for the gambler so that he or she can take advantage of the odds; nor is it merely a method for winning games like tic-tac-toe or matching pennies.

For example, when union and management sit down at the bargaining table to discuss contracts, each has definite strategies. Each will bluff, persuade, and try to discover the other’s strategy, while at the same time trying to prevent the discovery of his or her strategy. If enough information is known, results from the theory of games can help determine what is the best possible strategy for each player.

Another application of game theory can be made to politics. If two people are running for the same office, each has various campaign strategies. If it is possible to determine the impact of alternate strategies on the voters, the theory of games can help to find the best strategy (usually the one that gains the most votes, while losing the least votes).

Definition  Two-Person Game

Any conflict or competition between two people is called a two-person game.

EXAMPLE 1  Example of a Two-Person Game

In a game, player I picks heads or tails and player II attempts to guess the choice. Player I will pay player II $3 if both choose heads; player I will pay player II $2 if both choose tails. If player II guesses incorrectly, he will pay player I $5.
We use Example 1 to introduce some terminology. First, since two players are involved, this is a two-person game. Next, notice that no matter what outcome occurs (HH, HT, TH, TT), whatever is lost (or gained) by player I is gained (or lost) by player II. Such games are called two-person zero-sum games.

If we denote the gains of player I by positive entries and his losses by negative entries, we can display this game in a matrix as

\[
\begin{bmatrix} H & T \\ H & -3 & 5 \\ T & 5 & -2 \end{bmatrix}
\]

This matrix is called the game matrix or payoff matrix, and each entry \(a_{ij}\) of the matrix is termed a payoff.

Conversely, any \(m \times n\) matrix \(A = [a_{ij}]\) can be regarded as the game matrix for a two-person zero-sum game in which player I chooses any one of the \(m\) rows of \(A\) and simultaneously player II chooses any one of the \(n\) columns of \(A\). The entry at the intersection of the row and column chosen is the payoff and represents the net gain to player I.

We will assume that the game is played repeatedly, and that the problem facing each player is what choice he should make so that he gains the most benefit. Player I wishes to maximize his winnings and player II wishes to minimize his losses. By a strategy for a given matrix game \(A\), we mean the decision by player I to select rows of \(A\) and the decision of player II to select columns of \(A\).

**EXAMPLE 2** Discussing Strategies

Consider a two-person zero-sum game given by the matrix

\[
\begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix}
\]

in which the entries denote the net gain of player I. The game consists of player I choosing a row and player II simultaneously choosing a column, with the intersection of row and column giving the payoff for this play in the game. For example, if player I chooses row 1 and player II chooses column 2, then player I wins $6; if player I chooses row 2 and player II chooses column 1, then player I loses $2.

It should be evident from the matrix that this particular game is biased in favor of player I, who will always choose row 1 since she cannot lose by doing so. Similarly, player II, recognizing that player I will choose row 1, will always choose column 1, since her losses are then minimized.

The best strategy for player I is row 1 and the best strategy for player II is column 1. When both players employ their best strategy, the result is that player I wins $3. This amount is called the value of the game. Notice that the payoff $3 is the minimum of the entries in its row and is the maximum of the entries in its column.

NOW WORK PROBLEM 3.
Definition

Strictly Determined Game

A game defined by a matrix is said to be strictly determined if and only if there is an entry of the matrix that is the smallest element in its row and is also the largest element in its column. This entry is then called a saddle point and is the value of the game.

If a game has a positive value, the game favors player I. If a game has a negative value, the game favors player II. Any game with a value of 0 is termed a fair game.

If a game matrix has a saddle point, it can be shown that the row containing the saddle point is the best strategy for player I and the column containing the saddle point is the best strategy for player II. This is why such games are called strictly determined games. Such games are also called games of pure strategy.

Of course, a matrix may have more than one saddle point, in which case each player has more than one best strategy available. However, the value of the game is always the same no matter how many saddle points the matrix may have.

EXAMPLE 3 Identifying a Strictly Determined Game

Determine whether the game defined by the matrix below is strictly determined.

\[
\begin{pmatrix}
3 & 0 & -2 & -1 \\
2 & -3 & 0 & -1 \\
4 & 2 & 1 & 0
\end{pmatrix}
\]

SOLUTION

First, look at each row and find the smallest entry in each row:

Row 1: \(-2\)  \hspace{1cm} \text{Row 2: \(-3\)}  \hspace{1cm} \text{Row 3: 0}

Next, check to see if any of the above elements are also the largest in their column. The element \(-2\) in row 1 is not the largest entry in column 3; the element \(-3\) in row 2 is not the largest entry in column 2; however, the element 0 in row 3 is the largest entry in column 4. This game is strictly determined. Its value is 0, so the game is fair.

The game of Example 3 is represented by a \(3 \times 4\) matrix. This means that player I has three strategies, while player II can choose from four strategies.

NOW WORK PROBLEM 7.

EXAMPLE 4 Selecting the Best Business Site

Two competitive franchising firms, Alpha Products and Omega Industries, are each planning to add an outlet in a certain city. It is possible for the site to be located either
in the center of the city or in a large suburb of the city. If both firms decide to build in
the center of the city, Alpha Products will show an annual profit of $1000 more than the
profit of Omega Industries. If both firms decide to locate their outlets in the suburb, then
it is determined that Alpha Products’ profit will be $2000 less than the profit of Omega
Industries. If Alpha locates in the suburb and Omega in the city, then Alpha will show a
profit of $4000 more than Omega. Finally, if Alpha locates in the city and Omega in the
suburb, then Alpha will have a profit of $3000 less than Omega’s. Is there a best site for each
firm to locate its outlet? In this case, by best site we mean the one that produces the most
relative profit.

**SOLUTION** Assign rows as Alpha strategies and columns as Omega strategies and use positive entries
to denote the gain of Alpha over Omega and negative entries for the gain of Omega over
Alpha. Then the game matrix is

\[
\begin{array}{cc}
\text{Omega} \\
\text{City} & \text{Suburb} \\
\text{Alpha} & \\
\text{City} & 1 & -3 \\
\text{Suburb} & 4 & -2
\end{array}
\]

where the entries are in thousands of dollars.

This game is strictly determined and the saddle point is \(-2\), which is the value of the
game. If both firms locate in the suburb, this results in the best competition. This is so
since Omega will always choose to locate in the suburb, guaranteeing a larger profit than
Alpha’s. This being the case, Alpha, in order to minimize this larger profit of Omega, must
always choose the suburb. The game is not fair since it is favorable to Omega.

**EXERCISE 10.4** Answers Begin on Page AN–56.

**‘Are You Prepared?’ Problem** The answer is given at the end of these exercises. If you get a wrong answer, read the page listed in red.

1. What is the dimension of the following matrix?

\[
\begin{pmatrix}
2 & -3 & 4 & 0 \\
4 & 5 & -9 & 3
\end{pmatrix}
\]

(p. 105)

**Concepts and Vocabulary**

2. **True or False** In a strictly determined game, each player has a best strategy.

**Skill Building**

In Problems 3–6, write the game matrix that corresponds to each two-person conflict situation.

3. Katy and Colleen each simultaneously show one or two fingers. If they show the same number of fingers, Katy pays Colleen one dime. If they show a different number of fingers, Colleen pays Katy one dime.

4. Katy and Colleen each simultaneously show one or two fingers. If the total number of fingers shown is even, Katy pays Colleen that number of dimes. If the total number of fingers shown is odd, Colleen pays Katy that number of dimes.

5. Katy and Colleen, simultaneously and independently, each write down one of the numbers 1, 4, or 7. If the sum of the numbers is even, Katy pays Colleen that number of dimes. If the sum of the numbers is odd, Colleen pays Katy that number of dimes.

6. Katy and Colleen, simultaneously and independently, each write down one of the numbers 3, 6, or 8. If the sum of the numbers is even, Katy pays Colleen that number of dimes. If the sum of the numbers is odd, Colleen pays Katy that number of dimes.
In Problems 7–16, determine which of the two-person zero-sum games are strictly determined. For those that are, find the value of the game. All entries are the winnings of player I, who plays rows.

7. \[
\begin{bmatrix}
-2 & 4 \\
-3 & 6
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
8 & 0 \\
0 & -1
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
4 & 3 \\
3 & 2
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
-6 & -2 \\
0 & 0
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
2 & 0 & -1 \\
3 & 6 & 0 \\
1 & 3 & 7
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
2 & 3 & -2 \\
-2 & 0 & 4 \\
0 & -3 & 2
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
1 & 0 & 3 \\
-1 & 2 & 1 \\
2 & 2 & 3
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
1 & -3 & -2 \\
2 & 5 & 4 \\
2 & 3 & 2
\end{bmatrix}
\]

15. \[
\begin{bmatrix}
6 & 4 & -2 & 0 \\
-1 & 7 & 5 & 2 \\
1 & 0 & 4 & 4
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
8 & 6 & 4 & 0 \\
-1 & 6 & 5 & -2 \\
0 & 1 & 3 & 3
\end{bmatrix}
\]

Extensions

17. For what values of \( a \) is the matrix below strictly determined?

\[
\begin{bmatrix}
a & 8 & 3 \\
0 & a & -9 \\
-5 & 5 & a
\end{bmatrix}
\]

18. Show that the matrix below is strictly determined for any choice of \( a, b, \) or \( c \).

\[
\begin{bmatrix}
a & a \\
b & c
\end{bmatrix}
\]

19. Find conditions on \( a \) and \( b \) for the matrix below to be strictly determined.

\[
\begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}
\]

‘Are You Prepared?’ Answer

1. \( 2 \times 4 \)

10.5 Mixed Strategies

PREPARING FOR THIS SECTION Before getting started, review the following:

- Multiplication of Matrices (Section 3.2, pp. 121–131)
- Expected Value (Section 7.6, pp. 403–409)

NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 618.

OBJECTIVES

1. Find the expected payoff (p. 616)

EXAMPLE 1 Determining Whether a Game Is Strictly Determined

Consider a two-person zero-sum game given by the matrix

\[
\begin{bmatrix}
6 & 0 \\
-2 & 3
\end{bmatrix}
\]
in which the entries denote the winnings of player I. Is this game strictly determined? If so, find its value.

**SOLUTION**

The smallest entry in each row is

- Row 1: 0
- Row 2: -2

The entry 0 in row 1 is not the largest element in its column; similarly, the entry -2 in row 2 is not the largest element in its column. This game is not strictly determined.

Remember that a two-person game is not usually played just once. With this in mind, player I in Example 1 might decide always to play row 1, since she may win $6 at best and win $0 at worst. Does this mean she should always employ this strategy? If she does, player II would catch on and begin to choose column 2, since this strategy limits his losses to $0. However, after a while, player I would probably start choosing row 2 to obtain a payoff of $3. In a nonstrictly determined game it is advisable for the players to mix their strategies rather than to use the same one all the time. That is, a random selection is desirable. Indeed, to make certain that the other player does not discover the pattern of moves, it may be best not to have any pattern at all. For instance, player I may elect to play row 1 in 40% of the plays (that is, with probability 0.4), while player II elects to play column 2 in 80% of the plays (that is, with probability 0.8). This idea of mixing strategies is important and useful in game theory. Games in which each player’s strategies are mixed are termed **mixed-strategy games**.

### 1 Find the Expected Payoff

Suppose we know the probability for each player to choose a certain strategy. What meaning can be given to the term *payoff of a game* if mixed strategies are used? Since the payoff has been defined for a pair of pure strategies and in a mixed-strategy situation we do not know what strategy is being used, it is not possible to define a payoff for a single game. However, in the long run we do know how often each strategy is used, and we can use this information to compute the *expected payoff* of the game.

In Example 1 if player I chooses row 1 in 50% of the plays and row 2 in 50% of the plays, and if player II chooses column 1 in 30% of the plays and column 2 in 70% of the plays, the expected payoff of the game can be computed. For example, the strategy of row 1, column 1, is chosen (0.5)(0.3) = 0.15 of the time. This strategy has a payoff of $6, so that the expected payoff will be $(6)(0.15) = 0.90$. Table 2 summarizes the entire process. The expected payoff $E$ of this game, when the given strategies are employed, is $1.65$, which makes the game favorable to player I.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Payoff</th>
<th>Probability</th>
<th>Expected Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1—Column 1</td>
<td>6</td>
<td>0.15</td>
<td>$0.90$</td>
</tr>
<tr>
<td>Row 2—Column 1</td>
<td>-2</td>
<td>0.15</td>
<td>-0.30</td>
</tr>
<tr>
<td>Row 1—Column 2</td>
<td>0</td>
<td>0.35</td>
<td>0.00</td>
</tr>
<tr>
<td>Row 2—Column 2</td>
<td>3</td>
<td>0.35</td>
<td>1.05</td>
</tr>
<tr>
<td>Totals</td>
<td>1.00</td>
<td></td>
<td>$1.65$</td>
</tr>
</tbody>
</table>

**NOW WORK PROBLEM 3.**
If we look very carefully at the above derivation, we get a clue as to how the expected payoff of a game that is not strictly determined should be defined.

Let’s consider a game defined by the matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]

Let the strategy for player I, who plays rows, be denoted by the row vector

\[
P = [p_1 \ p_2]
\]

and the strategy for player II, who plays columns, be denoted by the column vector

\[
Q = \begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
\]

The probability that player I wins the amount \(a_{11}\) is \(p_1q_1\). Similarly, the probabilities that she wins the amounts \(a_{12}, a_{21}, \) and \(a_{22}\) are \(p_1q_2, p_2q_1, \) and \(p_2q_2\), respectively. If we denote the expectation of player I by \(E(P, Q)\), that is, the expected value of the amount she wins when she uses strategy \(P\) and player II uses strategy \(Q\), then

\[
E(P, Q) = p_1a_{11}q_1 + p_1a_{12}q_2 + p_2a_{21}q_1 + p_2a_{22}q_2
\]

Using matrix notation, the above can be expressed as

\[
E(P, Q) = PAQ
\]

In general, if \(A\) is an \(m \times n\) game matrix, we are led to the following definition:

**Definition**

**Expected Payoff**

The expected payoff \(E\) of player I in a two-person zero-sum game, defined by the matrix \(A\), in which the row vector \(P\) and column vector \(Q\) define the respective strategy probabilities of player I and player II, is

\[
E = PAQ
\]  

**EXAMPLE 2** Finding the Expected Payoff

Find the expected payoff of the game matrix

\[
A = \begin{bmatrix}
3 & -1 \\
-2 & 1 \\
1 & 0
\end{bmatrix}
\]
if player I and player II decide on the strategies

\[ P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad Q = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \]

**SOLUTION** By Equation (1), the expected payoff \( E \) of this game is

\[
E = PAQ = \begin{bmatrix} \frac{1}{3} & 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{2}{3} \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{2}{3} \end{bmatrix} = \frac{2}{9}
\]

The game is biased in favor of player I and has an expected payoff of \( \frac{2}{9} \). ■

**NOW WORK PROBLEM 9.**

Most games are not strictly determined. That is, most games do not give rise to best pure strategies for each player. Examples of games that are not strictly determined are matching pennies, bridge, poker, and so on. In the next section, we discuss a technique for finding optimal strategies for games that are not strictly determined.

---

**EXERCISE 10.5 Answers Begin on Page AN–57.**

**‘Are You Prepared?’ Problems** Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find the product

\[
\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & 4 & 0 \\ 4 & 5 & -9 & 3 \end{bmatrix}
\]

(p. 122)

2. In a certain game, you will win $6 with probability 0.15, lose $2 with probability 0.15, win $0 with probability 0.35, and win $3 with probability 0.35. What is the expected value of this game? (p. 405)

**Skill Building**

3. For the game of Example 1 find the expected payoff \( E \) if player I chooses row 1 in 30% of the plays and player II chooses column 1 in 40% of the plays.

4. For the game of Example 2 find the expected payoff \( E \) if player I chooses row 1 with probability 0.3 and row 2 with probability 0.4, while player II chooses column 1 half the time.
In Problems 5–8, find the expected payoff of the game matrix \[
\begin{bmatrix}
4 & 0 \\
2 & 3
\end{bmatrix}
\] for the given strategies.

5. \( P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \) \( Q = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \)

6. \( P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \) \( Q = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} \)

7. \( P = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix} \) \( Q = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \)

8. \( P = [0 & 1] \) \( Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

In Problems 9–12, find the expected payoff of each game matrix.

9. \[
\begin{bmatrix}
4 & 0 \\
-3 & 6
\end{bmatrix}; \quad P = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \quad Q = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}
\]

10. \[
\begin{bmatrix}
1 & -1 \\
-2 & 3
\end{bmatrix}; \quad P = \begin{bmatrix} 1/4 & 3/4 \\ 1/4 & 3/4 \end{bmatrix} \quad Q = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}
\]

11. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}; \quad P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \quad Q = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}
\]

12. \[
\begin{bmatrix}
4 & -1 & 0 \\
2 & 3 & 1
\end{bmatrix}; \quad P = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} \quad Q = \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix}
\]

Applications and Extensions

13. Matching Pennies In the children's game matching pennies, each player simultaneously shows one side of a penny, either a head or a tail. If the sides that are shown match, player I wins the penny. If the sides that are shown do not match, player II wins the penny. Model this game as a two-person zero-sum game, and find the expected payoff if each player chooses heads half the time.

14. Matching Pennies In the children's game matching pennies, each player simultaneously shows one side of a penny, either a head or a tail. If the sides that are shown match, player I wins the penny. If the sides that are shown do not match, player II wins the penny. Model this game as a two-person zero-sum game, and find the expected payoff if player I chooses heads 40% of the time and player II chooses heads 70% of the time.

15. Show that in a \( 2 \times 2 \) game matrix \[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\] the only games that are not strictly determined are those for which either

(a) \( a_{11} > a_{12} \quad a_{11} > a_{21} \quad a_{21} < a_{22} \quad a_{12} < a_{22} \)

or

(b) \( a_{11} < a_{12} \quad a_{11} < a_{21} \quad a_{21} > a_{22} \quad a_{12} > a_{22} \)

"Are You Prepared?" Answers

1. \[
\begin{bmatrix}
0 & -11 & 17 & -3 \\
8 & 21 & -35 & 9
\end{bmatrix}
\]

2. $1.65
We have already seen that the best strategy for two-person zero-sum games that are strictly determined is found in the row and column containing the saddle point. Suppose the game is not strictly determined.

In 1927, John von Neumann, along with E. Borel, initiated research in the theory of games and proved that, even in nonstrictly determined games, there is a single course of action that represents the best strategy. In practice this means that in order to avoid always using a single strategy, a player in a game may instead choose plays randomly according to a fixed probability. This has the effect of making it impossible for the opponent to know what the player will do, since even the player will not know until the final moment. That is, by selecting a strategy randomly according to the laws of probability, the actual strategy chosen at any one time cannot be known even to the one choosing it.

For example, in the Italian game of Morra each player shows one, two, or three fingers and simultaneously calls out his guess as to what the sum of his and his opponent’s fingers is. It can be shown that if he guesses four fingers each time and varies his own moves so that every 12 times he shows one finger 5 times, two fingers 4 times, and three fingers 3 times, he will, at worst, break even (in the long run).

### Example 1: Finding Optimal Strategies

Consider the nonstrictly determined game

\[
A = \begin{bmatrix}
1 & -1 \\
-2 & 3
\end{bmatrix}
\]

in which player I plays rows and player II plays columns. Find the optimal strategy for each player.

**Solution**

If player I chooses row 1 with probability \( p \), then she must choose row 2 with probability \( 1 - p \). If player II chooses column 1, player I then expects to earn

\[
E_I = p + (-2)(1 - p) = 3p - 2 \quad (1)
\]

Similarly, if player II chooses column 2, player I expects to earn

\[
E_I = (-1)p + 3(1 - p) = -4p + 3 \quad (2)
\]
10.6 Optimal Strategy in Two-Person Zero-Sum Games with $2 \times 2$ Matrices

We graph these two equations using $E_I$ as the vertical axis and $p$ as the horizontal axis. See Figure 7.

Player I wants to maximize her expected earnings, so she should maximize the minimum expected gain. This occurs when the two lines intersect since for any other choice of $p$, one or the other of the two expected earnings is less. Solving Equations (1) and (2) simultaneously, we find

$$3p - 2 = -4p + 3$$
$$7p = 5$$
$$p = \frac{5}{7}$$

The optimal strategy for player I is therefore

$$P = \begin{bmatrix} p & 1 - p \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \end{bmatrix}$$

Similarly, suppose player II chooses column 1 with probability $q$ (and therefore column 2 with probability $1 - q$). If player I chooses row 1, player II's expected earnings are

$$E_{II} = (-1)q + (1)(1 - q) = 1 - 2q \quad (3)$$

If player I chooses row 2, player II's expected earnings are

$$E_{11} = (2)q + (-3)(1 - q) = 5q - 3 \quad (4)$$

The optimal strategy for player II occurs where the lines given in Equations (3) and (4) intersect. That is,

$$1 - 2q = 5q - 3$$
$$-7q = -4$$
$$q = \frac{4}{7}$$

See Figure 8.

The optimal strategy for player II occurs

$$Q = \begin{bmatrix} q \\ 1 - q \end{bmatrix} = \begin{bmatrix} \frac{4}{7} \\ \frac{3}{7} \end{bmatrix}$$

The expected payoff $E$ corresponding to these optimal strategies is

$$E = PAQ = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{4}{7} \\ \frac{3}{7} \end{bmatrix} = \frac{1}{7}$$

NOW WORK PROBLEM 3.

Consider a two-person zero-sum game given by the $2 \times 2$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
in which player I chooses row strategies and player II chooses column strategies. It can be shown that the optimal strategy for player I is given as follows:

**Theorem: Optimal Strategy for Player I**

The optimal strategy for player I is

\[ P = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \]

where

\[ p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad p_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \]

with

\[ a_{11} + a_{22} - a_{12} - a_{21} \neq 0 \]

Similarly, the optimal strategy for player II is given as follows:

**Theorem: Optimal Strategy for Player II**

The optimal strategy for player II is

\[ Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \]

where

\[ q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad q_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \]

with

\[ a_{11} + a_{22} - a_{12} - a_{21} \neq 0 \]

**Theorem: Expected Payoff**

The expected payoff \( E \) corresponding to the optimal strategies is

\[ E = PAQ = \frac{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \]
10.6 Optimal Strategy in Two-Person Zero-Sum Games with $2 \times 2$ Matrices

**EXAMPLE 2** Finding Optimal Strategies

For the game matrix

$$
\begin{bmatrix}
1 & -1 \\
-2 & 3
\end{bmatrix}
$$

find the optimal strategies for player I and player II. Also find the expected payoff.

**SOLUTION** For player I, we use Formula (5) with $a_{11} = 1$, $a_{12} = -1$, $a_{21} = -2$, $a_{22} = 3$.

$$
p_1 = \frac{3 - (-2)}{1 + 3 - (-1) - (-2)} = \frac{5}{7}
p_2 = \frac{1 - (-1)}{1 + 3 - (-1) - (-2)} = \frac{2}{7}
$$

Player I’s optimal strategy is to select row 1 with probability $\frac{5}{7}$ and row 2 with probability $\frac{2}{7}$. For player II, we use Formula (6).

$$
q_1 = \frac{3 - (-1)}{1 + 3 - (-1) - (-2)} = \frac{4}{7}
q_2 = \frac{1 - (-2)}{1 + 3 - (-1) - (-2)} = \frac{3}{7}
$$

Player II’s optimal strategy is to select column 1 with probability $\frac{4}{7}$ and column 2 with probability $\frac{3}{7}$. Using Formula (7), the expected payoff is

$$
E = \frac{1 \cdot 3 - (-1)(-2)}{1 + 3 - (-1) - (-2)} = \frac{1}{7}
$$

In the long run the game is favorable to player I.

The results obtained in Example 2 are in agreement with those obtained earlier using the graphical technique.

**EXAMPLE 3** Finding Optimal Strategies

Find the optimal strategy for each player, and determine the expected payoff of the game given by the matrix

$$
\begin{bmatrix}
6 & 0 \\
-2 & 3
\end{bmatrix}
$$

**SOLUTION** Using the graphical technique or Formulas (5) and (6), we find player I’s optimal strategy to be

$$
p_1 = \frac{5}{11}, \quad p_2 = \frac{6}{11}
$$

Player II’s optimal strategy is

$$
q_1 = \frac{3}{11}, \quad q_2 = \frac{8}{11}
$$
EXAMPLE 4 Election Strategy

In a presidential campaign there are two candidates, a Democrat (D) and a Republican (R), and two types of issues, domestic issues and foreign issues. The units assigned to each candidate’s strategy are given in Table 3. We assume that positive entries indicate a strength for the Democrat, while negative entries indicate a weakness. We also assume that a strength of one candidate equals a weakness of the other so that the game is zero-sum. What is the best strategy for each candidate? What is the expected payoff of the game?

**TABLE 3**

<table>
<thead>
<tr>
<th></th>
<th>Domestic</th>
<th>Foreign</th>
</tr>
</thead>
<tbody>
<tr>
<td>Democrat</td>
<td>4</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>3</td>
</tr>
</tbody>
</table>

**SOLUTION**

Notice first that this game is not strictly determined. If D chooses to always play foreign issues, then R will counter with domestic issues, in which case D would also talk about domestic issues, in which case R would begin to talk about foreign issues, which would cause D to change over to foreign issues. There is no single strategy either can use. We compute that the optimal strategy for the Democrat is

\[
p_1 = \frac{3 - (-2)}{4 + 3 - (-2) - (-1)} = \frac{4}{10} = 0.4 \quad p_2 = \frac{4 - (-2)}{10} = 0.6
\]

The optimal strategy for the Republican is

\[
q_1 = \frac{3 - (-2)}{10} = 0.5 \quad q_2 = \frac{4 - (-1)}{10} = 0.5
\]

The best strategy for the Democrat is to spend 40% of her time on domestic issues and 60% on foreign issues, while the Republican should divide his time evenly between the two issues.

The expected payoff is

\[
E = \frac{3 \cdot 4 - (-1)(-2)}{10} = \frac{10}{10} = 1.0
\]

No matter what the Republican does, the Democrat gains at least 1.0 unit by employing her best strategy.

**NOW WORK PROBLEM 9.**
10.6 Optimal Strategy in Two-Person Zero-Sum Games with $2 \times 2$ Matrices

**EXAMPLE 5 War Game**

In a naval battle, attacking bomber planes are trying to sink ships in a fleet protected by an aircraft carrier with fighter planes. The bombers can attack either high or low, with a low attack giving more accurate results. Similarly, the aircraft carrier can send its fighters at high altitudes or low altitudes to search for the bombers. If the bombers avoid the fighters, credit the bombers with 8 points; if the bombers and fighters meet, credit the bombers with $-2$ points. Also, credit the bombers with 3 additional points for flying low (since this results in more accurate bombing). Find optimal strategies for the bombers and the fighters. What is the expected payoff of the game?

**SOLUTION**

First, we set up the game matrix. Designate the bombers as playing rows and the fighters as playing columns. Also, each entry of the matrix will denote winnings of the bombers. Then the game matrix is

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>High</td>
<td>8</td>
<td>-2</td>
</tr>
</tbody>
</table>

The reason for a 1 in row 1, column l, is that $-2$ points are credited for the planes meeting, but 3 additional points are credited to the bombers for a low flight.

Next, using Formulas (5) and (6), the optimal strategies for the bombers $[p_1 \ p_2]$ and for the fighters $[q_1 \ q_2]$ are

- $p_1 = \frac{-10}{-20} = \frac{1}{2}, \quad p_2 = \frac{-10}{-20} = \frac{1}{2}$
- $q_1 = \frac{-13}{-20} = \frac{13}{20}, \quad q_2 = \frac{-7}{-20} = \frac{7}{20}$

The expected payoff is

$$E = \frac{-2 - 88}{-20} = \frac{-90}{-20} = 4.5$$

The game is favorable to the bombers if both players employ their optimal strategies.

The bombers can decide whether to fly high or low by flipping a fair coin and flying high whenever heads appear. The fighters can decide whether to fly high or low by using a bowl with 13 black balls and 7 white balls. Each day, a ball should be selected at random and then replaced. If the ball is black, they will go low; if it is white, they will go high.

**EXERCISE 10.6 Answers Begin on Page AN–57.**

**'Are You Prepared?' Problem**

The answer is given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find the point of intersection of the two lines $y = 3x - 2$ and $y = -4x + 3$. (p. 25)

**Concepts and Vocabulary**

2. *True or False* In two-person zero-sum games whose game matrix is a $2 \times 2$ matrix, each player can utilize an optimal strategy.
Skill Building

In Problems 3–8, find the optimal strategy for each player and determine the expected payoff of each $2 \times 2$ game by using graphical techniques. Check your answers by using Formulas (5) and (6).

3. \[
\begin{pmatrix}
1 & 2 \\
4 & 1
\end{pmatrix}
\]

4. \[
\begin{pmatrix}
2 & 4 \\
3 & -2
\end{pmatrix}
\]

5. \[
\begin{pmatrix}
-3 & 2 \\
1 & 0
\end{pmatrix}
\]

6. \[
\begin{pmatrix}
3 & -2 \\
-1 & 2
\end{pmatrix}
\]

7. \[
\begin{pmatrix}
2 & -1 \\
-1 & 4
\end{pmatrix}
\]

8. \[
\begin{pmatrix}
5 & 4 \\
-3 & 7
\end{pmatrix}
\]

Applications and Extensions

9. **Election Strategy**  In Example 4 suppose the candidates are assigned the following weights for each issue:

<table>
<thead>
<tr>
<th>Issue</th>
<th>Republican</th>
<th>Democrat</th>
<th>Foreign</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domestic</td>
<td>4</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>Foreign</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

(a) What is each candidate’s best strategy?
(b) What is the expected payoff of the game and whom does it favor?

10. **War Game**  For the situation described in Example 5, credit the bomber with 4 points for avoiding the fighters and with $-6$ points for meeting the fighters. Also grant the bombers 2 additional points for flying low.

(a) What are the optimal strategies and the expected payoff of the game?
(b) Give instructions to the fighters and bombers as to how they should decide whether to fly high or low.

Discussion and Writing

14. In the game matrix

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

what happens if $a_{11} + a_{22} - a_{12} - a_{21} = 0$? Explain.

'Are You Prepared?' Answer

1. \[
\begin{pmatrix}
5 & 1 \\
7 & 7
\end{pmatrix}
\]
CHAPTER 10 REVIEW  OBJECTIVES

<table>
<thead>
<tr>
<th>Section</th>
<th>Examples</th>
<th>You should be able to</th>
<th>Review Exercises</th>
</tr>
</thead>
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<td>1 Find the probability distribution of a Markov chain (p. 578)</td>
<td>13a, 14a, 23a, 24a</td>
</tr>
<tr>
<td></td>
<td>3, 4</td>
<td>2 Solve applied problems involving a Markov chain (p. 582)</td>
<td>13b, c, 14b</td>
</tr>
<tr>
<td>10.2</td>
<td>1, 2, 3</td>
<td>1 Identify regular Markov chains (p. 587)</td>
<td>1–6</td>
</tr>
<tr>
<td></td>
<td>4, 5</td>
<td>2 Find the fixed probability vector of a regular Markov chain (p. 590)</td>
<td>7–12</td>
</tr>
<tr>
<td></td>
<td>6, 7, 8</td>
<td>3 Solve applied problems involving regular Markov chains (p. 593)</td>
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</tr>
<tr>
<td>10.3</td>
<td>1</td>
<td>1 Identify absorbing Markov chains (p. 600)</td>
<td>15e, b–22a, b</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2 Subdivide the transition matrix of an absorbing Markov chain (p. 601)</td>
<td>15c–22c</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3 Analyze the gambler’s ruin problem (p. 602)</td>
<td>23, 24</td>
</tr>
<tr>
<td>10.4</td>
<td>1–4</td>
<td>1 Identify a strictly determined game (p. 613)</td>
<td>25–30</td>
</tr>
<tr>
<td>10.5</td>
<td>1, 2</td>
<td>1 Find the expected payoff (p. 616)</td>
<td>31–36</td>
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<td>10.6</td>
<td>1, 2, 3</td>
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<tr>
<td></td>
<td>4, 5</td>
<td>2 Solve applied problems (p. 624)</td>
<td>39–42</td>
</tr>
</tbody>
</table>

THINGS TO KNOW

Markov chain (p. 578) A sequence of experiments each of which results in one of a finite number of states. The probability of being in a particular state depends only on the state previously occupied.

Initial probability distribution \( v^{(0)} \) (p. 578) A row vector whose \( i \)th entry is the probability of being in state \( i \) at the start of the experiment.

Probability distribution \( v^{(k)} \) after \( k \) trials (pp. 580 and 582) \( v^{(k)} = v^{(k-1)}P \) \( P \) is the transition matrix.

Regular Markov chain (p. 587) For some power of the transition matrix, all the entries are positive.

Fixed probability vector \( t \) (p. 590) For a regular Markov chain, \( tP = t \), where \( P \) is the transition matrix.

Absorbing Markov chain (p. 600) Contains at least one absorbing state (\( p_{ii} = 1 \)) and it is possible to go from any nonabsorbing state to an absorbing state in one or more trials.

Two-person game (p. 611) A conflict or competition between two people.

Strictly determined game (p. 613) There is an entry in the game matrix that is the smallest element in its row and is the largest element in its column.

Expected payoff (p. 617) \( E = PAQ \) where \( P \) is player I’s strategy, \( A \) is the game matrix, and \( Q \) is player II’s strategy.
In Problems 1–6, determine whether the Markov chain whose transition matrix $P$ is given is regular.

1. $P = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$
2. $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$
3. $P = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$
4. $P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{5} & \frac{3}{5} & 0 \end{bmatrix}$
5. $P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 5 \frac{3}{8} \frac{1}{8} \end{bmatrix}$
6. $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{3}{3} & \frac{3}{3} \frac{3}{3} \end{bmatrix}$

In Problems 7–10, find the fixed probability vector $\mathbf{t}$ of each regular Markov chain.

7. $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$
8. $P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$
9. $P = \begin{bmatrix} \frac{1}{2} & 0 \frac{1}{2} \\ 0 & 1 \\ \frac{1}{3} & \frac{1}{3} \frac{1}{3} \end{bmatrix}$
10. $P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} & \frac{3}{5} \end{bmatrix}$
11. $P = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.6 & 0.1 & 0.3 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}$
12. $P = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.9 & 0 & 0.1 \end{bmatrix}$

13. **Customer Loyalty**

Three beer distributors, A, B, and C, each presently hold $\frac{1}{3}$ of the beer market. Each wants to increase its share of the market, and to do so, each introduces a new brand. During the next year, it is learned that

(i) A keeps 50% of its business and loses 20% to B and 30% to C.
(ii) B keeps 40% of its business and loses 40% to A and 20% to C.
(iii) C keeps 25% of its business and loses 50% to A and 25% to B.

(a) Show that this situation forms a Markov chain and find the transition matrix that models it.
(b) Assuming this trend continues 1 more year, what share of the market does each distributor have?
(c) Assuming this trend continues 2 more years, what share of the market does each distributor have?
(d) In the long run, what is each distributor’s share?

14. If the current share of the market for each distributor A, B, and C in Problem 13 is A: 25%, B: 25%, C: 50%,

(a) Show that this situation forms a Markov chain and find the transition matrix that models it.
(b) Assuming this trend continues 1 more year, what share of the market does each distributor have?
(c) Assuming this trend continues 2 more years, what share of the market does each distributor have?
(d) In the long run, what is each distributor’s share?
In Problems 15–22, use the transition matrices $P$ of the following Markov chains to answer these questions:

(a) Determine whether the Markov chain is absorbing.

(b) If it is absorbing, identify the absorbing states.

(c) If it is absorbing, subdivide the transition matrix.

15. $P = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}$

16. $P = \begin{bmatrix} 0.75 & 0.25 \\ 0 & 1 \end{bmatrix}$

17. $P = \begin{bmatrix} 0 & 1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} \\ 3 & 1 & \frac{3}{8} \\ 8 & 4 & \frac{8}{3} \end{bmatrix}$

18. $P = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 5 & 5 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

19. $P = \begin{bmatrix} 0 & 0.2 & 0.8 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$

20. $P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$

21. $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.1 & 0.1 \\ 0.25 & 0.35 & 0.2 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

22. $P = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0.1 & 0.2 & 0.2 \end{bmatrix}$

23. **Gambler’s Ruin Problem** Suppose a man has $2, which he is going to bet $1 at a time until he either loses all his money or he wins $5. Assume he wins with a probability of 0.45 and he loses with a probability of 0.55.

(a) Construct the transition matrix for this game.

(b) On the average, how many times will the process be in nonabsorbing states?

(c) What is the expected length of the game?

(d) What is the probability the man loses all his money or wins $5?

*Hint:* The fundamental matrix $T$ of the transition matrix $P$ is

$$T = \begin{bmatrix} 1.584283 & 1.062332 & 0.635282 & 0.285877 \\ 1.298406 & 2.360738 & 1.411737 & 0.635282 \\ 0.949001 & 1.725456 & 2.360738 & 1.062332 \\ 0.52195 & 0.949001 & 1.298406 & 1.584283 \end{bmatrix}$$

24. **Gambler’s Ruin** Suppose a man has $10, and he will bet $5 at a time until he either loses all his money or wins $20. Assume he wins with a probability of 0.3 and loses with a probability of 0.7.

(a) Construct the transition matrix for this game.

(b) On the average, how many times will the process be in nonabsorbing states?

(c) What is the expected length of the game?

(d) What is the probability the man loses all his money or wins $20?

In Problems 25–30, determine which of the two-person zero-sum games are strictly determined. For those that are, find the value of the game.

25. $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$

26. $\begin{bmatrix} 29 & 15 \\ 79 & 3 \end{bmatrix}$

27. $\begin{bmatrix} 7 & 14 \\ 9 & 13 \end{bmatrix}$

28. $\begin{bmatrix} 50 & 75 \\ 30 & 15 \end{bmatrix}$

29. $\begin{bmatrix} 0 & 2 & 4 \\ 4 & 6 & 10 \\ 16 & 14 & 12 \end{bmatrix}$

30. $\begin{bmatrix} 8 & 1 & -2 \\ 2 & 5 & 1 \\ 5 & -3 & -1 \end{bmatrix}$
31. Find the expected payoff of the game
\[
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
\]
for the strategies
\[
P = \begin{bmatrix}
\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\quad Q = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]
32. Find the expected payoff of the game
\[
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
\]
for the strategies
\[
P = \begin{bmatrix}
0 & 1
\end{bmatrix}
\quad Q = \begin{bmatrix}
0.5 \\
0.5
\end{bmatrix}
\]
33. Find the expected payoff of the game
\[
\begin{bmatrix}
4 & 2 \\
-2 & 3
\end{bmatrix}
\]
for the strategies
\[
P = \begin{bmatrix}
0.5 & 0.5
\end{bmatrix}
\quad Q = \begin{bmatrix}
0.5 \\
0.5
\end{bmatrix}
\]
34. Find the expected payoff of the game
\[
\begin{bmatrix}
4 & 2 \\
-2 & 3
\end{bmatrix}
\]
for the strategies
\[
P = \begin{bmatrix}
0.8 & 0.2
\end{bmatrix}
\quad Q = \begin{bmatrix}
0.3 \\
0.7
\end{bmatrix}
\]
35. Find the expected payoff of the game
\[
\begin{bmatrix}
1 & 4 & -3 \\
2 & -5 & -1 \\
1 & -2 & 3
\end{bmatrix}
\]
for the strategies
\[
P = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\quad Q = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]
36. Find the expected payoff of the game
\[
\begin{bmatrix}
1 & 4 & -3 \\
2 & -5 & -1 \\
1 & -2 & 3
\end{bmatrix}
\]
for the strategies
\[
P = \begin{bmatrix}
0.4 & 0.2 & 0.4
\end{bmatrix}
\quad Q = \begin{bmatrix}
0.5 \\
0.25 \\
0.25
\end{bmatrix}
\]
37. Find the optimal strategy for the game described in Problem 31.
38. Find the optimal strategy for the game described in Problem 33.
39. Investment Strategy An investor has a choice of two investments, A and B. The percentage gain of each investment over the next year depends on whether the economy is “up” or “down.” This investment information is displayed in the following payoff matrix.

<table>
<thead>
<tr>
<th>Economy</th>
<th>Economy</th>
</tr>
</thead>
<tbody>
<tr>
<td>up</td>
<td>down</td>
</tr>
<tr>
<td>Invest in A</td>
<td>\begin{bmatrix} -5 &amp; 20 \ 18 &amp; 0 \end{bmatrix}</td>
</tr>
</tbody>
</table>

(a) Find the best investment allocation. That is, find the row player’s optimal strategy in the corresponding two-person zero-sum game.

(b) Find the percentage gain the investor expects when using this optimal strategy. That is, find the expected payoff of the corresponding two-person zero-sum game.

40. Investment Strategy There are two possible investments, A and B, and two possible states of the economy, “inflation” and “recession.” The estimated percentage increases in the value of the investments over the coming year for each possible state of the economy are shown in the following payoff matrix.

<table>
<thead>
<tr>
<th>Inflation</th>
<th>Recession</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invest in A</td>
<td>\begin{bmatrix} 10 &amp; 5 \ -5 &amp; 20 \end{bmatrix}</td>
</tr>
</tbody>
</table>

(a) Find the investor’s optimal strategy \( P = [p_1 \ p_2] \).

(b) Find the expected payoff of this game.

(c) What return can the investor expect if the optimal strategy is followed?

41. Real Estate Development A real estate developer has bought a large tract of land in Cook County. He is considering using some of the land for a shopping center and some for houses. It is not certain whether the Cook County government will build a highway near his property or not. His financial advisor provides an estimate for the percentage profit to be made in each case and these percentages are given in the following table:

<table>
<thead>
<tr>
<th>Government</th>
<th>Highway</th>
<th>No Highway</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shopping center</td>
<td>70%</td>
<td>30%</td>
</tr>
<tr>
<td>Houses</td>
<td>20%</td>
<td>80%</td>
</tr>
</tbody>
</table>
What percentage of the land should he use for the two categories, assuming that the government of Cook County is an active opponent?

42. **Betting Strategy**  The Pistons are going to play the Bulls in a basketball game. If you place your bet in Detroit, you can get $3 to 2$ odds for a bet on the Bulls; and if you place your bet in Chicago, you can get $1 to 1$ odds for a bet on the Pistons. This information is displayed in the following payoff matrix in which the entries represent the payoffs in dollars for each $1$ bet.

\[
\begin{pmatrix}
\text{Bet in Detroit} & \text{Bet in Chicago} \\
1.5 & 1 \\
-1 & 1
\end{pmatrix}
\]

Think of this situation as a game in which you are the row player and find your optimal strategy and the expected payoff of the game.

---

**Chapter 10 Project**

**EDUCATION, JOB SECURITY, AND SALARY**

Refer to the discussion given at the start of this chapter.

The table below represents the probabilities of the highest educational level of children based on the highest educational level of their parents. For example, the table shows that the probability $p_{21}$ is $12\%$ that parents with a high school education (row 2) will have children with a college education (column 1).

<table>
<thead>
<tr>
<th>Highest Education Level of Parents</th>
<th>College</th>
<th>High School</th>
<th>Elementary</th>
</tr>
</thead>
<tbody>
<tr>
<td>College</td>
<td>50%</td>
<td>48%</td>
<td>2%</td>
</tr>
<tr>
<td>High School</td>
<td>12%</td>
<td>76%</td>
<td>12%</td>
</tr>
<tr>
<td>Elementary</td>
<td>6%</td>
<td>68%</td>
<td>26%</td>
</tr>
</tbody>
</table>

*Source: Literacy NOW*

1. Convert the percentages to decimals.
2. What is the transition matrix?
3. Sum across the rows. What do you notice? Why do you think that you obtained this result?
4. What is the probability that a grandchild of a college graduate is a college graduate?  
5. What is the probability that the grandchild of a high school graduate finishes college?  
6. Go to the *Statistical Abstract of the United States* (http://www.census.gov) and determine the proportion of the U.S. population 25 years or older that has college, high school, and elementary school levels of education.
7. What is the long-run distribution of highest educational attainment of the population?  
8. Determine the distribution of highest educational attainment of the grandchildren of the current population.
A popular pastime with many people is trying to solve logic puzzles. Have you ever tried to solve one? They can be challenging! But they are also entertaining. The Chapter Project at the end of this chapter consists of three logic puzzles for you to solve. The information presented in this chapter will help you focus on a methodology for getting the solution. Have fun!
11.1 Propositions

**OBJECTIVES**

1. Determine whether a sentence is a proposition (p. 634)
2. Form compound propositions (p. 635)
3. Negate a proposition (p. 637)

---

**1 Determine Whether a Sentence Is a Proposition**

English and other natural languages are composed of various words and phrases with distinct functions that have a bearing on the meaning of the sentences in which they occur.

English sentences may be classified as *declarative*, *interrogative*, *exclamatory*, or *imperative*. In the study of logic, we assume that we are able to recognize a declarative sentence (or *statement*) and to form an opinion as to whether it is true or false.

**Definition**

**Proposition**

A proposition is a declarative sentence that can be meaningfully classified as either true or false.

---

**EXAMPLE 1** Determining Whether a Sentence Is a Proposition

The price of a Dell laptop computer was $899 on June 16, 2010.

This is a proposition, although few of us can say whether it is true or false.

---

**EXAMPLE 2** Determining Whether a Sentence Is a Proposition

The earth is round.

This sentence records a possible fact about reality and is a proposition. Some people would classify this proposition as true and others as false, depending on what the word *round* means to them. (Does it mean simply “curved” or “perfectly spherical”?) Only in an ideal situation can we unequivocally state to which truth category a proposition belongs.

---

**EXAMPLE 3** Determining Whether a Sentence Is a Proposition

What is the exchange rate from United States dollars to Euros?

This is not a proposition—it is an interrogative sentence.
EXAMPLE 4 Determining Whether a Sentence Is a Proposition

The prices of most stocks on the New York Stock Exchange rose during the period 1929–1931.

This is a proposition. It happens to be false.

In an ideal situation, a proposition could be easily and decisively classified as true or false. However, as Examples 1 and 2 illustrate, it is often difficult to classify propositions in this way because of unclear meanings, ambiguous situations, differences of opinion, etc. In the mathematical treatment of logic, we avoid these difficulties by assuming a proposition is either true or false. We do this so we can use symbolic logic to draw conclusions about other propositions.

NOW WORK PROBLEM 5.

2 Form Compound Propositions

Consider the sentence “Jones is handsome and Smith is selfish.” This sentence is obtained by joining the two propositions “Jones is handsome,” “Smith is selfish” by the word “and.” It is called a compound proposition. In working with propositions, we usually use lowercase letters such as $p$ or $q$ to represent a proposition.

Definition Compound Proposition; Connectives

A compound proposition is a proposition formed by connecting two or more propositions or by negating a single proposition. The words and phrases (or symbols) used to form compound propositions are called connectives.

Some of the connectives used in English are or; either . . . or; and; but; if . . . , then.

Definition Conjunction

Let $p$ and $q$ denote propositions. The compound proposition $p$ and $q$ is called the conjunction of $p$ and $q$ and is denoted symbolically by $p \land q$.

We define $p \land q$ to be true when both $p$ is true and $q$ is true and to be false otherwise.

EXAMPLE 5 Forming a Compound Proposition

Consider the two statements

$p$: Washington, D.C., is the capital of the United States.
$q$: Hawaii is the fiftieth state of the United States.

The conjunction of $p$ and $q$ is
$p \land q$: Washington, D.C., is the capital of the United States and Hawaii is the fiftieth state of the United States.

Since both $p$ and $q$ are true statements, we conclude that the compound statement $p \land q$ is true.

**Example 6** Forming a Compound Proposition

Consider the two statements

$p$: Washington, D.C., is the capital of the United States.
$q$: Vermont is the largest of the fifty states.

The compound proposition $p \land q$ is

$p \land q$: Washington, D.C., is the capital of the United States and Vermont is the largest of the fifty states.

Since the statement $q$ is false, the compound statement $p \land q$ is false—even though $p$ is true.

**Definition**

**Inclusive Disjunction**

Let $p$ and $q$ be any propositions. The compound proposition $p$ or $q$ is called the **inclusive disjunction of $p$ and $q$**, read as “$p$ or $q$ or both,” and is denoted symbolically by

$p \lor q$

We define $p \lor q$ to be true if **at least one** of the propositions $p$, $q$ is true. That is, $p \lor q$ is true if both $p$ and $q$ are true, if $p$ is true and $q$ is false, or if $p$ is false and $q$ is true. It is false only if both $p$ and $q$ are false.

**Example 7** Forming a Compound Proposition

Consider the propositions

$p$: XYZ Company is the largest producer of nails in the world.
$q$: Mines, Ltd. has three uranium mines in Nevada.

The compound proposition $p \lor q$ is

$p \lor q$: XYZ Company is the largest producer of nails in the world or Mines, Ltd. has three uranium mines in Nevada.

The English word “or” can be used in two different ways—as an inclusive (“and/or”) or exclusive (“either/or”) disjunction. The correct meaning is usually inferred from the context in which the word is used. However, when it is important to be precise (as it often
is in mathematics, business, science, etc.), we must carefully distinguish between the two meanings of “or.”

Definition

**Exclusive Disjunction**

Let \( p \) and \( q \) be any proposition. The exclusive disjunction of \( p \) and \( q \), read as “either \( p \) or \( q \), but not both,” is denoted symbolically by

\[
p \lor q
\]

We define \( p \lor q \) to be true if exactly one of the propositions \( p \) or \( q \) is true. That is, \( p \lor q \) is true if \( p \) is true and \( q \) is false, or if \( p \) is false and \( q \) is true. It is false if both \( p \) and \( q \) are false or if both \( p \) and \( q \) are true.

**EXAMPLE 8** Forming a Compound Proposition

Consider the propositions

\[
p: \text{XYZ Company earned $3.20 per share in 2007.}
q: \text{XYZ Company paid a dividend of $1.20 per share in 2007.}
\]

The inclusive disjunction of \( p \) and \( q \) is

\[
p \lor q: \text{XYZ Company earned $3.20 per share in 2007 or XYZ Company paid a dividend of $1.20 per share in 2007, or both.}
\]

The exclusive disjunction of \( p \) and \( q \) is

\[
p \lor q: \text{XYZ Company earned $3.20 per share in 2007 or XYZ Company paid a dividend of $1.20 per share in 2007, but not both.}
\]

**EXAMPLE 9** Analyzing a Compound Proposition Using the Connective “or”

Consider the compound proposition

\[
p: \text{This weekend I will meet Caryl or Mary.}
\]

The use of the connective “or” is not clear here. If the “or” means inclusive disjunction, then I will be meeting at least one and possibly both persons. If the “or” means exclusive disjunction, then I will be meeting only one person.

**Negate a Proposition**

Definition

**Negation**

If \( p \) is any proposition, the negation of \( p \), denoted by

\[
\neg p
\]

and read as “not \( p \),” is a proposition that is false when \( p \) is true and true when \( p \) is false.
The negation of \( p \) is sometimes called the *denial* of \( p \). The symbol \( \sim \) is called the *negation operator*.

The definition of \( \sim p \) assumes that \( p \) and \( \sim p \) cannot both be true. In logic this assumption is known as the **Law of Contradiction**.

### Example 10  
**Negating a Proposition**

Consider the proposition

\[
p: \text{One share of IBM stock is worth less than } \$85.
\]

The negation of \( p \) is

\[
\sim p: \text{One share of IBM stock is worth at least } \$85.
\]

The sentence “A share of IBM stock is worth more than \$85” is *not* a correct statement of the negation of \( p \). Do you see why?

**Now Work Problem 15.**

A *quantifier* is a word or phrase telling how many (Latin *quantus*). English quantifiers include “all,” “none,” “some,” and “not all.” The quantifiers “all,” “every,” and “each” are interchangeable. The quantifiers “some,” “there exist(s),” and “at least one” are also interchangeable.

### Example 11  
**Using Different Quantifiers to Give the Same Meaning**

The following propositions all have the same meaning:

\[
p: \text{All people are intelligent.}
\]

\[
q: \text{Every person is intelligent.}
\]

\[
r: \text{Each person is intelligent.}
\]

\[
s: \text{Any person is intelligent.}
\]

### Example 12  
**Forming the Negation of a Proposition**

(a) The negation of the proposition

\[
p: \text{All students are intelligent.}
\]

can be stated in any of the following ways:

\[
\sim p: \text{Some students are not intelligent.}
\]

\[
\sim p: \text{There exists a student who is not intelligent.}
\]

\[
\sim p: \text{At least one student is not intelligent.}
\]

Note that “No student is intelligent” is *not* a negation of \( p \).

(b) The negation of the proposition

\[
q: \text{No student is intelligent.}
\]

may be stated as:

\[
\sim q: \text{At least one student is intelligent.}
\]

Note that “All students are intelligent” is *not* a negation of \( q \).
EXERCISE 11.1 Answers Begin on Page AN–59.

Concepts and Vocabulary

1. A declarative sentence that can be meaningfully classified as either true or false is called a(n) __________.

3. True or False The inclusive disjunction of two statements \( p \) and \( q \) is true provided both \( p \) and \( q \) are true.

2. The conjunction of two statements \( p \) and \( q \) is symbolized as __________.

4. True or False The exclusive disjunction of two statements \( p \) and \( q \) is false if both \( p \) and \( q \) are true.

Skill Building

In Problems 5–12, determine which are propositions.

5. The cost of shell egg futures was up on June 18, 1990.

7. What a portfolio!

9. The earnings of XYZ Company doubled last year.

11. Jones is guilty of murder in the first degree.

In Problems 13–20, negate each proposition.

13. A fox is an animal.

15. I am buying stocks.

17. No one wants to buy my house.

19. Some people have no car.

In Problems 21–28, let \( p \) denote “John is an economics major” and let \( q \) denote “John is a sociology minor.” State each compound proposition as an English sentence.

21. \( p \lor q \)

22. \( p \lor \neg q \)

23. \( p \land q \)

24. \( \neg p \)

25. \( \neg p \lor \neg q \)

26. \( \neg (\neg q) \)

27. \( \neg p \lor q \)

28. \( \neg p \land q \)

11.2 Truth Tables

PREPARING FOR THIS SECTION Before getting started, review the following:

• Properties of Real Numbers (Appendix A, Section A.1, pp. A–8 to A–14)

NOW WORK THE ‘ARE YOU PREPARED?’ PROBLEMS ON PAGE 647.

OBJECTIVES

1. Construct truth tables (p. 639)

2. Use truth tables to show statements are equivalent (p. 642)

3. Work with properties of propositions (p. 643)

1. Construct Truth Tables

The truth value of a proposition is either true (denoted by T) or false (denoted by F). A truth table is a table that shows the truth value of a compound proposition for all possible cases.
For example, consider the conjunction of any two propositions \( p \) and \( q \). Recall that
\[ p \land q \]
is false if either \( p \) is false, or \( q \) is false, or if both \( p \) and \( q \) are false. There are four possible cases.

1. \( p \) is true and \( q \) is true.
2. \( p \) is true and \( q \) is false.
3. \( p \) is false and \( q \) is true.
4. \( p \) is false and \( q \) is false.

These four cases are listed in the first two columns of Table 1, which is the truth table for \( p \land q \). For convenience, the cases for \( p \) and \( q \) will always be listed in this order.

The truth table for \( \neg p \) is given in Table 2.

Using the previous definitions of inclusive disjunction and exclusive disjunction, we obtain truth tables for \( p \lor q \) and \( p \not\lor q \). See Table 3.

### Table 1

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( q )</td>
<td>( p \land q )</td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>Case 2</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>Case 3</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>Case 4</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

### Table 2

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( \neg p )</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

### Table 3

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( q )</td>
<td>( p \lor q )</td>
<td>( p \not\lor q )</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(( p \lor q ) ( \lor (\neg p) ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Besides using the connectives \( \land, \lor, \lor, \neg \) one at a time to form compound propositions, we can use them together to form more complex statements. For example, Table 4 is the truth table for \( (p \lor q) \lor (\neg p) \). The parentheses are used as grouping symbols (just like in algebra) to indicate that \( \lor \) and \( \neg \) are applied before \( \lor \). The parentheses around \( \neg p \) are not required as the negation symbol behaves like a minus sign in algebra.

To obtain Table 4, list each component of the proposition on the top row, to the right of any intermediate steps; there is a column underneath each component or connective. Truth values are then entered in the truth table, one step at a time. See the steps shown in Table 5.

### Table 5 Step 1

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th>(( p \lor q ) ( \lor (\neg p) ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td>T</td>
</tr>
</tbody>
</table>

### Table 5 Step 2

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th>(( p \lor q ) ( \lor (\neg p) ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
11.2 Truth Tables

STEP 3

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∨ q</th>
<th>~q</th>
<th>(p ∨ q) ∨ (~p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

The truth table of the compound proposition then consists of the original columns under p and q and the fifth column entered into the table in the last step, as shown earlier in Table 4.

**EXAMPLE 1** Constructing a Truth Table

Construct the truth table for (p ∨ ~q) ∧ p. The component parts of this proposition are p, ~q, and p ∨ ~q.

**SOLUTION** Construct Table 6.

**TABLE 6**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>~q</th>
<th>(p ∨ ~q)</th>
<th>(p ∨ ~q) ∧ p</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Notice that the truth table is the same as for p alone.

**EXAMPLE 2** Constructing a Truth Table

Construct the truth table for ~(p ∨ q) ∨ (~p ∧ ~q).

**SOLUTION** Construct Table 7.

**TABLE 7**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>~p</th>
<th>~q</th>
<th>p ∨ q</th>
<th>~(p ∨ q)</th>
<th>(~p ∧ ~q)</th>
<th>~(p ∨ q) ∨ (~p ∧ ~q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
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<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Notice that the statements ~(p ∨ q), ~p ∧ ~q, and ~(p ∨ q) ∨ (~p ∧ ~q) all have the same truth table.

**NOW WORK PROBLEM 17.**
The next example is of a truth table involving three components, \( p, q, \) and \( r \).

**EXAMPLE 3** Constructing a Truth Table

Construct the truth table for \( p \land (q \lor r) \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( q \lor r )</th>
<th>( p \land (q \lor r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
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<tr>
<td>( T )</td>
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<td>( F )</td>
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<td>( F )</td>
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<td>( F )</td>
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<tr>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

**NOW WORK PROBLEM 21.**

**2 Use Truth Tables to Show Statements Are Equivalent**

Very often, two propositions stated in different ways have the same meaning. For example, in law, “Jones agreed and is obligated to paint Smith’s house” has the same meaning as “It was agreed and contracted by Jones that Jones would paint the house belonging to Smith and Jones is therefore required to paint the aforesaid house.”

**Definition** Logically Equivalent

If two propositions \( a \) and \( b \) have the same truth values in every possible case, the propositions are called **logically equivalent** and we write \( a \equiv b \).

**EXAMPLE 4** Using Truth Tables to Show That Two Statements Are Logically Equivalent

Show that \( \neg(p \land q) \) is logically equivalent to \( \neg p \lor \neg q \).

Construct the truth table in Table 9.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \land q )</th>
<th>( \neg(p \land q) )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( \neg p \lor \neg q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
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<td>( F )</td>
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<td>( F )</td>
<td>( T )</td>
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</tr>
</tbody>
</table>
Since the entries in columns 4 and 7 of the truth table are the same, the two propositions are logically equivalent.

EXAMPLE 5 Using Truth Tables to Show That Two Statements Are Logically Equivalent

Show that \( \sim p \land \sim q \) is logically equivalent to \( \sim(p \lor q) \).

SOLUTION

Construct the truth table in Table 10.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( p \lor q )</th>
<th>( \sim p )</th>
<th>( \sim q )</th>
<th>( \sim p \land \sim q )</th>
<th>( \sim(p \lor q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
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</tr>
</tbody>
</table>

The entries under \( \sim p \land \sim q \) and \( \sim(p \lor q) \) are the same, so the two propositions are logically equivalent.

NOW WORK PROBLEM 33.

Work with Properties of Propositions

The following properties are especially useful in the study of logic circuits. They can be proved using truth tables.

**Idempotent Properties**

For any proposition \( p \),

\[
p \land p = p \quad p \lor p = p
\]

Here, \( p \land p = p \) since \( p \land p \) is true when \( p \) is true, and is false when \( p \) is false. A similar argument shows that \( p \lor p = p \).

**Commutative Properties**

For any two propositions \( p \) and \( q \),

\[
p \land q = q \land p \quad p \lor q = q \lor p
\]

The commutative properties state that, if two propositions are combined using the connective and, changing the order in which the components are connected does not change the meaning of the compound proposition. The same is true of the connective or.
For example, if we consider the two statements

\[ p: \text{ Mrs. Jones designs houses.} \]
\[ q: \text{ Mr. Jones is a banker.} \]

the compound propositions

\[ p \land q: \text{ Mrs. Jones designs houses and Mr. Jones is a banker.} \]
\[ q \land p: \text{ Mr. Jones is a banker and Mrs. Jones designs houses.} \]

have the same meaning.

**Associative Properties**

For any three propositions \( p, q, r \),

\[
(p \land q) \land r = p \land (q \land r) \quad (p \lor q) \lor r = p \lor (q \lor r)
\]

Because of the associative properties, it is possible to omit the parentheses when using the same connective more than once. For instance, we can write

\[ p \land q \land r \quad \text{for} \quad (p \land q) \land r \]

and

\[ p \land q \land r \land s \quad \text{for} \quad [(p \land q) \land r] \land s \]

The same comments hold for \( \lor \). Note, however, that the parentheses cannot be omitted when using \( \land \) and \( \lor \) together.

**Distributive Properties**

For any three propositions \( p, q, r \),

\[
p \lor (q \land r) = (p \lor q) \land (p \lor r) \\
p \land (r \lor q) = (p \land r) \lor (p \land q)
\]

Here, \( p \lor (q \land r) = (p \lor q) \land (p \lor r) \) means that \( p \lor (q \land r) \) has the same meaning as \( (p \lor q) \land (p \lor r) \). Also, \( p \land (r \lor q) = (p \land r) \lor (p \land q) \) means that the compound proposition \( p \land (r \lor q) \) has the same meaning as \( (p \land r) \lor (p \land q) \).

**EXAMPLE 6** Using the Distributive Property to Show That Two Statements Are Logically Equivalent

Consider the three propositions

\[ p: \text{ Betsy will do her homework.} \]
\[ q: \text{ Betsy will wash her car.} \]
\[ r: \text{ Betsy will read a book.} \]
The first distributive property, namely,
\[ p \lor (q \land r) = (p \lor q) \land (p \lor r) \]
says that the following two statements are logically equivalent:

1. Betsy will do her homework, or she will wash her car and read a book.
2. Betsy will do her homework or wash her car, and Betsy will do her homework or read a book.

**De Morgan’s Properties**

For any two propositions \( p \) and \( q \),

\[
\sim(p \land q) = \sim p \lor \sim q \quad \sim(p \land q) = \sim p \lor \sim q
\]

De Morgan’s properties are proved using the truth tables in Examples 4 and 5.

**EXAMPLE 7** Using De Morgan’s Properties to Negate Statements

Negate the compound statements:

(a) The first child is a girl and the second child is a boy.
(b) Tonight I will study or I will go bowling.

**SOLUTION** (a) Let \( p \) and \( q \) represent the components:

\( p \): The first child is a girl.
\( q \): The second child is a boy.

To negate the statement \( p \land q \) is to find \( \sim(p \land q) \). By De Morgan’s property,

\[ \sim(p \land q) = \sim p \lor \sim q \]

The negation can be stated as:

The first child is not a girl or the second child is not a boy.

(b) We represent \( p \) and \( q \) as

\( p \): I will study.
\( q \): I will go bowling.

To negate the statement \( p \lor q \), use De Morgan’s property,

\[ \sim(p \lor q) = \sim p \land \sim q \]

The negation can be stated as:

Tonight I will not study and I will not go bowling.

**NOW WORK PROBLEM 43.**
Absorption Properties

For any two propositions $p$ and $q$,

$$p \lor (p \land q) = p \quad p \land (p \lor q) = p$$

You are asked to prove these propositions in Problem 31.

**Example 8** Using Truth Tables to Prove a Distributive Property

Use truth tables to prove the Distributive Property: $p \lor (q \land r) = (p \lor q) \land (p \lor r)$.

**Solution** Construct Table 11. Since the entries in the last two columns of the truth table in Table 11 are the same, the two propositions are logically equivalent.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$q \land r$</th>
<th>$p \lor (q \land r)$</th>
<th>$(p \lor q)$</th>
<th>$p \lor r$</th>
<th>$(p \lor q) \land (p \lor r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<tr>
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<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**Example 9** Using Truth Tables to Prove an Idempotent Property

Use truth tables to prove the Idempotent Property: $p \land p = p$.

**Solution** See Table 12.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$p \land p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
### EXERCISE 11.2 Answers Begin on Page AN–59.

#### 'Are You Prepared?' Problems
Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The associative property of addition of real numbers states that __________. (pp. A–8 to A–14)
2. The distributive property of real numbers states that __________. (pp. A–8 to A–14)

#### Concepts and Vocabulary

3. True or False Truth tables can be used to show two statements are logically equivalent.
4. For propositions $p$ and $q$, the commutative properties state that __________ and __________.
5. For a proposition $p$, the idempotent properties state that __________ and __________.
6. True or False For two statements $p$ and $q$, either $p$ and $q$ are both true or $p$ and $q$ are both false.

#### Skill Building

In Problems 7–26, construct a truth table for each compound proposition.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$p \lor \neg q$</td>
<td>8</td>
<td>$\neg p \lor \neg q$</td>
<td>9</td>
</tr>
<tr>
<td>11</td>
<td>$\neg (\neg p \land q)$</td>
<td>12</td>
<td>$(p \lor \neg q) \land \neg p$</td>
<td>13</td>
</tr>
<tr>
<td>15</td>
<td>$(p \lor q) \land p$</td>
<td>16</td>
<td>$p \land (q \lor \neg q)$</td>
<td>17</td>
</tr>
<tr>
<td>19</td>
<td>$(p \land q) \lor (\neg p \land \neg q)$</td>
<td>20</td>
<td>$(p \land q) \lor (p \land r)$</td>
<td>21</td>
</tr>
<tr>
<td>23</td>
<td>$p \land (q \land \neg r)$</td>
<td>24</td>
<td>$(p \land q) \lor p$</td>
<td>25</td>
</tr>
</tbody>
</table>

In Problems 27–32, construct a truth table for each property.

27. Idempotent properties
28. Commutative properties
29. Associative properties
30. Distributive properties
31. Absorption properties
32. De Morgan’s properties

In Problems 33–36, show that the given propositions are logically equivalent.

33. $p \land (\neg q \lor q)$ and $p$
34. $p \lor (q \land \neg q)$ and $p$
35. $\neg (\neg p)$ and $p$
36. $p \land q$ and $q \land p$

In Problems 37 and 38 use the propositions

$p$: Smith is an ex-convict. $q$: Smith is rehabilitated.

to give examples, using English sentences, of each property.

37. Commutative properties
38. De Morgan’s properties

39. Use the Distributive Property to show that

$(p \lor q) \land r = (p \land r) \lor (q \land r)$

40. Use the Distributive Property to show that

$(p \land q) \lor r = (p \lor r) \land (q \lor r)$

In Problems 41–44, use De Morgan’s properties to negate each proposition.

41. Mike can hit the ball well and he can pitch strikes.
42. Katy is a good volleyball player and is not conceited.
43. The baby is crying or talking all the time.
44. Billy is happy and he is not hungry.
45. The statement
   
   The actor is intelligent or handsome and talented.
   
   could be interpreted to mean either
   
   a: The actor is intelligent, or he is handsome and talented.
   
   or
   
   b: The actor is intelligent or handsome, and he is talented.

   Describe a situation in which a is true and b is false.

46. The statement
   
   Michael will sell his car and buy a bicycle or rent a truck.
   
   could be interpreted to mean either
   
   a: Michael will sell his car, and he will buy a bicycle or rent a truck.
   
   or
   
   b: Michael will sell his car and buy a bicycle, or he will rent a truck.

   Describe a situation in which b is true and a is false.

11.3 Implications; The Biconditional Connective; Tautologies

**OBJECTIVES**

1. Understand implications (p. 648)
2. Form the converse of an implication (p. 650)
3. Form the contrapositive of an implication (p. 651)
4. Form the inverse of an implication (p. 651)
5. Use the biconditional connective (p. 652)

1. **Understand Implications**

   Consider the following compound proposition: “If I get an A in math, then I will continue to study.” The above sentence states a condition under which I will continue to study. Another example of such a proposition is: “If we get an offer of $100,000, we will sell the house.”
11.3 Implications; The Biconditional Connective; Tautologies

Definition

Implication; Conditional Connective

If \( p \) and \( q \) are any two propositions, then we call the proposition

\[
\text{If } p, \text{ then } q
\]

an implication or conditional statement and the connective \textit{if . . . , then} the conditional connective.

We denote the conditional connective symbolically by \( \Rightarrow \) and the implication \( If \ p, \ then \ q \) by \( p \Rightarrow q \). In the implication \( p \Rightarrow q \), \( p \) is called the \textbf{hypothesis} or \textbf{premise} and \( q \) is called the \textbf{conclusion}. The implication \( p \Rightarrow q \) can also be read as follows:

1. \( p \) implies \( q \).
2. \( p \) is sufficient for \( q \).
3. \( p \) only if \( q \).
4. \( q \) is necessary for \( p \).
5. \( q \), if \( p \).

To arrive at the truth table for implication, consider the following example. Suppose we make the statement

\[
\text{If IBM common stock reaches $180 per share, it will be sold.}
\]

When is the statement true and when is it false? Clearly, if IBM stock reaches $180 per share and it is not sold, the implication is false. It is also clear that if IBM stock reaches $180 per share and it is sold, the implication is true. In other words, if the hypothesis and conclusion are both true, the implication is true; if the hypothesis is true and the conclusion false, the implication is false.

But what happens when the hypothesis is false? If IBM stock does not reach $180 a share, the stock might be sold or it might not be sold. In either case we would not say that a person making the above statement was wrong. That is, we would not accuse the person of making a false statement. For this reason we say that an implication with a false hypothesis is not false and, therefore, is true. The truth table for implication is given in Table 13.

As in the previous section, we express a compound proposition symbolically by replacing each component statement and connective by an appropriate symbol.

For example, denoting “I study” and “I will pass” by \( a \) and \( b \), respectively, the proposition “If I study, then I will pass” is written as \( a \Rightarrow b \). This proposition can also be read as “A sufficient condition for passing is to study.”

To understand “implication” better, look at an implication as a conditional promise. If the promise is broken, the implication is false; otherwise, it is true. For this reason the only circumstance under which the implication \( p \Rightarrow q \) is false is when \( p \) is true and \( q \) is false.

The word \textit{then} in an implication merely serves to separate the conclusion from the hypothesis—it can be, and often is, omitted.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \Rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>
The implication \( p \implies q \) can be expressed in the symbols defined previously. In fact, Table 14 shows that \( p \implies q \) is logically equivalent to the compound proposition \( \sim p \lor q \).

### TABLE 14

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \sim p )</th>
<th>( \sim p \lor q )</th>
<th>( p \implies q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
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</tbody>
</table>

#### Form the Converse of an Implication

Suppose we start with the implication \( p \implies q \) and then interchange the roles of \( p \) and \( q \), obtaining the implication, “If \( q \), then \( p \).”

**Definition**

**Converse**

The implication *if \( q \), then \( p \)* is called the *converse* of the implication *if \( p \), then \( q \)*. That is, \( q \implies p \) is the converse of \( p \implies q \).

### TABLE 15

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<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \implies q )</th>
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</table>

The truth tables for the implication \( p \implies q \) and its converse \( q \implies p \) are compared in Table 15.

Notice that \( p \implies q \) and \( q \implies p \) are not equivalent. As an illustration, consider the following example.

### EXAMPLE 1 Forming the Converse of an Implication

Consider the statements

\[
p: \text{You are a thief.}
q: \text{You go to jail.}
\]

The implication \( p \implies q \) states that

If you are a thief, you go to jail.

The converse of this implication, namely, \( q \implies p \), states that

If you go to jail, you are a thief.

To say that thieves go to jail is not the same as saying that everyone who goes to jail is a thief.

This example illustrates that the truth of an implication does not imply the truth of its converse. Many common fallacies in thinking arise from confusing an implication with its converse.
11.3 Implications; The Biconditional Connective; Tautologies

3 Form the Contrapositive of an Implication

**Definition**

**Contrapositive**

The implication *If not q, then not p*, written as \( \sim q \Rightarrow \sim p \), is called the **contrapositive** of the implication \( p \Rightarrow q \).

**EXAMPLE 2** Forming the Contrapositive of an Implication

Consider the statements

\[ p: \text{ You are a thief.} \]
\[ q: \text{ You go to jail.} \]

The implication \( p \Rightarrow q \) states “If you are a thief, then you go to jail.” The contrapositive, namely \( \sim q \Rightarrow \sim p \), is “If you do not go to jail, then you are not a thief.”

4 Form the Inverse of an Implication

**Definition**

**Inverse**

The implication *If not p, then not q*, written as \( \sim p \Rightarrow \sim q \), is called the **inverse** of the implication \( p \Rightarrow q \).

**EXAMPLE 3** Forming the Inverse of an Implication

Consider the statements

\[ p: \text{ You are a thief.} \]
\[ q: \text{ You go to jail.} \]

The implication \( p \Rightarrow q \) states “If you are a thief, then you go to jail.” The inverse of \( p \Rightarrow q \), namely, \( \sim p \Rightarrow \sim q \), is “If you are not a thief, then you do not go to jail.”

**EXAMPLE 4** Constructing a Truth Table

The truth tables for \( p \Rightarrow q \), \( q \Rightarrow p \), \( \sim p \Rightarrow \sim q \), and \( \sim q \Rightarrow \sim p \) are given in Table 16.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Implication</th>
<th>Converse</th>
<th>Inverse</th>
<th>Contrapositive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( q )</td>
<td>( p \Rightarrow q )</td>
<td>( q \Rightarrow p )</td>
<td>( \sim p )</td>
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</table>
Notice that the entries under Implication and Contrapositive are the same; also, the entries under Converse and Inverse are the same. We conclude that

\[ p \implies q = \neg q \implies \neg p \quad \neg p \implies \neg q \]

We have shown that an implication and its contrapositive are logically equivalent. Also, the converse and inverse of an implication are logically equivalent, but neither of these is logically equivalent to the original implication.

5 Use the Biconditional Connective

The compound statement “\( p \) if and only if \( q \)” is another way of stating the conjunction of two implications: “if \( p \), then \( q \) and if \( q \), then \( p \).” For example, consider

\[ p: \text{Jill is happy.} \]
\[ q: \text{Jack is attentive.} \]

If we say “Jill is happy if, and only if, Jack is attentive,” we mean that if Jill is happy, then Jack is attentive, and if Jack is attentive, then Jill is happy. In symbols, we could write this as

\[ (p \implies q) \land (q \implies p) \]

Definition

Biconditional Connective

The connective if and only if is called the biconditional connective and is denoted by the symbol \( \iff \). The compound proposition

\[ p \iff q \quad (p \text{ if and only if } q) \]

is equivalent to

\[ (p \implies q) \land (q \implies p) \]

The statement \( p \iff q \) also may be read as “\( p \) is necessary and sufficient for \( q \)” or as “\( p \) implies \( q \) and \( q \) implies \( p \).” (The abbreviation “iff” for “if and only if” is also sometimes used.)

We can restate the definition of the biconditional connective by using its truth table, Table 17.

<table>
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<tr>
<th>( p )</th>
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<th>( p \implies q )</th>
<th>( q \implies p )</th>
<th>( (p \implies q) \land (q \implies p) )</th>
<th>( p \iff q )</th>
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</table>
11. 3 Implications; The Biconditional Connective; Tautologies

**TABLE 18**

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \sim p )</th>
<th>( p \lor \sim p )</th>
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**TABLE 19**

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \lor q )</th>
<th>( p \Rightarrow (p \lor q) )</th>
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</table>

**EXAMPLE 5 Using the Biconditional Connective**

Let \( p \) and \( q \) denote the statements

\[
p: \quad \text{Joe is happy.}
\]

\[
q: \quad \text{Joe is not studying.}
\]

Then “A necessary condition for Joe to be happy is that Joe is not studying” means “If Joe is happy, he is not studying.” Moreover, “A sufficient condition for Joe to be happy is that Joe is not studying” means “If Joe is not studying, he is happy.”

If both implications \( p \Rightarrow q, q \Rightarrow p \) are true statements, then the biconditional is true.

In other words, Joe is happy if and only if he is not studying.

**Tautologies**

In Section 11.1 we saw how compound propositions can be obtained from simple propositions by using connectives. By using symbols of grouping (parentheses and brackets), we can form more complicated propositions. We denote a compound proposition by \( P(p, q, \ldots) \), where \( p, q, \ldots \) are the components of the compound proposition. Some examples of such propositions are

\[
\neg(p \land q) \quad p \lor \neg q \quad p \land \neg q \quad (\neg p \land q) \lor \neg p \lor (\neg q \land \neg r)
\]

Ordinarily, when we write a compound proposition, we cannot be certain of the truth of that proposition unless we know the truth or falsity of the component propositions. A compound proposition that is true regardless of the truth values of its components is called a tautology.

**Definition**

**Tautology**

A tautology is a compound proposition \( P(p, q, \ldots) \) that is true in every possible case.

Examples of tautologies are

\[
p \lor \neg p \quad p \Rightarrow (p \lor q)
\]

Tables 18 and 19 show the truth tables of these tautologies.
Many other examples of tautologies are provided by the logical equivalences in Section 11.2. For instance, to say that

\[ \sim (p \lor q) = \sim p \land \sim q \]

is to say that the biconditional proposition

\[ \sim (p \lor q) \iff \sim p \land \sim q \]

is a tautology.

**NOW WORK PROBLEM 25.**

### Substitution

Let \( P(p, q, \ldots) \) be a compound proposition with \( p \) as one of its components. If we replace \( p \) with another proposition \( h \) having the same truth values, then we obtain a new compound proposition \( P(h, q, \ldots) \) that has the same truth value as \( P(p, q, \ldots) \). This leads to the following principle:

**The Law of Substitution**

Suppose \( p \) and \( h \) are propositions and \( h = p \). If \( h \) is substituted for \( p \) in the compound proposition \( P(p, q, \ldots) \), a logically equivalent proposition is obtained:

\[ P(p, q, \ldots) \equiv P(h, q, \ldots) \]

The law of substitution is often used to obtain new versions of existing tautologies. It is particularly useful in mathematics for constructing tautologies of a certain kind, known as valid arguments or proofs. This is the topic of Section 11.4.

### EXERCISE 11.3 Answers Begin on Page AN–60.

#### Concepts and Vocabulary

1. In the implication if \( p \) then \( q \), \( p \) is called the ________ and \( q \) is called the ________.
2. A(n) ________ is a proposition that is true in every possible case.
3. **True or False** If an implication is true, then its converse is also true.
4. **True or False** If an implication is true, then its contrapositive is also true.

#### Skill Building

In Problems 5–14, write the converse, contrapositive, and inverse of each statement.

5. \( \sim p \iff q \)
6. \( \sim p \iff \sim q \)
7. \( \sim q \iff \sim p \)
8. \( p \iff \sim q \)
9. If it is raining, the grass is wet.
10. It is raining if it is cloudy.
11. If it is not raining, it is not cloudy.
12. If it is not cloudy, then it is not raining.
13. Rain is sufficient for it to be cloudy.
14. Rain is necessary for it to be cloudy.
In Problems 15–24, construct a truth table for each statement.

15. \( \neg p \lor (p \land q) \)  
16. \( \neg p \land (p \lor q) \)  
17. \( p \lor (\neg p \land q) \)  
18. \( (p \lor q) \land \neg q \)  
19. \( \neg p \Rightarrow q \)  
20. \( (p \lor q) \Rightarrow p \)  
21. \( \neg p \lor p \)  
22. \( p \land \neg p \)  
23. \( p \land (p \Rightarrow q) \)  
24. \( p \lor (p \Rightarrow q) \)

In Problems 25–28, prove that the following are tautologies.

25. \( p \land (q \land r) \iff (p \land q) \land r \)  
26. \( p \lor (q \lor r) \iff (p \lor q) \lor r \)  
27. \( p \lor (p \lor q) \iff p \)  
28. \( p \land (p \lor q) \iff p \)

In Problems 29–34, let \( p \) be “The examination is hard” and \( q \) be “The grades are low.” Write each symbolically.

29. If the examination is hard, the grades are low.
30. The examination is not hard nor are the grades low.
31. The grades are not low if and only the examination is not hard.
32. The examination is not hard, and the grades are low.
33. The grades are low only if the examination is hard.
34. The grades are low if the examination is hard.

Extensions

35. Show that
   \[ p \Rightarrow q = \neg q \Rightarrow \neg p \]
   using the fact that \( p \Rightarrow q = \neg p \lor q \).
36. Show that
   \[ p \Rightarrow (q \land r) = (p \land \neg q) \Rightarrow r \]
   (a) using a truth table  
   (b) using De Morgan’s properties and the fact that \( p \Rightarrow q = \neg p \lor q \).

37. Show that
   \[ ((p \land q) \Rightarrow r) = [(p \land \neg r) \Rightarrow \neg q] \]
   (a) Using a truth table  
   (b) Using De Morgan’s properties and the fact that \( p \Rightarrow q = \neg p \lor q \).

Discussion and Writing

38. Give a verbal sentence that describes
   (a) \( p \Rightarrow q \)  
   (b) \( q \Rightarrow p \)  
   (c) \( \neg p \Rightarrow q \)
   using the statements:
   \( p \): Jack studies psychology. 
   \( q \): Mary studies sociology.

11.4 Arguments

OBJECTIVES

1. Use a direct proof (p. 656)
2. Use an indirect proof (p. 659)

Valid Arguments

In an argument or a proof we are not concerned with the truth of the conclusion but rather with whether the conclusion does or does not follow from the premises. If the
conclusion follows from the premises, we say that our reasoning is valid; if it does not, we say that our reasoning is invalid. For example, from the two

**Premises:** (1) All college students are intelligent.  
(2) All freshmen are college students.

follows the

**Conclusion:** All freshmen are intelligent.

Now the last statement certainly is not regarded generally as true, but the reasoning leading to it is valid. *If both of the premises are true, the conclusion is also true.*

**Definition**

**Argument**

An argument or proof consists of a set of propositions $p_1, p_2, \ldots, p_n$, called the premises or hypotheses, and a proposition $q$, called the conclusion. An argument is valid if and only if the conclusion is true whenever the premises are all true. An argument that is not valid is called a fallacy or an invalid argument.

We limit our discussion to two types of argument: direct proof and indirect proof.

**1 Use a Direct Proof**

In a direct proof we go through a chain of propositions, beginning with the hypotheses and leading to the desired conclusion.

**EXAMPLE 1 Using a Direct Proof to Prove a Statement**

Suppose each of the following statements are true:

Either John obeyed the law or John was punished, but not both

and

John was not punished.

Prove that

John obeyed the law.

**SOLUTION** Let $p$ and $q$ denote the propositions

$p$: John obeyed the law.

$q$: John was punished.

The premises can be symbolized as

$p \lor q$ and $\neg q$

Since $\neg q$ is true, then, by the Law of Contradiction (see page 638), $q$ is false. Since $p \lor q$ is true, either $p$ is true or $q$ is true. Since $q$ is false, $p$ must be true. John obeyed the law.
More examples of direct proof are given later, but before we look at them, let’s discuss two laws of logic that are useful in direct proofs.

**Law of Detachment**
If the implication $p \implies q$ is true and $p$ is true, then $q$ must be true.

See Table 13 on page 649 for an illustration of this law.

**Law of Syllogism**
Let $p$, $q$, $r$ be three propositions. If $p \implies q$ and $q \implies r$ are both true, then $p \implies r$ is true.

Table 20 proves this law.

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<th>$p$</th>
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<th>$r$</th>
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<th>$q \implies r$</th>
<th>$p \implies r$</th>
<th>$(p \implies q) \land (q \implies r)$</th>
<th>$(p \implies q) \land (q \implies r) \implies (p \implies r)$</th>
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**EXAMPLE 2** Using a Direct Proof to Prove a Statement

Suppose each of the following statements are true:

- It is snowing
- If it is warm, then it is not snowing
- If it is not warm, then I cannot go swimming.

Prove that

- I cannot go swimming.
SOLUTION Let $p, q, r$ represent the statements

$p$: It is snowing.
$q$: It is warm.
$r$: I can go swimming.

The premises are that the propositions

\[ p \implies \neg q \quad \neg q \implies \neg r \]

are true. We want to prove that $\neg r$ is true.

Since $q \implies \neg p$ is true, its contrapositive

\[ p \implies \neg q \]

is also true. Using the Law of Syllogism, since $p \implies \neg q$ and $\neg q \implies \neg r$, we see that

\[ p \implies \neg r \]

But $p$ is true. By the Law of Detachment $\neg r$ is true. That is, “I cannot go swimming” is true.

EXAMPLE 3 Using a Direct Proof to Prove a Statement

Suppose each of the following statements is true:

If Dan is happy, so is Bill

and

If Sandy is not happy, then Bill is not happy.

Show that

If Dan is happy, then Sandy is happy.

SOLUTION Let $p, q, r$ denote the propositions

$p$: Dan is happy
$q$: Bill is happy
$r$: Sandy is happy

The premises are that

\[ p \implies q \quad \text{and} \quad \neg r \implies \neg q \]

are true. Since $\neg r \implies \neg q$ is true, then the contrapositive $q \implies r$ is also true. Since

\[ p \implies q \quad \text{and} \quad q \implies r \]

are true, by the Law of Syllogism it is true that

\[ p \implies r \]

That is, if Dan is happy, then Sandy is happy.

NOW WORK PROBLEM 3 USING A DIRECT PROOF.
EXAMPLE 4  Using a Direct Proof to Prove a Statement

Suppose each the following statements is true:

If I enjoy studying, then I will study

and

I will do my homework or I will not study

and

I will not do my homework.

Show that

I do not enjoy studying

is true.

SOLUTION  Let \( p, q, r \) denote the three propositions

\[ p: \text{ I enjoy studying.} \]
\[ q: \text{ I will study.} \]
\[ r: \text{ I will do my homework.} \]

Then

\[ p \implies q \quad r \lor \sim q \quad \sim r \]

are true. We want to prove that \( \sim p \) is true.

Since \( \sim r \) is true, then \( r \) is false. Either \( r \) or \( \sim q \) is true. Since \( r \) is false then \( \sim q \) is true.

Since \( p \implies q \) is true, the contrapositive

\[ \sim q \implies \sim p \]

is true. Since \( \sim q \) is true, then \( \sim p \) must be true. That is, “I do not enjoy studying” is true. ■

2  Use an Indirect Proof

In any valid proof one shows that an implication \( p \implies q \) is a tautology. To do this by the method of indirect proof, we tentatively suppose that \( q \) is false (equivalently, that \( \sim q \) is true) and show that we are thereby led to a contradiction—that is, a logically impossible situation. This contradiction can be resolved only by abandoning the supposition that the conclusion is false. Therefore, it must be true.

An indirect proof is also called a proof by contradiction.

EXAMPLE 5  Using an Indirect Proof to Prove a Statement

Prove the result of Example 1 using an indirect proof.

SOLUTION  To prove that the conclusion \( p \) is true, we suppose instead that \( p \) is false, hoping to be able to show that this leads to a contradiction.
We assume that
\[ \sim p \quad p \lor q \quad \sim q \]
all are true. Now \( p \lor q \) is true if either \( p \) is true or \( q \) is true. But both \( p \) and \( q \) are false. This is impossible. We have reached a contradiction, which means that \( p \) must be true.

**NOW WORK PROBLEM 3 USING AN INDIRECT PROOF.**

**EXAMPLE 6 Using an Indirect Proof to Prove a Statement**

Suppose each of the following statements is true:
- If I am lazy, I do not study
- and
- I study or I enjoy myself
- and
- I do not enjoy myself.

Prove that
- I am not lazy

is true.

**SOLUTION** Let \( p, q, r \) be the statements

\[ p: \text{I am lazy.} \]
\[ q: \text{I study.} \]
\[ r: \text{I enjoy myself.} \]

Then
\[ p \implies \sim q \quad q \lor r \quad \sim r \]
are true.

We want to show that \( \sim p \), the conclusion, is true. In an indirect proof we assume that \( \sim p \) is false—that is, that \( p \) is true. If \( p \) and \( p \implies \sim q \) are true, then \( \sim q \) is true (by the Law of Detachment). That is, \( q \) is false. Since \( \sim r \) is true, then \( r \) is also false. But \( q \lor r \) is true, which is impossible if \( q \) and \( r \) are both false. This contradiction tells us that we have incorrectly assumed that \( \sim p \) is false. So, \( \sim p \) is true. That is, “I am not lazy” is true.

**EXERCISE 11.4 Answers Begin on Page AN–62.**

**Concepts and Vocabulary**

1. There are two types of proof: _________ and _________.

**Skill Building**

Prove the statements in Problems 3–6 first by using a direct proof and then by using an indirect proof.

3. When it rains, John does not go to school. John is going to school. Show that it is not raining.

4. If I do not go to work, I will go fishing. I will not go fishing. Show that I will go to work.
5. If Smith is elected president, Kuntz will be elected secretary. If Kuntz is elected secretary, then Brown will not be elected treasurer. Smith is elected president. Show that Brown is not elected treasurer.

6. Either Laura is a good girl or Rob is a good boy. If Danny cries, then Laura is not a good girl. Rob is not a good boy. Does Danny cry?

In Problems 7–10, determine whether the arguments are valid and justify your reasoning.

7. Hypotheses: When students study, they receive good grades. These students do not study.
   Conclusion: These students do not receive good grades.

8. Hypotheses: If Danny is affluent, he is either a snob or a hypocrite, but not both. Danny is a snob and is not a hypocrite.
   Conclusion: Danny is not affluent.

9. Hypotheses: If Tami studies, she will not fail this course. If she does not play with her dolls too often, she will study.
   Conclusion: She played with her dolls too often.

10. Hypotheses: If John invests his money wisely, he will be rich. If John is rich, he will buy an expensive car. John bought an expensive car.
    Conclusion: John invested his money wisely.

11.5 Logic Circuits

OBJECTIVES
1 Construct a logic circuit (p. 661)
2 Redesign a logic circuit (p. 662)

1 Construct a Logic Circuit

A logic circuit is a type of electrical circuit widely used in computers and other electronic devices (such as calculators, cell phones, and mp3 players). The simplest logic circuits, called gates, have the following properties.

1. Current flows to the circuit through one or two connectors called input lines and from the circuit through a connector called the output line.
2. The current in any of the input or output lines may have either of two possible voltage levels. The higher level is denoted by 1 and the lower level by 0. (Level 1 is sometimes referred to as On or True; Level 0 may be referred to as Off or False.)
3. The voltage level of the output line depends on the voltage level(s) of the input line(s), according to rules of logic similar to the principles discussed in Section 11.2.

The three most basic types of gates are the inverter (or NOT gate), the AND gate, and the OR gate.

The standard symbols for inverters, AND gates, and OR gates appear in Figure 1. In each case, the input lines appear at the left of the diagram and the output line at the right. The label on each line denotes the voltage level or truth value of that line; the symbols \( \sim p, pq \) (or \( p \land q \)), and \( p \oplus q \) (or \( p \lor q \)) are defined by the output tables in Tables 21, 22, and 23. Note that \( \sim p = 1 - p \) and

\[
p \oplus q = \begin{cases} p & \text{if } p \geq q \\ q & \text{if } q > p \end{cases} = \max \{p, q\}
\]
Computer designers and electrical engineers frequently use one of the symbols $\sim p$ or $p'$ in place of $\sim p$.

More complex logic circuits are constructed by connecting two or more gates.

**EXAMPLE 1 Constructing a Logic Circuit**

Construct a logic circuit whose inputs are $p$, $q$, $r$ and whose output is $\overline{(pq) \oplus \sim r}$. When is this output equal to 1?

**SOLUTION**

We need to connect an AND gate, an inverter, and an OR gate. Figure 2 shows how.

Table 24 shows the output table for $pq \oplus \sim r$. This table shows that the output is 1 when $p = q = 1$ (regardless of whether $r$ is 0 or 1) and when $r = 0$ (regardless of the values of $p$ and $q$).

**NOW WORK PROBLEM 1.**

**Redesign a Logic Circuit**

Since the output tables for circuits have the same form as the truth tables for propositions (with 1 and 0 corresponding to T and F, $pq$ corresponding to $p \land q$, and $p \oplus q$ corresponding to $p \lor q$), the principles of logic formulated in Section 11.2 apply. We can translate these principles as follows:
These properties are often useful in simplifying the design of circuits. If two circuits perform the same function and one has fewer gates and connectors than the other, the simpler circuit is preferred because it is less expensive to manufacture or purchase, and also because it is less likely to fail due to overheating or physical stress. Very often it will also be more efficient in terms of power consumption and speed of operation.

**EXAMPLE 2** Redesigning a Logic Circuit

Redesign the circuit of Example 1 so that \( r = p \); simplify the circuit.

**SOLUTION**

Modify the circuit given in Figure 2 as the one shown in Figure 3. The solid dot is used in circuit diagrams to show where a connector branches.

Using the distributive properties, we can simplify:

\[
pq \oplus \sim p = (p \oplus \sim p)(q \oplus \sim p) = 1 \cdot (q \oplus \sim p) = q \oplus \sim p
\]

The circuit in Figure 3 can be replaced with the one in Figure 4.
EXAMPLE 3  Redesigning a Logic Circuit

Find the output of the circuit in Figure 5 and simplify its design.

FIGURE 5

\[ p \quad q \]

\[ (A) \quad (B) \quad (C) \]

SOLUTION  The output is \( B \oplus C \), where

\[ A = (\neg p)q \quad B = p(\neg q) \quad C = p \oplus A \]

Then, substituting \( (\neg p)q \) for \( A \) in \( C = p \oplus A \) and simplifying, we get

\[ C = p \oplus [(\neg p)q] = (p \oplus \neg p)(p \oplus q) = 1 \cdot (p \oplus q) = p \oplus q \]

The output is

\[ B \oplus C = [p(\neg q)] \oplus (p \oplus q) \]

Using an associative property followed by an absorption property, this expression can be simplified to

\[ B \oplus C = [p(\neg q)] \oplus (p \oplus q) = \{[p(\neg q)] \oplus p\} \oplus q = p \oplus q \]

The circuit can be replaced by a single OR gate.

NAND, NOR, and XOR

The standard symbols for NAND, NOR, and XOR gates, and their outputs, are shown in Figure 6. (The names are abbreviations for NOT-AND, NOT-OR, and Exclusive-OR, respectively.) Table 25 is the output table for XOR.

FIGURE 6

\[ p \quad q \quad \neg (pq) \]

\[ p \quad q \quad \neg (p \oplus q) \]

\[ p \quad q \quad p \oplus q \]

TABLE 25

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \oplus q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
EXERCISE 11.5 Answers Begin on Page AN–63.

Skill Building

In Problems 1–4, determine when the output of each circuit is 1; use a truth table if necessary.

1. 2.

3. 4.

In Problems 5–8, construct a circuit corresponding to each expression.

5. 6. 7. 8.

9. Design simpler circuits having the same outputs as the circuits in Problems 1, 3, 5, and 7.

Applications and Extensions

11. Design a logic circuit that can be turned on or off from either of two switches. (That is, if the inputs are $p$ and $q$, the output can be changed from 1 to 0 or from 0 to 1 by changing the level of either $p$ or $q$, but not both.)

12. Design a circuit, consisting of at least two gates, whose output is always 1.

13. Design a circuit, consisting of at least two gates, whose output is always 0.

14. Using an OR gate, an AND gate, and an inverter, design a circuit that will work as an XOR gate. [Hint: What combination of $\lor$, $\land$, and $\neg$ has the same truth table as $\oplus$?]

15. Show that a circuit whose output is

$$pq \oplus pr \oplus q(\neg r)$$

can be replaced by one whose output is

$$pr \oplus q(\neg r)$$

[HINT: $pq = pq(r \oplus \neg r) = pqr \oplus pq(\neg r)$]

16. Show that

(a) $(p \oplus q)(p \oplus r)(q \oplus \neg r) = (p \oplus r)(q \oplus \neg r)$

(b) $(p \oplus r)(q \oplus \neg r) = p(\neg r) \oplus qr$

CHAPTER 11 REVIEW OBJECTIVES

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<th>Section</th>
<th>Examples</th>
<th>You should be able to</th>
<th>Review Exercises</th>
</tr>
</thead>
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<td>1–8</td>
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<td></td>
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<td>Use truth tables to show statements are equivalent (p. 642)</td>
<td>23, 24, 41, 42</td>
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<tr>
<td></td>
<td>6–9</td>
<td>Work with properties of propositions (p. 643)</td>
<td>23, 24</td>
</tr>
</tbody>
</table>
In Problems 1–8, determine which sentences are propositions.

1. A cow is a mammal.
2. The third planet is Mars.
3. Did you see the game?
4. Is Jack going to the party?
5. Fifteen people were injured in the fire.
6. The music ended suddenly.
7. Get the car.
8. Close the door.

In Problems 9–16, use the statements below and the given logic statement to write compound sentences in English.

\[ p: \text{I go to the math learning center.} \]
\[ q: \text{I complete my math homework.} \]

9. \( p \lor q \)
10. \( p \land q \)
11. \( p \Rightarrow q \)
12. \( p \Leftrightarrow q \)
13. \( \sim p \Rightarrow \sim q \)
14. \( p \lor \sim q \)
15. \( \sim p \land \sim q \)
16. \( q \Rightarrow p \)

In Problems 17–20, negate each proposition.

17. Some people are rich.
18. All professional basketball players are tall.
19. Danny is not tall and Mary is short.
20. Neither Mike nor Katy is big.

In Problems 21–24, circle each correct answer; some questions may have more than one correct answer.

21. Which of the following negate the statement below?
\[ p: \text{All people are rich.} \]
(a) Some people are rich.
(b) Some people are poor.
(c) Some people are not rich.
(d) No person is rich.

22. Which of the following negate the statement below?
\[ p: \text{It is either hot or humid.} \]
(a) It is neither hot nor humid.
(b) Either it is not hot or it is not humid.
(c) It is not hot and it is not humid.
(d) It is hot but not humid.

23. Which of the following statements are logically equivalent to the statement below?
\[ \sim (p \lor q) \land r \]
(a) \( p \Rightarrow q \land r \)
(b) \( \sim p \lor (q \land r) \)
(c) \( (\sim p \Rightarrow q) \land r \)
(d) \( (\sim p \lor r) \land (q \land r) \)

24. Which of the following statements are logically equivalent to the statement below?
\[ p \land \sim q \]
(a) \( q \land p \)
(b) \( p \lor q \)
(c) \( \sim p \lor q \)
(d) \( q \Rightarrow \sim p \)
In Problems 25–28, construct a truth table for each compound proposition.

25. \((p \land q) \lor \sim p\)  
26. \(\sim p \lor (p \lor \sim q)\)  
27. \((p \lor q) \land p\)  
28. \(\sim p \Rightarrow (p \lor q)\)

In Problems 29–32, let \(p\) stand for “I pass the course” and let \(q\) stand for “I do homework regularly.” Put each statement into symbolic form.

29. I pass the course if I do homework regularly.
30. Passing this course is a sufficient condition for me to do homework regularly.
31. I pass this course if and only if I do homework regularly.
32. If I do not do homework regularly, then I do not pass the course.

In Problems 33–40, (a) Write each compound statement symbolically.  
(b) Write the converse of the statement both symbolically and in words.  
(c) Write the contrapositive of the statement both symbolically and in words.  
(d) Write the inverse of the statement both symbolically and in words.

33. If the temperature outside is below 30°, then I wear gloves.  
34. If Peter works 20 hours this week, he earns $300.  
35. Stu works on the project if Julie helps.  
36. Ali needs to study hard to pass the test.  
37. Kurt goes to the club if Jessica comes to town.  
38. Peter works 20 hours only if he gets paid $300.  
39. Brian comes to the gym if Mike works out.  
40. I go to the game only if Kathleen gets tickets.

41. Show that the statements, If Patrick goes to practice then he starts the game and Either Patrick does not go to practice or he starts the game, are equivalent by writing each symbolically and by constructing a truth table.

42. Show that the statements, If Ted is accepted at Loyola, then he goes there, and Either Ted is not accepted at Loyola or he goes there, are equivalent by writing each symbolically and by constructing a truth table.

43. Show that the following are logically equivalent:
\[
\sim p \lor q \quad p \Rightarrow q
\]
That is, show \((\sim p \lor q) \iff [p \Rightarrow q]\).

44. Show that the two statements below are logically equivalent:
\[
(p \Rightarrow q) \land (\sim q \lor p) \quad p \Leftrightarrow q
\]
That is, show \([(p \Rightarrow q) \land (\sim q \lor p)] \iff (p \Leftrightarrow q)\).

45. Prove the following first by using a direct proof and then by using an indirect proof: If I do not paint the house, I will go bowling. I will not go bowling. Show that I will paint the house.

46. Show that the following argument is valid first by using a direct proof and then by using an indirect proof: If I do not go to Florida, I refinish my floor. I do not refinish my floor. Show that I go to Florida.

47. Show that the following argument is valid by using either a direct or an indirect proof: If John is in town, then Mark gets tickets. If we don’t go to the game, Mark didn’t get tickets. John is in town. Therefore, we went to the game.

48. Show that the following argument is valid by using either a direct or an indirect proof: If Ed doesn’t call, then he is not depressed. If Ed is overworked, then he is depressed. Ed is overworked. Therefore, Ed calls.

49. Show that the following argument is valid by using either a direct or an indirect proof: I pay a finance charge if my payment is late. Either Colleen sends the mail or my payment is late. Colleen did not send the mail. Conclude that I pay a finance charge.

50. Show that the following argument is valid by using either a direct or an indirect proof: Either the team loses or they go to the tournament. If the team loses, the coach is fired. If the coach is fired, the athletic director quits. The team did not go to the tournament. Conclude that the athletic director quits.

51. Using reasons from logic with the following premises, give a valid argument to answer the question “Is Laura a good girl?”
(a) Rob is a bad boy or Danny is crying.  
(b) If Laura is a good girl, then Rob is not a bad boy.  
(c) Danny is not crying.

52. Using logical reasoning with the following premises, give a valid argument to answer the question “Is Marissa crying?”
(a) Kevin got his license or Michael is driving.  
(b) If Marissa is crying, Kevin did not get his license.  
(c) Michael is not driving.

53. Design a nor gate using only inverter and and gates. That is, if \(p\) and \(q\) are inputs, then \(\sim (p \oplus q)\) is the output.
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54. Design a NAND gate using only inverter and OR gates. That is, if \( p \) and \( q \) are inputs, then \( \sim(pq) \) is the output.

55. Construct an XOR gate by connecting two NOR gates and an AND gate.

56. Design an inverter using only a NAND gate.

57. Show that the output of an XOR gate is equal to

\[
(p \oplus q)[\sim(pq)]
\]

Use this fact to construct an XOR gate by connecting an OR gate, an AND gate, and a NAND gate.

58. Simplify the following design, making it a simpler logic circuit. [Hint: Separate the circuit into two parts.]

\[
p \quad q \quad r \quad ?
\]

Chapter 11 Project

LOGIC PUZZLES

Logic puzzles are giving crossword puzzles a run for their money as the most popular pastime of many people. Now that you’ve finished this chapter on logic, you have skills that will serve you well in solving logic puzzles. Try the three puzzles below and written by Kevin Stone.* Write down your time to reach the solution and compare it with the times of others in your class.

1. At a recent downhill mountain bike race, four entrants entered the challenging slalom event. Alan came first. The entrant wearing number 2 wore red, whereas John didn’t wear yellow. The loser wore blue, and Steve wore number 1. Key beat Steve, and the person who came second wore number 3. The entrant in yellow beat the entrant in green. Only one of the entrants wore the same number as his final position. Can you determine who finished where as well as the number and color they wore?

2. At a recent Pets Anonymous reunion, the attendees were discussing which pets they had recently owned. James used to have a dog. The person who used to own a mouse now owns a cat, but the person who used to have a cat does not have a mouse. Kevin has now or used to have a dog, I can’t remember which. Becky has never owned a mouse. Only one person now owns the pet they previously had. Rebecca kept quiet throughout the meeting, and nobody mentioned the hamster. Can you determine who owns which pet and what they used to own?

3. Five patients, all potential blood donors, are waiting in the doctor’s surgery and are sitting on the bench from left to right. Can you determine the position of each patient along with their blood group, age, height, and weight? Their ages are 5, 9, 30, 46, and 60. Their heights are 40, 48, 60, 65, and 74. Their weights are 40, 75, 96, 125, and 165.

The person on the far right is 37 years older than Jason and is 60 inches tall.

Jason weighs 56 pounds more than his height.

Alan weighs 75 pounds and is 74 inches tall.

John is type AB and weighs 56 pounds less than Jason.

The person in the center is 9 years old, is blood type AO, and weighs 96 pounds.

Adam, who is the first, is 65 inches tall and weighs 100 pounds more than his height.

The person who is blood type O is 25 years older than the person to his left.

Kevin is 60 years old.

The person who is blood type A is 55 years younger than Kevin and is not next to the person who is type AO.

The person who is next to the 9-year-old, but not next to the person who is 65 inches tall, is blood type B and weighs 125 pounds.

*Puzzles © Kevin Stone.
A.1 Real Numbers

PREPARING FOR THIS BOOK Before getting started, read “To the Student” on the inside front cover of this book.

OBJECTIVES

1. Work with Sets (p. A–1)
2. Classify Numbers (p. A–2)
3. Evaluate Numerical Expressions (p. A–6)

1. Work with Sets

A set is a well-defined collection of distinct objects. The objects of a set are called its elements. By well-defined, we mean that there is a rule that enables us to determine whether a given object is an element of the set. If a set has no elements, it is called the empty set, or null set, and is denoted by the symbol $\emptyset$.

For example, the set of digits consists of the collection of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. If we use the symbol $D$ to denote the set of digits, then we can write

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

In this notation, the braces $\{\}$ are used to enclose the objects, or elements, in the set. This method of denoting a set is called the roster method. A second way to denote a set is to use set-builder notation, where the set $D$ of digits is written as

$$D = \{x \mid x \text{ is a digit}\}$$

Read as “$D$ is the set of all $x$ such that $x$ is a digit.”

EXAMPLE 1 Using Set-Builder Notation and the Roster Method

(a) $E = \{x \mid x \text{ is an even digit}\} = \{0, 2, 4, 6, 8\}$
(b) $O = \{x \mid x \text{ is an odd digit}\} = \{1, 3, 5, 7, 9\}$
Because the elements of a set are distinct, we never repeat elements. For example, we would never write \( \{1, 2, 3, 2\} \); the correct listing is \( \{1, 2, 3\} \). Because a set is a collection, the order in which the elements are listed is immaterial. \( \{1, 2, 3\}, \{3, 2, 1\}, \{2, 1, 3\} \), and so on, all represent the same set.

If every element of a set \( A \) is also an element of a set \( B \), then we say that \( A \) is a subset of \( B \) and write \( A \subseteq B \). If two sets \( A \) and \( B \) have the same elements, then we say that \( A \) equals \( B \) and write \( A = B \).

For example, \( \{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\} \) and \( \{1, 2, 3\} = \{2, 3, 1\} \).

### 2 Classify Numbers

It is helpful to classify the various kinds of numbers that we deal with as sets. The counting numbers, or natural numbers, are the numbers in the set \( \{1, 2, 3, 4, \ldots\} \). (The three dots, called an ellipsis, indicate that the pattern continues indefinitely.) As their name implies, these numbers are often used to count things. For example, there are 26 letters in our alphabet; there are 100 cents in a dollar. The whole numbers are the numbers in the set \( \{0, 1, 2, 3, \ldots\} \), that is, the counting numbers together with 0.

**Definition**

The integers are the numbers in the set \( \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \).

These numbers are useful in many situations. For example, if your checking account has $10 in it and you write a check for $15, you can represent the current balance as $-5. Notice that the set of counting numbers is a subset of the set of whole numbers. Each time we expand a number system, such as from the whole numbers to the integers, we do so in order to be able to handle new, and usually more complicated, problems. The integers allow us to solve problems requiring both positive and negative counting numbers, such as profit/loss, height above/below sea level, temperature above/below 0°F, and so on.

But integers alone are not sufficient for all problems. For example, they do not answer the question “What part of a dollar is 38 cents?” To answer such a question, we enlarge our number system to include rational numbers. For example, \( \frac{38}{100} \) answers the question “What part of a dollar is 38 cents?”

**Definition**

A rational number is a number that can be expressed as a quotient \( \frac{a}{b} \) of two integers. The integer \( a \) is called the numerator, and the integer \( b \), which cannot be 0, is called the denominator. The rational numbers are the numbers in the set \( \{x \mid x = \frac{a}{b}, \text{ where } a, b \text{ are integers and } b \neq 0\} \).

Examples of rational numbers are \( \frac{3}{4}, \frac{5}{2}, \frac{0}{4}, -\frac{2}{3}, \frac{100}{3} \). Since \( \frac{a}{1} = a \) for any integer \( a \), it follows that the set of integers is a subset of the set of rational numbers.

Rational numbers may be represented as decimals. For example, the rational numbers \( \frac{3}{4}, \frac{5}{2}, -\frac{2}{3}, \frac{7}{66} \) may be represented as decimals by merely carrying out the indicated division:
A.1 Real Numbers

\[
\frac{3}{4} = 0.75 \quad \frac{5}{2} = 2.5 \quad -\frac{2}{3} = -0.66\ldots = -0.\overline{6} \quad \frac{7}{66} = 0.1060606\ldots = 0.1\overline{06}
\]

Notice that the decimal representations of \(\frac{3}{4}\) and \(\frac{5}{2}\) terminate, or end. The decimal representations of \(-\frac{2}{3}\) and \(\frac{7}{66}\) do not terminate, but they do exhibit a pattern of repetition. For \(-\frac{2}{3}\), the 6 repeats indefinitely, as indicated by the bar over the 6; for \(\frac{7}{66}\), the block 06 repeats indefinitely, as indicated by the bar over the 06. It can be shown that every rational number may be represented by a decimal that either terminates or is nonterminating with a repeating block of digits, and vice versa.

On the other hand, some decimals do not fit into either of these categories. Such decimals represent **irrational numbers**. Every irrational number may be represented by a decimal that neither repeats nor terminates. In other words, irrational numbers cannot be written in the form \(\frac{a}{b}\), where \(a, b\) are integers and \(b \neq 0\).

Irrational numbers occur naturally. For example, consider the isosceles right triangle whose legs are each of length 1. See Figure 1. The length of the hypotenuse is \(\sqrt{2}\), an irrational number.

Also, the number that equals the ratio of the circumference \(C\) to the diameter \(d\) of any circle, denoted by the symbol \(\pi\) (the Greek letter pi), is an irrational number. See Figure 2.

**Definition**

Together, the rational numbers and the irrational numbers form the set of **real numbers**.

Figure 3 shows the relationship of various types of numbers.
EXAMPLE 2  Classifying the Numbers in a Set

List the numbers in the set
\[ \left\{ -3, \frac{4}{3}, 0.12, \sqrt{2}, \pi, 10, 2.151515 \ldots = 2.\overline{15} \right\} \]
that are
(a) Natural numbers  (b) Integers  (c) Rational numbers
(d) Irrational numbers  (e) Real numbers

SOLUTION
(a) 10 is the only natural number.
(b) $-3$ and 10 are integers.
(c) $-3, 10, \frac{4}{3}, 0.12, \text{and } 2.151515 \ldots = 2.\overline{15}$ are rational numbers.
(d) $\sqrt{2}$ and $\pi$ are irrational numbers.
(e) All the numbers listed are real numbers.

NOW WORK PROBLEM 11.

Approximations

Every decimal may be represented by a real number (either rational or irrational), and
every real number may be represented by a decimal.

In practice, the decimal representation of an irrational number is given as an
approximation. For example, using the symbol $\approx$ (read as “approximately equal to”),
we can write
\[ \sqrt{2} \approx 1.4142 \quad \pi \approx 3.1416 \]

In approximating decimals, we either round off or truncate to a given number of
decimal places.* The number of places establishes the location of the final digit in the
decimal approximation.

**Truncation:** Drop all the digits that follow the specified final digit in the decimal.
**Rounding:** Identify the specified final digit in the decimal. If the next digit is 5
or more, add 1 to the final digit; if the next digit is 4 or less, leave the final digit
as it is. Then truncate following the final digit.

EXAMPLE 3  Approximating a Decimal to Two Places

Approximate 20.98752 to two decimal places by
(a) Truncating
(b) Rounding

SOLUTION
For 20.98752, the final digit is 8, since it is two decimal places from the decimal point.
(a) To truncate, we remove all digits following the final digit 8. The truncation of
20.98752 to two decimal places is 20.98.
(b) The digit following the final digit 8 is the digit 7. Since 7 is 5 or more, we add 1 to
the final digit 8 and truncate. The rounded form of 20.98752 to two decimal places
is 20.99.

* Sometimes we say “correct to a given number of decimal places” instead of “truncate.”


**EXAMPLE 4** Approximating a Decimal to Two and Four Places

<table>
<thead>
<tr>
<th>Number</th>
<th>Rounded to Two Decimal Places</th>
<th>Rounded to Four Decimal Places</th>
<th>Truncated to Two Decimal Places</th>
<th>Truncated to Four Decimal Places</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 3.14159</td>
<td>3.14</td>
<td>3.1416</td>
<td>3.14</td>
<td>3.1415</td>
</tr>
<tr>
<td>(b) 0.056128</td>
<td>0.06</td>
<td>0.0561</td>
<td>0.05</td>
<td>0.0561</td>
</tr>
<tr>
<td>(c) 893.46125</td>
<td>893.46</td>
<td>893.4613</td>
<td>893.46</td>
<td>893.4612</td>
</tr>
</tbody>
</table>

**NOW WORK PROBLEM 15.**

**Calculators**

Calculators are finite machines. As a result, they are incapable of displaying decimals that contain a large number of digits. For example, some calculators are capable of displaying only eight digits. When a number requires more than eight digits, the calculator either truncates or rounds. To see how your calculator handles decimals, divide 2 by 3. How many digits do you see? Is the last digit a 6 or a 7? If it is a 6, your calculator truncates; if it is a 7, your calculator rounds.

There are different kinds of calculators. An **arithmetic** calculator can only add, subtract, multiply, and divide numbers; therefore, this type is not adequate for this course. **Scientific** calculators have all the capabilities of arithmetic calculators and also contain **function keys** labeled ln, log, sin, cos, tan, $x^2$, inv, and so on. **Graphing** calculators have all the capabilities of scientific calculators and contain a screen on which graphs can be displayed.

For those who have access to a graphing calculator, we have included comments, examples, and exercises marked with a , indicating that a graphing calculator is required. These comments, examples, and exercises may be omitted without loss of continuity, if so desired. We have also included Appendix C that explains some of the capabilities of a graphing calculator.

**Operations**

In algebra, we use letters such as $x$, $y$, $a$, $b$, and $c$ to represent numbers. The symbols used in algebra for the operations of addition, subtraction, multiplication, and division are $\div$, $\times$, $\cdot$, and $/$. The words used to describe the results of these operations are **sum**, **difference**, **product**, and **quotient**. Table 1 summarizes these ideas.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Symbol</th>
<th>Words</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>$a + b$</td>
<td>Sum: $a$ plus $b$</td>
</tr>
<tr>
<td>Subtraction</td>
<td>$a - b$</td>
<td>Difference: $a$ minus $b$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$a \cdot b$, $(a) \cdot b$, $a \cdot (b)$, $(a) \cdot (b)$</td>
<td>Product: $a$ times $b$</td>
</tr>
<tr>
<td>Division</td>
<td>$a/b$ or $\frac{a}{b}$</td>
<td>Quotient: $a$ divided by $b$</td>
</tr>
</tbody>
</table>

In algebra, we generally avoid using the multiplication sign $\times$ and the division sign $\div$ so familiar in arithmetic. Notice also that when two expressions are placed next to each other without an operation symbol, as in $ab$, or in parentheses, as in $(a)(b)$, it is understood that the expressions, called **factors**, are to be multiplied.
We also prefer not to use mixed numbers in algebra. When mixed numbers are used, addition is understood; for example, \(2 \frac{3}{4}\) means \(2 + \frac{3}{4}\). In algebra, use of a mixed number may be confusing because the absence of an operation symbol between two terms is generally taken to mean multiplication. The expression \(2 \frac{3}{4}\) is therefore written instead as \(2.75\) or as \(\frac{11}{4}\).

The symbol \(=\), called an equal sign and read as “equals” or “is,” is used to express the idea that the number or expression on the left of the equal sign is equivalent to the number or expression on the right.

**EXAMPLE 5** Writing Statements Using Symbols

(a) The sum of 2 and 7 equals 9. In symbols, this statement is written as \(2 + 7 = 9\).

(b) The product of 3 and 5 is 15. In symbols, this statement is written as \(3 \cdot 5 = 15\).

**NOW WORK PROBLEM 27.**

**3 Evaluate Numerical Expressions**

Consider the expression \(2 + 3 \cdot 6\). It is not clear whether we should add 2 and 3 to get 5, and then multiply by 6 to get 30; or first multiply 3 and 6 to get 18, and then add 2 to get 20. To avoid this ambiguity, we have the following agreement.

We agree that whenever the two operations of addition and multiplication separate three numbers, the multiplication operation always will be performed first, followed by the addition operation.

For \(2 + 3 \cdot 6\), we have

\[2 + 3 \cdot 6 = 2 + 18 = 20\]

**EXAMPLE 6** Finding the Value of an Expression

Evaluate each expression.

(a) \(3 + 4 \cdot 5\)  
(b) \(8 \cdot 2 + 1\)  
(c) \(2 + 2 \cdot 2\)

**SOLUTION**

(a) \(3 + 4 \cdot 5 = 3 + 20 = 23\)  
(b) \(8 \cdot 2 + 1 = 16 + 1 = 17\)  
(c) \(2 + 2 \cdot 2 = 2 + 4 = 6\)

**NOW WORK PROBLEM 39.**

To first add 3 and 4 and then multiply the result by 5, we use parentheses and write \((3 + 4) \cdot 5\). Whenever parentheses appear in an expression, it means “perform the operations within the parentheses first!”
EXAMPLE 7  Finding the Value of an Expression

(a) \((5 + 3) \cdot 4 = 8 \cdot 4 = 32\)  \hspace{1cm} (b) \((4 + 5) \cdot (8 - 2) = 9 \cdot 6 = 54\)

When we divide two expressions, as in

\[
\frac{2 + 3}{4 + 8}
\]

it is understood that the division bar acts like parentheses; that is,

\[
\frac{2 + 3}{4 + 8} = \frac{2 + 3}{4 + 8}
\]

The following list gives the rules for the order of operations.

**Rules for the Order of Operations**

1. Begin with the innermost parentheses and work outward. Remember that in dividing two expressions the numerator and denominator are treated as if they were enclosed in parentheses.
2. Perform multiplications and divisions, working from left to right.
3. Perform additions and subtractions, working from left to right.

EXAMPLE 8  Finding the Value of an Expression

Evaluate each expression.

(a) \(8 \cdot 2 + 3\)  \hspace{1cm} (b) \(5 \cdot (3 + 4) + 2\)

(c) \(\frac{2 + 5}{2 + 4 \cdot 7}\)  \hspace{1cm} (d) \(2 + [4 + 2 \cdot (10 + 6)]\)

**SOLUTION**

(a) \(8 \cdot 2 + 3 = 16 + 3 = 19\)

Multiply first.

(b) \(5 \cdot (3 + 4) + 2 = 5 \cdot 7 + 2 = 35 + 2 = 37\)

Parentheses first  \hspace{1cm} Multiply before adding.

(c) \(\frac{2 + 5}{2 + 4 \cdot 7} = \frac{2 + 5}{2 + 28} = \frac{7}{30}\)

(d) \(2 + [4 + 2 \cdot (10 + 6)] = 2 + [4 + 2 \cdot 16]\)

\[= 2 + [4 + 32] = 2 + [36] = 38\]

Be careful if you use a calculator. For Example 8(c), you need to use parentheses. See Figure 4.* If you don’t, the calculator will compute the expression

\[\frac{2 + \frac{5}{2} + 4 \cdot 7}{2 + 2.5 + 28} = 32.5\]

giving a wrong answer.

**NOW WORK PROBLEMS 45 AND 53.**

* Notice that we converted the decimal to its fraction form. Consult your manual to see how your calculator does this.
4  Work with Properties of Real Numbers

We have used the equal sign to mean that one expression is equivalent to another. Four important properties of equality are listed next. In this list, \(a\), \(b\), and \(c\) represent real numbers.

1. The **reflexive property** states that a number always equals itself; that is, \(a = a\).
2. The **symmetric property** states that if \(a = b\) then \(b = a\).
3. The **transitive property** states that if \(a = b\) and \(b = c\) then \(a = c\).
4. The **principle of substitution** states that if \(a = b\) then we may substitute \(b\) for \(a\) in any expression containing \(a\).

Now, let’s review some other properties of real numbers.

**EXAMPLE 9**  Commutative Properties

\[
\begin{align*}
(\text{a}) & \quad 3 + 5 = 8 \\
& \quad 5 + 3 = 8 \\
& \quad 3 + 5 = 5 + 3
\end{align*}
\]

\[
\begin{align*}
(\text{b}) & \quad 2 \cdot 3 = 6 \\
& \quad 3 \cdot 2 = 6 \\
& \quad 2 \cdot 3 = 3 \cdot 2
\end{align*}
\]

This example illustrates the **commutative property** of real numbers, which states that the order in which addition or multiplication takes place will not affect the final result.

**Commutative Properties**

\[
\begin{align*}
\text{(1a)} & \quad a + b = b + a \\
\text{(1b)} & \quad a \cdot b = b \cdot a
\end{align*}
\]

Here, and in the properties listed on pages A-9 to A-12, \(a\), \(b\), and \(c\) represent real numbers.

**EXAMPLE 10**  Associative Properties

\[
\begin{align*}
(\text{a}) & \quad 2 + (3 + 4) = 2 + 7 = 9 \\
& \quad (2 + 3) + 4 = 5 + 4 = 9 \\
& \quad 2 + (3 + 4) = (2 + 3) + 4
\end{align*}
\]

\[
\begin{align*}
(\text{b}) & \quad 2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24 \\
& \quad (2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24 \\
& \quad 2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4
\end{align*}
\]

The way we add or multiply three real numbers will not affect the final result. Expressions such as \(2 + 3 + 4\) and \(3 \cdot 4 \cdot 5\) present no ambiguity, even though addition and multiplication are performed on one pair of numbers at a time. This property is called the **associative property**.
A.1 Real Numbers

### Associative Properties

\[
\begin{align*}
 a + (b + c) &= (a + b) + c = a + b + c & \text{(2a)} \\
 a \cdot (b \cdot c) &= (a \cdot b) \cdot c = a \cdot b \cdot c & \text{(2b)} 
\end{align*}
\]

The next property is perhaps the most important.

### Distributive Property

\[
\begin{align*}
 a \cdot (b + c) &= a \cdot b + a \cdot c & \text{(3a)} \\
 (a + b) \cdot c &= a \cdot c + b \cdot c & \text{(3b)} 
\end{align*}
\]

The **Distributive Property** may be used in two different ways.

**EXAMPLE 11** Distributive Property

\begin{itemize}
  \item \textbf{(a)} \(2 \cdot (x + 3) = 2 \cdot x + 2 \cdot 3 = 2x + 6\) \quad \text{Use to remove parentheses.}
  \item \textbf{(b)} \(3x + 5x = (3 + 5)x = 8x\) \quad \text{Use to combine two expressions.}
  \item \textbf{(c)} \((x + 2)(x + 3) = x(x + 3) + 2(x + 3) = (x^2 + 3x) + (2x + 6)
  \quad = x^2 + (3x + 2x) + 6 = x^2 + 5x + 6\)
\end{itemize}

**NOW WORK PROBLEM 73.**

The number 0, called the **additive identity**, and the number 1, called the **multiplicative identity**, have unique properties, called the **identity properties**.

**EXAMPLE 12** Identity Properties

\begin{itemize}
  \item \textbf{(a)} \(4 + 0 = 0 + 4 = 4\)
  \item \textbf{(b)} \(3 \cdot 1 = 1 \cdot 3 = 3\)
\end{itemize}

**Identity Properties**

\[
\begin{align*}
 0 + a &= a + 0 = a & \text{(4a)} \\
 a \cdot 1 &= 1 \cdot a = a & \text{(4b)} 
\end{align*}
\]
For each real number $a$, there is a real number $-a$, called the **additive inverse** of $a$, having the following property:

**Additive Inverse Property**

$$a + (-a) = -a + a = 0$$ \hspace{1cm} (5a)

**EXAMPLE 13 Finding an Additive Inverse**

(a) The additive inverse of 6 is $-6$, because $6 + (-6) = 0$.

(b) The additive inverse of $-8$ is $-(-8) = 8$, because $-8 + 8 = 0$.

The additive inverse of $a$, that is, $-a$, is often called the **negative of $a$** or the **opposite** of $a$. The use of such terms can be dangerous, because they suggest that the additive inverse is a negative number, which may not be the case. For example, the additive inverse of $-3$, namely $-(-3)$, equals 3, a positive number.

For each **nonzero** real number $a$, there is a real number $\frac{1}{a}$, called the **multiplicative inverse** of $a$, having the following property:

**Multiplicative Inverse Property**

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \quad \text{if } a \neq 0$$ \hspace{1cm} (5b)

The multiplicative inverse $\frac{1}{a}$ of a nonzero real number $a$ is also referred to as the **reciprocal** of $a$.

**EXAMPLE 14 Finding a Reciprocal**

(a) The reciprocal of 6 is $\frac{1}{6}$, because $6 \cdot \frac{1}{6} = 1$.

(b) The reciprocal of $-3$ is $\frac{1}{-3}$, because $-3 \cdot \frac{1}{-3} = 1$.

(c) The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, because $\frac{2}{3} \cdot \frac{3}{2} = 1$.

With these properties for adding and multiplying real numbers, we can now define the operations of subtraction and division as follows:

**Definition**

The **difference** $a - b$, also read “$a$ less $b$” or “$a$ minus $b$,” is defined as

$$a - b = a + (-b)$$ \hspace{1cm} (6)

To subtract $b$ from $a$, add the opposite of $b$ to $a$. 
Definition
If \( b \) is a nonzero real number, the quotient \( \frac{a}{b} \), also read as “\( a \) divided by \( b \)” or “the ratio of \( a \) to \( b \),” is defined as

\[
\frac{a}{b} = a \cdot \frac{1}{b} \quad \text{if} \quad b \neq 0
\]  

(7)

EXAMPLE 15 Working with Differences and Quotients

(a) \( 8 - 5 = 8 + (-5) = 3 \)
(b) \( 4 - 9 = 4 + (-9) = -5 \)
(c) \( \frac{5}{8} = 5 \cdot \frac{1}{8} \)

For any number \( a \), the product of \( a \) times 0 is always 0; that is,

\[
a \cdot 0 = 0
\]  

(8)

For a nonzero number \( a \),

\[
\frac{0}{a} = 0 \quad \frac{a}{a} = 1 \quad \text{if} \quad a \neq 0
\]  

(9)

NOTE Division by 0 is not defined. One reason is to avoid the following difficulty: \( \frac{2}{0} = x \) means to find \( x \) such that \( 0 \cdot x = 2 \). But \( 0 \cdot x \) equals 0 for all \( x \); so there is no unique number \( x \) such that \( \frac{2}{0} = x \).

Rules of Signs

\[
\begin{align*}
  a(-b) &= -(ab) \\
  (-a)b &= -(ab) \\
  (-a)(-b) &= ab \\
  -(-a) &= a \\
  \frac{a}{-b} &= -\frac{a}{b} \\
  \frac{-a}{b} &= -\frac{a}{b} \\
  \frac{-a}{-b} &= \frac{a}{b}
\end{align*}
\]  

(10)
EXAMPLE 16  Applying the Rules of Signs

(a) \(2(-3) = -(2 \cdot 3) = -6\)  \hspace{1cm} (b) \((-3)(-5) = 3 \cdot 5 = 15\)

(c) \(\frac{3}{-2} = \frac{-3}{2} = -\frac{3}{2}\)  \hspace{1cm} (d) \(-\frac{4}{-9} = \frac{4}{9}\)

(e) \(\frac{x}{-2} = \frac{1}{-2} \cdot x = -\frac{1}{2}x\)

Cancellation Properties

\[
ac = bc \quad \text{implies} \quad a = b \quad \text{if} \quad c \neq 0
\]

\[
ac = \frac{a}{bc} \quad \text{if} \quad b \neq 0, c \neq 0
\]

EXAMPLE 17  Using the Cancellation Properties

(a) If \(2x = 6\), then

\[
\begin{align*}
2x &= 6 \\
2x &= 2 \cdot 3 \\
x &= 3
\end{align*}
\]

Factor 6.  Cancel the 2’s.

(b) \(\frac{18}{12} = \frac{3 \cdot 6}{2 \cdot 6} = \frac{3}{2}\)

\[
\text{Cancel the 6’s.}
\]

NOTE  We follow the common practice of using slash marks to indicate cancellations.

Zero-Product Property

If \(ab = 0\), then \(a = 0\), or \(b = 0\), or both.  \hspace{1cm} (12)

In other words, if a product equals 0, then one or both of the factors is 0.

EXAMPLE 18  Using the Zero-Product Property

If \(2x = 0\), then either \(2 = 0\) or \(x = 0\). Since \(2 \neq 0\), it follows that \(x = 0\).

Arithmetic of Quotients

\[
\begin{align*}
\frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd} & \text{if} \quad b \neq 0, d \neq 0 \\
\frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} & \text{if} \quad b \neq 0, d \neq 0 \\
\frac{a}{b} &= \frac{a \cdot d}{b \cdot c} & \text{if} \quad b \neq 0, c \neq 0, d \neq 0
\end{align*}
\]
EXAMPLE 19 Adding, Subtracting, Multiplying, and Dividing Quotients

(a) \(\frac{2}{3} + \frac{5}{2} = \frac{2 \cdot 2}{3 \cdot 2} + \frac{3 \cdot 5}{3 \cdot 2} = \frac{2 \cdot 2 + 3 \cdot 5}{3 \cdot 2} = \frac{4 + 15}{6} = \frac{19}{6}\)

\(\uparrow\) By equation (13)

(b) \(\frac{3}{5} - \frac{2}{3} = \frac{3 \cdot 3 - 2 \cdot 3}{5 \cdot 3} = \frac{3}{5} - \frac{2}{3}\)

\(\uparrow\) By equation (8) By equation (10)

\[= \frac{3 \cdot 3 + 5 \cdot (-2)}{5 \cdot 3} = \frac{9 + (-10)}{15} = \frac{-1}{15} = -\frac{1}{15}\]

\(\uparrow\) By equation (13)

NOTE Slanting the cancellation marks in different directions for different factors, as shown here, is a good practice to follow, since it will help in checking for errors.

(c) \(\frac{8}{3} \cdot \frac{15}{4} = \frac{8 \cdot 15}{3 \cdot 4} = \frac{2 \cdot 4 \cdot 3 \cdot 5}{3 \cdot 4 \cdot 1} = \frac{2 \cdot 5}{1} = 10\)

\(\uparrow\) By equation (14) By equation (11)

(d) \(\frac{3}{5} \cdot \frac{9}{7} = \frac{3 \cdot 9}{5 \cdot 7} = \frac{27}{35}\)

\(\uparrow\) By equation (14)

By equation (15)

NOTE In writing quotients, we shall follow the usual convention and write the quotient in lowest terms. That is, we write it so that any common factors of the numerator and the denominator have been removed using the cancellation properties, equations (11). As examples,

\[\frac{90}{24} = \frac{15 \cdot 6}{4 \cdot 6} = \frac{15}{4}\]

\[\frac{24x^2}{18x} = \frac{4 \cdot 6 \cdot x \cdot x}{3 \cdot 6 \cdot x} = \frac{4x^2}{3} \quad x \neq 0\]

NOW WORK PROBLEMS 55, 59, AND 69.

Sometimes it is easier to add two fractions using least common multiples (LCM). The LCM of two numbers is the smallest number that each has as a common multiple.

EXAMPLE 20 Finding the Least Common Multiple of Two Numbers

Find the least common multiple of 15 and 12.

SOLUTION To find the LCM of 15 and 12, we look at multiples of 15 and 12.

15, 30, 45, 60, 75, 90, 105, 120, ...

12, 24, 36, 48, 60, 72, 84, 96, 108, 120, ...

The common multiples are in blue. The least common multiple is 60.
1. The numbers in the set \( \left\{ x \mid x = \frac{a}{b}, a, b \text{ are integers and } b \neq 0 \right\} \), are called _____ numbers.

2. The value of the expression \( 4 + 5 \cdot 6 - 3 \) is _____.

3. The fact that \( 2x + 3x = (2 + 3)x \) is a consequence of the _____ Property.

4. “The product of 5 and \( x + 3 \) equals 6” may be written as _____.

5. True or False Rational numbers have decimals that either terminate or are nonterminating with a repeating block of digits.

6. True or False The Zero-Product Property states that the product of any number and zero equals zero.

7. True or False The least common multiple of 12 and 18 is 6.

8. True or False No real number is both rational and irrational.

EXAMPLE 21 Using the Least Common Multiple to Add Two Fractions

Find: \( \frac{8}{15} + \frac{5}{12} \)

SOLUTION We use the LCM of the denominators of the fractions and rewrite each fraction using the LCM as a common denominator. The LCM of the denominators (12 and 15) is 60. Rewrite each fraction using 60 as the denominator.

\[
\frac{8}{15} + \frac{5}{12} = \frac{8 \cdot 4}{15 \cdot 4} + \frac{5 \cdot 5}{12 \cdot 5} = \frac{32 + 25}{60} = \frac{57}{60} = \frac{19}{20}
\]

NOW WORK PROBLEM 63.

EXERCISE A.1 Answers Begin on Page AN–66.

Concepts and Vocabulary

1. The numbers in the set \( \left\{ x \mid x = \frac{a}{b}, a, b \text{ are integers and } b \neq 0 \right\} \), are called _____ numbers.

2. The value of the expression \( 4 + 5 \cdot 6 - 3 \) is _____.

3. The fact that \( 2x + 3x = (2 + 3)x \) is a consequence of the _____ Property.

4. “The product of 5 and \( x + 3 \) equals 6” may be written as _____.

5. True or False Rational numbers have decimals that either terminate or are nonterminating with a repeating block of digits.

6. True or False The Zero-Product Property states that the product of any number and zero equals zero.

7. True or False The least common multiple of 12 and 18 is 6.

8. True or False No real number is both rational and irrational.

Skill Building

In Problems 9–14, list the numbers in each set that are (a) natural numbers, (b) integers, (c) rational numbers, (d) irrational numbers, (e) real numbers.

9. \( A = \left\{ -6, -\frac{1}{2}, -1.333 \ldots = -1.\overline{3} \text{ (the 3s repeat)}, \pi, 2, 5 \right\} \)

10. \( B = \left\{ -\frac{5}{3}, 2.060606 \ldots = 2.\overline{06} \text{ (the block 06 repeats)}, 1.25, 0, 1, \sqrt{3} \right\} \)

11. \( C = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right\} \)

12. \( D = \{-1, -1.1, -1.2, -1.3\} \)

13. \( E = \left\{ \sqrt{2}, \pi, \sqrt{2} + 1, \pi + \frac{1}{2} \right\} \)

14. \( F = \left\{ -\sqrt{2}, \pi + \sqrt{2}, \frac{1}{2} + 10.3 \right\} \)
In Problems 15–26, approximate each number (a) rounded and (b) truncated to three decimal places.

15. 18.9526  
16. 25.86134  
17. 28.65319  
18. 99.05249  
19. 0.06291  
20. 0.05388  
21. 9.9985  
22. 1.0006  
23. \( \frac{3}{7} \)  
24. \( \frac{5}{9} \)  
25. \( \frac{521}{15} \)  
26. \( \frac{81}{5} \)  

In Problems 27–36, write each statement using symbols.

27. The sum of 3 and 2 equals 5.  
28. The product of 5 and 2 equals 10.  
29. The sum of x and 2 is the product of 3 and 4.  
30. The sum of 5 and 2 equals 10.  
31. The product of 3 and y is the sum of 1 and 2.  
32. The product of 2 and x is the product of 4 and 6.  
33. The difference x less 2 equals 6.  
34. The difference 2 less y equals 6.  
35. The quotient x divided by 2 is 6.  
36. The quotient 2 divided by x is 6.

In Problems 37–70, evaluate each expression.

37. 9 − 4 + 2  
38. 6 − 4 + 3  
39. −6 + 4 · 3  
40. 8 − 4 · 2  
41. 4 + 5 − 8  
42. 8 − 3 − 4  
43. 4 + 1  
44. 2 − 1  
45. 6 − [3 · 5 + 2 · (3 − 2)]  
46. 2 · [8 − 3(4 + 2)] − 3  
47. 2 · (3 − 5) + 8 · 2 − 1  
48. 1 − (4 · 3 − 2 + 2)  
49. 10 − [6 − 2 · 2 + (8 − 3)] · 2  
50. 2 − 5 · 4 − [6 · (3 − 4)]  
51. \( \frac{5 − 3}{2} \)  
52. \( \frac{5 + 4}{3} \)  
53. \( \frac{4 + 8}{5 − 3} \)  
54. \( \frac{2 − 4}{5 − 3} \)  
55. \( \frac{3}{5} \)  
56. \( \frac{3}{10} \)  
57. \( \frac{6}{10} \)  
58. \( \frac{21}{100} \)  
59. \( \frac{3}{4} + \frac{2}{5} \)  
60. \( \frac{4}{3} + \frac{1}{2} \)  
61. \( \frac{5}{6} + \frac{9}{5} \)  
62. \( \frac{8}{9} + \frac{15}{2} \)  
63. \( \frac{5}{18} + \frac{1}{12} \)  
64. \( \frac{2}{15} + \frac{8}{9} \)  
65. \( \frac{1}{30} − \frac{7}{18} \)  
66. \( \frac{3}{14} − \frac{2}{21} \)  
67. \( \frac{3}{20} − \frac{2}{15} \)  
68. \( \frac{6}{35} − \frac{3}{14} \)  
69. \( \frac{18}{11} \)  
70. \( \frac{21}{35} \)

In Problems 71–82, use the Distributive Property to remove the parentheses.

71. 6(x + 4)  
72. 4(2x − 1)  
73. x(x − 4)  
74. 4x(x + 3)  
75. (x + 2)(x + 4)  
76. (x + 5)(x + 1)  
77. (x − 2)(x + 1)  
78. (x − 4)(x + 1)  
79. (x − 8)(x − 2)  
80. (x − 4)(x − 2)  
81. (x + 2)(x − 2)  
82. (x − 3)(x + 3)  

Discussion and Writing

83. Explain to a friend how the Distributive Property is used to justify the fact that 2x + 3x = 5x.

84. Explain why 2 + 3 · 4 = 14, whereas (2 + 3) · 4 = 20.

85. Explain why 2(3 · 4) is not equal to (2 · 3) · (2 · 4).

86. Explain why \( \frac{4 + 3}{2 + 5} \) is not equal to \( \frac{4}{2} + \frac{3}{5} \).

87. Is subtraction commutative? Support your conclusion with an example.

88. Is subtraction associative? Support your conclusion with an example.

89. Is division commutative? Support your conclusion with an example.

90. Is division associative? Support your conclusion with an example.

91. If 2 = x, why does x = 2?

92. If x = 5, why does \( x^2 + x = 30 \)?

93. Are there any real numbers that are both rational and irrational? Are there any real numbers that are neither? Explain your reasoning.

94. Explain why the sum of a rational number and an irrational number must be irrational.

95. What rational number does the repeating decimal 0.9999\ldots equal?
The Real Number Line

There is a one-to-one correspondence between real numbers and points on a line. That is, every real number corresponds to a point on the line, and each point on the line has a unique real number associated with it.

Pick a point on the line somewhere in the center, and label it \( O \). This point, called the origin, corresponds to the real number 0. See Figure 5. The point 1 unit to the right of \( O \) corresponds to the number 1. The distance between 0 and 1 determines the scale of the number line. For example, the point associated with the number 2 is twice as far from \( O \) as 1. Notice that an arrowhead on the right end of the line indicates the direction in which the numbers increase. Points to the left of the origin correspond to the real numbers \(-1, -2, \) and so on. Figure 5 also shows the points associated with the rational numbers \(-\frac{1}{2}\) and \(\frac{1}{2}\) and with the irrational numbers \(\sqrt{2}\) and \(\pi\).

Definition

The real number associated with a point \( P \) is called the coordinate of \( P \), and the line whose points have been assigned coordinates is called the **real number line**.

The real number line consists of three classes of real numbers, as shown in Figure 6.

Definition

1. The **negative real numbers** are the coordinates of points to the left of the origin \( O \).
2. The real number **zero** is the coordinate of the origin \( O \).
3. The **positive real numbers** are the coordinates of points to the right of the origin \( O \).
Negative and positive numbers have the following multiplication properties:

**Multiplication Properties of Positive and Negative Numbers**
1. The product of two positive numbers is a positive number.
2. The product of two negative numbers is a positive number.
3. The product of a positive number and a negative number is a negative number.

**NOW WORK PROBLEM 11.**

**Graph Inequalities**
An important property of the real number line follows from the fact that, given two numbers (points) \( a \) and \( b \), either \( a \) is to the left of \( b \), or \( a \) is at the same location as \( b \), or \( a \) is to the right of \( b \). See Figure 7.

If \( a \) is to the left of \( b \), we say that “\( a \) is less than \( b \)” and write \( a < b \). If \( a \) is to the right of \( b \), we say that “\( a \) is greater than \( b \)” and write \( a > b \). If \( a \) is at the same location as \( b \), then \( a = b \). If \( a \) is either less than or equal to \( b \), we write \( a \leq b \). Similarly, \( a \geq b \) means that \( a \) is either greater than or equal to \( b \). Collectively, the symbols \(<, >, \leq, \text{ and } \geq \) are called *inequality symbols*.

Note that \( a < b \) and \( b > a \) mean the same thing. It does not matter whether we write \( 2 < 3 \) or \( 3 > 2 \).

Furthermore, if \( a < b \) or if \( b > a \), then the difference \( b - a \) is positive. Do you see why?

**EXAMPLE 1  Using Inequality Symbols**

\[
\begin{align*}
\text{(a)} & \quad 3 < 7 & \quad \text{(b)} & \quad -8 > -16 & \quad \text{(c)} & \quad -6 < 0 \\
\text{(d)} & \quad -8 < -4 & \quad \text{(e)} & \quad 4 > -1 & \quad \text{(f)} & \quad 8 > 0
\end{align*}
\]

In Example 1(a), we conclude that \( 3 < 7 \) either because 3 is to the left of 7 on the real number line or because the difference, \( 7 - 3 = 4 \), is a positive real number.

Similarly, we conclude in Example 1(b) that \( -8 > -16 \) either because \( -8 \) lies to the right of \( -16 \) on the real number line or because the difference, \( -8 - (-16) = -8 + 16 = 8 \), is a positive real number.

Look again at Example 1. Note that the inequality symbol always points in the direction of the smaller number.

An *inequality* is a statement in which two expressions are related by an inequality symbol. The expressions are referred to as the *sides* of the inequality. Statements of the form \( a < b \) or \( b > a \) are called *strict inequalities*, whereas statements of the form \( a \leq b \) or \( b \geq a \) are called *nonstrict inequalities*.

Based on the discussion so far, we conclude that

\[
\begin{align*}
a > 0 & \quad \text{is equivalent to } a \text{ is positive} \\
a < 0 & \quad \text{is equivalent to } a \text{ is negative}
\end{align*}
\]

We sometimes read \( a > 0 \) by saying that “\( a \) is positive.” If \( a \geq 0 \), then either \( a > 0 \) or \( a = 0 \), and we may read this as “\( a \) is nonnegative.”

**NOW WORK PROBLEMS 15 AND 25.**
EXAMPLE 2 Graphing Inequalities

(a) On the real number line, graph all numbers \( x \) for which \( x > 4 \).
(b) On the real number line, graph all numbers \( x \) for which \( x \leq 5 \).

**SOLUTION**

(a) See Figure 8. Notice that we use a left parenthesis to indicate that the number 4 is *not* part of the graph.

(b) See Figure 9. Notice that we use a right bracket to indicate that the number 5 *is* part of the graph.

NOW WORK PROBLEM 31.

| 2 |

Find Distance on the Real Number Line

The *absolute value* of a number \( a \) is the distance from 0 to \( a \) on the number line. For example, \(-4\) is 4 units from 0, and 3 is 3 units from 0. See Figure 10. So, the absolute value of \(-4\) is 4, and the absolute value of 3 is 3.

**Definition**

The *absolute value* of a real number \( a \), denoted by the symbol \( |a| \), is defined by the rules:

\[
|a| = a \quad \text{if} \quad a \geq 0 \quad \text{and} \quad |a| = -a \quad \text{if} \quad a < 0
\]

For example, since \(-4 < 0\), the second rule must be used to get \( |-4| = -(−4) = 4 \).

EXAMPLE 3 Computing Absolute Value

(a) \(|8| = 8\) \quad (b) \(|0| = 0\) \quad (c) \(|-15| = -(-15) = 15\)

Look again at Figure 10. The distance from \(-4\) to 3 is 7 units. This distance is the difference \(3 - (-4)\), obtained by subtracting the smaller coordinate from the larger. However, since \(|3 - (-4)| = |7| = 7\) and \(|-4 - 3| = |-7| = 7\), we can use absolute value to calculate the distance between two points without being concerned about which is smaller.

**Definition**

If \(P\) and \(Q\) are two points on a real number line with coordinates \(a\) and \(b\) respectively, the *distance between* \(P\) and \(Q\), denoted by \(d(P, Q)\), is

\[
d(P, Q) = |b - a|
\]

Since \(|b - a| = |a - b|\), it follows that \(d(P, Q) = d(Q, P)\).
**EXAMPLE 4** Finding Distance on a Number Line

Let \( P \), \( Q \), and \( R \) be points on a real number line with coordinates \(-5\), \( 7 \), and \(-3\), respectively. Find the distance

(a) between \( P \) and \( Q \)  
(b) between \( Q \) and \( R \)

**SOLUTION** See Figure 11.

\[
\begin{array}{c|ccccc}
\text{FIGURE 11} & P & R & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
P & & & & & & & & & & & & & & & & \hline
Q & & & & & & & & & & & & & & & & \\
\end{array}
\]

(a) \( d(P, Q) = |7 - (-5)| = |12| = 12 \)
(b) \( d(Q, R) = |-3 - 7| = |-10| = 10 \)

**NOW WORK PROBLEM 37.**

3 Evaluate Algebraic Expressions

Remember, in algebra we use letters such as \( x, y, a, b, \) and \( c \) to represent numbers. If the letter used is to represent any number from a given set of numbers, it is called a **variable**. A **constant** is either a fixed number, such as \( 5 \) or \( \sqrt{3} \), or a letter that represents a fixed (possibly unspecified) number.

Constants and variables are combined using the operations of addition, subtraction, multiplication, and division to form **algebraic expressions**. Examples of algebraic expressions include

\[ x + 3 \quad \frac{3}{1 - t} \quad 7x - 2y \]

To evaluate an algebraic expression, substitute for each variable its numerical value.

**EXAMPLE 5** Evaluating an Algebraic Expression

Evaluate each expression if \( x = 3 \) and \( y = -1 \).

(a) \( x + 3y \)  
(b) \( 5xy \)  
(c) \( \frac{3y}{2 - 2x} \)  
(d) \( |-4x + y| \)

**SOLUTION**

(a) Substitute \( 3 \) for \( x \) and \(-1\) for \( y \) in the expression \( x + 3y \).

\[ x + 3y = 3 + 3(-1) = 3 + (-3) = 0 \]

\( x = 3, \ y = -1 \)

(b) If \( x = 3 \) and \( y = -1 \), then

\[ 5xy = 5(3)(-1) = -15 \]

(c) If \( x = 3 \) and \( y = -1 \), then

\[ \frac{3y}{2 - 2x} = \frac{3(-1)}{2 - 2(3)} = \frac{-3}{2 - 6} = \frac{-3}{-4} = \frac{3}{4} \]
(d) If \( x = 3 \) and \( y = -1 \), then

\[
|4x + y| = |-4(3) + (-1)| = |-12 + (-1)| = |-13| = 13
\]

NOW WORK PROBLEMS 39 AND 47.

4 Determine the Domain of a Variable

In working with expressions or formulas involving variables, the variables may be allowed to take on values from only a certain set of numbers. For example, in the formula for the area \( A \) of a circle of radius \( r \), \( A = \pi r^2 \), the variable \( r \) is necessarily restricted to the positive real numbers. In the expression \( \frac{1}{x} \), the variable \( x \) cannot take on the value 0, since division by 0 is not defined.

**Definition**

The set of values that a variable may assume is called the **domain of the variable**.

**EXAMPLE 6** Finding the Domain of a Variable

The domain of the variable \( x \) in the expression

\[
\frac{5}{x - 2}
\]

is \( \{ x | x \neq 2 \} \), since, if \( x = 2 \), the denominator becomes 0, which is not defined.

**EXAMPLE 7** Circumference of a Circle

In the formula for the circumference \( C \) of a circle of radius \( r \),

\[
C = 2\pi r
\]

the domain of the variable \( r \), representing the radius of the circle, is the set of positive real numbers. The domain of the variable \( C \), representing the circumference of the circle, is also the set of positive real numbers.

In describing the domain of a variable, we may use either set notation or words, whichever is more convenient.

NOW WORK PROBLEM 57.

5 Use the Laws of Exponents

Integer exponents provide a shorthand device for representing repeated multiplications of a real number. For example,

\[
3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81
\]

Additionally, many formulas have exponents. For example,
• The formula for the horsepower rating \( H \) of an engine is

\[
H = \frac{D^2N}{2.5}
\]

where \( D \) is the diameter of a cylinder and \( N \) is the number of cylinders.

• A formula for the resistance \( R \) of blood flowing in a blood vessel is

\[
R = \frac{C}{r^4}
\]

where \( L \) is the length of the blood vessel, \( r \) is the radius, and \( C \) is a positive constant.

**Definition**

If \( a \) is a real number and \( n \) is a positive integer, then the symbol \( a^n \) represents the product of \( n \) factors of \( a \). That is,

\[
a^n = a \cdot a \cdot \cdots \cdot a
\]

Here it is understood that \( a^1 = a \).

Then \( a^2 = a \cdot a \), \( a^3 = a \cdot a \cdot a \), and so on. In the expression \( a^n \), \( a \) is called the base and \( n \) is called the exponent, or power. We read \( a^n \) as “\( a \) raised to the power \( n \)” or as “\( a \) to the \( n \)th power.” We usually read \( a^2 \) as “\( a \) squared” and \( a^3 \) as “\( a \) cubed.”

In working with exponents, the operation of raising to a power is performed before any other operation. As examples,

\[
4 \cdot 3^2 = 4 \cdot 9 = 36 \quad 2^2 + 3^2 = 4 + 9 = 13
\]

\[
-2^4 = -16 \quad 5 \cdot 3^2 + 2 \cdot 4 = 5 \cdot 9 + 2 \cdot 4 = 45 + 8 = 53
\]

Parentheses are used to indicate operations to be performed first. For example,

\[
(-2)^4 = (-2)(-2)(-2)(-2) = 16 \quad (2 + 3)^2 = 5^2 = 25
\]

**Definition**

We define

\[
a^0 = 1 \quad \text{if} \quad a \neq 0
\]

**Definition**

If \( n \) is a positive integer, we define

\[
a^{-n} = \frac{1}{a^n} \quad \text{if} \quad a \neq 0
\]

Whenever you encounter a negative exponent, think “reciprocal.”
EXAMPLE 8 Evaluating Expressions Containing Negative Exponents

(a) \(2^{-3} = \frac{1}{2^3} = \frac{1}{8}\)
(b) \(x^{-4} = \frac{1}{x^4}\)
(c) \(\left(\frac{1}{5}\right)^{-2} = \frac{1}{\left(\frac{1}{5}\right)^2} = \frac{1}{\frac{1}{25}} = 25\)

NOW WORK PROBLEMS 75 AND 91.

The following properties, called the Laws of Exponents, can be proved using the preceding definitions. In the list, \(a\) and \(b\) are real numbers, and \(m\) and \(n\) are integers.

**Theorem**

**Laws of Exponents**

\[
\begin{align*}
a^m a^n &= a^{m+n} \\
(a^m)^n &= a^{mn} \\
(ab)^n &= a^n b^n \\
\frac{a^m}{a^n} &= a^{m-n} \quad \text{if } a \neq 0 \\
\left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n} \quad \text{if } b \neq 0
\end{align*}
\]

EXAMPLE 9 Using the Laws of Exponents

(a) \(x^{-3} \cdot x^5 = x^{-3+5} = x^2 \quad x \neq 0\)
(b) \((x^{-3})^2 = x^{-3\cdot2} = x^{-6} = \frac{1}{x^6} \quad x \neq 0\)
(c) \((2x)^3 = 2^3 \cdot x^3 = 8x^3\)
(d) \(\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}\)
(e) \(\frac{x^{-2}}{x^{-3}} = x^{-2-(-3)} = x^1 \quad x \neq 0\)

NOW WORK PROBLEMS 77 AND 87.

**Evaluate Square Roots**

A real number is squared when it is raised to the power 2. The inverse of squaring is finding a **square root**. For example, since \(6^2 = 36\) and \((-6)^2 = 36\), the numbers 6 and −6 are square roots of 36.

The symbol \(\sqrt{\ }\), called a **radical sign**, is used to denote the **principal**, or nonnegative, square root. For example, \(\sqrt{36} = 6\).

**Definition**

If \(a\) is a nonnegative real number, the nonnegative number \(b\), such that \(b^2 = a\), is the **principal square root** of \(a\) and is denoted by \(b = \sqrt{a}\).
A.2 Algebra Essentials

The following comments are noteworthy:

1. Negative numbers do not have square roots (in the real number system), because the square of any real number is nonnegative. For example, \( \sqrt{-4} \) is not a real number, because there is no real number whose square is \(-4\).
2. The principal square root of \(0\) is \(0\), since \(0^2 = 0\). That is, \(\sqrt{0} = 0\).
3. The principal square root of a positive number is positive.
4. If \(c \geq 0\), then \((\sqrt{c})^2 = c\). For example, \((\sqrt{2})^2 = 2\) and \((\sqrt{3})^2 = 3\).

**EXAMPLE 10** Evaluating Square Roots

(a) \(\sqrt{64} = 8\)  
(b) \(\sqrt{\frac{1}{16}} = \frac{1}{4}\)  
(c) \((\sqrt{1.4})^2 = 1.4\)

Examples 10(a) and (b) are examples of square roots of perfect squares, since \(64 = 8^2\) and \(\frac{1}{16} = (\frac{1}{4})^2\).

Consider the expression \(\sqrt{a^2}\). Since \(a^2 \geq 0\), the principal square root of \(a^2\) is defined whether \(a > 0\) or \(a < 0\). However, since the principal square root is nonnegative, we need an absolute value to ensure the nonnegative result. That is,

\[\sqrt{a^2} = |a|\quad a \text{ any real number} \] (2)

**EXAMPLE 11** Using Equation (2)

(a) \(\sqrt{(2.3)^2} = |2.3| = 2.3\)  
(b) \(\sqrt{(-2.3)^2} = |-2.3| = 2.3\)  
(c) \(\sqrt{x^2} = |x|\)

**NOW WORK PROBLEM 83.**

**7** Use a Calculator to Evaluate Exponents

Your calculator has either a caret key, \(^\wedge\) or an \(x^y\) key, which is used for computations involving exponents.

**EXAMPLE 12** Exponents on a Graphing Calculator

Evaluate: \((2.3)^5\)

**SOLUTION** Figure 12 shows the result using a TI-84 Plus graphing calculator.

**NOW WORK PROBLEM 107.**
Solve Equations and Inequalities

An **equation in one variable** is a statement in which two expressions, at least one containing the variable, are equal. The expressions are called the sides of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variable. Unless otherwise restricted, the admissible values of the variable are those in the domain of the variable. The admissible values of the variable, if any, that result in a true statement are called **solutions**, or **roots**, of the equation.

To solve an equation means to find all the solutions of the equation. Usually, we will write the solution of an equation in set notation. This set is called the **solution set** of the equation. For example, the solution set of the equation \( x - 5 = 9 \) is \{14\}. We solve equations by using properties of real numbers.

### EXAMPLE 13 Solving an Equation

Solve the equation: \( 2x + 3(x + 2) = 1 \)

**SOLUTION**

\[
\begin{align*}
2x + 3(x + 2) &= 1 \\
2x + 3x + 6 &= 1 \\
(2x + 3x) + 6 &= 1 \\
x + 6 &= 1 \\
x &= -5
\end{align*}
\]

Remove parentheses using the Distributive Property.

Use the Associative Property.

Combine \( 2x + 3x = (2 + 3)x = 5x \).

Subtract 6 from each side.

Divide each side by 5.

✔ CHECK:

\[
\begin{align*}
2x + 3(x + 2) &= 2(-1) + 3[(-1) + 2] = -2 + 3(1) = -2 + 3 = 1
\end{align*}
\]

The solution set is \{-1\}.

**NOW WORK PROBLEM 117**

An **inequality in one variable** is a statement involving two expressions, at least one containing the variable, separated by one of the inequality symbols \( (<, \leq, >, \geq) \). To solve an inequality means to find all values of the variable for which the statement is true. The set of all such values is called the **solution set** of the inequality. We solve inequalities by applying the following procedures:

### Procedures That Leave the Inequality Symbol Unchanged

1. Simplify both sides of the inequality by combining like terms and eliminating parentheses:

   Replace \( x + 2 + 6 > 2x + 5(x + 1) \)

   by \( x + 8 > 7x + 5 \)

2. Add or subtract the same expression on both sides of the inequality:

   Replace \( 3x - 5 < 4 \)

   by \( (3x - 5) + 5 < 4 + 5 \)

3. Multiply or divide both sides of the inequality by the same positive expression:

   Replace \( 4x > 16 \)

   by \( \frac{4x}{4} > \frac{16}{4} \)
1. A(n) _____ is a letter used in algebra to represent any number from a given set of numbers.

2. On the real number line, the real number zero is the coordinate of the _____.

3. An inequality of the form \( a > b \) is called a(n) _____ inequality.

4. In the expression \( 2^4 \), the number 2 is called the _____ and 4 is called the _____.

5. True or False The product of two negative real numbers is always greater than zero.

6. True or False The distance between two distinct points on the real number line is always greater than zero.

7. True or False The absolute value of a real number is always greater than zero.

8. True or False To multiply two expressions having the same base, retain the base and multiply the exponents.

9. The solution set of the equation \( 2x + 1 = 9 \) is _____.

10. True or False If both sides of an inequality are multiplied by \(-1\), the inequality remains the same.

11. On the real number line, label the points with coordinates 0, 1, \(-1\), \(\frac{5}{2}\), \(-2.5\), \(\frac{3}{4}\), and 0.25.

12. On the real number line, label the points with coordinates \(0\), \(-2\), \(2\), \(-1.5\), \(\frac{3}{2}\), \(\frac{1}{3}\), and \(\frac{2}{3}\).

**Procedures That Reverse the Sense or Direction of the Inequality Symbol**

1. Interchange the two sides of the inequality:
   
   \[ 3 < x \quad \text{by} \quad x > 3 \]

2. Multiply or divide both sides of the inequality by the same negative expression:

   \[ -2x > 6 \quad \text{by} \quad \frac{-2x}{-2} < \frac{6}{-2} \]

---

**EXAMPLE 14 Solving an Inequality**

Solve the inequality: \( x + 2 \leq 3x - 5 \). Graph the solution set.

**SOLUTION**

\[
\begin{align*}
    x + 2 & \leq 3x - 5 \\
    -2x & \leq -7 \\
    x & \geq \frac{7}{2}
\end{align*}
\]

Subtract 2 and then 3x from each side.

Multiply by \(-1\) and remember to reverse the inequality because we multiplied by a negative number.

The solution set is all real numbers to the right of \(\frac{7}{2}\), including \(\frac{7}{2}\), namely \( \{ x | x \geq \frac{7}{2} \} \).

See Figure 13 for the graph of the solution set.

NOW WORK PROBLEM 123.
A–26  Appendix A  Review

In Problems 13–22, replace the question mark by <, >, or =, whichever is correct.

13. $\frac{1}{2} \neq 0$
14. $5 \neq 6$  
15. $-1 < -2$  
16. $-3 < \frac{5}{2}$
17. $\pi \neq 3.14$
18. $\sqrt{2} < 1.41$
19. $\frac{1}{2} \neq 0.5$
20. $\frac{1}{3} < 0.33$
21. $\frac{2}{3} > 0.67$
22. $\frac{1}{4} > 0.25$

In Problems 23–28, write each statement as an inequality.

23. $x$ is positive
24. $x$ is negative
25. $x$ is less than 2
26. $y$ is greater than $-5$
27. $x$ is less than or equal to 1
28. $x$ is greater than or equal to 2

In Problems 29–32, graph the numbers $x$ on the real number line.

29. $x \geq -2$
30. $x < 4$
31. $x > -1$
32. $x \leq 7$

In Problems 33–38, use the given real number line to compute each distance.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$</td>
<td>$-3$</td>
<td>$-2$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

33. $d(C, D)$  
34. $d(C, A)$  
35. $d(D, E)$  
36. $d(C, E)$  
37. $d(A, E)$  
38. $d(D, B)$

In Problems 39–46, evaluate each expression if $x = -2$ and $y = 3$.

39. $x + 2y$
40. $3x + y$
41. $5xy + 2$
42. $-2x + xy$
43. $\frac{2x}{x - y}$
44. $\frac{x + y}{x - y}$
45. $\frac{3x + 2y}{2 + y}$
46. $\frac{2x - 3}{y}$

In Problems 47–56, find the value of each expression if $x = 3$ and $y = -2$.

47. $|x + y|$  
48. $|x - y|$  
49. $|x| + |y|$  
50. $|x| - |y|$  
51. $|x| + |y|$
52. $\frac{|y|}{y}$
53. $|4x - 5y|$  
54. $|3x + 2y|$  
55. $|4x| - |3y|$  
56. $3|x| + 2|y|$

In Problems 57–64, determine which of the value(s) (a) through (d), if any, must be excluded from the domain of the variable in each expression:

(a) $x = 3$  
(b) $x = 1$  
(c) $x = 0$  
(d) $x = -1$

57. $\frac{x^2 - 1}{x}$  
58. $\frac{x^2 + 1}{x}$  
59. $\frac{x}{x^2 - 9}$  
60. $\frac{x}{x^2 + 9}$
61. $\frac{x^2}{x^2 + 1}$  
62. $\frac{x^3}{x^2 - 1}$  
63. $\frac{x^2 + 5x - 10}{x^3 - x}$  
64. $\frac{-9x^2 - x + 1}{x^3 + x}$

In Problems 65–68, determine the domain of the variable $x$ in each expression.

65. $\frac{4}{x - 5}$  
66. $\frac{-6}{x + 4}$  
67. $\frac{x}{x + 4}$  
68. $\frac{x - 2}{x - 6}$

In Problems 69–72, use the formula $C = \frac{5}{9}(F - 32)$ for converting degrees Fahrenheit into degrees Celsius to find the Celsius measure of each Fahrenheit temperature.

69. $F = 32^\circ$  
70. $F = 212^\circ$  
71. $F = 77^\circ$  
72. $F = -4^\circ$

In Problems 73–84, simplify each expression.

73. $(-4)^2$  
74. $-4^2$  
75. $4^{-2}$  
76. $-4^{-2}$  
77. $3^{-6} \cdot 3^4$  
78. $4^{-2} \cdot 4^3$
79. $(3^{-2})^{-1}$  
80. $(2^{-1})^{-3}$  
81. $\sqrt{25}$  
82. $\sqrt{36}$  
83. $\sqrt{(-4)^2}$  
84. $\sqrt{(-3)^2}$

In Problems 85–90, simplify each expression. Express the answer so that all exponents are positive. Whenever an exponent is 0 or negative, we assume that the base is not 0.

85. $(8x^3)^2$  
86. $(-4x^2)^{-1}$  
87. $(x^2y)^{1/2}$  
88. $(x^{-1}y)^3$  
89. $\frac{x^2y^3}{xy^4}$  
90. $\frac{x^2y^3}{xy^4}$
A.2 Algebra Essentials A–27

In Problems 91–102, find the value of each expression if \( x = 2 \) and \( y = -1 \).

91. \( 2xy^{-1} \)
92. \(-3x^{-1}y \)
93. \( x^2 + y^2 \)
94. \( x^2y^2 \)
95. \( (xy)^2 \)
96. \((x + y)^2 \)
97. \( \sqrt{x^2} \)
98. \( (\sqrt{x})^2 \)
99. \( \sqrt{x^2 + y^2} \)
100. \( \sqrt{x^2 + \sqrt{y^2}} \)
101. \( x^y \)
102. \( y^x \)
103. What is the value of \( (666)^4 \)?
104. What is the value of \( (0.1)^3(20)^3 \)?

In Problems 105–112, use a calculator to evaluate each expression. Round your answer to three decimal places.

105. \( (8.2)^6 \)
106. \( (3.7)^5 \)
107. \( (6.1)^{-3} \)
108. \( (2.2)^{-5} \)
109. \( (-2.8)^6 \)
110. \(-(-2.8)^6 \)
111. \( (-3.11)^{-4} \)
112. \(-(-3.11)^{-4} \)

In Problems 113–120, solve each equation.

113. \( x + 5 = 7 \)
114. \( x + 6 = 2 \)
115. \( 6 - x = 0 \)
116. \( 6 + x = 0 \)
117. \( 3(2 - x) = 9 \)
118. \( 5(x + 1) = 10 \)
119. \( 4x + 3 = 2x - 5 \)
120. \( 5x - 8 = 2x + 1 \)

In Problems 121–128, solve each inequality. Graph the solution set.

121. \( x + 5 \leq 2 \)
122. \( 4 - x \geq 3 \)
123. \( 3x + 5 \geq 2 \)
124. \( 14x - 21x + 16 \leq 3x - 2 \)
125. \(-3x + 5 \leq 2 \)
126. \( 8 - 2x \leq 5x - 6 \)
127. \( 6x - 3 \geq 8x + 5 \)
128. \(-3x \leq 2x + 5 \)

Applications and Extensions

In Problems 129–138, express each statement as an equation involving the indicated variables.

129. **Area of a Rectangle** The area \( A \) of a rectangle is the product of its length \( l \) and its width \( w \).

\[
\begin{align*}
\text{Area} & = \text{length} \times \text{width} \\
A & = lw
\end{align*}
\]

130. **Perimeter of a Rectangle** The perimeter \( P \) of a rectangle is twice the sum of its length \( l \) and its width \( w \).

\[
P = 2(l + w)
\]

131. **Circumference of a Circle** The circumference \( C \) of a circle is the product of \( \pi \) and its diameter \( d \).

\[
C = \pi d
\]

132. **Area of a Triangle** The area \( A \) of a triangle is one-half the product of its base \( b \) and its height \( h \).

\[
A = \frac{1}{2}bh
\]

133. **Area of an Equilateral Triangle** The area \( A \) of an equilateral triangle is \( \frac{\sqrt{3}}{4} \) times the square of the length \( x \) of one side.

\[
A = \frac{\sqrt{3}}{4}x^2
\]

134. **Perimeter of an Equilateral Triangle** The perimeter \( P \) of an equilateral triangle is 3 times the length \( x \) of one side.

\[
P = 3x
\]

135. **Volume of a Sphere** The volume \( V \) of a sphere is \( \frac{4}{3}\pi \) times the cube of the radius \( r \).

\[
V = \frac{4}{3}\pi r^3
\]

136. **Surface Area of a Sphere** The surface area \( S \) of a sphere is \( 4\pi \) times the square of the radius \( r \).

\[
S = 4\pi r^2
\]

137. **Volume of a Cube** The volume \( V \) of a cube is the cube of the length \( x \) of a side.

\[
V = x^3
\]

138. **Surface Area of a Cube** The surface area \( S \) of a cube is 6 times the square of the length \( x \) of a side.

\[
S = 6x^2
\]

139. **Manufacturing Cost** The weekly production cost \( C \) of manufacturing \( x \) watches is given by the formula \( C = 4000 + 2x \), where the variable \( C \) is in dollars.

(a) What is the cost of producing 1000 watches?
(b) What is the cost of producing 2000 watches?
140. **Balancing a Checkbook** At the beginning of the month, Mike had a balance of $210 in his checking account. During the next month, he deposited $80, wrote a check for $120, made another deposit of $25, wrote two checks: one for $60 and the other for $32. He was also assessed a monthly service charge of $5. What was his balance at the end of the month?

141. **U.S. Voltage** In the United States, normal household voltage is 110 volts. It is acceptable for the actual voltage $x$ to differ from normal by at most 5 volts. A formula that describes this is

$$\left|x - 110\right| \leq 5$$

(a) Show that a voltage of 108 volts is acceptable.
(b) Show that a voltage of 104 volts is not acceptable.

142. **Foreign Voltage** In some countries, normal household voltage is 220 volts. It is acceptable for the actual voltage $x$ to differ from normal by at most 8 volts. A formula that describes this is

$$\left|x - 220\right| \leq 8$$

(a) Show that a voltage of 214 volts is acceptable.
(b) Show that a voltage of 209 volts is not acceptable.

143. **Making Precision Ball Bearings** The FireBall Company manufactures ball bearings for precision equipment. One of its products is a ball bearing with a stated radius of 3 centimeters (cm). Only ball bearings with a radius within 0.01 cm of this stated radius are acceptable. If $x$ is the radius of a ball bearing, a formula describing this situation is

$$\left|x - 3\right| \leq 0.01$$

(a) Is a ball bearing of radius $x = 2.999$ acceptable?
(b) Is a ball bearing of radius $x = 2.89$ acceptable?

144. **Body Temperature** Normal human body temperature is 98.6°F. A temperature $x$ that differs from normal by at least 1.5°F is considered unhealthy. A formula that describes this is

$$\left|x - 98.6\right| \geq 1.5$$

(a) Show that a temperature of 97°F is unhealthy.
(b) Show that a temperature of 100°F is not unhealthy.

145. Does $\frac{1}{3}$ equal 0.333? If not, which is larger? By how much?

146. Does $\frac{2}{3}$ equal 0.666? If not, which is larger? By how much?

147. Is there a positive real number “closest” to 0?

148. **Number Game** I’m thinking of a number! It lies between 1 and 10; its square is rational and lies between 1 and 10. The number is larger than $\pi$. Correct to two decimal places (that is, truncated to two decimal places) name the number. Now think of your own number, describe it, and challenge a fellow student to name it.

149. Write a brief paragraph that illustrates the similarities and differences between “less than” ($<$) and “less than or equal to” ($\leq$).

150. Give a reason why the statement $5 < 8$ is true.

---

**A.3 Exponents and Logarithms**

**PREPARING FOR THIS SECTION** Before getting started, review the following:

- Exponents and Square Roots (Appendix A, Section A.2, pp. A–20 to A–23)

**NOW WORK THE “ARE YOU PREPARED” PROBLEMS ON PAGE A–33**

**OBJECTIVES**

1. Evaluate Expressions Containing Exponents (p. A–29)
2. Change Exponential Expressions to Logarithmic Expressions (p. A–31)
3. Change Logarithmic Expressions to Exponential Expressions (p. A–31)
4. Evaluate Logarithmic Expressions (p. A–32)
5. Evaluate Logarithms Using the Change-of-Base Formula (p. A–32)
Evaluate Expressions Containing Exponents

We begin by reviewing the definition of the symbol $a^n$.

**Definition**

If $a$ is a real number and $n$ is a positive integer, then the **symbol $a^n$** represents the product of $n$ factors of $a$. That is,

$$a^n = a \cdot a \cdot \ldots \cdot a$$

where it is understood that $a^1 = a$.

In the expression $a^n$, read “$a$ raised to the power $n$,” $a$ is called the **base** and $n$ is called the **exponent** or **power**.

If $n$ is a positive integer we define

$$a^{-n} = \frac{1}{a^n} \quad a \neq 0$$

Also, we define

$$a^0 = 1 \quad a \neq 0$$

To extend the definition of $a^n$ to include rational exponents requires **principal $n^{th}$ roots**.

**Definition**

The **principal $n^{th}$ root of a real number $a$, written $\sqrt[n]{a}$**, is defined as follows:

(a) If $a > 0$ and $n$ is even, then $\sqrt[n]{a}$ is the positive number $x$ for which $x^n = a$.
(b) If $a < 0$ and $n$ is even, then $\sqrt[n]{a}$ does not exist.
(c) If $n$ is odd, then $\sqrt[n]{a}$ is the number $x$ for which $x^n = a$.
(d) $\sqrt[n]{a^0} = 0$ for any $n$.

If $n = 2$, we write $\sqrt{a}$ in place of $\sqrt[n]{a}$.

**EXAMPLE 1** Evaluating Principal $n^{th}$ Roots

(a) $\sqrt{8} = 2$ because $2^3 = 8$
(b) $\sqrt{64} = 8$ because $8^2 = 64$ and $8 > 0$
(c) $\sqrt{-27} = -3$ because $(−3)^3 = −27$

**NOW WORK PROBLEM 7.**
We use principal roots to define exponents that are rational.

**Definition**

If \( a \) is a real number and \( n \geq 2 \) is an integer, then

\[ a^{1/n} = \sqrt[n]{a} \]

provided \( \sqrt[n]{a} \) exists.

If \( a \) is a real number and \( m, n \) are integers containing no common factors with \( n \geq 2 \), then

\[ a^{m/n} = \sqrt[m]{a^m} = (\sqrt[n]{a})^m \]

provided \( \sqrt[n]{a} \) exists.

**EXAMPLE 2** Evaluating Rational Exponents

(a) \( 4^{3/2} = (\sqrt{4})^3 = 2^3 = 8 \)

(b) \( (-8)^{4/3} = (\sqrt[3]{-8})^4 = (-2)^4 = 16 \)

(c) \( (32)^{-2/5} = (\sqrt[5]{32})^{-2} = 2^{-2} = \frac{1}{4} \)

**NOW WORK PROBLEM 13.**

But what is the meaning of \( a^x \), where the base \( a \) is a positive real number and the exponent \( x \) is an irrational number? Although a rigorous definition requires methods discussed in calculus, the basis for the definition is easy to follow: Select a rational number \( r \) that is formed by truncating (removing) all but a finite number of digits from the irrational number \( x \). Then it is reasonable to expect that

\[ a^x \approx a^r \]

For example, take the irrational number \( \pi \approx 3.14159 \ldots \) An approximation to \( a^\pi \) is

\[ a^\pi \approx a^{3.14} \]

where the digits after the hundredths position have been removed from the value for \( \pi \). A better approximation would be

\[ a^\pi \approx a^{3.14159} \]

where the digits after the hundred-thousandths position have been removed. Continuing in this way, we can obtain approximations to \( a^x \) to any desired degree of accuracy.

Scientific calculators and graphing utilities have an \( [x^r] \) key or a \( \sqrt[n]{ } \) key for working with exponents.

**EXAMPLE 3** Evaluating Exponents with a Calculator

Evaluate

(a) \( (1.05)^4 \) \hspace{1cm} (b) \( (1 + 0.08)^3 \) \hspace{1cm} (c) \( \left( 1 + \frac{0.10}{4} \right)^8 \) \hspace{1cm} (d) \( \left( 1 + \frac{0.08}{12} \right)^{-24} \)

Round answers to four decimal places.

**SOLUTION**

(a) \( (1.05)^4 \approx 1.2155 \)

(b) \( (1 + 0.08)^3 \approx 1.2597 \)
FIGURE 14

(c) \[ \left( 1 + \frac{0.10}{4} \right)^8 \approx 1.2184 \]

(d) Figure 14 shows the result using a TI-84 Plus graphing calculator. Rounded to four decimal places,

\[ \left( 1 + \frac{0.08}{12} \right)^{-24} \approx 0.8526 \]

NOW WORK PROBLEMS 19 AND 23.

Logarithms

The definition of a logarithm is based on an exponential relationship.

**Definition**

Suppose \( N = a^x, a > 0, a \neq 1, x \) real. The **logarithm to the base** \( a \) **of** \( N \) **is symbolized by** \( \log_a N \) **and is defined by**

\[ \log_a N = x \quad \text{if and only if} \quad N = a^x \]

As this definition illustrates, a **logarithm** is a name for a certain exponent.

**EXAMPLE 4** Relating Logarithms to Exponents

(a) If \( x = \log_3 N \), then \( N = 3^x \). For example, \( 4 = \log_3 81 \) is equivalent to \( 81 = 3^4 \).

(b) If \( x = \log_5 N \), then \( N = 5^x \). For example, \( -1 = \log_5 \left( \frac{1}{5} \right) \) is equivalent to \( \frac{1}{5} = 5^{-1} \).

**EXAMPLE 5** Changing Exponential Expressions to Logarithmic Expressions

Change each exponential expression to an equivalent expression involving a logarithm.

(a) \( 1.2^3 = m \)  \hspace{1cm} (b) \( e^b = 9 \)  \hspace{1cm} (c) \( a^4 = 24 \)

**SOLUTION** We use the fact that \( x = \log_a N \) and \( N = a^x, a > 0, a \neq 1 \), are equivalent.

(a) If \( 1.2^3 = m \), then \( 3 = \log_{1.2} m \).

(b) If \( e^b = 9 \), then \( b = \log_e 9 \).

(c) If \( a^4 = 24 \), then \( 4 = \log_a 24 \).

NOW WORK PROBLEM 29.

**EXAMPLE 6** Changing Logarithmic Expressions to Exponential Expressions

Change each logarithmic expression to an equivalent expression involving an exponent.

(a) \( \log_a 4 = 5 \)  \hspace{1cm} (b) \( \log_e b = -3 \)  \hspace{1cm} (c) \( \log_3 5 = c \)
SOLUTION

(a) If \( \log_a 4 = 5 \), then \( a^5 = 4 \).

(b) If \( \log_b b = -3 \), then \( e^{-3} = b \).

(c) If \( \log_3 5 = c \), then \( 3^c = 5 \).

NOW WORK PROBLEM 37.

4 Evaluate Logarithmic Expressions

To find the exact value of a logarithm, we write the logarithm in exponential notation and use the fact that if \( a^u = a^v \) then \( u = v \).

EXAMPLE 7 Finding the Exact Value of a Logarithmic Expression

Find the exact value of:

(a) \( \log_2 16 \)

(b) \( \log_3 \frac{1}{27} \)

SOLUTION

(a) \( y = \log_2 16 \)

Change to exponential form. \( 2^y = 16 \)

Equate exponents. \( 2^y = 2^4 \)

\( y = 4 \)

\( \log_2 16 = 4 \).

(b) \( y = \log_3 \frac{1}{27} \)

Change to exponential form. \( 3^y = \frac{1}{27} \)

Equate exponents. \( 3^y = 3^{-3} \)

\( y = -3 \)

\( \log_3 \frac{1}{27} = -3 \).

NOW WORK PROBLEM 45.

5 Evaluate Logarithms Using the Change-of-Base Formula

Logarithms to the base 10, common logarithms, were used to facilitate arithmetic computations before the widespread use of calculators. Common logarithms are usually abbreviated by writing \( \log \), with the base understood to be 10. Most calculators have a \( \log \) key to calculate the common logarithm of a number. To approximate logarithms having a base other than 10, we use the Change-of-Base formula.

Theorem

Change-of-Base Formula

If \( M > 0 \), \( a > 0 \) and \( a \neq 1 \), then

\[
\log_a M = \frac{\log M}{\log a}
\]

EXAMPLE 8 Using the Change-of-Base Formula

(a) \( \log_{1.1} 100 = \frac{\log 100}{\log 1.1} \approx 48.3177 \)
A.3 Exponents and Logarithms

\[(b) \quad \log_{1.25} 350 = \frac{\log 350}{\log 1.25} \approx 26.2519\]

\[(c) \quad \log_{1.005} 1000 = \frac{\log 1000}{\log 1.005} \approx 1385.0021\]

NOW WORK PROBLEMS 61 AND 65.

**EXERCISE A.3**  Answers Begin on Page AN–66.

‘Are You Prepared?’ Problems  Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. (a) \(5^3 = \) (pp. A-20 to A-23)
   (b) \(4^{-2} = \) (pp. A-20 to A-23)

2. (a) \(\sqrt[4]{-4} = \) (pp. A-20 to A-23)
   (b) \(\sqrt[9]{9} = \) (pp. A-20 to A-23)

3. Express as the power of an integer: \(27^n = \)

4. Use a calculator to find \((1.12)^8 = \). Round your answer to two decimal places.

5. Change \(1.8^4 = x\) to a logarithmic expression.

6. Use a calculator to find \(\log_4 89 = \). Round your answer to three decimal places.

**Concepts and Vocabulary**

7. \(\log_{1.005} 1000 = \frac{\log 1000}{\log 1.005} \approx 1385.0021\)

8. \(\log_{1.25} 350 = \frac{\log 350}{\log 1.25} \approx 26.2519\)

**Skill Building**

In Problems 7–24, evaluate each expression.

7. \(\sqrt[6]{16} = \)
8. \(\sqrt[3]{9} = \)
9. \(\sqrt[27]{27} = \)
10. \(\sqrt[16]{-1} = \)
11. \(\sqrt[16]{16} = \)
12. \(\sqrt[0]{0} = \)
13. \(8^{2/3} = \)
14. \(9^{3/2} = \)
15. \(16^{-3/2} = \)
16. \(8^{-2/3} = \)
17. \((-8)^{-2/3} = \)
18. \((-27)^{-4/3} = \)

In Problems 19–28, approximate each number using a calculator. Round answers to four decimal places.

19. \(e^{2.2} = \)
20. \(-2.2^{2.2} = \)
21. \((1.08)^5 = \)
22. \((1 + 0.02)^6 = \)
23. \((1 + 0.06)^{12} = \)
24. \((-1 + 0.12)^6 = \)
25. \((1 + 0.02)^{12} = \)
26. \((1 + 0.05)^{-8} = \)
27. \((1 + 0.04)^{-20} = \)
28. \((-1 + 0.06)^{-7} = \)

In Problems 29–36, change each exponential expression to an equivalent expression involving a logarithm.

29. \(9 = 3^2\)
30. \(16 = 4^2\)
31. \(a^2 = 1.6\)
32. \(a^3 = \sqrt[3]{2}\)
33. \(2^3 = 7.2\)
34. \(3^5 = 4.6\)
35. \(e^5 = 8\)
36. \(e^{\sqrt{2}} = M\)

In Problems 37–44, change each logarithmic expression to an equivalent expression involving an exponent.

37. \(\log_2 8 = 3\)
38. \(\log_3 \left(\frac{1}{9}\right) = -2\)
39. \(\log_a 3 = 6\)
40. \(\log_6 4 = 2\)
41. \(\log_3 2 = x\)
42. \(\log_6 6 = x\)
43. \(\log_4 4 = x\)
44. \(\log x = 4\)

In Problems 45–56, find the exact value of each logarithm without using a calculator.

45. \(\log_2 1 = \)
46. \(\log_8 8 = \)
47. \(\log_5 25 = \)
48. \(\log_3 \left(\frac{1}{9}\right) = \)
49. \(\log_{1/2} 16 = \)
50. \(\log_{1/3} 9 = \)
51. \(\log_{10} \sqrt{10} = \)
52. \(\log_3 \sqrt{25} = \)
53. \(\log_{\sqrt{5}} 4 = \)
54. \(\log_{\sqrt{3}} 9 = \)
55. \(\log_e \sqrt{2} = \)
56. \(\log_e 8 = \)

In Problems 57–64, use a calculator to evaluate each expression. Round your answer to three decimal places.

57. \(\log_5 9 = \)
58. \(\log_5 3 = \)
59. \(\log_3 10 = \)
60. \(\log_3 \frac{2}{3} = \)
61. \(\log_5 9 = \)
62. \(\log_8 5 = \)
63. \(\log_4 10 = \)
64. \(\log_1 2 = \)
65. \(\log_{1.02} 5 = \)
66. \(\log_{1.05} 8 = \)
67. \(\log_{1.06} 4 = \)
68. \(\log_{1.12} 10 = \)
In Problems 65–74, evaluate each logarithm using the Change-of-Base formula. Round your answer to three decimal places.

65. \( \log_{1.006} 200 \)
66. \( \log_{1.005} 200 \)
67. \( \log_{1.001} 1000 \)
68. \( \log_{1.0001} 10 \)
69. \( \log_{1.002} 20 \)
70. \( \log_{1.0015} 30 \)
71. \( \log_{1.0005} 500 \)
72. \( \log_{1.00001} 1000 \)
73. \( \log_{1.003} 1000 \)
74. \( \log_{1.006} 500 \)

‘Are You Prepared?’ Answers

1. (a) 125 (b) \( \frac{1}{16} \)
2. (a) 4 (b) 3

A.4 Recursively Defined Sequences: Geometric Sequences

OBJECTIVES

1. Write the First Several Terms of a Sequence (p. A–34)
2. Write the Terms of a Sequence Defined by a Recursive Formula (p. A–36)
3. Find the First Term and Common Ratio of a Geometric Sequence (p. A–38)
4. Adding the First \( n \) Terms of a Geometric Sequence (p. A–39)

1. Write the First Several Terms of a Sequence

Definition

A sequence is a rule that assigns a real number to each positive integer.

A sequence is often represented by listing its values in order. For example, the sequence whose rule is to assign to each positive integer its reciprocal may be represented as

\[
s_1 = 1, \quad s_2 = \frac{1}{2}, \quad s_3 = \frac{1}{3}, \quad s_4 = \frac{1}{4}, \ldots, \quad s_n = \frac{1}{n}, \ldots
\]

or merely as the list

\[
1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \ldots, \quad \frac{1}{n}, \ldots
\]

The list never ends, as the dots indicate. The real numbers in this ordered list are called the terms of the sequence.

Notice that we use subscripted letters to represent the terms of a sequence. In the sequence above, \( s_1 \) is the 1st term, \( s_2 \) is the 2nd term, and so on. For the sequence above we have a rule for the \( n \)th term, namely, \( s_n = \frac{1}{n} \), so it is easy to find any term of the sequence. In this case we represent the sequence by placing braces around the formula for the \( n \)th term, writing \( \{s_n\} = \left\{ \frac{1}{n} \right\} \).
A.4 Recursively Defined Sequences: Geometric Sequences

For example, the sequence whose $n$th term is $b_n = \left(\frac{1}{2}\right)^n$ may be represented as

$$\{b_n\} = \left\{\left(\frac{1}{2}\right)^n\right\}$$

or by

$$b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{4}, \quad b_3 = \frac{1}{8}, \ldots, \quad b_n = \left(\frac{1}{2}\right)^n, \ldots$$

**EXAMPLE 1 Writing the Terms of a Sequence**

Write the first six terms of the sequence below.

$$\{a_n\} = \left\{\frac{n - 1}{n}\right\}$$

**SOLUTION**

$$a_1 = \frac{1 - 1}{1} = 0 \quad a_2 = \frac{2 - 1}{2} = \frac{1}{2}$$

$$a_3 = \frac{2}{3} \quad a_4 = \frac{3}{4}$$

$$a_5 = \frac{4}{5} \quad a_6 = \frac{5}{6}$$

**COMMENT** Graphing utilities can be used to write the terms of a sequence. Figure 15 shows the sequence given in Example 1 generated on a T1-84 Plus graphing calculator. We can see the first few terms of the sequence on the viewing window. You need to press the right arrow key to scroll right to see the remaining terms of the sequence. The TABLE feature can also be used to generate the terms of the sequence. See Table 2.

**FIGURE 15**

**TABLE 2**

NOW WORK PROBLEM 5.
EXAMPLE 2 Writing the Terms of a Sequence

Write the first six terms of the sequence below.

\[ \{ b_n \} = \left\{ (-1)^{n-1} \left( \frac{2}{n} \right) \right\} \]

**SOLUTION**

\[ b_1 = (-1)^0 \frac{2}{1} = 2 \quad b_2 = (-1)^1 \frac{2}{2} = -1 \]

\[ b_3 = (-1)^2 \frac{2}{3} = \frac{2}{3} \quad b_4 = (-1)^3 \frac{2}{4} = -\frac{1}{2} \]

\[ b_5 = (-1)^4 \frac{2}{5} = \frac{2}{5} \quad b_6 = (-1)^5 \frac{2}{6} = -\frac{1}{3} \]

Notice in the sequence \( \{ b_n \} \) in Example 2 that the signs of the terms **alternate**. Such sequences use factors such as \((-1)^{n-1}\), which equals 1 if \( n \) is odd and \(-1\) if \( n \) is even, or \((-1)^n\), which equals \(-1\) if \( n \) is odd and 1 if \( n \) is even.

**NOW WORK PROBLEM 7.**

Sometimes a sequence is indicated by an observed pattern in the first few terms that makes it possible to infer the makeup of the \( n \)th term. In the example that follows, a sufficient number of terms of the sequence is given so that a natural choice for the \( n \)th term is suggested.

EXAMPLE 3 Determining a Sequence from a Pattern

(a) \( e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots \quad a_n = \frac{e^n}{n} \)

(b) \( 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \quad b_n = \frac{1}{3^{n-1}} \)

(c) \( 1, 3, 5, 7, \ldots \quad c_n = 2n - 1 \)

(d) \( 1, 4, 9, 16, 25, \ldots \quad d_n = n^2 \)

(e) \( 1, \frac{-1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \quad e_n = (-1)^{n+1} \left( \frac{1}{n} \right) \)

**NOW WORK PROBLEM 15.**

Write the Terms of a Sequence Defined by a Recursive Formula

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the \( n \)th term by a formula or equation that involves one or more of the terms preceding it. Sequences defined this way are said to be defined **recursively**, and the rule or formula is called a **recursive formula**.
A.4 Recursively Defined Sequences: Geometric Sequences  

EXAMPLE 4 Writing the Terms of a Recursively Defined Sequence

Write the first five terms of the recursively defined sequence given below.

\[ s_1 = 1, \quad s_n = 4s_{n-1} \]

**SOLUTION** The first term is given as \( s_1 = 1 \). To get the second term, we use \( n = 2 \) in the formula to get \( s_2 = 4s_1 = 4 \cdot 1 = 4 \). To get the third term, we use \( n = 3 \) in the formula to get \( s_3 = 4s_2 = 4 \cdot 4 = 16 \). To get the new term requires that we know the value of the preceding term. The first five terms are

\[
\begin{align*}
  s_1 &= 1 \\
  s_2 &= 4 \cdot 1 = 4 \\
  s_3 &= 4 \cdot 4 = 16 \\
  s_4 &= 4 \cdot 16 = 64 \\
  s_5 &= 4 \cdot 64 = 256
\end{align*}
\]

NOW WORK PROBLEM 23.

EXAMPLE 5 Writing the Terms of a Recursively Defined Sequence

Write the first five terms of the recursively defined sequence given below.

\[ u_1 = 1, \quad u_2 = 1, \quad u_{n+2} = u_n + u_{n+1} \]

**SOLUTION** We are given the first two terms. To get the third term requires that we know each of the previous two terms.

\[
\begin{align*}
  u_1 &= 1 \\
  u_2 &= 1 \\
  u_3 &= u_1 + u_2 = 2 \\
  u_4 &= u_2 + u_3 = 1 + 2 = 3 \\
  u_5 &= u_3 + u_4 = 2 + 3 = 5
\end{align*}
\]

The sequence defined in Example 5 is called a **Fibonacci sequence**, and the terms of this sequence are called **Fibonacci numbers**. These numbers appear in a wide variety of applications.

EXAMPLE 6 Writing the Terms of a Recursively Defined Sequence

Write the first five terms of the recursively defined sequence given below.

\[ f_1 = 1, \quad f_{n+1} = (n + 1)f_n \]

**SOLUTION** Here,

\[
\begin{align*}
  f_1 &= 1 \\
  f_2 &= 2f_1 = 2 \cdot 1 = 2 \\
  f_3 &= 3f_2 = 3 \cdot 2 = 6 \\
  f_4 &= 4f_3 = 4 \cdot 6 = 24 \\
  f_5 &= 5f_4 = 5 \cdot 24 = 120
\end{align*}
\]

NOW WORK PROBLEM 31.
3 Find the First Term and Common Ratio of a Geometric Sequence

When the ratio of successive terms of a sequence is always the same nonzero number, the sequence is called geometric.

**Definition**

A geometric sequence* may be defined recursively as or as:

\[ a_1 = a, \quad a_{n+1} = ra_n \]

where \( a_1 = a \) and \( r \neq 0 \) are real numbers. The number \( a_1 \) is the first term, and the nonzero number \( r \) is called the common ratio.

The terms of a geometric sequence with first term \( a_1 \) and common ratio \( r \) follow the pattern

\[ a_1, \quad a_1r, \quad a_1r^2, \quad a_1r^3, \ldots \]

**EXAMPLE 7** Identifying a Geometric Sequence

The sequence

\[ 2, \quad 6, \quad 18, \quad 54, \quad 162, \ldots \]

is geometric since the ratio of successive terms is 3. \( \frac{6}{2} = 3, \quad \frac{18}{6} = 3, \quad \frac{54}{18} = 3, \ldots \) The first term is \( a_1 = 2 \), and the common ratio is 3.

**EXAMPLE 8** Finding the First Term and Common Ratio of a Geometric Sequence

Show that the sequence below is geometric. Find the first term and the common ratio.

\[ \{ t_n \} = \{ 4^n \} \]

**SOLUTION** The first term is \( t_1 = 4^1 = 4 \). The \((n + 1)\)st term and the \(n\)th term are

\[ t_{n+1} = 4^{n+1} \quad \text{and} \quad t_n = 4^n \]

Their ratio is

\[ \frac{t_{n+1}}{t_n} = \frac{4^{n+1}}{4^n} = 4 \]

Because the ratio of successive terms is a nonzero number, the sequence \( \{ t_n \} \) is a geometric sequence with common ratio 4.

NOW WORK PROBLEMS 37(a) AND (b).

* Sometimes called a geometric progression.
A.4 Recursively Defined Sequences: Geometric Sequences

**EXAMPLE 9** Finding the First Term and Common Ratio of a Geometric Sequence

Show that the sequence below is geometric. Find the first term and the common ratio.

\[ \{ s_n \} = \{ 2^{-n} \} \]

**SOLUTION**

The first term is \( s_1 = 2^{-1} = \frac{1}{2} \). The \((n + 1)\text{st}\) term and the \(n\text{th}\) term of the sequence \( \{ s_n \} \) are

\[ s_{n+1} = 2^{-(n+1)} \quad \text{and} \quad s_n = 2^{-n} \]

Their ratio is

\[ \frac{s_{n+1}}{s_n} = \frac{2^{-(n+1)}}{2^{-n}} = 2^{-n-1+n} = 2^{-1} = \frac{1}{2} \]

Because the ratio of successive terms is a nonzero number, the sequence \( \{ s_n \} \) is geometric with common ratio \( \frac{1}{2} \).

---

**Adding the First \( n \) Terms of a Geometric Sequence**

**Theorem**

**Sum of the First \( n \) Terms of a Geometric Sequence**

Let \( \{ a_n \} \) be a geometric sequence with first term \( a_1 \) and common ratio \( r \), where \( r \neq 0 \) and \( r \neq 1 \). The sum \( S_n \) of the first \( n \) terms of \( \{ a_n \} \) is

\[ S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = a_1 \left( \frac{1 - r^n}{1 - r} \right) \quad r \neq 0, 1 \quad \text{(1)} \]

**PROOF**

Formula (1) may be derived as follows. The sum \( S_n \) of the first \( n \) terms of \( \{ a_n \} \) is

\[ S_n = a_1 + a_1r + \cdots + a_1r^{n-1} \quad \text{(2)} \]

Multiply each side by \( r \) to obtain

\[ rS_n = a_1r + a_1r^2 + \cdots + a_1r^n \quad \text{(3)} \]

Now subtract (3) from (2). The result is

\[ S_n - rS_n = a_1 - a_1r^n \]

\[ (1 - r)S_n = a_1(1 - r^n) \quad \text{Factor.} \]

Since \( r \neq 1 \), we can solve for \( S_n \):

\[ S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right) \]
EXAMPLE 10  Adding the Terms of a Geometric Sequence

Find the sum \( S_n \) of the first \( n \) terms of the sequence \( \left\{ \left( \frac{1}{2} \right)^n \right\} \); that is, find

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left( \frac{1}{2} \right)^n
\]

**SOLUTION**  The sequence \( \left\{ \left( \frac{1}{2} \right)^n \right\} \) is a geometric sequence with \( a_1 = \frac{1}{2} \) and \( r = \frac{1}{2} \). Refer to Example 9. The sum \( S_n \) we seek is the sum of the first \( n \) terms of the sequence, so we use Formula (1) to get

\[
S_n = \frac{1}{2} \left[ \frac{1 - \left( \frac{1}{2} \right)^n}{1 - \frac{1}{2}} \right] = \frac{1 - \left( \frac{1}{2} \right)^n}{1 - \frac{1}{2}} = 1 - \left( \frac{1}{2} \right)^n
\]

**NOW WORK PROBLEM 37(c).**


Concepts and Vocabulary

1. For the sequence \( \{s_n\} = \{4n - 1\} \), the first term is \( s_1 = \) and the fourth term is \( s_4 = \).

Skill Building

In Problems 3–14, write down the first five terms of each sequence.

3. \( \{s_n\} = \{ n \} \)  
4. \( \{s_n\} = \{ n^2 + 1 \} \)  
5. \( \{a_n\} = \{ \frac{n}{n + 2} \} \)  
6. \( \{b_n\} = \{ \frac{2n + 1}{2n} \} \)  
7. \( \{c_n\} = \{ (-1)^n + 1 \} \)  
8. \( \{d_n\} = \{ (-1)^{n-1} \left( \frac{n}{2n - 1} \right) \} \)  
9. \( \{s_n\} = \{ \frac{2^n}{3^n + 1} \} \)  
10. \( \{s_n\} = \{ \frac{4^n}{3^n} \} \)  
11. \( \{t_n\} = \{ \frac{(-1)^n}{(n + 1)(n + 2)} \} \)  
12. \( \{a_n\} = \{ \frac{2^n}{n} \} \)  
13. \( \{b_n\} = \{ \frac{n}{n^2} \} \)  
14. \( \{c_n\} = \{ \frac{n^2}{2^n} \} \)
A.4 Recursively Defined Sequences: Geometric Sequences

In Problems 15–22, the given pattern continues. Write down the nth term of a sequence \( \{ a_n \} \) suggested by the pattern.

15. \[ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \]
16. \[ \frac{1}{1 \cdot 2}, \frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \ldots \]
17. \[ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots \]
18. \[ \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots \]
19. \[ 1, -1, 1, -1, 1, -1, \ldots \]
20. \[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots \]
21. \[ 1, -2, 3, -4, 5, -6, \ldots \]
22. \[ 2, -4, 6, -8, 10, \ldots \]

In Problems 23–36, a sequence is defined recursively. Write down the first five terms.

23. \( a_1 = 1; a_{n+1} = 2 + a_n \)
24. \( a_1 = 3; a_{n+1} = 5 - a_n \)
25. \( a_1 = -2; a_{n+1} = n + a_n \)
26. \( a_1 = 1; a_{n+1} = n - a_n \)
27. \( a_1 = 5; a_{n+1} = 2a_n \)
28. \( a_1 = 2; a_{n+1} = -a_n \)
29. \( a_1 = 3; a_{n+1} = \frac{a_n}{n} \)
30. \( a_1 = -2; a_{n+1} = n + 3a_n \)
31. \( a_1 = 1; a_2 = 2; a_{n+2} = a_n a_{n+1} \)
32. \( a_1 = -1; a_2 = 1; a_{n+2} = a_{n+1} + na_n \)
33. \( a_1 = 1; a_{n+1} = a_n + d \)
34. \( a_1 = 1; a_{n+1} = r a_n, \ r \neq 0 \)
35. \( a_1 = \sqrt{2}; a_{n+1} = \sqrt{2 + a_n} \)
36. \( a_1 = \sqrt{2}; a_{n+1} = \sqrt{\frac{a_n}{2}} \)

In Problems 37–46, a geometric sequence is given.

(a) Find the first term and the common ratio.
(b) Write out the first four terms.
(c) Find the sum of the first n terms.

37. \( \{ 2^n \} \)
38. \( \{ (-4)^n \} \)
39. \( \left\{ -\frac{3}{2} \right\} \)
40. \( \left\{ \frac{5}{2} \right\} \)
41. \( \left\{ \frac{2^{n-1}}{4} \right\} \)
42. \( \left\{ \frac{3^n}{9} \right\} \)
43. \( \left\{ 2^{n/3} \right\} \)
44. \( \left\{ 3^{2n} \right\} \)
45. \( \left\{ \frac{2^{n-1}}{2^n} \right\} \)
46. \( \left\{ \frac{2^n}{3^{n-1}} \right\} \)

Applications and Extensions

47. **Credit Card Debt** John has a balance of $3000 on his Discover card that charges 1% interest per month on any unpaid balance. John can afford to pay $100 toward the balance each month. His balance each month after making a $100 payment is given by the recursively defined sequence

\[ B_0 = 3000, \quad B_n = 1.01B_{n-1} - 100 \]

Determine John’s balance after making the first payment. That is, determine \( B_1 \).

48. **Trout Population** A pond currently has 2000 trout in it. A fish hatchery decides to add an additional 20 trout each month. In addition, it is known that the trout population is growing 3% per month. The size of the population after \( n \) months is given by the recursively defined sequence

\[ p_0 = 2000, \quad p_n = 1.03p_{n-1} + 20 \]

How many trout are in the pond after two months? That is, what is \( p_2 \)?

49. **Car Loans** Phil bought a car by taking out a loan for $18,500 at 0.5% interest per month. Phil’s normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

\[ B_0 = 18500, \quad B_n = 1.005B_{n-1} - 534.47 \]

Determine Phil’s balance after making the first payment. That is, determine \( B_1 \).

50. **Environmental Control** The Environmental Protection Agency (EPA) determines that Maple Lake has 250 tons of pollutant as a result of industrial waste and that 10% of the pollutant present is neutralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 tons of new pollutant entering the lake each year. The amount of pollutant in the lake after \( n \) years is given by the recursively defined sequence

\[ p_0 = 250, \quad p_n = 0.9p_{n-1} + 15 \]

Determine the amount of pollutant in the lake after 2 years. That is, determine \( p_2 \).
51. Growth of a Rabbit Colony
A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?

[Hint: A Fibonacci sequence models this colony. Do you see why?]

52. Fibonacci Sequence
Let
\[ u_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \]
define the nth term of a sequence.
(a) Show that \( u_1 = 1 \) and \( u_2 = 1 \).
(b) Show that \( u_{n+2} = u_{n+1} + u_n \).
(c) Draw the conclusion that \( \{u_n\} \) is a Fibonacci sequence.

53. Pascal’s Triangle
Divide the triangular array shown (called Pascal’s triangle) using diagonal lines as indicated. Find the sum of the numbers in each diagonal row. Do you recognize this sequence?

A.5 The Binomial Theorem

OBJECTIVES
1. Use the Binomial Theorem (p. A–42)
2. Establish Properties of Binomial Coefficients (p. A–45)

1. Use the Binomial Theorem
The Binomial Theorem deals with the problem of expanding an expression of the form \((x + y)^n\), where \( n \) is a positive integer.

Expressions such as \((x + y)^2\) and \((x + y)^3\) are not too difficult to expand. For example,
\[
(x + y)^2 = x^2 + 2xy + y^2 \\
(x + y)^3 = (x + y)^2(x + y) = (x^2 + 2xy + y^2)(x + y) = x^3 + 3x^2y + 3xy^2 + y^3
\]
However, expanding expressions such as \((x + y)^6\) or \((x + y)^8\) by the normal process of multiplication would be tedious and time consuming. It is here that the Binomial Theorem is especially useful.

Since \( \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1^* \), \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \), and \( \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1 \), we can write the expression
\[
(x + y)^2 = x^2 + 2xy + y^2
\]

* Recall the notation \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).
The Binomial Theorem generalizes this pattern.

**Theorem: Binomial Theorem**

Let $x$ and $y$ be real numbers. If $n$ is a positive integer, we have

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{1}
\]

Observe that the powers of $x$ begin at $n$ and decrease by 1, while the powers of $y$ begin with 0 and increase by 1. Also, the coefficient of the term involving $y^k$ is always $\binom{n}{k}$. Because of this, the notation $\binom{n}{k}$ is also called a **binomial coefficient**.

**Example 1: Expanding a Binomial**

Expand $(x + y)^6$ using the Binomial Theorem.

**Solution**

We apply Formula (1) with $n = 6$,

\[
(x + y)^6 = \binom{6}{0} x^6 + \binom{6}{1} x^5 y + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} xy^5 + \binom{6}{6} y^6
\]

\[
= x^6 + 6x^5 y + 15x^4 y^2 + 20x^3 y^3 + 15x^2 y^4 + 6xy^5 + y^6
\]

The coefficients in the expansion of $(x + y)^6$ are the entries in the Pascal triangle for $n = 6$. See Figure 24, page 475 and insert the row $n = 6$.

**Now work Problem 5.**
EXAMPLE 2 Expanding a Binomial

Expand \((x + 2y)^4\) using the Binomial Theorem.

**SOLUTION** Here, we let “2y” play the role of “y” in the Binomial Theorem. We then get

\[
(x + 2y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3(2y) + \binom{4}{2}x^2(2y)^2 + \binom{4}{3}x(2y)^3 + \binom{4}{4}(2y)^4
\]

\[=
 x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4\]

NOW WORK PROBLEM 7.

EXAMPLE 3 Finding a Particular Coefficient

Find the coefficient of \(x^3y^4\) in the expansion of \((x + y)^7\).

**SOLUTION** The expansion of \((x + y)^7\) is

\[
(x + y)^7 = \binom{7}{0}x^7 + \binom{7}{1}x^6y + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4
\]

\[+ \binom{7}{5}x^2y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7
\]

The coefficient of \(x^3y^4\) is

\[
\binom{7}{4} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35
\]

NOW WORK PROBLEM 11.

To explain why the Binomial Theorem is true, we take a closer look at what happens when we expand \((x + y)^3\). Write \((x + y)^3\) as the product of three factors, namely,

\[
(x + y)^3 = (x + y)(x + y)(x + y)
\]

Were we to multiply these three factors together without any attempt at simplification or collecting of terms, we would get

\[
(x + y)(x + y)(x + y) = xxx + xyx + yxx + yyx + xxy + yxy + xyx + yyy
\]

\[
\text{Factor} \quad \text{Factor} \quad \text{Factor}
\]

\[
1 \quad 2 \quad 3
\]

Notice that the terms on the right yield all possible products that can be formed by picking either an \(x\) or \(y\) from each of the three factors on the left. For example,

\[xyx\] results from choosing an \(x\) from factor 1, an \(y\) from factor 2, and an \(x\) from factor 3.
Now, the number of terms on the right that will simplify to, say, \(xy^2\), will be those terms that resulted from choosing \(y\)'s from two of the factors and an \(x\) from the remaining factor. How many such terms are there? There are as many as there are ways of choosing two of the three factors to contribute \(y\)'s—that is, there are \(C(3, 2) = \binom{3}{2}\) such terms.

This is why the coefficient of \(xy^2\) in the expansion of \((x + y)^3\) is \(\binom{3}{2}\).

In general, if we write \((x + y)^n\) as the product of \(n\) factors,

\[
(x + y)^n = (x + y) \cdot (x + y) \cdot \ldots \cdot (x + y)
\]

then, upon multiplying out and simplifying, there will be as many terms of the form \(x^{n-k}y^k\) as there are ways of choosing \(k\) of the \(n\) factors to contribute \(y\)'s (and the remaining \(n - k\) to contribute \(x\)'s). There are \(C(n, k)\) ways of making this choice. So the coefficient of \(x^{n-k}y^k\) is \(\binom{n}{k}\), and this is the assertion of the Binomial Theorem.

### 2 Establish Properties of Binomial Coefficients

Binomial coefficients have some interesting properties. For example, it turns out that \(\binom{n}{k}\) and \(\binom{n}{n-k}\) are equal. We can explain this equality as follows: Suppose we wanted to pick a team of \(k\) players from \(n\) people. Then choosing those \(k\) who will play is the same as choosing those \(n - k\) who will not. So the number of ways of choosing the players equals the number of ways of choosing the nonplayers. The players can be chosen in \(C(n, k) = \binom{n}{k}\) ways, while those to be left out can be chosen in \(\binom{n}{n-k}\) ways, and the equality follows.

### Example 4 Proving a Property of Binomial Coefficients

Show that 

\[
\binom{n}{k} = \binom{n}{n-k}
\]

**Solution** By definition,

\[
\binom{n}{k} = \frac{n!}{(n-k)!k!}
\]

while

\[
\binom{n}{n-k} = \frac{n!}{(n-k)![(n-(n-k))!]
\]

Since \(n - (n - k) = k\), a comparison of the expressions shows that they are equal.  ■
Based on Example 4, we have
\[
\binom{5}{3} = \binom{5}{2} = \binom{10}{2} = \binom{10}{8}
\]
and so on. This identity accounts for the symmetry in the rows of Pascal’s triangle.

**EXAMPLE 5** Proving a Property of Binomial Coefficients

Show that
\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

**SOLUTION** We could expand both sides of the above identity using the definition of binomial coefficients and, after some algebra, demonstrate the equality. But we choose the route of posing a problem that we solve two different ways. Equating the two solutions will prove the identity.

A committee of \(k\) is to be chosen from \(n\) people. The total number of ways this can be done is \(C(n, k) = \binom{n}{k}\).

Now count the total number of committees a different way. Assume that one of the \(n\) people is Jennifer. We compute

1. those committees not containing Jennifer
   and
2. those committees containing Jennifer

The number of committees of type 1 is \(\binom{n - 1}{k}\) since the \(k\) people must be chosen from the \(n - 1\) people who are not Jennifer. The number of committees of type 2 will correspond to the number of ways we can choose the \(k - 1\) people other than Jennifer to be on the committee. So the number of committees of type 2 is given by \(\binom{n - 1}{k - 1}\). Since the number of committees of type 1 plus the number of committees of type 2 equals the total number of committees, our identity follows.

For example, \(\binom{8}{5} = \binom{7}{5} + \binom{7}{4}\). It is precisely this identity that explains the reason why an entry in Pascal’s triangle can be obtained by adding the nearest two entries in the row above it. Due to its recursive character, the identity in Example 5 is sometimes used in computer programs that evaluate binomial coefficients.

**EXAMPLE 6** Establishing a Relationship of Specific Binomial Coefficients

Show that
\[
\binom{6}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}
\]
We make repeated use of the identity in Example 5. So
\[
\binom{6}{3} = \binom{5}{3} + \binom{5}{2}
\]
\[
= \left[ \binom{4}{3} + \binom{4}{2} \right] + \binom{5}{2}
\]
Apply the identity to \(\binom{5}{3}\).
\[
= \left[ \binom{3}{3} + \binom{3}{2} \right] + \left[ \binom{4}{2} + \binom{5}{2} \right]
\]
Apply the identity to \(\binom{4}{3}\).
\[
= \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}
\]
Since \(\binom{2}{2} = \binom{3}{3} = 1\).
EXAMPLE 8 Proving a Property of Binomial Coefficients

Show that

\[
\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0 \quad (2)
\]

(The last term will be preceded by a plus or minus sign depending on whether \(n\) is even or odd.)

**SOLUTION** We use the Binomial Theorem. This time we let \(x = 1\) and \(y = -1\). This gives

\[
[1 + (-1)]^n = \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} \cdot (-1) + \binom{n}{2} 1^{n-2} \cdot (-1)^2 + \binom{n}{3} 1^{n-3} \cdot (-1)^3 + \cdots + \binom{n}{n} (-1)^n
\]

\[
0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}
\]

For example, when \(n = 5\), property (2) states that

\[
\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} = 0
\]

Rearranging some terms yields

\[
\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = \binom{5}{1} + \binom{5}{3} + \binom{5}{5}
\]

This says that a set with 5 elements has as many subsets containing an even number of elements as it has subsets containing an odd number of elements.


**Concepts and Vocabulary**

1. \(\binom{6}{2} = \) 
2. \((x + y)^4 = \) 
3. True or False \(\binom{n}{j} = \frac{j!}{(n-j)!n!} \)
4. The _____ _____ can be used to expand expressions like \((2x + 3)^6\).

In Problems 5–10, use the Binomial Theorem to expand each expression.

5. \((x + y)^5\) 
6. \((x + y)^4\) 
7. \((x + 3y)^3\)

8. \((2x + y)^3\) 
9. \((2x - y)^4\) 
10. \((x - y)^4\)

11. What is the coefficient of \(x^2y^3\) in the expansion of \((x + y)^5\)?
12. What is the coefficient of \(x^2y^6\) in the expansion of \((x + y)^8\)?
13. What is the coefficient of \(x^8\) in the expansion of \((x + 3)^{10}\)?
14. What is the coefficient of \(x^3\) in the expansion of \((x + 2)^7\)?
15. How many different subsets does a set with 8 elements have?
16. How many different subsets does a set with 50 elements have?
Applications and Extensions

17. Use the Binomial Theorem to find the numerical value of $(1.001)^5$ correct to five decimal places.
   [Hint: $(1.001)^5 = (1 + 10^{-3})^5$]

18. Use the Binomial Theorem to find the numerical value of $(0.998)^6$ correct to five decimal places.

19. How many nonempty subsets does a set with 10 elements have?

20. How many subsets with an even number of elements does a set with 10 elements have?

21. How many subsets with an odd number of elements does a set with 10 elements have?

22. Show that
   \[
   \binom{8}{5} = \binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}
   \]

23. Show that
   \[
   \binom{10}{7} = \binom{6}{6} + \binom{7}{6} + \binom{8}{6} + \binom{9}{6}
   \]

24. Show that
   \[
   \binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7} = 2^6
   \]

25. Replace \(\binom{11}{6} + \binom{11}{5}\) by a single binomial coefficient.

26. Replace \(\binom{8}{8} + \binom{9}{8} + \binom{10}{8}\) by a single binomial coefficient.

27. \[
   \binom{5}{0} \left(\frac{1}{4}\right)^5 + \binom{5}{1} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right) + \binom{5}{2} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 \\
   + \binom{5}{3} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^3 + \binom{5}{4} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^4 + \binom{5}{5} \left(\frac{3}{4}\right)^5 = ?
   \]

28. **Stirling's Formula**  
   An approximation for \(n!\), when \(n\) is large, is given by
   \[
   n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - 1}\right)
   \]
   Calculate 12!, 20!, and 25! on your calculator. Then use Stirling's formula to approximate 12!, 20!, and 25!.
A number of available software packages can be used to solve linear programming problems. Because of the popularity of LINDO, we have chosen to include a brief section illustrating how it can be used to solve some linear programming problems. LINDO was designed to allow users to do simple problems in an easy, cost efficient manner. At the other extreme, LINDO has been used to solve real industrial programs involving more than 10,000 rows and several thousand variables. In solving such problems, LINDO uses a revised simplex method—one that exploits the special character of large linear programming problems. As a result, the intermediate tableaus obtained using LINDO may differ from those obtained manually. The latest version of LINDO, 6.1, is available for downloading at www.lindo.com.

EXAMPLE 1 Using LINDO

Use LINDO to Maximize

\[ P = 3x_1 + 4x_2 \]

subject to

\[ 2x_1 + 4x_2 \leq 120 \]
\[ 2x_1 + 2x_2 \leq 80 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

SOLUTION

The objective function

\[ \text{MAX } 3x_1 + 4x_2 \]

The constraints:

\[ 2x_1 + 4x_2 \leq 120 \]
\[ 2x_1 + 2x_2 \leq 80 \]

The initial tableau. SLK 2 is the first constraint equation. ROW 1 ART is the objective row. The column for P is not written since pivoting does not affect it. The column on the far right is RHS. The bottom row ART is added to LINDO to represent the pivot objective of minimizing the sum of the infeasibilities.
The pivot element is in row 3, column X1. X1 will become a basic variable.

After pivoting, SLK 2 and X1 are basic variables. \( P = 120 \)

at this stage; SLK 2 = 40;

X1 = 40; SLK 3 = 0; X2 = 0.

The pivot element is in row 2, column X2. After pivoting, X2 and X1 are basic variables.

This is a final tableau. The solution is \( P = 140 \),

\( X1 = 20 \), \( X2 = 20 \).

Example 2 Using LINDO

Use LINDO to Maximize

\[ P = 20x_1 + 15x_2 \]

subject to

\[ \begin{align*}
  x_1 + x_2 & \geq 7 \\
  9x_1 + 5x_2 & \leq 45 \\
  2x_1 + x_2 & \geq 8 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0
\end{align*} \]

Solution

\[ \text{MAX} \quad 20 \times 1 + 15 \times 2 \]

\[ \text{SUBJECT TO} \]

\[ \begin{align*}
  x_1 + x_2 & \geq 7 \\
  9x_1 + 5x_2 & \leq 45 \\
  2x_1 + x_2 & \geq 8 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0
\end{align*} \]

\[ \text{END} \]

The Tableau

\[ \begin{align*}
  \text{ROW} & \quad \text{BASIS} & \quad x_1 & \quad x_2 & \quad \text{SLK} 2 & \quad \text{SLK} 3 & \quad \text{SLK} 4 \\
  1 & \text{ART} & -20.000 & -15.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
  2 & \text{SLK} 2 & -1.000 & -1.000 & 1.000 & 0.000 & 0.000 & -7.000 \\
  3 & \text{SLK} 3 & 9.000 & 5.000 & 0.000 & 1.000 & 0.000 & 45.000 \\
  4 & \text{SLK} 4 & -2.000 & -1.000 & 0.000 & 0.000 & 1.000 & -8.000 \\
  \text{ART} & \text{ART} & -3.000 & -2.000 & 0.000 & 0.000 & 0.000 & -15.000
\end{align*} \]
Appendix B  Using LINDO to Solve Linear Programming Problems  B–3

**X1 ENTERS AT VALUE 4.0000 IN ROW 4 OBJ. VALUE = 80.00**

**THE TABLEAU**

<table>
<thead>
<tr>
<th>ROW (BASIS)</th>
<th>X1</th>
<th>X2</th>
<th>SLK 2</th>
<th>SLK 3</th>
<th>SLK 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ART</td>
<td>0.000</td>
<td>-5.000</td>
<td>0.000</td>
<td>0.000</td>
<td>-10.000</td>
</tr>
<tr>
<td>2 SLK 2</td>
<td>0.000</td>
<td>-0.500</td>
<td>1.000</td>
<td>0.000</td>
<td>-0.500</td>
</tr>
<tr>
<td>3 SLK 3</td>
<td>0.000</td>
<td>0.500</td>
<td>0.000</td>
<td>1.000</td>
<td>4.500</td>
</tr>
<tr>
<td>4 x1</td>
<td>1.000</td>
<td>0.500</td>
<td>0.000</td>
<td>0.000</td>
<td>-0.500</td>
</tr>
<tr>
<td>ART</td>
<td>0.000</td>
<td>-0.500</td>
<td>0.000</td>
<td>0.000</td>
<td>-0.500</td>
</tr>
</tbody>
</table>

**X2 ENTERS AT VALUE 8.000 IN ROW 4 OBJ. VALUE = 120.00**

**THE TABLEAU**

<table>
<thead>
<tr>
<th>ROW (BASIS)</th>
<th>X1</th>
<th>X2</th>
<th>SLK 2</th>
<th>SLK 3</th>
<th>SLK 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ART</td>
<td>10.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>-15.000</td>
</tr>
<tr>
<td>2 SLK 2</td>
<td>1.000</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
<td>-1.000</td>
</tr>
<tr>
<td>3 SLK 3</td>
<td>-1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>5.000</td>
</tr>
<tr>
<td>4 x2</td>
<td>2.000</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>-1.000</td>
</tr>
<tr>
<td>ART</td>
<td>10.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>-15.000</td>
</tr>
</tbody>
</table>

**SLK 4 ENTERS AT VALUE 1.0000 IN ROW 3 OBJ. VALUE = 135.00**

**THE TABLEAU**

<table>
<thead>
<tr>
<th>ROW (BASIS)</th>
<th>X1</th>
<th>X2</th>
<th>SLK 2</th>
<th>SLK 3</th>
<th>SLK 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ART</td>
<td>7.000</td>
<td>0.000</td>
<td>0.000</td>
<td>3.000</td>
<td>0.000</td>
</tr>
<tr>
<td>2 SLK 2</td>
<td>0.800</td>
<td>0.000</td>
<td>1.000</td>
<td>0.200</td>
<td>0.000</td>
</tr>
<tr>
<td>3 SLK 4</td>
<td>-0.200</td>
<td>0.000</td>
<td>0.000</td>
<td>0.200</td>
<td>1.000</td>
</tr>
<tr>
<td>4 x2</td>
<td>1.800</td>
<td>1.000</td>
<td>0.000</td>
<td>0.200</td>
<td>0.000</td>
</tr>
</tbody>
</table>

LP OPTIMUM FOUND AT STEP  3

**OBJECTIVE FUNCTION VALUE**

1)  135.000000

**VARIABLE**  **VALUE**  **REDUCED COST**

<table>
<thead>
<tr>
<th></th>
<th><strong>VALUE</strong></th>
<th><strong>REDUCED COST</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>0.000000</td>
<td>7.000000</td>
</tr>
<tr>
<td>X2</td>
<td>9.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

**ROW**  **SLACK OR SURPLUS**  **DUAL PRICES**

2)  2.000000  0.000000
3)  0.000000  3.000000
4)  1.000000  0.000000

**NO. ITERATIONS =  3**

Compare the final tableau found using LINDO with the final tableau of Example 1, Section 5.4, namely,

\[
<table>
<thead>
<tr>
<th>BV</th>
<th>P</th>
<th>x_1</th>
<th>x_2</th>
<th>s_1</th>
<th>s_2</th>
<th>s_3</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_2</td>
<td>0</td>
<td>\frac{9}{5}</td>
<td>1</td>
<td>0</td>
<td>\frac{1}{5}</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>\frac{s_3}{5}</td>
<td>0</td>
<td>-\frac{1}{5}</td>
<td>0</td>
<td>0</td>
<td>\frac{1}{5}</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>\frac{s_1}{5}</td>
<td>0</td>
<td>\frac{4}{5}</td>
<td>0</td>
<td>1</td>
<td>\frac{1}{5}</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>135</td>
</tr>
</tbody>
</table>
\]
The solution, in both cases, is $P = 135$, $x_2 = 9$, $x_1 = 0$, $s_1 = \text{SLK } 2 = 2$, $s_2 = \text{SLK } 3 = 0$, $s_3 = \text{SLK } 4 = 1$.

Notice that the intermediate tableaus are different. This is because LINDO uses a different methodology of the simplex method.

EXAMPLE 3 Using LINDO

Use LINDO to Minimize

$$z = 5x_1 + 6x_2$$

subject to

$$x_1 + x_2 \leq 10$$
$$x_1 + 2x_2 \geq 12$$
$$2x_1 + x_2 \geq 12$$
$$x_1 \geq 3$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

SOLUTION

MIN $5x_1 + 6x_2$
SUBJECT TO

$$x_1 + x_2 \leq 10$$
$$x_1 + 2x_2 \geq 12$$
$$2x_1 + x_2 \geq 12$$
$$x_1 \geq 3$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

X1 ENTERS AT VALUE 3.0000 IN ROW 5 OBJ. VALUE = -15.000

X2 ENTERS AT VALUE 6.0000 IN ROW 4 OBJ. VALUE = -51.000
**Appendix B  Using LINDO to Solve Linear Programming Problems  B–5**

SLK 5 ENTERS AT VALUE 1.0000 IN ROW 3 OBJ. VALUE = -44.000

THE TABLEAU

<table>
<thead>
<tr>
<th>ROW (BASIS)</th>
<th>X1</th>
<th>X2</th>
<th>SLK 2</th>
<th>SLK 3</th>
<th>SLK 4</th>
<th>SLK 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ART</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>2.333</td>
<td>1.333</td>
<td>0.000</td>
</tr>
<tr>
<td>2 SLK 2</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>0.333</td>
<td>0.333</td>
</tr>
<tr>
<td>3 SLK 5</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.333</td>
<td>-0.667</td>
<td>1.000</td>
</tr>
<tr>
<td>4 X2</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
<td>-0.667</td>
<td>0.333</td>
<td>0.000</td>
</tr>
<tr>
<td>5 X1</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.333</td>
<td>-0.667</td>
<td>0.000</td>
</tr>
</tbody>
</table>

LP OPTIMUM FOUND AT STEP 3
OBJECTIVE FUNCTION VALUE
1) 44.000000

VARIABLE VALUE REDUCED COST
| X1 | 4.0000000 | 0.000000 |
| X2 | 4.0000000 | 0.000000 |

ROW SLACK OR SURPLUS DUAL PRICES
2) 2.0000000 | -2.333333 |
3) 0.0000000 | -1.333333 |
4) 0.0000000 | -0.666667 |
5) 1.0000000 | 0.000000 |

NO. ITERATIONS = 3

Compare this result, using LINDO, to the result obtained in Example 2, Section 5.4. Note again that the same solution is found: $z = 44, x_1 = 4, x_2 = 4, s_1 = SLK 2 = 2, s_2 = SLK 3 = 0, s_3 = SLK 4 = 0, s_4 = SLK 5 = 1$. Also, note again the different intermediate tableaus.
Graphing Utilities

C.1 The Viewing Rectangle

All graphing utilities, that is, all graphing calculators and all computer software graphing packages, graph equations by plotting points on a screen. The screen itself actually consists of small rectangles, called pixels. The more pixels the screen has, the better the resolution. Most graphing calculators have 48 pixels per square inch; most computer screens have 32 to 108 pixels per square inch. When a point to be plotted lies inside a pixel, the pixel is turned on (lights up). The graph of an equation is a collection of pixels. Figure 1 shows how the graph of $y = 2x$ looks on a TI-84 Plus graphing calculator.

The screen of a graphing utility will display the coordinate axes of a rectangular coordinate system. However, you must set the scale on each axis. You must also include the smallest and largest values of $x$ and $y$ that you want included in the graph. This is called setting the viewing rectangle or viewing window. Figure 2 illustrates a typical viewing window.

To select the viewing window, we must give values to the following expressions:

- $X_{\text{min}}$: the smallest value of $x$
- $X_{\text{max}}$: the largest value of $x$
- $X_{\text{scl}}$: the number of units per tick mark on the $x$-axis
- $Y_{\text{min}}$: the smallest value of $y$
- $Y_{\text{max}}$: the largest value of $y$
- $Y_{\text{scl}}$: the number of units per tick mark on the $y$-axis

Figure 3 illustrates these settings and their relation to the Cartesian coordinate system.
If the scale used on each axis is known, we can determine the minimum and maximum values of \( x \) and \( y \) shown on the screen by counting the tick marks. Look again at Figure 2. For a scale of 1 on each axis, the minimum and maximum values of \( x \) are \(-10\) and \(10\), respectively; the minimum and maximum values of \( y \) are also \(-10\) and \(10\). If the scale is 2 on each axis, then the minimum and maximum values of \( x \) are \(-20\) and \(20\), respectively; and the minimum and maximum values of \( y \) are \(-20\) and \(20\), respectively.

Conversely, if we know the minimum and maximum values of \( x \) and \( y \), we can determine the scales being used by counting the tick marks displayed. We shall follow the practice of showing the minimum and maximum values of \( x \) and \( y \) in our illustrations so that you will know how the viewing window was set. See Figure 4.

**EXAMPLE 1** Finding the Coordinates of a Point Shown on a Graphing Utility Screen

Find the coordinates of the point shown in Figure 5. Assume that the coordinates are integers.

**Solution**

First we note that the viewing window used in Figure 5 is
\[
X \text{ min } = -3 \quad Y \text{ min } = -4
\]
\[
X \text{ max } = 3 \quad Y \text{ max } = 4
\]
\[
X \text{ scl } = 1 \quad Y \text{ scl } = 2
\]

The point shown is 2 tick units to the left on the horizontal axis (scale \(=1\)) and 1 tick up on the vertical axis (scale \(=2\)). The coordinates of the point shown are \((-2, 2)\). 

**EXERCISE C.1** Answers Begin on Page AN–67.

1. In Problems 1–4, determine the coordinates of the points shown. Tell in which quadrant each point lies. Assume that the coordinates are integers.

2. In Problems 5–10, determine the viewing window used.
C.2 Using a Graphing Utility to Graph Equations

The graph of an equation in two variables can usually be obtained by plotting points in a rectangular coordinate system and connecting them. Graphing utilities perform these same steps when graphing an equation. For example, the TI-84 Plus determines 95 evenly spaced input values,* starting at $X_{\text{min}}$ and ending at $X_{\text{max}}$, uses the equation to determine the output values, plots these points on the screen, and finally (if in the connected mode) draws a line between consecutive points.

To graph an equation in two variables $x$ and $y$ using a graphing utility requires that the equation be written in the form $y = \{\text{expression in } x\}$. If the original equation is not in this form, replace it by equivalent equations until the form $y = \{\text{expression in } x\}$ is obtained. In general, there are four ways to obtain equivalent equations.

### Procedures That Result in Equivalent Equations

1. **Interchange the two sides of the equation:**
   
   Replace $3x + 5 = y$ by $y = 3x + 5$

2. **Simplify the sides of the equation by combining like terms, eliminating parentheses, and so on:**
   
   Replace $(2y + 2) + 6 = 2x + 5(x + 1)$ by $2y + 8 = 7x + 5$

3. **Add or subtract the same expression on both sides of the equation:**
   
   Replace $y + 3x - 5 = 4$ by $y + 3x - 5 + 5 = 4 + 5$

4. **Multiply or divide both sides of the equation by the same nonzero expression:**
   
   Replace $3y = 6 - 2x$ by $\frac{1}{3} \cdot 3y = \frac{1}{3}(6 - 2x)$

---

*These input values depend on the values of $X_{\text{min}}$ and $X_{\text{max}}$. For example, if $X_{\text{min}} = -10$ and $X_{\text{max}} = 10$, then the first input value will be $-10$ and the next input value will be $-10 + \frac{10 - (-10)}{94} = -9.7872$, and so on.*
EXAMPLE 1 Expressing an Equation in the Form $y = \{\text{expression in } x\}$

Solve for $y$: $2y + 3x - 5 = 4$

**SOLUTION** We replace the original equation by a succession of equivalent equations.

\[
\begin{align*}
2y + 3x - 5 &= 4 \\
2y + 3x - 5 + 5 &= 4 + 5 & \text{Add 5 to both sides.} \\
2y + 3x &= 9 & \text{Simplify.} \\
2y + 3x - 3x &= 9 - 3x & \text{Subtract 3x from both sides.} \\
2y &= 9 - 3x & \text{Simplify.} \\
\frac{2y}{2} &= \frac{9 - 3x}{2} & \text{Divide both sides by 2.} \\
y &= \frac{9 - 3x}{2} & \text{Simplify.}
\end{align*}
\]

Now we are ready to graph equations using a graphing utility. Most graphing utilities require the following steps:

**Steps for Graphing an Equation Using a Graphing Utility**

**STEP 1** Solve the equation for $y$ in terms of $x$.

**STEP 2** Get into the graphing mode of your graphing utility. The screen will usually display $Y_1 = $, prompting you to enter the expression involving $x$ that you found in Step 1. (Consult your manual for the correct way to enter the expression; for example, $y = x^2$ might be entered as $x^2$ or as $x \times x$ or as $xx^2$ 2).

**STEP 3** Select the viewing window. Without prior knowledge about the behavior of the graph of the equation, it is common to select the **standard viewing window**$^*$ initially. In this text the standard viewing window is

\[
\begin{align*}
X \text{ min} &= -10 & Y \text{ min} &= -10 \\
X \text{ max} &= 10 & Y \text{ max} &= 10 \\
X \text{ scl} &= 1 & Y \text{ scl} &= 1
\end{align*}
\]

**STEP 4** Graph.

**STEP 5** Adjust the viewing window until a complete graph is obtained.

$^*$ Some graphing utilities have a **ZOOM-STANDARD** feature that automatically sets the viewing window to the standard viewing window and graphs the equation.
EXAMPLE 2  Graphing an Equation on a Graphing Utility

Graph the equation: $6x^2 + 3y = 36$

**SOLUTION**

**STEP 1** We solve for $y$ in terms of $x$.

\[
6x^2 + 3y = 36 \\
3y = -6x^2 + 36 \\
y = -2x^2 + 12
\]

Subtract $6x^2$ from both sides of the equation. Divide both sides of the equation by 3 and simplify.

**STEP 2** From the $Y_1 =$ screen enter the expression $-2x^2 + 12$ after the prompt.

**STEP 3** Set the viewing window to the standard viewing window.

**STEP 4** Graph. The screen should look like Figure 6.

**STEP 5** The graph of $y = -2x^2 + 12$ is not complete. The value of $Y_{\text{max}}$ must be increased so that the top portion of the graph is visible. After increasing the value of $Y_{\text{max}}$ to 12, we obtain the graph in Figure 7. The graph is now complete.

Look again at Figure 7. Although a complete graph is shown, the graph might be improved by adjusting the values of $X_{\text{min}}$ and $X_{\text{max}}$. Figure 8 shows the graph of $y = -2x^2 + 12$ using $X_{\text{min}} = -4$ and $X_{\text{max}} = 4$. Do you think this is a better choice for the viewing window?

EXAMPLE 3  Creating a Table and Graphing an Equation

Create a table and graph the equation: $y = x^3$

**SOLUTION** Most graphing utilities have the capability of creating a table of values for an equation. (Check your manual to see if your graphing utility has this capability.) Table 1 illustrates a table of values for $y = x^3$ on a TI-84 Plus. See Figure 9 for the graph.
In Problems 1–16, graph each equation using the following viewing windows:

(a) \( X_{\text{min}} = -5 \) \( X_{\text{max}} = 5 \) \( X_{\text{scl}} = 1 \) \( Y_{\text{min}} = -4 \) \( Y_{\text{max}} = 4 \) \( Y_{\text{scl}} = 1 \)
(b) \( X_{\text{min}} = -10 \) \( X_{\text{max}} = 10 \) \( X_{\text{scl}} = 2 \) \( Y_{\text{min}} = -8 \) \( Y_{\text{max}} = 8 \) \( Y_{\text{scl}} = 2 \)
(c) \( X_{\text{max}} = -10 \) \( X_{\text{max}} = 10 \) \( X_{\text{scl}} = 2 \) \( Y_{\text{min}} = -20 \) \( Y_{\text{max}} = 20 \) \( Y_{\text{scl}} = 5 \)
(d) \( X_{\text{max}} = -5 \) \( X_{\text{max}} = 5 \) \( X_{\text{scl}} = 1 \) \( Y_{\text{min}} = -2 \) \( Y_{\text{max}} = 2 \) \( Y_{\text{scl}} = 5 \)

1. \( y = x + 2 \)
2. \( y = x - 2 \)
3. \( y = -x + 2 \)
4. \( y = -x - 2 \)
5. \( y = 2x + 2 \)
6. \( y = 2x - 2 \)
7. \( y = -2x + 2 \)
8. \( y = -2x - 2 \)
9. \( y = x^2 + 2 \)
10. \( y = x^2 - 2 \)
11. \( y = -x^2 + 2 \)
12. \( y = -x^2 - 2 \)
13. \( 3x + 2y = 6 \)
14. \( 3x - 2y = 6 \)
15. \(-3x + 2y = 6 \)
16. \(-3x - 2y = 6 \)

17–32. For each of the above equations, create a table, \(-3 \leq x \leq 3\), and list points on the graph.

**C.3 Square Screens**

Most graphing utilities have a rectangular screen. Because of this, using the same settings for both \( x \) and \( y \) will result in a distorted view. For example, Figure 10 shows the graph of the line \( y = x \) connecting the points \((-4, -4)\) and \((4, 4)\).

We expect the line to bisect the first and third quadrants, but it doesn’t. We need to adjust the selections for \( X_{\text{min}} \), \( X_{\text{max}} \), \( Y_{\text{min}} \), and \( Y_{\text{max}} \) so that a square screen results. On most graphing utilities, this is accomplished by setting the ratio of \( x \) to \( y \) at 3:2.* For example, if

\[
\begin{align*}
X_{\text{max}} - X_{\text{min}} &= 6 - (-6) \\
Y_{\text{max}} - Y_{\text{min}} &= 4 - (-4)
\end{align*}
\]

then the ratio of \( x \) to \( y \) is

\[
\frac{X_{\text{max}} - X_{\text{min}}}{Y_{\text{max}} - Y_{\text{min}}} = \frac{12}{8} = \frac{3}{2}
\]

for a ratio of 3:2, resulting in a square screen.

**EXAMPLE 1**

**Examples of Viewing Rectangles That Result in Square Screens**

\[\begin{array}{ccc}
(a) & (b) & (c) \\
X_{\text{min}} &= -3 & X_{\text{min}} &= -6 & X_{\text{min}} &= -6 \\
X_{\text{max}} &= 3 & X_{\text{max}} &= 6 & X_{\text{max}} &= 6 \\
X_{\text{scl}} &= 1 & X_{\text{scl}} &= 1 & X_{\text{scl}} &= 2 \\
Y_{\text{min}} &= -2 & Y_{\text{min}} &= -2 & Y_{\text{min}} &= -4 \\
Y_{\text{max}} &= 2 & Y_{\text{max}} &= 6 & Y_{\text{max}} &= 4 \\
Y_{\text{scl}} &= 1 & Y_{\text{scl}} &= 1 & Y_{\text{scl}} &= 1 \\
\end{array}\]

Figure 11 shows the graph of the line \( y = x \) on a square screen using the viewing rectangle given in Example 1(b). Notice that the line now bisects the first and third quadrants. Compare this illustration to Figure 10.

* Some graphing utilities have a built-in function that automatically squares the screen. For example, the TI-85 has a ZSQR function that does this. Some graphing utilities require a ratio other than 3:2 to square the screen. For example, the HP 48G requires the ratio of \( x \) to \( y \) to be 2:1 for a square screen. Consult your manual.
EXERCISE C.3  Answers Begin on Page AN–68.

In Problems 1–8, determine which of the given viewing rectangles result in a square screen.

1. \( X_{\min} = -3 \) \( X_{\max} = 3 \) \( X_{\text{scl}} = 2 \)
   \( Y_{\min} = -2 \) \( Y_{\max} = 2 \) \( Y_{\text{scl}} = 2 \)

2. \( X_{\min} = -5 \) \( X_{\max} = 5 \) \( X_{\text{scl}} = 1 \)
   \( Y_{\min} = -4 \) \( Y_{\max} = 4 \) \( Y_{\text{scl}} = 1 \)

3. \( X_{\min} = 0 \) \( X_{\max} = 9 \) \( X_{\text{scl}} = 3 \)
   \( Y_{\min} = -2 \) \( Y_{\max} = 4 \) \( Y_{\text{scl}} = 2 \)

4. \( X_{\min} = -6 \) \( X_{\max} = 6 \) \( X_{\text{scl}} = 2 \)
   \( Y_{\min} = -6 \) \( Y_{\max} = 6 \) \( Y_{\text{scl}} = 2 \)

5. \( X_{\min} = -6 \) \( X_{\max} = 6 \) \( X_{\text{scl}} = 1 \)
   \( Y_{\min} = -4 \) \( Y_{\max} = 4 \) \( Y_{\text{scl}} = 1 \)

6. \( X_{\min} = -6 \) \( X_{\max} = 6 \) \( X_{\text{scl}} = 2 \)
   \( Y_{\min} = -2 \) \( Y_{\max} = 4 \) \( Y_{\text{scl}} = 1 \)

7. \( X_{\min} = 0 \) \( X_{\max} = 9 \) \( X_{\text{scl}} = 1 \)
   \( Y_{\min} = -2 \) \( Y_{\max} = 4 \) \( Y_{\text{scl}} = 2 \)

8. \( X_{\min} = -6 \) \( X_{\max} = 12 \) \( X_{\text{scl}} = 2 \)
   \( Y_{\min} = -6 \) \( Y_{\max} = 6 \) \( Y_{\text{scl}} = 2 \)

9. If \( X_{\min} = -4 \), \( X_{\max} = 8 \), and \( X_{\text{scl}} = 1 \), how should \( Y_{\min} \), \( Y_{\max} \), and \( Y_{\text{scl}} \) be selected so that the viewing rectangle contains the point \((4, 8)\) and the screen is square?

10. If \( X_{\min} = -6 \), \( X_{\max} = 12 \), and \( X_{\text{scl}} = 2 \), how should \( Y_{\min} \), \( Y_{\max} \), and \( Y_{\text{scl}} \) be selected so that the viewing rectangle contains the point \((4, 8)\) and the screen is square?

C.4 Using a Graphing Utility to Graph Inequalities

EXAMPLE 1  Graphing an Inequality Using a Graphing Utility

Use a graphing utility to graph: \( 3x + y - 6 \leq 0 \)

**SOLUTION**  We begin by graphing the equation \( 3x + y - 6 = 0 \) \((Y_1 = -3x + 6)\). See Figure 12.  

As with graphing by hand, we need to test points selected from each region and determine whether they satisfy the inequality. To test the point \((-1, 2)\), for example, enter \(3(-1) + 2 - 6 \leq 0\). See Figure 13(a). The 1 that appears indicates that the statement entered \((the\ inequality)\ is\ true. When\ the\ point\ \((5, 5)\)\ is tested, a 0 appears, indicating that the statement entered is false. So, \((-1, 2)\) is a part of the graph of the inequality and \((5, 5)\) is not. Figure 13(b) shows the graph of the inequality on a TI-84 Plus.*

* Consult your owner’s manual for shading techniques.
Steps for Graphing an Inequality Using a Graphing Utility

STEP 1 Replace the inequality symbol by an equal sign, solve the resulting equation for \( y \), and graph, \( Y_1 = \).

STEP 2 In each of the regions, select a test point \( P \).
   (a) Use a graphing utility to determine if the test point \( P \) satisfies the inequality. If the test point satisfies the inequality, then so do all the points in the region. Indicate this by using the graphing utility to shade the region.
   (b) If the coordinates of \( P \) do not satisfy the inequality, then none of the points in that region do.
CHAPTER 1  Linear Equations

Exercise 1.1 (p. 16)

3. True  
4. $x = -3$  
5. Vertical  
6. $m = -4$; $y$-intercept: $(0, 6)$  
7. $y - y_1 = m(x - x_1)$  
8. negative  
9. $A = (4, 2); B = (6, 2); C = (5, 3); D = (-2, 1); E = (-2, -3); F = (3, -2); G = (6, -2); H = (5, 0)$  
11. The set of points of the form $(2, y)$, where $y$ is a real number, is a vertical line passing through $(2, 0)$ on the $x$-axis.

13. $x$ $0$ $-2$ $2$ $-2$ $4$ $-4$  
$y$ $4$ $0$ $8$ $0$ $12$ $-4$

15. $x$ $0$ $3$ $2$ $-2$ $4$ $-4$  
$y$ $-6$ $0$ $-2$ $-10$ $2$ $-14$

17. (a) Vertical line: $x = 2$  
(b) Horizontal line: $y = -3$  
(c) $5x - y = 13$

19. (a) Vertical line: $x = -4$  
(b) Horizontal line: $y = 1$  
(c) $5x - y = -21$

21. (a) Vertical line: $x = 0$  
(b) Horizontal line: $y = 3$  
(c) $5x - y = -3$

23. $m = \frac{1}{2}$  
A slope of $\frac{1}{2}$ means that for every 2 unit change in $x$, $y$ changes 1 unit.

25. $m = -1$  
A slope of $-1$ means that for every 1 unit change in $x$, $y$ changes by $(1)$ unit.

27. $m = 3$  
A slope of 3 means that for every 1 unit change in $x$, $y$ will change 3 units.

29. $m = -\frac{1}{2}$  
A slope of $-\frac{1}{2}$ means that for every 2 unit change in $x$, $y$ will change $(1)$ unit.

31. $m = 0$  
A slope of 0 means that regardless how $x$ changes, $y$ remains constant.

Answers to Odd-Numbered Problems
AN–2  Answers to Odd-Numbered Problems

33. The slope is not defined.

35. \( y = 2x + 3 \)

37. \( y = -x - 2 \)

39. \( y = 0 \)

41. \( y = -x + 1 \)

43. \( x - 2y = 0 \)

45. \( x + y = 2 \)

47. \( 2x - y = -9 \)

49. \( 2x + 3y = -1 \)

51. \( x - 2y = -5 \)

53. \( 2x + y = 3 \)

55. \( 3x - y = -12 \)

57. \( 4x - 5y = 0 \)

59. \( x - 2y = 2 \)

61. \( x = 1 \)

63. \( y = 4 \)

65. slope: \( m = -2 \)

\( y \)-intercept: \((0, 3)\)

67. slope: \( m = 2 \)

\( y \)-intercept: \((0, -3)\)

69. slope: \( m = -\frac{2}{3} \)

\( y \)-intercept: \((0, -2)\)

71. slope: \( m = -1 \)

\( y \)-intercept: \((0, 1)\)

73. Slope is not defined; there is no \( y \)-intercept.

75. slope: \( m = 0 \)

\( y \)-intercept: \((0, 5)\)

77. slope: \( m = 1 \)

\( y \)-intercept: \((0, 0)\)

79. slope: \( m = \frac{3}{2} \)

\( y \)-intercept: \((0, 0)\)

81. Window:

\[ X\text{ min} = -10; X\text{ max} = 10 \]

\[ Y\text{ min} = -10; Y\text{ max} = 10 \]

83. Window:

\[ X\text{ min} = -10; X\text{ max} = 10 \]

\[ Y\text{ min} = -10; Y\text{ max} = 10 \]

85. Window:

\[ X\text{ min} = -10; X\text{ max} = 10 \]

\[ Y\text{ min} = -10; Y\text{ max} = 10 \]

x-intercept: \((1.67, 0)\)

y-intercept: \((0, 2.50)\)
1. Exercise 1.2 (p. 28)

The slope indicates that the reservoir gains 0.0467 million gallons of water per day.

(a) (b) 

101. (a) \( C = \frac{5}{2}(F - 32) \)  (b) \( C = 20^\circ \)

103. (a) \( A = 0.0467t + 102.7 \)  (b) There were 103.167 billion gallons in the reservoir on December 31.

(c) The slope indicates that the reservoir gains 0.0467 million gallons of water per day.  (d) On January 31, 2010 we predict 104.6147 billion gallons of water in the reservoir.  (e) The reservoir will be full after 3586 days.  105. (a) \( \$80,000.00 \)  (b) \( \$95,000.00 \)  (c) In 2012 sales are predicted to be $110,000.00.  (d) In 2015 sales are predicted to be $125,000.00.  107. (a) \( S = 0.05x + 400 \)  (b) Dan earns $600.00.  (c) Dan must generate a profit of $9146.20 to make the median earnings.

109. (a) \( S = 2.1(t - 1999) + 475 \)  (b) In 2011 the average mathematics SAT score is predicted to be 500.  111. (a) \( P = 0.5(t - 1998) + 24.4 \)  (b) If the trend continues, in 2011 about 30.9% of people over 25 years of age will have a bachelor's degree or higher.  (c) The slope is the annual average increase in the percent of people over 25 years of age who hold bachelor's degrees or higher.

113. (a) \( C = -6543(t - 2008) + 94823 \)  (b) \( \$75,194 \)  115. (a) \( S = 50.817(t - 2007) + 214.091 \)  (b) The predicted total sales and operating income of Chevron Corporation is predicted to be $417.359 billion in 2011.  117. (a) \( C = \frac{2.739}{17}x \)  (b) \( C = \frac{1.847}{17}x \)  (c) The Smiths spend $2417 on gasoline at the 2010 price.  (d) The Smiths spend $1630 on gasoline at the 2009 price.  (e) They spend $737 more at the 2010 price than at the 2009 price.  119. (a) \( N = 64.3t + 954.7 \)  (b) The slope indicates that the number of credit and debit cards in force is increasing at an average rate of 64.3 million cards per year.  (c) There will be an estimated 1.276 billion credit and debit cards in force at the end of the first quarter of 2011.

121. (a), (c), (f), (g)  123. Vertical lines cannot be written in slope–intercept form.  125. Two lines with equal slopes and equal y-intercepts have the same graphs.

127. If two lines have the same slope but different x-intercepts, they cannot have the same y-intercepts.

Exercise 1.2 (p. 28)


17. (3, 1)  19. (1, 0)  21. (2, 1)
AN–4  Answers to Odd-Numbered Problems

23. \((-1, 1)\)

25. \((4, -2)\)

27. \(y = 2x - 3\) or \(2x - y = 3\)

29. \(y = 4x + 6\) or \(4x - y = -6\)

31. \(y = 2x\) or \(2x - y = 0\)

33. \(x = 4\)

35. \(y = \frac{-19}{5}x - \frac{63}{5}\) or \(19x + 5y = -63\)

37. To break even, put 20 caramels and 30 creams into each box; increase the number of caramels to obtain a profit.

39. Mr. Nicholson should invest $50,000 in AA Bonds and $100,000 in Savings and Loan Certificates.

41. Mix 25 pounds of Kona coffee with 75 pounds of Colombian coffee to obtain a blend worth $10.80 per pound.

43. Mix 30 cubic centimeters of the 15% solution with 70 cubic centimeters of the 5% solution to obtain a solution that is 8% acid.

45. (a) \(y = 562.44x\)  (b) A purchase of 17.8 ounces of gold would have resulted in a gain of $10,000.00.

(b) \(x = 536.450\); the 2000th station is predicted to have aired on July 22, 2009.

Exercise 1.3 (p. 34)

1. False  2. True

3. The break-even point is \(x = 30\).

5. The break-even point is \(x = 500\).

7. The market price is $1.00.

9. The market price is $10.00.

11. The break-even point occurs when \(x = 1200\) game-day pennants are produced and sold.

13. (a) Market price: $1.00  
(b) Supply demanded at market price: 1.1 million pounds

(c) The point of intersection is called market equilibrium. It is the price where supply produced equals the amount demanded.

15. \(D = -4p + 23\)

17. At most 16 DVDs can be ordered from the club to keep the price lower than that of the discount retailer.

Exercise 1.4 (p. 39)

1. True  2. Correlation coefficient

9. (a) A relation exists, and it appears to be linear.

(c) A relation exists, and it appears to be linear.

(d) A relation exists, and it appears to be linear.

(e) Using the LinReg program, the line of best fit is: \(y = 2.0357x - 2.3571\).
Exercise 1.4

11. (a) 
(b) \( y = \frac{9}{4} x + \frac{1}{2} \)

13. (a) 
(b) \( y = -\frac{3}{4} x + 115 \)

15. (a) 
(b) \( y = 4x + 180 \)

17. (a) 
(c) The slope of this line indicates that a family will spend $23 of every extra $30 of disposable income.
(d) A family with a disposable income of $42,000 is predicted to consume $32,200 worth of goods.
(e) Using the LinReg function on the graphing utility, the line of best fit is: \( y = 0.75489x + 0.62663 \).

19. (a) 
(b) Using the LinReg function, the line of best fit is: \( y = 0.07818x + 59.0909 \).
(c) 
(d) The slope indicates the change in apparent temperature in a 65°F room for every percent increase in relative humidity.
(e) The apparent temperature of a room with an actual temperature of 65°F remains at 65° when the relative humidity is 75%.
AN–6  Answers to Odd-Numbered Problems

21. (a) Using the LinReg function, the line of best fit is \( y = 0.0651t + 10.6049 \).

The predicted energy use in 2015, \( t = 20 \), is \( y = 0.0651(20) + 10.6049 \approx 11.91 \) quadrillion BTU. The predicted energy use in 2030, \( t = 35 \), is \( y = 0.0651(35) + 10.6049 \approx 12.88 \) quadrillion BTU.

(b) When we graph the scatterplot and the line of best fit, we see that the points are not strictly linear. Therefore, estimates based on the line of the best fit are approximations. The line of best fit may differ depending on which points are chosen. The U.S. Dept. of Energy may have used more or less data points to make its predictions. They also may not have used a linear model.

CHAPTER 1  Review

Review Exercises (p. 43)

1. \( y = -2x + 5 \)

3. \( 2y = 3x + 6 \)

5. (a) \( m = -\frac{1}{2} \): A slope = -\( \frac{1}{2} \) means that for every 2 units \( x \) moves to the right, \( y \) moves down 1 unit.

(b) \( y = \frac{1}{2}x + \frac{5}{2} \) or \( x + 2y = 5 \)

(c) \( x + 2y = 5 \)

7. (a) \( m = 2 \); A slope = 2 means that for every 1 unit change in \( x \), \( y \) increases 2 units.

(b) \( y = 2x + 7 \) or \( 2x - y = -7 \)

(c) \( 2x - y = -7 \)

9. \( y = -3x + 5 \) or \( 3x + y = 5 \)

11. \( y = 4 \)

13. \( x = 8 \)

15. \( y = \frac{5}{2}x + 5 \) or \( 5x + 2y = 10 \)

17. \( y = -\frac{4}{3}x - 4 \) or \( 4x + 3y = -12 \)
19. \( y = \frac{2}{3}x - \frac{1}{3} \) or \( 2x + 3y = -1 \)

21. \( m = -\frac{9}{2} \) \( y \)-intercept \((0,9)\)

23. \( m = -2 \) \( y \)-intercept: \((0, \frac{9}{2})\)

25. \( m = -\frac{3}{2} \) \( y \)-intercept: \((0, \frac{1}{2})\)

27. parallel
29. intersecting
31. coincident
33. \((5,1)\)
35. \((1,3)\)

37. \((-2, 1)\)

39. Invest $78,571.43 in B-rated bonds and $11,428.57 in the well-known bank.

41. (a) 120 people need to attend for the group to break even.
(b) 300 people need to attend to achieve the $900 profit.
(c) If tickets are sold for $12.00 each, 86 people must attend to break even, and 215 people must attend to achieve a profit of $900.

43. Relation does not appear to be linear.

45. (a) The market price for corn is $1.33.
(b) 1.264 million bushels will be supplied at the market price.
(d) At a price of $1.33 per bushel the supply and the demand for corn are equal.

47. (a)

(b) The slope is 11,700.
(c) The mean price of houses sold in the United States increased by an average of $11,700 per year from 1998 through 2002.
(d) The slope is 10,650.
(e) The mean price of houses sold in the United States increased by an average of $10,650 per year from 2002 through 2008.
(f) The slope of the line of best fit is 13,960.
(g) The mean price of houses sold in the United States increased by an average of $13,960 per year from 1998 through 2008.
(h) The average annual increase in mean price of houses sold in the United States is not constant. The slope depends on the points used to calculate it.
(i) The average price of houses sold in the United States is starting to decline.
AN–8 Answers to Odd-Numbered Problems

49. (a)

(b) The data do not appear to be linearly related.

51. (a) The graph of \( x = 0 \) is the \( y \)-axis. It is a vertical line; the slope is not defined.

(b) The graph of \( y = 0 \) is the \( x \)-axis. It is a horizontal line and has a slope of zero.

(c) The graph of \( x + y = 0 \) is a decreasing line with a slope of \(-1\) and a \( y \)-intercept of \((0, 0)\). It passes through the origin.

Mathematical Questions from Professional Exams

1. b  2. d  3. d  4. b
5. c  6. c  7. b  8. b
**Exercise 2.1 (p. 65)**

5. False  
6. Inconsistent  
7. Yes  
9. No  
11. Yes  
13. Yes  

15. The solution of the system is \( x = 3 \) and \( y = 2 \) or \((3, 2)\).
17. The solution of the system is \( x = \frac{1}{3} \) and \( y = -\frac{1}{6} \) or \( \left( \frac{1}{3}, -\frac{1}{6} \right) \).
19. The system is inconsistent.

21. The solutions of the system are \( y = -\frac{1}{2}x + 2 \) where \( x \) is any real number, or as \( x = -2y + 4 \) where \( y \) is any real number or, using ordered pairs, \( \{(x, y) | x = -2y + 4, y \text{ any real number}\} \) or \( \{(x, y) | y = -\frac{1}{2}x + 2, x \text{ any real number}\} \).  
23. The solution of the system is \( x = 1 \) and \( y = 1 \) or \((1, 1)\).
25. The solution of the system is \( x = \frac{3}{2} \) and \( y = 1 \) or \( \left( \frac{3}{2}, 1 \right) \).
27. The solution of the system is \( x = 4 \) and \( y = 3 \) or \((4, 3)\).

29. The solution of the system is \( x = 8, y = 2, \) and \( z = 0 \) or \((8, 2, 0)\).
31. The solution of the system is \( x = 2, y = -1, \) and \( z = 1 \) or \((2, -1, 1)\).  
33. The system is inconsistent.  
35. The solutions of the system are \( \begin{cases} x = 5z - \frac{2}{2} \\ y = 4z - 3 \end{cases} \) where \( z \) is any real number.  
37. The system is inconsistent.  
39. The solution of the system is \( x = 1, y = 3, \) and \( z = -2 \) or \((1, 3, -2)\).
41. The solution of the system is \( x = -3, y = \frac{1}{2}, \) and \( z = 1 \) or \( \left( -3, \frac{1}{2}, 1 \right) \).
43. The dimensions of the floor are 30 ft \( \times \) 15 ft.
45. They should plant 130.8 acres of corn and 237.2 acres of soybeans.  
47. 22.5 pounds of cashews should be mixed with the peanuts.  
49. A package of tofu costs 156 yen and a carton of milk costs 230 yen.
51. The amount of the refund should be $5.56.
53. 50 mg of the first liquid (20% vitamin C and 30% vitamin D) should be mixed with 75 mg of the second liquid.
55. Use 9.16 pounds of rolled oats and 8.73 pounds of molasses.
57. The theater has 100 orchestra seats, 210 main seats, and 190 balcony seats.

59. There is not sufficient information to determine the price of each food item. Possible prices include:

<table>
<thead>
<tr>
<th>Price of a Hamburger</th>
<th>Price of Fries</th>
<th>Price of Cola</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.15</td>
<td>$0.88</td>
<td>$0.60</td>
</tr>
<tr>
<td>$2.00</td>
<td>$0.93</td>
<td>$0.75</td>
</tr>
<tr>
<td>$1.95</td>
<td>$0.95</td>
<td>$0.80</td>
</tr>
<tr>
<td>$1.85</td>
<td>$0.98</td>
<td>$0.90</td>
</tr>
</tbody>
</table>

61. (a) The equilibrium price is $16.00.  
(b) The equilibrium quantity is 600 T-shirts.  
(c) The dimensions of the floor are 30 ft \( \times \) 15 ft.

63. For this IS-LM model the equilibrium level of income \( Y \) is $9000 million = $9,000,000,000.

65. (a)

<table>
<thead>
<tr>
<th>Mutual Shares, ( x )</th>
<th>Strategic Income, ( y )</th>
<th>Small Cap Value, ( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$30,000</td>
<td>$40,000</td>
</tr>
<tr>
<td>$5,000</td>
<td>$20,000</td>
<td>$45,000</td>
</tr>
<tr>
<td>$10,000</td>
<td>$10,000</td>
<td>$50,000</td>
</tr>
<tr>
<td>$15,000</td>
<td>0</td>
<td>$55,000</td>
</tr>
</tbody>
</table>

(b) Answers will vary.
AN–10  Answers to Odd-Numbered Problems

Exercise 2.2 (p. 84)

1. matrix 2. augmented 3. True 4. True 5. \[
\begin{bmatrix}
2 & -3 & 5 \\
1 & -1 & 3 \\
\end{bmatrix}
\]
7. \[
\begin{bmatrix}
2 & 1 & -6 \\
3 & 1 & -1 \\
\end{bmatrix}
\]
9. \[
\begin{bmatrix}
2 & -1 & -1 & 0 \\
1 & 1 & 1 & 1 \\
3 & -1 & 0 & 2 \\
\end{bmatrix}
\]
11. \[
\begin{bmatrix}
2 & -3 & 1 & 7 \\
1 & 1 & -1 & 1 \\
2 & 2 & -3 & -4 \\
\end{bmatrix}
\]
13. \[
\begin{bmatrix}
4 & -1 & 2 & -1 & 4 \\
1 & 1 & 0 & 0 & -6 \\
0 & 2 & -1 & 1 & 5 \\
\end{bmatrix}
\]
15. \[
\begin{bmatrix}
1 & -1 & 1 & -1 & 0 \\
2 & 3 & -1 & 4 & 5 \\
\end{bmatrix}
\]
17. \[
\begin{bmatrix}
1 & -3 & -2 \\
0 & 1 & 9 \\
\end{bmatrix}
\]
19. (a) \[
\begin{bmatrix}
1 & -3 & 4 & 3 \\
0 & 1 & -2 & 0 \\
-3 & 3 & 4 & 6 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & -3 & 4 & 3 \\
2 & -5 & 6 & 6 \\
0 & -6 & 16 & 15 \\
\end{bmatrix}
\]
21. (a) \[
\begin{bmatrix}
1 & -3 & 2 & -6 \\
0 & 1 & -1 & 8 \\
-3 & -6 & 2 & 6 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & -3 & 2 & -6 \\
2 & -5 & 3 & -4 \\
0 & -15 & 8 & -12 \\
\end{bmatrix}
\]
23. (a) \[
\begin{bmatrix}
1 & -3 & 1 & -2 \\
0 & 1 & 4 & 2 \\
-3 & 1 & 4 & 6 \\
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & -3 & 1 & -2 \\
2 & -5 & 6 & -2 \\
0 & -8 & 7 & 0 \\
\end{bmatrix}
\]
25. (a) \[
\begin{align*}
x + 2y &= 5 \\
y &= -1
\end{align*}
\]
(b) The system is consistent and the solution is \( x = 7 \) and \( y = -1 \) or \( (7, -1) \).

27. (a) \[
\begin{align*}
x + 2y + 3z &= 1 \\
y + 4z &= 2 \\
o &= 3
\end{align*}
\]
(b) The system is inconsistent. 29. (a) \[
\begin{align*}
x + 2z &= -1 \\
y - 4z &= -2 \\
o &= 0
\end{align*}
\]
(b) The system is consistent and has an infinite number of solutions. The solutions are \( x = -2z - 1 \), \( y = 4x - 2 \), where \( z \) is any real number.

31. (a) \[
\begin{align*}
x_1 + 2x_2 - x_3 + x_4 &= 1 \\
x_2 + 4x_3 + x_4 &= 2 \\
x_3 + 2x_4 &= 3 \\
x_4 &= 4
\end{align*}
\]
(b) The system is consistent. The solutions are \( x_1 = -44 \), \( x_2 = 18 \), \( x_3 = -5 \), \( x_4 = 4 \) or \( (-44, 18, -5, 4) \).

33. (a) \[
\begin{align*}
x_1 + 2x_2 + 4x_4 &= 2 \\
x_2 + x_3 + 3x_4 &= 3 \\
x_3 &= 0 \\
o &= 0
\end{align*}
\]
(b) The system is consistent. The solutions are \( x_1 = 2x_4 - 4 \), \( x_2 = -3x_4 + 3 \), \( x_3 = 0 \) where \( x_4 \) is any real number.

35. (a) \[
\begin{align*}
x_1 - 2x_2 + x_4 &= -2 \\
x_2 - 3x_3 + 2x_4 &= 2 \\
x_3 - x_4 &= 0
\end{align*}
\]
(b) The system is consistent. The solutions are \( x_1 = x_4 + 2 \), \( x_2 = x_4 + 2 \), \( x_3 = x_4 \) where \( x_4 \) is any real number.

37. The system is inconsistent. 39. The solution of the system is \( x = \frac{1}{3} \) and \( y = \frac{1}{3} \) or \( \left( \frac{1}{3}, \frac{1}{3} \right) \).

41. The solutions of the system are \( y = -\frac{1}{3}x + \frac{2}{3} \) where \( x \) is any real number, or \( x = -3y + 2 \) where \( y \) is any real number, or written as ordered pairs, \( \{(x, y) | y = -\frac{1}{3}x + \frac{2}{3}, x \text{ any real number}\} \) or \( \{(x, y) | x = -3y + 2, y \text{ any real number}\} \). 43. The solution of the system is \( x = \frac{2}{3}, y = \frac{1}{3} \) or \( \left( \frac{2}{3}, \frac{1}{3} \right) \).

45. The solution of the system is \( x = 1 \), \( y = 4 \), \( z = 0 \) or \( (1, 4, 0) \). 47. The solution of the system is \( x = \frac{8}{5}, y = -\frac{12}{5}, z = 6 \) or \( \left( \frac{8}{5}, -\frac{12}{5}, 6 \right) \).

49. The solution of the system is \( x = 2, y = -1, z = 1 \) or \( (2, -1, 1) \).

51. The system is inconsistent. 53. The solution of the system is \( x = \frac{1}{3}, y = \frac{2}{3}, z = 1 \) or \( \left( \frac{1}{3}, \frac{2}{3}, 1 \right) \).

55. Row echelon form (REF):  Reduced row echelon form (RREF):  The solution of the system is \( x = \frac{2}{9}, y = -\frac{2}{3}, z = \frac{2}{9} \) or \( \left( \frac{2}{9}, -\frac{2}{3}, \frac{2}{9} \right) \).
57. Row echelon form (REF):

\[
\begin{bmatrix}
1 & 1 & 1 & 4 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Reduced row echelon form (RREF):

\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]

The solution of the system is \( x = 2, y = -1, z = 3 \) or \( (2, -1, 3) \).

59. Row echelon form (REF):

\[
\begin{bmatrix}
1 & 0 & 0 & 21 \\
0 & 1 & 0 & 13 \\
0 & 0 & 1 & 41
\end{bmatrix}
\]

Reduced row echelon form (RREF):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

The solution of the system is \( x_1 = 20, x_2 = -13, x_3 = 17, x_4 = -4 \) or \( (20, -13, 17, -4) \).

61. Carla should invest $22,500 in treasury bills, $65,000 in bank CDs, and $32,500 in corporate bonds.

63. The meal should consist of 1 serving of chicken, 1 serving of potatoes, and 2 servings of spinach.

65. 20 cases of orange juice, 12 cases of tomato juice, and 6 cases of pineapple juice are prepared.

67. A mezzanine ticket costs $66; a lower balcony seat costs $49; a middle balcony seat costs $39.

69. The teacher should order 2 of the first package (20 white, 15 blue, 1 red), 10 of the second package (3 blue, 1 red), and 4 of the third package.

71. The recreation center should purchase 4 assorted cartons, 8 mixed cartons, and 5 single cartons.

73. To fill the order use 2 large cans, 1 mammoth can, and 4 giant cans.

75. Karen should buy 3200 shares of NCC.B, 800 shares of IBM, and 800 shares of ESRX.

Exercise 2.3 (p. 95)

1. False  2. True  3. Yes  5. No, the leftmost 1 in the 2nd row is not to the right of the leftmost 1 in the 1st row. 7. Yes

9. Infinitely many solutions. \( \begin{cases} x = 2z + 6 \\ y = -3z + 1 \end{cases} \) where \( z \) is the parameter.

11. One solution, \( x = -1, y = 3, z = 4 \).

13. Infinitely many solutions. \( \begin{cases} x = z + 1 \\ y = -2z + 1 \end{cases} \) where \( z \) is the parameter.

15. Infinitely many solutions, \( \begin{cases} x_1 = x_4 + 4 \\ x_2 = -2x_3 - 3x_4 \end{cases} \), where \( x_3 \) and \( x_4 \) are the parameters.

17. \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) The system is inconsistent.

19. \( \begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \) The solutions of the system are \( x = 2y + 4 \), where \( y \) is the parameter.

21. \( \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \) The solutions of the system are \( \begin{cases} x = -3 \\ y = -3z + 5 \end{cases} \), where \( z \) is the parameter.

23. \( \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) The solution of the system is \( x_1 = -17, x_2 = 24, x_3 = 33, \) and \( x_4 = 14 \).

25. \( \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & -2 & 3 \end{bmatrix} \) The solutions are \( \begin{cases} x = z + 8 \\ y = 2z + 3 \end{cases} \) where \( z \) is the parameter.

27. \( \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) The solution of the system is \( x_1 = 1, x_2 = 2, x_3 = 0, \) and \( x_4 = 1 \).
AN–12  Answers to Odd-Numbered Problems

31. (a) Let \( x_1 \) be the servings of turkey bologna, \( x_2 \) be the servings of bananas, \( x_3 \) be the servings of low-fat cottage cheese, and \( x_4 \) be the servings of chocolate milk that Steven eats.

\[
\begin{align*}
184x_1 + 92x_2 + 90x_3 + 72x_4 &= 700 \\
13.2x_1 + 0.48x_2 + 1.93x_3 + 2x_4 &= 20 \\
4.85x_1 + 23.43x_2 + 0.01x_3 + 10.4x_4 &= 100
\end{align*}
\]

(b) where \( x_4 \) is the parameter

\[
\begin{align*}
1 & -0.117 & 1.188 \\
-0.117 & 1 & 0.420 & 4.022 \\
0 & 0 & 1.32 & 1.238
\end{align*}
\]

\[
\begin{align*}
x_1 &= -0.117x_4 + 1.188 \\
x_2 &= -0.420x_4 + 4.022 \\
x_3 &= -0.132x_4 + 1.238
\end{align*}
\]

33. (a) Let \( x_1 \) be the amount invested in John Hancock Large Cap Equity Growth fund, \( x_2 \) be the amount invested in T Rowe Price Emerging Markets fund, \( x_3 \) be the amount invested in Templeton China World fund, and \( x_4 \) be the amount invested in TCW Small Cap growth fund.

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 50,000 \\
0.09x_1 + 0.13x_2 + 0.14x_3 + 0.07x_4 &= 6,000
\end{align*}
\]

(b) where \( x_3 \) and \( x_4 \) are parameters

35. (a) Possible investment strategies for a couple with $20,000.00 to invest.

<table>
<thead>
<tr>
<th>Amount Invested</th>
<th>High Income</th>
<th>Global Discovery</th>
<th>Global Bond</th>
<th>Small Cap Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>$0</td>
<td>$10,000</td>
<td>$10,000</td>
<td></td>
</tr>
<tr>
<td>$5714</td>
<td>$6667</td>
<td>$0</td>
<td>$13,333</td>
<td></td>
</tr>
</tbody>
</table>

(b) Possible investment strategies for a couple with $25,000.00 to invest.

<table>
<thead>
<tr>
<th>Amount Invested</th>
<th>High Income</th>
<th>Global Discovery</th>
<th>Global Bond</th>
<th>Small Cap Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>$0</td>
<td>$25,000</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>$10,000</td>
<td>$5000</td>
<td>$0</td>
<td>$5000</td>
<td></td>
</tr>
<tr>
<td>$13,333</td>
<td>$0</td>
<td>$16,667</td>
<td>$10,000</td>
<td></td>
</tr>
</tbody>
</table>

(c) Possible investment strategies for a couple with $30,000.00 to invest.

<table>
<thead>
<tr>
<th>Amount Invested</th>
<th>High Income</th>
<th>Global Discovery</th>
<th>Global Bond</th>
<th>Small Cap Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>$26,667</td>
<td>$0</td>
<td>$3333</td>
<td></td>
</tr>
<tr>
<td>$0</td>
<td>$20,000</td>
<td>$10,000</td>
<td>$0</td>
<td></td>
</tr>
<tr>
<td>$22,857</td>
<td>$0</td>
<td>$0</td>
<td>$7143</td>
<td></td>
</tr>
<tr>
<td>$13,333</td>
<td>$0</td>
<td>$16,667</td>
<td>$0</td>
<td></td>
</tr>
</tbody>
</table>

37. Bacteria 1  Bacteria 2  Bacteria 3

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>6000</td>
<td>0</td>
</tr>
<tr>
<td>2000</td>
<td>4000</td>
<td>1000</td>
</tr>
<tr>
<td>2000</td>
<td>0</td>
<td>3000</td>
</tr>
</tbody>
</table>
Review Exercises (p. 99)

1. \(x = 2, y = -1\) or \((2, -1)\)  
3. \(x = 2, y = -1\) or \((2, -1)\)  
5. No solution, the system is inconsistent.  
7. \(x = -1, y = 2, z = -3\) or \((-1, 2, -3)\)

9. \[
\begin{align*}
\begin{cases}
x = \frac{7}{4} + \frac{39}{4} \\
y = \frac{9}{8} + \frac{69}{8}
\end{cases}
\end{align*}
\]
11. \[
\begin{align*}
\begin{cases}
3x + 2y = 8 \\
x + 4y = -1
\end{cases}
\end{align*}
\]
13. \(x = 4, y = 6, z = -1\) or \((4, 6, -1)\)
15. \(x = \frac{14}{9}, y = \frac{26}{9} \) or \(\left(\frac{14}{9}, \frac{26}{9}\right)\)

17. \(x = -103, y = 32, z = 9\) or \((-103, 32, 9)\)
19. \(x = 29, y = -10, z = -1\) or \((29, -10, -1)\)
21. No solution, the system is inconsistent.

23. \(x = 9, y = -\frac{56}{3}, z = -\frac{37}{3}\) or \(\left(-\frac{56}{3}, -\frac{37}{3}\right)\)
25. \(x = 29, y = 8, z = -24\) or \((29, 8, -24)\)

27. \[
\begin{align*}
\begin{cases}
x = \frac{3}{7}z + \frac{10}{7} \\
y = \frac{6}{7}z - \frac{9}{7}
\end{cases}
\end{align*}
\]
where \(z\) is a parameter. Answers will vary. Three possible solutions are \(x = \frac{10}{7}, y = -\frac{9}{7}\), when \(z = 0\); \(x = \frac{13}{7}, y = -\frac{4}{7}\), when \(z = 1\); and \(x = 1, y = -2\), when \(z = -1\).  
29. \[
\begin{align*}
\begin{cases}
x = -0.6z + 1 \\
y = 0.8z + 2
\end{cases}
\end{align*}
\]
Answers will vary. Three solutions are: \(x = 1, y = 2\), when \(z = 0\); \(x = 0.4, y = 2.8\), when \(z = 1\); \(x = 1.6, y = 1.2\), when \(z = -1\).  
31. No solution, the system is inconsistent.  
33. One solution, \(x = 5, y = -1\), and \(z = 1\); or \((5, -1, 1)\)
35. Infinitely many solutions, \(\begin{cases}x_1 = -2x_4 + 1 \\
x_2 = -2x_4 - 1\end{cases}\) where \(x_4\) is a parameter.

37. Each box should contain 20 caramels and 30 creams.  
To obtain a profit, increase the number of caramels in each box (decreasing the number of creams).

<table>
<thead>
<tr>
<th>Almonds</th>
<th>Cashews</th>
<th>Peanuts</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>60</td>
<td>35</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>35</td>
<td>20</td>
<td>45</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>

41. (a) To attain $2500 per year in income:
(b) To attain $3000 per year in income:
(c) To attain $3500 per year in income:

Mathematical Questions from Professional Exams

1. b  2. b  3. d  4. c
Exercise 3.1 (p. 117)

3. $2 \times 3$  True 5. True 6. False 7. $2 \times 2$, a square matrix 9. $3 \times 2$ 11. $3 \times 2$ 13. $2 \times 1$, a column matrix 15. False; the dimensions must be the same for two matrices to be equal.

17. True 19. True 21. True 23. True 25. False; the dimensions must be the same for two matrices to be equal.

27. $\begin{bmatrix} 1 & 1 \\ 6 & 7 \end{bmatrix}$
29. $\begin{bmatrix} 6 & 18 & 0 \\ 12 & -6 & 3 \end{bmatrix}$
31. $\begin{bmatrix} 2 & 4 & -8 \\ 1 & -4 & -1 \end{bmatrix}$
33. $\begin{bmatrix} 13a & 54 \\ -2b & -7 \\ -2c & -6 \end{bmatrix}$
35. $\begin{bmatrix} 1 & -1 & 4 \\ -5 & 1 & 1 \end{bmatrix}$
37. $\begin{bmatrix} 13 & -6 & -7 \\ -6 & 1 & -7 \end{bmatrix}$
39. $\begin{bmatrix} 9 & -5 & -6 \\ 1 & 1 & 3 \end{bmatrix}$

43. $A + B = \begin{bmatrix} 3 & -5 & 4 \\ 5 & 3 & 3 \end{bmatrix} = B + A$
45. $A + (-A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$
47. $2B + 3B = \begin{bmatrix} 5 & -10 & 0 \\ 25 & 5 & 10 \end{bmatrix} = 5B$

49. $x = -4$ and $z = 4$
51. $x = 5$ and $y = 1$
53. $x = 4, y = -6$, and $z = 6$
57. $[C -3 \times [A] + [B]] = \begin{bmatrix} 1 & -2 & 1 \\ 5 & 6 & 3 \\ 12 & -1 & 4 \end{bmatrix}$

59. Democrats | Republicans | Independents
Under $25,000 | 351 | 271 | 73
Over $25,000 | 203 | 215 | 55

61. LAS | ENG | EDUC
Male | 250 | 225 | 80
Female | 250 | 75 | 120

63. (a) M | 275,000 | 633,000 | 275,000 | 27,600
F | 460,000 | 949,000 | 418,000 | 27,300
(b) The rows represent the distribution of the projected postsecondary degrees for each gender. Row 1 represents the degrees projected to be earned by males, and row 2 represents the projections for females. The columns represent the distribution of the projected number of each postsecondary degree by gender. Column 1 represents associate degrees, column 2 bachelor’s degrees, column 3 master’s degrees, and column 4 doctoral degrees.
(c) M | 275,000 | 460,000
F | 633,000 | 949,000
(d) The rows of this matrix represent the numbers of various postsecondary degrees, associate, bachelor, master, and doctoral, respectively, projected to be earned distributed between the two genders: male (column 1) and female (column 2). Column 1 represents the numbers of the various degrees projected to be earned by males, and column 2 represents the numbers of the various degrees projected to be earned by females.

65. Local | State | Federal
Male | 685,790 | 1,307,706 | 187,996
Female | 99,766 | 101,460 | 13,284

67. (a) January Sales | February Sales | (b) Combined January/February Sales
City | Sub | Int | SUV
350 | 225 | 80
300 | 175 | 40
650 | 700 | 120
Suburban | City | Sub | Int | SUV
375 | 200 | 75
325 | 150 | 50
700 | 350 | 125
(c) January Sales | February Sales | (d) Combined January/February Sales
City | Sub | Int
Suburban | City | Sub | Int
350 | 375 | 300 | 325 | 650 | 700
Int | 225 | 200 | 175 | 150 | 400 | 350
SUV | 80 | 50 | 40 | 50 | 120 | 125

69. (a) Doctoral degrees earned by gender:

(b) Doctoral degrees earned over 2 years by gender:

(c) Difference in the numbers of doctoral degrees awarded from 2007 to 2008:

71. $A = \begin{bmatrix} 0.74 & 0.77 & 0.70 \end{bmatrix}$
Exercise 3.2 (p. 131)

\begin{bmatrix}
4 & 2 \\
2 & 8
\end{bmatrix}
\] 15. [4 6] 17. \[
\begin{bmatrix}
4 & -14 \\
12 & -32
\end{bmatrix}
\] 19. \[
\begin{bmatrix}
4 & 2 \\
2 & 8
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
9 & 1 \\
5 & 4 \\
11 & 7
\end{bmatrix}
\] 23. \(BA\) is defined and is a \(3 \times 4\) matrix. 25. \(AB\) is not defined. 27. \((BA)C\) is not defined. 29. \(BA + A\) is defined and is a \(3 \times 4\) matrix. 31. \(CB - A\) is not defined. 33. \[
\begin{bmatrix}
-1 & 10 & -1 \\
-4 & 16 & -8
\end{bmatrix}
\] 35. \[
\begin{bmatrix}
11 & 5 \\
13 & -9
\end{bmatrix}
\] 37. \[
\begin{bmatrix}
6 & 10 \\
8 & 2
\end{bmatrix}
\] 39. \[
\begin{bmatrix}
3 & -1 \\
4 & 2
\end{bmatrix}
\] 41. \[
\begin{bmatrix}
8 & 4 \\
32 & 16
\end{bmatrix}
\]

43. \[
\begin{bmatrix}
-14 & 7 \\
-20 & -6
\end{bmatrix}
\] 45. \[
\begin{bmatrix}
10 & 30 & 37 \\
15 & 16 & 50 \\
-6 & 20 & -8
\end{bmatrix}
\]

47. \(D(AB) = \begin{bmatrix}
-6 & 42 & -9 \\
7 & 20 & 6
\end{bmatrix} = (DC)B\) 49. \(AB = \begin{bmatrix}
5 & -2 \\
6 & 4 \\
7 & -3
\end{bmatrix} \neq \begin{bmatrix}
7 & -3 \\
6 & 2
\end{bmatrix} = BA\)

51. \[
\begin{bmatrix}
0.5 & 16 & -30 & 25 \\
21 & 14 & 28 & 64 \\
19.5 & 8 & -23 & 33 \\
-9.5 & 45 & -9 & 83
\end{bmatrix}
\]

53. \[
\begin{bmatrix}
31.5 & 251 & -31.5 & 143 \\
861 & 350 & 791 & 420 \\
-71 & 13 & 106 & 28 \\
158 & -3 & 152 & 46
\end{bmatrix}
\]

55. \[
\begin{bmatrix}
667 & 74 & 94 & 38 \\
71 & 13 & 106 & 28 \\
165 & 124.5 & 79 & 52 \\
-346 & 279 & -249 & 187
\end{bmatrix}
\]

59. (a) Matrix A represents the price at the close of the trading on July 7, 2009, of a share of Chevron Corporation, Conoco Phillips, and Exxon Mobil Corporation, respectively. A is \(1 \times 3\). (b) \(AB = [37,899.50 \quad 44,595.60 \quad 27,623.30]\); matrix AB represents the amount each of the three persons spent on the three stocks on July 7, 2009, at their closing price. (c) \(BC = \begin{bmatrix}
690 \\
540 \\
680
\end{bmatrix}\); matrix BC represents the total number shares of Chevron Corporation, Conoco Phillips, and Exxon Mobil Corporation, respectively, purchased by the three persons at the close of trading on July 7, 2009.

61. (a) \(A\) is \(2 \times 2\). The rows are John and Marsha; the column is cost of tuition. (b) The entries in \(AB\) are the total amounts (in dollars) that each student spends on tuition.

63. (a) \(A = \begin{bmatrix}
12 & 4 \\
8 & 8
\end{bmatrix}\) (b) \(A\) is \(3 \times 3\). (c) \(B\) is \(3 \times 2\). (d) \(AB = \begin{bmatrix}
4570 & 7180 \\
3950 & 6710 \\
4440 & 7250
\end{bmatrix}\)

71. \(A^2 = \begin{bmatrix}
4 & 2 \\
2 & 8
\end{bmatrix}
\] \(A^3 = \begin{bmatrix}
17 & 5 \\
-10 & -3
\end{bmatrix}
\] \(A^4 = \begin{bmatrix}
41 & 12 \\
-24 & -7
\end{bmatrix}
\] 73. \(AB = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

75. \(A^2 = \begin{bmatrix}
7 & 2 \\
-4 & -1
\end{bmatrix}
\] \(A^3 = \begin{bmatrix}
17 & 5 \\
-10 & -3
\end{bmatrix}
\]

77. \(A^2 = \begin{bmatrix}
3 & 5 \\
8 & 8
\end{bmatrix}
\] \(A^3 = \begin{bmatrix}
11 & 21 \\
32 & 32
\end{bmatrix}
\] \(A^4 = \begin{bmatrix}
43 & 85 \\
128 & 128
\end{bmatrix}
\]

79. Answers will vary, but as \(n\) gets larger \(A^n\) becomes approximately equal to

\[
\begin{bmatrix}
1 & 2 \\
3 & 2
\end{bmatrix}
\]
AN–16  Answers to Odd-Numbered Problems

81.  
\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

Exercise 3.3 (p. 148)

3. inverse  4. False  5. \[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix}
\]

7. \[
\begin{pmatrix}
2 & 1 & 2 \\
3 & 4 & 3
\end{pmatrix}
\]

9. \[
\begin{pmatrix}
2 & 1 \\
3 & 2
\end{pmatrix}
\]

11. \[
\begin{pmatrix}
-2 & -3 \\
-4 & -5
\end{pmatrix}
\]

13. \[
\begin{pmatrix}
5 & 6 \\
7 & 8
\end{pmatrix}
\]

15. \[
\begin{pmatrix}
-2 & 1 \\
-5 & 4
\end{pmatrix}
\]

17. \[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

19. \[
\begin{pmatrix}
-1 & 1 \\
-2 & 3
\end{pmatrix}
\]

21. \[
\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}
\]

23. \[
\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}
\]

33. Inverse does not exist.  35. \[A^{-1} = \begin{pmatrix}
1 & 2 \\
2 & 3
\end{pmatrix}
\]

37. \[B^{-1} = \begin{pmatrix}
1 & 2 \\
2 & 3
\end{pmatrix}
\]

39. \[x = 36, y = -14 \text{ or } (36, -14)
\]

41. \[x = 2, y = 1 \text{ or } (2, 1)
\]

43. \[x = 88, y = -36 \text{ or } (88, -36)
\]

45. \[x = \frac{14}{9}, y = \frac{26}{3}, z = \frac{65}{9} \text{ or } \left(\frac{14}{9}, \frac{26}{3}, \frac{65}{9}\right)
\]

47. \[x = \frac{20}{3}, y = 24, z = \frac{56}{3} \text{ or } \left(\frac{20}{3}, 24, \frac{56}{3}\right)
\]

49. \[x = -\frac{14}{9}, y = -\frac{10}{3}, z = \frac{16}{9} \text{ or } \left(-\frac{14}{9}, -\frac{10}{3}, \frac{16}{9}\right)
\]

51. \[
\begin{pmatrix}
0.00545 & 0.0509 & -0.0066 \\
0.01036 & -0.0186 & 0.0095 \\
-0.0193 & 0.0116 & 0.0344
\end{pmatrix}
\]

53. \[
\begin{pmatrix}
0.0249 & -0.0360 & -0.0057 & 0.0059 \\
-0.0171 & 0.0521 & 0.0292 & -0.0305 \\
0.0206 & 0.0081 & -0.0421 & 0.0005 \\
-0.0175 & 0.0570 & 0.0657 & 0.0619
\end{pmatrix}
\]

57. \[x = 4.567, y = -6.444, z = -24.075
\]

59. \[x = -1.187, y = 2.457, z = 8.265
\]

61. (a) Invest $2291.22 in the mid cap fund and $708.79 in the bond fund for a 5% rate of return.  (b) Invest $1467.03 in the mid cap fund and $1532.97 in the bond fund for an 5% rate of return.  (c) Invest $642.86 in the mid cap fund and $2357.14 in the bond fund for a 6% rate of return.  63. (a) Invest $4132.54 in the small cap growth fund, $3132.54 in the large cap value fund, and $734.92 in the international fund for a 4% rate of return.  (b) Invest $3736.89 in the small cap growth fund, $2736.89 in the large cap value fund, and $734.92 in the international fund for a 4% rate of return.  (c) Invest $3341.25 in the small cap growth fund, $2341.25 in the large cap value fund, and $2317.50 in the international fund for a 5% rate of return.
Exercise 3.4: Application in Economics (p. 159)

1. If $27,000 is allotted to the financial planner, then the physician earns $19,500 and the attorney earns $16,500.

Exercise 3.4: Application in Accounting (p. 163)

1. 3. 5. 7.

<table>
<thead>
<tr>
<th>Department</th>
<th>Total Costs (in Dollars)</th>
<th>Direct Costs (in Dollars)</th>
<th>Indirect Costs for Services from Departments (in Dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>3109.09</td>
<td>2000</td>
<td>345.45, 763.64</td>
</tr>
<tr>
<td>S2</td>
<td>2290.91</td>
<td>1000</td>
<td>1036.36, 254.55</td>
</tr>
<tr>
<td>P1</td>
<td>3354.54</td>
<td>2500</td>
<td>345.45, 509.09</td>
</tr>
<tr>
<td>P2</td>
<td>2790.91</td>
<td>1500</td>
<td>1036.36, 254.55</td>
</tr>
<tr>
<td>P3</td>
<td>3854.54</td>
<td>3000</td>
<td>345.45, 509.09</td>
</tr>
<tr>
<td>Totals</td>
<td>15,399.99</td>
<td>10,000</td>
<td>3109.07, 2290.92</td>
</tr>
</tbody>
</table>

Total of the service charges allocated to P1, P2, and P3: $2999.99
Sum of the direct costs of the service departments, S1, and S2: $3000

Exercise 3.4: Application in Statistics (p. 168)

1. 3. 5. 7.

(a) Not symmetric  (b) Symmetric  (c) Not symmetric  Yes, for two matrices to be equal, they must have the same dimensions.

9. (a) $y = \frac{54}{35}x + \frac{27}{5}$  (b) 17,743 units will be supplied.

11. $y = 1.498x + 36.135$

13. $N = 0.93t + 6.75$; 22.56 million persons are predicted to have diagnosed diabetes in the year 2011.
Review Exercises (p. 170)

1. The dimension is $3 \times 2$. 3. The dimension is $3 \times 2$. 5. The dimension is $3 \times 3$.

7. The dimension is $3 \times 3$. 9. The dimension is $3 \times 3$. 11. The dimension is $3 \times 2$.

13. $\begin{bmatrix} 3 & -4 \\ 1 & 5 \\ 5 & -2 \end{bmatrix} = C$. The dimension is $3 \times 2$. 15. The dimension is $3 \times 3$. 17. $\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 3 & 4 \end{bmatrix}$

19. Inverse does not exist. 21. $\begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 3 & 16 \\ 16 & 32 \end{bmatrix}$

23. $AB = BA$ when $x = w$ and $y = z$.

25. (a) Coded with matrix $A$: 79 82 29 50 41 58 70 52 89 90 29 30 29 34  Coded with matrix $B$: 111 42 8 93 92 76 82 76 70 68 32 52 14 4
(b) Coded with matrix $A$: 98 100 60 60 64 72 49 50 44 32 94 92 60 60  Coded with matrix $B$: 158 78 60 115 74 88 92 52 56 142 48 88 77 32 4
(c) Message: I forgot my pin.

27. (a) $A = \text{Price} \begin{bmatrix} 28.50 & 19.50 & 13.30 \end{bmatrix}$  (b) $B = \begin{bmatrix} 20 & 15 & 10 \\ 30 & 25 & 20 \\ 10 & 20 & 25 \end{bmatrix}$  (c) $AB = \begin{bmatrix} 1288.00 & 1181.00 & 1007.50 \end{bmatrix}$. The entries of matrix $AB$ represent the cost of Juanita’s, Debbie’s, and Dawn’s purchases, respectively, in dollars.
CHAPTER 4  Linear Programming with Two Variables

Exercise 4.1 (p. 187)

6. half-plane  7. True  8. True

9. graph of the system, and are not.

11. graph of the system, is not.

13. graph of the system, and are part of the graph of the system.

15. graph of the system. 21. 23. 25. 27. 29. 31. 33. 35. 37. 41. 43. 45.
45. (a) Bounded. Corner points: (2, 0), (4, 0), (0, 4), (0, 2).

\[
\begin{align*}
\frac{x}{4} + \frac{y}{2} &\leq 60 \\
\frac{x}{2} + \frac{y}{4} &\leq 90
\end{align*}
\]

(b) Corner points: (2, 0), (4, 0), (0, 4), (0, 2).

51. (a) Bounded. Corner points: (0, 0), (120, 0), (60, 90), (0, 120).

57. (a) \(x =\) number of units of grain 1
\(y =\) number of units of grain 2

\[
\begin{align*}
x + 2y &\geq 5 \\
5x + y &\geq 16 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

Corner points: (0, 16), (3, 1), (5, 0).

59. (a) \(x =\) number of oz. of food A
\(y =\) number of oz. of food B

\[
\begin{align*}
5x + 4y &\geq 85 \\
3x + 3y &\geq 70 \\
2x + 3y &\geq 50 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

Corner points: \(\left(\frac{70}{3}, \frac{20}{3}\right), (0, \frac{70}{3}), (25, 0)\).
Exercise 4.3

19. 

\[ x + y \leq 30,000 \]
\[ 0.0435x + 0.0505y \leq 500 \]
\[ x \geq 3,000 \]
\[ y \geq 5,000 \]

Corner points: (5690, 5000), (25000, 5000), (3000, 7317), (3000, 27000).

(c) Each corner point represents an amount \( x \) deposited in the ICON bond fund and an amount \( y \) deposited in ING Mid Cap Opportunities fund that satisfy the conditions given.

Exercise 4.2 (p. 198)

1. Objective function 

2. Feasible point 

3. False 

4. True 

5. Maximum of 38 at (7, 8). Minimum of 10 at (2, 2). 


9. Maximum of 55 at (7, 8). Minimum of 14 at any point on the line segment between (2, 2) and (8, 1). 

11. Maximum of 53 at (7, 8). Minimum of 14 at (2, 2). 

12. Maximum of 81 at (8, 1). Minimum of 22 at (2, 2). 

13. Corner points: (0, 0), (15, 0), (5, 10), (0, 10). 

19. Corner points: (3, 0), (10, 0), (10, 8), (0, 8), (0, 4). 

21. Maximum of 14 at (0, 2). 

23. Maximum of 15 at (3, 0). 

25. Maximum of 56 at (0, 8). 

27. Maximum of 58 at (6, 4). 

29. Minimum of at (2, 2). 

3. Minimum of 4 at (2, 0). 

35. Minimum of \( \frac{3}{2} \) at \( \left( 0, \frac{1}{2} \right) \). 

37. Maximum of 10 at any point on the line \( x + y = 10 \) between (0, 10) and (10, 0).

Minimum of \( \frac{20}{3} \) at \( \left( \frac{10}{3}, 0 \right) \). 

39. Maximum of 50 at (10, 0). Minimum of 20 at (0, 10). 

41. Maximum of 40 at (0, 10). Minimum of \( \frac{70}{3} \) at \( \left( \frac{10}{3}, \frac{10}{3} \right) \). 

43. Maximum of 100 at (10, 0). Minimum of 10 at (0, 10). 

45. Maximum of 192 at (4, 4). Minimum of 54 at (3, 0). 

47. Maximum of 58 at (4, 5). Minimum of 12 at (0, 2). 

49. Maximum of 240 at (3, 10). 

51. Maximum of 216 at (2, 10).

Exercise 4.3 (p. 206)

1. (a) 24 acres of soy beans and 12 acres of wheat should be planted to maximize profit. 

(b) Maximum profit is $5520.00. 

(c) Maximum profit is $7200.00 if preparation constraint is raised to $2400.00. 

9. Eric should eat 2 turkey breast sandwiches and 5 V eggie Delite® sandwiches to meet his dietary requirements and minimize his fat intake at 24 grams.

11. The factory should manufacture 15 of the first product and 25 of the second product, maximizing the profit at $2100.00. 

13. The minimum cost is $22.50. It can be attained using 7.5 oz of Supplement A and 11.25 oz of Supplement B or using 15 oz of Supplement A and no Supplement B. Other solutions are on the line segment joining (7, 5, 11.25) and (15, 0). 

15. Fremont Bank should allocate $24 million to 15-year mortgages and $48 million to 30-year mortgages to maximize the month's interest income at $292,500. 

17. The factory should manufacture 10 pairs of racing skates and 15 pairs of figure skates to obtain a maximum profit of $280.00. 

19. The chemical plant should produce 700 units of compound A and 400 units of compound B to obtain a maximum profit of $1450.00 while remaining below government pollution standards. 

21. The maximum profit of $1280.00 is made when 4 tons of Type 1 steel and 4 tons of Type 2 steel are produced. 

23. The minimum cost is $0.19 when Danny adds 5 ounces of Supplement I and 1 ounce of Supplement II to every 100 ounces of feed. 

25. The couple should deposit $15,000 in the ICON Bond fund and $30,000 in the ING Mid Cap fund. Their average annual interest income will be $2167.50. 

27. Eighteen sales representatives from New York City and 22 from Chicago should be sent to the meeting to minimize the total airfare. The minimum airfare is $13,412.00. 

29. If high-grade carpet is priced between $520 and $620 per roll, some of each type of carpet will be manufactured to maximize income.
AN–22 Answers to Odd-Numbered Problems

Review Exercises (p. 210)

1. Bounded. Corner points: (0, 2) (0, 6).

3. Bounded. Corner points: (0, 2) \(\left(\frac{8}{5}, \frac{6}{5}\right)\), (0, 6).

9. Bounded. Corner points: (4, 0), (0, 4), (0, 6).

11. Bounded. Corner points: (2, 0), (4, 0), (2, 3), (0, 3), (0, 4).

13. Bounded. Corner points: (2, 0), (4, 0), (2, 3), (0, 3), (0, 4).

15. Maximum of \(\frac{80}{3}\) at \(\left(\frac{40}{3}, \frac{40}{3}\right)\).

17. Minimum of 20 at (0, 10).

19. Maximum of 40 at (20, 0), \(\left(\frac{40}{3}, \frac{40}{3}\right)\), and at any point on the line segment \(2x + y = 40\) connecting them.

21. Minimum of 20 at (10, 0).

23. Maximum of 235 at (5, 8). Minimum of 60 at (4, 0), (0, 3) and at all points on the line segment \(3x + 4y = 12\), connecting them.

25. Maximum of 155 at (5, 4). Minimum of 0 at (0, 0).

27. Maximum of 42 at (9, 8).

29. Maximum of 24 at (8, 8).

31. Minimum of \(\frac{48}{5}\) at \(\left(\frac{4}{5}, \frac{18}{5}\right)\).

33. (a) \[
\begin{align*}
\frac{1}{4}x + \frac{1}{2}y &\leq 75 \\
\frac{3}{4}x + \frac{1}{2}y &\leq 120 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

(b) Produce 8 pairs of downhill skis and 24 pairs of cross-country skis for a maximum profit of $1760.

37. Give the child 1.58 servings of Gerber Banana Oatmeal and Peach and no Gerber Mixed Fruit Juice for a minimum cost of $1.25.

39. Manufacture 550,000 high-potency vitamins and 250,000 thousand calcium-enriched vitamins for a maximum profit of $67,500.

41. To maximize the projected dividend, the fund should purchase 3125 shares of Boeing and 7500 shares of Honeywell. The projected dividend is $3562.50.

(c) Bill should prepare 90 packages of economy blend and 105 packages of superior blend to obtain a maximum profit of $69.00.

Mathematical Questions from Professional Exams

CHAPTER 5 Linear Programming: Simplex Method

Exercise 5.1 (p. 232)

17. Cannot be modified 19. Cannot be modified

21. Can be modified. The constraints become

\[
\begin{align*}
x_1 - x_2 - x_3 &\leq 6 \\
-2x_1 + 3x_2 &\leq 12 \\
x_3 &\leq 2 \\
x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]

The initial simplex tableau is

\[
\begin{array}{cccccc}
BV & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \\
s_1 & 0 & 2 & 2 & -1 & 0 & 0 \\
s_2 & 0 & 0 & 8 & 1 & 1 & 0 \\
s_3 & 0 & 0 & 1 & 0 & 0 & 1 \\
P & 1 & -2 & -1 & -3 & 0 & 0
\end{array}
\]

25. \[
P - 3x_1 - 5x_2 = 0 \\
2.2x_1 - 1.8x_2 + s_1 = 5 \\
0.8x_1 + 1.2x_2 + s_2 = 2.5 \\
x_1 + x_2 + s_3 = 0.1 \\
x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\]

The initial simplex tableau is

\[
\begin{array}{cccccc}
BV & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\
s_1 & 0 & 2.2 & -1.8 & 1 & 0 & 0 \\
s_2 & 0 & 0.8 & 1.2 & 0 & 1 & 0 \\
s_3 & 0 & 0 & 1 & 0 & 0 & 1 \\
P & 1 & -2 & -1 & -3 & 0 & 0
\end{array}
\]

29. \[
P - 3x_1 - 4x_2 - 2x_3 = 0 \\
3x_1 + x_2 + 4x_3 + s_1 = 5 \\
x_1 - x_2 + s_2 = 5 \\
2x_1 - x_2 + x_3 + s_3 = 6 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\]

The initial tableau is

\[
\begin{array}{cccccc}
BV & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \\
s_1 & 0 & 3 & 1 & 4 & 1 & 0 & 0 \\
s_2 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\
s_3 & 0 & 2 & -1 & 0 & 0 & 1 & 6 \\
P & 1 & -3 & -4 & -2 & 0 & 0 & 0
\end{array}
\]

23. \[
P - 2x_1 - x_2 - 3x_3 = 0 \\
5x_1 + 2x_2 + x_3 + s_1 = 20 \\
6x_1 - x_2 + 4x_3 + s_2 = 24 \\
x_1 + x_2 + 4x_3 + s_3 = 16 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\]

The initial simplex tableau is

\[
\begin{array}{cccccc}
BV & P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \\
s_1 & 0 & 5 & 2 & 1 & 1 & 0 & 0 & 20 \\
s_2 & 0 & 6 & -1 & 4 & 0 & 1 & 0 & 24 \\
s_3 & 0 & 1 & 1 & 4 & 0 & 0 & 1 & 16 \\
P & 1 & -2 & -1 & -3 & 0 & 0 & 0 & 0
\end{array}
\]

27. \[
P - 2x_1 - 3x_2 - x_3 = 0 \\
x_1 + x_2 + x_3 + s_1 = 50 \\
3x_1 + 2x_2 + x_3 + s_2 = 10 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0
\]

The initial simplex tableau is

\[
\begin{array}{cccccc}
BV & P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \\
s_1 & 0 & 1 & 1 & 1 & 1 & 0 & 50 \\
s_2 & 0 & 3 & 2 & 1 & 0 & 1 & 10 \\
P & 1 & -2 & -3 & -1 & 0 & 0 & 0
\end{array}
\]

31. Maximize \( P = x_1 + 2x_2 + 5x_3 \) subject to the constraints

\[
\begin{align*}
x_1 - 2x_2 - 3x_3 &\leq 10 \\
-3x_1 - x_2 + x_3 &\leq 12 \\
x_1 &\geq 0, x_2 &\geq 0, x_3 &\geq 0
\end{align*}
\]

System with slack variables:

\[
\begin{align*}
P - x_1 - 2x_2 - 5x_3 &\leq 0 \\
x_1 - 2x_2 - 3x_3 + s_1 &\leq 10 \\
-3x_1 - x_2 + x_3 + s_2 &\leq 12 \\
x_1 \geq 0, x_2 &\geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0
\end{align*}
\]

Initial tableau:

\[
\begin{array}{cccccc}
BV & P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \\
s_1 & 0 & 1 & -2 & -3 & 1 & 0 & 10 \\
s_2 & 0 & -3 & -1 & 1 & 0 & 1 & 12 \\
P & 1 & -1 & -2 & -5 & 0 & 0 & 0
\end{array}
\]
### Answers to Odd-Numbered Problems

33. Maximize \( P = 2x_1 + 3x_2 + x_3 + 6x_4 \)
subject to the constraints
\[
\begin{align*}
-x_1 + 2x_2 + 3x_4 & \leq 10 \\
-x_1 + 2x_2 + x_3 + 4x_4 & \leq 8 \\
x_1 + 2x_3 + x_4 & \leq 9 \\
x_1 & \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
\end{align*}
\]
System with slack variables:
\[
\begin{align*}
P - 2x_1 - 3x_2 - x_3 - 6x_4 & = 0 \\
-x_1 + 2x_2 + 3x_4 + s_1 & = 10 \\
-x_1 + 2x_2 + x_3 + 4x_4 + s_2 & = 8 \\
x_1 + 2x_3 + x_4 + s_3 & = 9 \\
x_3 & \geq 0, x_2 \geq 0, x_4 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\end{align*}
\]
Initial tableau:

<table>
<thead>
<tr>
<th>( s_1 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>RHS</th>
</tr>
</thead>
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</table>

37. (a) Tableau after pivot:

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<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
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<td>3</td>
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<td>1</td>
<td>-2</td>
</tr>
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<td>1</td>
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<tr>
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</tr>
<tr>
<td>( P )</td>
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<td>-3</td>
<td>-1</td>
<td>-6</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

39. (a) Tableau after pivot:

<table>
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<th>( s_1 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
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</tbody>
</table>

41. (a) \( P \) is the profit, \( x_1 \) is the number of sleeping dolls, \( x_2 \) is the number of walking dolls, and \( x_3 \) is the number of talking dolls.

(b) Maximize
\[
P = 6x_1 + 6x_2 + 8x_3
\]
subject to the constraints
\[
\begin{align*}
x_1 + x_2 + x_3 & \leq 60 \\
6x_1 + 7.5x_2 + 9x_3 & \leq 405 \\
x_1 & \geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]
43. (a) $P$ is the daily profit, $x_1$ is the number of shirts produced, $x_2$ is the number of jackets produced, and $x_3$ is the number of pairs of pants produced each day.

(b) Maximize

$$P = 19x_1 + 34x_2 + 15x_3$$

subject to the constraints

$$10x_1 + 20x_2 + 20x_3 \leq 500$$
$$20x_1 + 40x_2 + 30x_3 \leq 900$$
$$x_1 + x_2 + x_3 \leq 30$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

(c) System with slack variables added:

$$10x_1 + 20x_2 + 20x_3 + s_1 = 500$$
$$20x_1 + 40x_2 + 30x_3 + s_2 = 900$$
$$x_1 + x_2 + x_3 + s_3 = 30$$

$$P - 19x_1 - 34x_2 - 15x_3 = 0$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \quad s_3 \geq 0$$

(d) Initial tableau:

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</tbody>
</table>

45. (a) $P$ is the return on her investments; $x_1$ is the amount invested in the money market, $x_2$ is the amount invested in the mutual fund, and $x_3$ is the amount invested in the CD.

(b) Maximize

$$P = 0.0105x_1 + 0.05x_2 + 0.0166x_3$$

subject to the constraints

$$x_1 + x_2 + x_3 \leq 25,000$$
$$x_2 + x_3 \leq 10,000$$
$$-x_1 + x_3 \leq 1,500$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

(c) System with slack variables added:

$$x_1 + x_2 + x_3 + s_1 = 25,000$$
$$x_2 + x_3 + s_2 = 10,000$$
$$-x_1 + x_3 + s_3 = 1,500$$

$$P - 0.0105x_1 - 0.05x_2 - 0.0166x_3 = 0$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0 \quad s_3 \geq 0$$

(d) Initial tableau:

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Exercise 5.2 (p. 252)

1. False  2. True  3. False  4. Column  5. (b); the pivot element is 1 in row s_1, column x_1
7. (a); the solution is $P = \frac{256}{7}, x_1 = \frac{32}{7}, x_2 = 0$
9. (c) 11. (b); the pivot element is 1 in row x_1, column s_1
13. The maximum is $P = \frac{204}{7} = \frac{29}{7}$ when $x_1 = \frac{24}{7}, x_2 = \frac{12}{7}$. 15. The maximum is $P = 8$ when $x_1 = \frac{2}{3}, x_2 = \frac{2}{3}$. 17. The maximum is $P = 6$ when $x_1 = 2, x_2 = 0$.
19. There is no maximum for P; the feasible region is unbounded.
21. The maximum is $P = 30$ when $x_1 = 0, x_2 = 0, x_3 = 10$. 23. The maximum is $P = 42$ when $x_1 = 1, x_2 = 10, x_3 = 0, x_4 = 0$.
25. The maximum is $P = 40$ when $x_1 = 20, x_2 = 0, x_3 = 0$. 27. The maximum is $P = 50$ when $x_1 = 0, x_2 = 15, x_3 = 5, x_4 = 0$.

29. (a) Let $P$ be the daily profit, $x_1$ be the number of pairs of Jean I produced daily, $x_2$ be the number of pairs of Jean II produced daily, and $x_3$ be the number of pairs of Jean III produced daily.
Maximize

$$P = 4x_1 + 4.5x_2 + 6x_3$$

subject to the constraints

$$8x_1 + 12x_2 + 18x_3 \leq 5200$$
$$12x_1 + 18x_2 + 24x_3 \leq 6000$$
$$4x_1 + 8x_2 + 12x_3 \leq 2200$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

(b) Maximum $P = 2000$, when $x_1 = 500, x_2 = 0, x_3 = 0, s_1 = 1200, s_2 = 0, s_3 = 200$

(c) A maximum daily profit of $2000 is made when 500 pairs of Jean I are produced and neither Jean II nor Jean III is produced.
31. (a) Let $P$ be the weekly profit, $x_1$ be the number of rolls of birthday wrapping paper sold per week, $x_2$ be the number of rolls of holiday wrapping paper sold weekly, and $x_3$ be the number of rolls of wedding wrapping paper sold per week. Maximize $P = x_1 + x_2 + 2x_3$ subject to the constraints
\[ \begin{align*}
3x_1 + 5x_2 + 4x_3 & \leq 500 \\
10x_1 + 15x_2 + 12x_3 & \leq 1800 \\
\frac{1}{2}x_1 + x_2 & \leq 75 \\
x_1 & \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*} \]
(b) Maximum $P = 190$, when $x_1 = 0$, $x_2 = 40$, $x_3 = 75$, $s_1 = 0$, $s_2 = 300$, and $s_3 = 0$.
(c) The lacrosse team makes a maximum profit of $190 a week when constrained as in the problem by selling no rolls of birthday wrapping paper, 40 rolls of holiday wrapping paper, and 75 rolls of wedding wrapping paper.

33. (a) Let $P$ denote the revenue, $x_1$ denote the number of Can I nuts packaged, $x_2$ denote the number of Can II nuts packaged, and $x_3$ denote the number of Can III nuts packaged. Maximize $P = 28x_1 + 24x_2 + 21x_3$ subject to the constraints
\[ \begin{align*}
3x_1 + 4x_2 + 5x_3 & \leq 500 \\
x_1 + \frac{1}{2}x_2 & \leq 100 \\
x_1 + \frac{1}{2}x_2 & \leq 50 \\
x_1 & \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*} \]
(b) Maximum $P = 2870$, when $x_1 = 50$, $x_2 = 0$, $x_3 = 70$, $s_1 = 0$, $s_2 = 50$, and $s_3 = 0$.
(c) Revenue is maximized at $2870.00 when 50 packages of Can I nuts, 70 packages of Can III nuts, and no packages of Can II nuts are produced.

35. (a) The store should order 45 sleeping dolls, 15 walking dolls, and no talking dolls to maximize profit.
(b) The maximum profit is $390.00.

37. (a) Steven should produce 15 shirts, 15 jackets, and no pants to maximize his profit.
(b) The maximum profit is $795.00.

39. (a) Let $P$ denote the company’s revenue and let $x_1$, $x_2$, and $x_3$ represent the number of gallons of regular, premium, and super premium gasoline, respectively, to be refined. The company wants to refine amounts that will maximize its revenue subject to its available resources. Maximize $P = 2.80x_1 + 2.97x_2 + 3.08x_3$ subject to the constraints
\[ \begin{align*}
0.6x_1 + 0.7x_2 + 0.8x_3 & \leq 140,000 \\
0.4x_1 + 0.3x_2 + 0.2x_3 & \leq 120,000 \\
x_1 + x_2 + x_3 & \leq 225,000 \\
x_1 & \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0
\end{align*} \]
(b) The maximum is $P = 638,500$, obtained when $x_1 = 175,000$, $x_2 = 50,000$, and $x_3 = 0$.
(c) To achieve a maximum revenue of $638,500, the company should refine 175,000 gallons of regular gasoline, 50,000 gallons of premium gasoline, and no super premium gasoline.
41. (a) Let $P$ be the total yield, $x_1$ be the amount invested in stocks, $x_2$ be the amount invested in corporate bonds, and $x_3$ be the amount invested in municipal bonds.

Maximize

\[ P = 0.08x_1 + 0.05x_2 + 0.03x_3 \]

subject to the constraints

\[ \begin{align*}
  x_1 + x_2 + x_3 & \leq 90,000 \\
  x_1 & \leq 45,000 \\
  x_1 - x_2 & \leq 18,000 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0 \\
  x_3 & \geq 0
\end{align*} \]

(b) The maximum is $P = 5580$, obtained when $x_1 = 45,000$, $x_2 = 31,500$, and $x_3 = 13,500$.

(c) The financial consultant can maximize her client’s investment income while maintaining her investment strategy by investing $45,000 in stocks, $31,500 in corporate bonds, and $13,500 in municipal bonds. The maximum return will be $5580.

43. (a) Let $P$ denote the profit, $x_1$ denote the number of acres of soybeans planted, $x_2$ denote the number of acres of corn planted, and $x_3$ denote the number of acres of wheat planted.

Maximize:

\[ P = 70x_1 + 90x_2 + 50x_3 \]

subject to the constraints

\[ \begin{align*}
  x_1 + x_2 + x_3 & \leq 200 \\
  40x_1 + 50x_2 + 30x_3 & \leq 18,000 \\
  20x_1 + 30x_2 + 15x_3 & \leq 4,200 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0 \\
  x_3 & \geq 0
\end{align*} \]

(b) Maximum $P = 14,400$, when $x_1 = 180$, $x_2 = 20$, $x_3 = 0$, $s_1 = 0$, $s_2 = 9800$, and $s_3 = 0$

(c) The farmer realizes a maximum profit of $14,400 when planting 180 acres of soybeans, 20 acres of corn, and no wheat.

45. (a) Let $P$ denote the profit, $x_1$ denote the number of televisions shipped from Chicago, $x_2$ denote the number shipped from New York, and $x_3$ denote the number shipped from Denver.

Maximize:

\[ P = 70x_1 + 80x_2 + 40x_3 \]

subject to the constraints

\[ \begin{align*}
  x_1 + x_2 + x_3 & \leq 400 \\
  50x_1 + 40x_2 + 80x_3 & \leq 20,000 \\
  6x_1 + 8x_2 + 4x_3 & \leq 3,000 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0 \\
  x_3 & \geq 0
\end{align*} \]

(b) The maximum $P = 31,000$, obtained when $x_1 = 100$, $x_2 = 300$, $x_3 = 0$.

(c) The manufacturer will obtain a maximum profit of $31,000 if it ships 100 televisions from Chicago, 300 televisions from New York, and none from Denver.

47. (a) Kami should invest $15,000 in the money market, $10,000 in the mutual fund, and nothing in the CD.

(b) The maximum return on Kami’s investments is $657.50.

(c) The financial consultant can maximize her client’s investment income while maintaining her investment strategy by investing $45,000 in stocks, $31,500 in corporate bonds, and $13,500 in municipal bonds. The maximum return will be $5580.

Exercise 5.3 (p. 266)

3. True
4. \begin{bmatrix} 8 & 1 & -2 \\ 6 & 4 & 2 \end{bmatrix}
5. True
6. $\geq$
7. Duality principle
8. False
9. Standard form
11. Not in standard form
13. Not in standard form
AN–28 Answers to Odd-Numbered Problems

15. Maximize

\[ P = 2y_1 + 6y_2 \]

subject to the constraints

\[ y_1 + 2y_2 \leq 2 \]
\[ y_1 + 3y_2 \leq 3 \]
\[ y_1 \geq 0 \quad y_2 \geq 0 \]

17. Maximize

\[ P = 5y_1 + 4y_2 \]

subject to the constraints

\[ y_1 + 2y_2 \leq 3 \]
\[ y_1 + y_2 \leq 1 \]
\[ y_1 \geq 0 \quad y_2 \geq 0 \]

19. Maximize

\[ P = 60y_1 + 90y_2 \]

subject to the constraints

\[ y_1 + 3y_2 \leq 3 \]
\[ y_1 + 2y_2 \leq 4 \]
\[ y_1 + y_2 \leq 1 \]
\[ 2y_1 + 2y_2 \leq 2 \]
\[ y_1 \geq 0 \quad y_2 \geq 0 \]

21. The minimum is \( C = 6 \) when \( x_1 = 0, x_2 = 2 \).

23. The minimum is \( C = 12 \) when \( x_1 = 0, x_2 = 4 \).

25. The minimum is \( C = \frac{21}{5} \) when \( x_1 = \frac{8}{5}, x_2 = 0, x_3 = \frac{13}{5} \).

27. The minimum is \( C = 5 \) when \( x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0 \).

29. Minimize the cost by assigning 7 employees to work on Friday, 15 to work on Saturday, and 8 to work on Sunday. The minimum cost is $4150. No one is scheduled to work a Friday–Saturday schedule.

31. (a) Let \( C \) be the cost of the order; let \( x_1 \) be the number of Lunch #1 ordered, \( x_2 \) be the number of Lunch #2 ordered, and \( x_3 \) the number of Lunch #3 ordered.

Minimize

\[ C = 6.2x_1 + 7.4x_2 + 9.1x_3 \]

subject to the constraints

\[ x_1 \geq 4 \]
\[ x_1 + x_2 + x_3 \geq 9 \]
\[ x_1 \geq 4 \]
\[ x_2 + x_3 \geq 5 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \]

(b) Minimum \( C = 65.2 \) when \( x_1 = 4, x_2 = 3, \) and \( x_3 = 2 \).

(c) Mrs. Mintz spends the least amount, $65.20, when she orders 4 of Lunch #1, 3 of Lunch #2, and 2 of Lunch #3.

33. (a) Let \( C \) be the cost of the supplements, \( x_1 \) be the number of pill \( P \) needed, and \( x_2 \) the number of pill \( Q \) needed.

Minimize

\[ C = 3x_1 + 4x_2 \]

subject to the constraints

\[ 3x_1 + 10x_2 \leq 50 \]
\[ 2x_1 + x_2 \geq 8 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

(b) Minimum \( C = 22 \) when \( x_1 = 2 \) and \( x_2 = 3 \).

(c) Mr. Jones minimizes his cost at $0.22 when he adds 2 \( P \) pills and 4 \( Q \) pills to his diet.

35. Katy should mix 1 kilogram of dried peaches and 1 kilogram of dried pears to meet her requirements and to minimize the sodium content. The minimum sodium content is 230 milligrams.

Exercise 5.4 (p. 281)

1. False 2. –z 3. True 4. False 5. The maximum is \( P = 44 \) when \( x_1 = 4, x_2 = 8 \).

7. The maximum is \( P = 27 \) when \( x_1 = 9, x_2 = 0, x_3 = 0 \).

9. The maximum is \( P = 7 \) when \( x_1 = 1, x_2 = 2 \).

11. \( x_1 = 0, x_2 = 0, x_3 = \frac{20}{3}, x = \frac{20}{3} \)

13. (a) Minimize

\[ C = 400x_1 + 100x_2 + 200x_3 + 200x_4 \]

subject to the constraints

\[ x_1 + x_2 \leq 600 \]
\[ x_3 + x_4 \leq 400 \]
\[ x_2 + x_4 \leq 500 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \]

(b) Minimum \( C = 150,000 \) when \( x_1 = 100, x_2 = 300, x_3 = 400, x_4 = 0 \)

(c) Private Motors minimizes shipping charges at $150,000 by shipping 100 engines from M1 to A1, 300 engines from M1 to A2, 400 engines from M2 to A1, and no engines from M2 to A2.
15. (a) Let $C$ denote the total shipping cost, and $x_1$ the number of GPS systems shipped from W1 to D1; $x_2$ the number shipped from W1 to D2; $x_3$ the number shipped from W2 to D1; and $x_4$ the number shipped from W2 to D2.

The manufacturer wants to fill the orders at the lowest possible cost.

Minimize

$$C = 8x_1 + 12x_2 + 13x_3 + 7x_4$$

subject to the constraints

$$x_2 + x_3 = 20$$
$$x_2 + x_4 = 30$$
$$x_1 + x_2 \leq 40$$
$$x_3 + x_4 \leq 15$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0$$
$$x_4 \geq 0$$

(b) Minimum is $C = 445$ when $x_1 = 20$, $x_2 = 15$, $x_3 = 0$, $x_4 = 15$.

(c) The GPS systems manufacturer minimizes shipping costs at $445$ by shipping $20$ GPS systems from warehouse $W1$ to dealer $D1$, shipping $15$ GPS systems from $W1$ to $D2$, and $15$ GPS systems from $W2$ to $D2$.

17. (a) Let $x_1$ represent the amount spent on newspaper advertising, and $x_2$ represent the amount spent on radio advertising. $C$ is the total cost of advertising.

Minimize

$$C = x_1 + x_2$$

subject to the constraints

$$50x_1 + 70x_2 \leq 100,000$$
$$40x_1 + 20x_2 \leq 120,000$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

(b) Minimum is $C = 3000$ when $x_1 = 3000$ and $x_2 = 0$.

(c) The appliance store will spend the least on advertising, $3000$, and reach the intended audience, if it spends all $3000$ on newspaper advertising and nothing on radio advertising.

Review Exercises (p. 284)

1. In standard form

3. In standard form

5. Not in standard form

7. Not in standard form

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</tbody>
</table>

15. (a) The pivot element 2 is found in row $s_2$, column $x_2$. The new tableau after pivoting is

<table>
<thead>
<tr>
<th>RV</th>
<th>P</th>
<th>x1</th>
<th>x2</th>
<th>s1</th>
<th>s2</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>-5</td>
<td>2</td>
</tr>
<tr>
<td>x2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) The resulting system of equations is

$$x_1 = 15 + 4s_1 + \frac{5}{2}s_2$$
$$x_2 = 5 - s_1 - \frac{1}{2}s_2$$
$$P = 125 - s_1 - \frac{7}{2}s_2$$

(c) The new tableau is the final tableau. The solution is maximum $P = 125$ when $x_1 = 15$ and $x_2 = 5$.

21. Purchase 615.0 shares of Duke Energy, 434.9 shares of H. J. Heinz, 534.5 shares of General Electric, and 432.2 shares of Ferrellgas Partners.

The minimum average price/earnings ratio is

$$\frac{45345.21}{2016.6} = 22.49$$

The annual yield for this optimal investment strategy is 2398.14.
17. (a) The pivot element 1 is found in row \( s_1 \), column \( s_2 \). The new tableau after pivoting is

\[
\begin{array}{cccccc}
BV & P & x_1 & x_2 & s_1 & s_2 & RHS \\
1 & 0 & 1 & 1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 5 & 2 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 4 \\
\end{array}
\]

(b) The resulting system of equations is

\[
\begin{align*}
&x_1 = 10 - x_2 + x_3 - s_1 \\
&s_2 = 4 - x_2 - x_3 \\
&P = 20 - x_2 + 5s_2 - 2s_1
\end{align*}
\]

(c) The pivot element 0.5 is found in row \( s_3 \), column \( x_1 \). The new pivot element 1 is found in row \( s_2 \), column \( x_3 \).

19. (a) The pivot element 0.5 is found in row \( s_1 \), column \( x_1 \). The new tableau after pivoting is

\[
\begin{array}{cccccc}
BV & P & x_1 & x_2 & s_1 & s_2 & RHS \\
1 & 0 & 0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0.5 & -2 & 1 \\
1 & 0 & 0 & 0 & 5 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

(b) The resulting system of equations is

\[
\begin{align*}
&s_1 = 2 - x_2 - 2s_1 \\
&s_2 = 1 - 0.5s_2 + 2s_1 \\
&P = 5 - 0.5s_2 - 5s_1
\end{align*}
\]

(c) The new tableau is the final tableau.

21. (a) The pivot element 1 is found in row \( s_3 \), column \( x_1 \). The new tableau after pivoting is

\[
\begin{array}{cccccc}
BV & P & x_1 & x_2 & s_1 & s_2 & RHS \\
1 & 0 & 0 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 1 & 8 & 0 \\
1 & 0 & 1 & 0 & 0 & 3 & 0 \\
0 & 1 & 0 & 13 & 0 & 15 & 3 \\
0 & 0 & 0 & 14 & 0 & 5 & 1 \\
\end{array}
\]

(b) The resulting system of equations is

\[
\begin{align*}
&s_1 = 10 - 4s_2 + 6s_3 - s_2 \\
&s_2 = 8 - 8s_3 + 4s_2 - 5s_3 \\
&s_3 = 3 - 3s_3 + 5s_2 - 3s_3 \\
&P = 14 - 13s_3 + 15s_2 - 3s_3
\end{align*}
\]

(c) No solution exists since in the pivot column \( s_2 \) all entries are negative.

23. The maximum is \( P = 22,500 \) when \( x_1 = 0, x_2 = 100, x_3 = 50 \).

25. The maximum is \( P = 352 \) when \( x_1 = 0, x_2 = \frac{6}{5}, x_3 = \frac{28}{5} \).

27. In standard form

29. Not in standard form, constraints are not written as greater than or equal to inequalities

31. Not in standard form, constraints are not written as greater than or equal to inequalities

33. Maximize \( P = 8y_1 + 2y_2 \)

subject to the constraints

\[
\begin{align*}
2y_1 + y_2 & \leq 2 \\
2y_1 - y_2 & \leq 1 \\
y_1 & \geq 0 \quad y_2 & \geq 0
\end{align*}
\]

35. Maximize \( P = 100y_1 + 50y_2 \)

subject to the constraints

\[
\begin{align*}
y_1 + 2y_2 & \leq 5 \\
y_1 + y_2 & \leq 4 \\
y_1 & \leq 2 \\
y_1 & \geq 0 \quad y_2 & \geq 0
\end{align*}
\]

37. Minimum is \( C = 7 \) when \( x_1 = 3, x_2 = 1 \)

39. Minimum is \( C = 275 \) when \( x_1 = 25, x_2 = 0, x_3 = 75 \)

41. Maximum is \( P = 20 \) when \( x_1 = 0, x_2 = 4 \)

43. Minimum is \( C = 6 \) when \( x_1 = 3, x_2 = 0 \)

45. Maximum is \( P = 12,250 \) when \( x_1 = 0, x_2 = 5, x_3 = 25 \)

47. (a) Let \( P \) denote the profit, and let \( x_1 \) represent the number of cocktail tables and \( x_2 \) the number of end tables made each day.

Sanding, staining, and varnishing times, given in hours, have been changed to minutes.

Maximize

\[
P = 20x_1 + 15x_2
\]

subject to the constraints

\[
\begin{align*}
4x_1 + 8x_2 & \leq 360 \\
4x_1 + 10x_2 & \leq 360 \\
8x_1 + 4x_2 & \leq 360 \\
x_1 & \geq 0 \quad x_2 & \geq 0 \quad x_3 & \geq 0
\end{align*}
\]

(b) Maximum \( P = \frac{2025}{2} = 1012.5 \), obtained when \( x_1 = \frac{135}{4} = 33.75 \), and \( x_2 = \frac{45}{2} = 22.5 \).

(c) The furniture maker can obtain a maximum profit of $1012.50 while maintaining the time constraints if 33.75 cocktail tables and 22.5 end tables are manufactured each day. That is, profit is maximized by producing 135 cocktail tables and 90 end tables over a four-day period. The maximum profit for the four days is $4050.
49. (a) \( C \) denotes cost of a pound of meat, \( x_1 \) denotes the amount of beef in the pound, and \( x_2 \) denotes the amount of pork in the pound.

Minimize
\[ C = 1.89x_1 + 1.29x_2 \]
subject to the constraints
\[ x_1 + x_2 = 1 \]
\[ 0.75x_1 + 0.60x_2 \geq 0.70 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]

(b) Minimum \( C = 1.69 \) when \( x_1 = \frac{2}{3} \) and \( x_2 = \frac{1}{3} \)

(c) The minimum cost of the meat loaf is $1.69 per pound when the butcher mixes \( \frac{2}{3} \) pound of ground beef with \( \frac{1}{3} \) pound of ground pork.

Mathematical Questions from Professional Exams (p. 290)

CHAPTER 6  Finance

Exercise 6.1 (p. 299)

1. Prt 2. Discounted 3. False 4. True 5. 60% 6. 7. 110% 8. 9% 9. 0.25% 10. 0.25 11. 1.00 12. 0.065 13. 0.0005 14. 150
15. 18 25. 105 27. 5% 29. 160% 31. 250 33. \( \frac{1000}{3} \approx 333.33 \) 35. \( \$10 \) 37. \( \$45 \) 39. \( \$150 \) 41. 10% 43. 33.3% 45. 13.3%
47. \( \$1140 \) 49. \( \$1680 \) 51. \( \$1263.16 \); 10.53% 53. \( \$2380.95 ; 9.52\% \) 55. \( \$489.00 \) 57. The list price is reduced by 35%. 59. She should get a loan of \( \$5779.34 \).
61. (a) The interest charged is \( \$1377.60 \). (b) \( \$9377.60 \) is the total loan. (c) The monthly payment is \( \$260.49 \). 67. 0.0032 = 0.32%.
69. \( \$675 \) in interest was received. The annual simple interest rate is 0.027 = 2.7%.

Exercise 6.2 (p. 311)

5. True 6. False 7. Compounded continuously 8. Present value 9. \( \$108.29 \) 11. \( \$690.50 \) 13. \( \$697.09 \) 15. \( \$10.41 \) 17. \( \$106.98 \)
19. \( \$96.08 \) 21. \( \$860.72 \) 23. \( \$554.09 \) 25. \( \$74.97 \) 27. \( \$834.32 \) 29. 6-1/4% compounded annually yields more in a year than 6% compounded quarterly.
31. 9% compounded monthly yields more in a year than 8.8% compound daily. 33. Effective rate of interest is 5.095%.
35. Effective rate of interest is 5.127%. 37. To double an investment in 3 years requires an interest rate of 25.99%.
39. To triple an investment in 5 years requires an interest rate of 24.573%. 41. (a) It will take 8.69 years to double money invested at 8% compounded monthly. (b) It will take 8.66 years to double money invested at 8% compounded continuously.
43. 6.823% compounded quarterly has an effective rate of 7%. 45. (a) After 3 years, \( A = \$1124.86 \); \$124.86 interest has been earned. (b) After 3 years, \( A = \$1127.27 \); \$127.27 interest has been earned. 47. (a) \( A = \$1040.77 \) after 2 years. (b) \( A = \$1061.68 \) after 3 years. (c) \( A = \$1083.07 \) after 4 years. 49. (a) Deposit \( \$4438.56 \) to have \$5000 in 4 years. (b) Deposit \( \$3940.16 \) to have \$5000 in 8 years.
51. The loan at 10% compounded monthly results in less interest due. 53. At 3% compounded monthly, it will take 13.53 years for \$100 to grow to \$150. At 3% compounded continuously, it will take 13.52 years to reach \$150. 55. It takes 15.27 years for \$10,000 to grow to \$25,000 at 6% compounded continuously.
57. The house will appreciate to \$104,335 in 5 years. 59. Jerome should ask his parents for \$12,910.62. 61. George’s stock should be worth \$3017.04. 63. Jim will not have enough money; he will be \$2.40 short. The second investment is a better deal.
65. 64. At the end of 20 years, Will will have \$450.92 more than Harry. 67. They should deposit \$35,492.71 into the savings account.
69. The account will be worth \$12,631.45 on the child’s 25th birthday. 71. It takes 30.54 years for \$10,000 to grow to \$25,000 at 3% compounded daily.
73. After 30 years, the IRA will be worth \$12,973.59. 75. Tom and Anita should invest \$10,288.11 to have the money needed in 18 years.
77. (a) 7.87%  (b) 26.26 trillion dollars. 79. Option (a) is best with interest of \$30,000. 81. If \$1000 purchases \$950 after 2 years, the inflation rate is 2.5%. 83. At 2% annual inflation rate, money’s value is halved in 34.3 years.
85. (a) Pay \$3686.45 if the interest rate is 5% compounded monthly. (b) Pay \$3678.79 if the interest rate is 5% compounded continuously. 87. The bond should be sold for \$6755.64 for an interest rate of 4% per annum.
89. (a) At 4% annual interest, it takes 17.7 years to double an investment. (b) At 3% annual interest, it takes 36.8 years to triple an investment.
(c) \( mP = P \left( 1 + \frac{r}{n} \right)^{nt} = m \left( 1 + \frac{r}{n} \right)^{nt} \). \( m \cdot \ln n = n \cdot \ln \left( 1 + \frac{r}{n} \right) = r \).
91. (a) Average annual inflation was 2.54%. (b) In 2022 the CPI will reach 300. 93. It will take 30.5 years.

Exercise 6.3 (p. 326)

3. Annuity 4. Amount; annuity 5. \$1593.74 7. \$4844.25 9. \$7524.11 11. \$6629.90 13. \$113,201.03 15. \$154.69 per month 17. \$1955.41 per quarter 19. \$4119.18 per month 21. \$200.46 per month 23. \$2088.11 per year 25. \$53,946.41 27. \$972.80 29. \$851.51 per month 31. \$23,902.70 per year

<table>
<thead>
<tr>
<th>Payment $</th>
<th>Sinking Fund Deposits $</th>
<th>Cumulative Deposits</th>
<th>Accumulated Interest</th>
<th>Total $</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23,902.70</td>
<td>23,902.70</td>
<td>0</td>
<td>23,902.70</td>
</tr>
<tr>
<td>2</td>
<td>23,902.70</td>
<td>47,805.40</td>
<td>717.081</td>
<td>48,522.48</td>
</tr>
<tr>
<td>3</td>
<td>23,902.70</td>
<td>71,708.10</td>
<td>2172.76</td>
<td>73,880.88</td>
</tr>
<tr>
<td>4</td>
<td>23,902.70</td>
<td>95,610.80</td>
<td>4389.18</td>
<td>99,999.98</td>
</tr>
</tbody>
</table>
### Exercise 6.5 (p. 348)

33. The investor should pay $193,484 for the well. 35. $78,848.79 per quarter 37. (a) $180,611.12 (b) $2990.15 semiannually

39. It will take about 34 years to accumulate. 41. Angie should deposit $385.50 monthly. 43. The school district should deposit $44,320.39 per quarter

45. The account will be worth $30,068.42. 47. After 25 years, the lump sum will be worth $10,192,646.88; the annual payments will be worth $8,635,455.48. Dan should select the lump sum payment. 49. (a) The projected price of the Honda is $27,061.24. (b) The projected price including tax is $29,564.40. (c) Monthly payments of $583.37 are needed in the sinking fund.

(b) To pay for the projected four years of college, a quarterly payment of $1307.35 should be made.

### Exercise 6.4 (p. 341)


17. (a) $248.48 monthly (b) Total payment is $8945.28. (c) Interest is $945.28. 19. (a) The monthly payment for the 9% loan is $1342.71 and the monthly payment for the 8% loan is $1338.30. The monthly payment for the 9% loan is larger. (b) The total interest paid is larger for the 9% loan, $242,813 compared to $161,192. (c) After 10 years the equity from the 8% loan is larger, $89,695.33 compared to $67,617.64. 21. John should deposit $171.33 per month. 23. (a) After 12 weeks, Dan has $1201.27. (b) Dan can withdraw $35.45 per week for 34 weeks. 25. (a) $40,000 is the down payment (b) $60,000 is the amount of loan (c) $1287.40 per month (d) $303,464 is the total interest on the loan (e) 22 years, 4 months to pay off the loan (f) $211,823.20 total interest with payments of $1387.40 per month. 27. $332.79 per month 29. (a) $474.01 per month (b) interest paid: $4732.48 31. 30 year: monthly payment—$671.50; total interest—$129,740; 15 year: monthly payment—$945.12; total interest—$58,121.60 33. In about 4 years and 8 months the IRA will be depleted. 35. Jeremy needs $14,910.21 in his college fund on August 1. 37. The foundation must invest $821,530.33 to meet the project’s goal. 39. Monthly payments are reduced by $128.07. They will pay $38,361 less in interest. 41. 61.9 months; $50,870.77 in interest is saved by prepaying the loan. 43. The present value of the money, $6529.84, is more than the purchase price, so purchasing is preferable. 45. The present value of the money, $177,297.53, is greater than the purchase price, so purchasing is preferable.

47. (a) If the time value of money is 10%, then Machine A is preferable. (b) If the time value of money is 14%, then Machine A is still preferable. 49. A price of $9845.95 will yield a true interest of 6.5%. 51. The bond sold for $999.76. 53. The bond sold for $998.14.

### Exercise 6.5 (p. 348)

3. (a) After 1 payment, the balance is $2930. (b) After 14 payments the balance is less than $2000. (c) John pays off the balance on the 36th payment. (d) He pays a total of $584.62 in interest. 5. (a) 2162 trout are in the pond after 2 months. (b) In the 26th month the population reaches 5000 trout. 7. (a) $A_0 = 500$, $A_n = 1.02A_{n-1} + 500$. (b) After 81 quarters, there is more than $100,000 in the IRA. (c) $159,738 in 25 years.

9. (a) $A_0 = 150,000$, $A_n = \left(1 + \frac{.06}{12}\right)A_{n-1} - 899.33$. (b) $149,850.67 remains after the 1st payment.

(c) $A_0 = 150,000$, $A_n = \left(1 + \frac{.06}{12}\right)A_{n-1} - 899.33$

(d) After 4 years and 10 months (58 payments) the balance is below $140,000.

(e) With the 360th payment (30 years) the loan is paid.

(f) A total of $173,754.57 in interest was paid.

(g) a. $A_0 = 150,000$, $A_n = \left(1 + \frac{.06}{12}\right)A_{n-1} - 999.33$  b. After the 1st payment, $149,750.67 is owed

(d) The balance is below $140,000 after 3 years and 1 month (37 payments).

(e) With the 279th payment (23 yrs. 3 mos.), the loan is paid.

(f) Total interest paid is $128,169.17.
ANSWERS TO ODD-NUMBERED PROBLEMS

Review Exercises (p. 350)

1. 15  3. 350  5. 21  7. 80  9. 2200  11. Dan paid $23.10 in sales tax.  13. $52.50 interest is charged. Dan must repay $552.50.

15. Warren must pay $19,736.84 to settle his debt.  17. $106.97  19. The 10% per annum compounded monthly loan will cost Mike less.

21. Katy should deposit $73.88.  23. 5.95%  25. 5.87% compounded quarterly has an effective rate of 6%.  27. The Coreys should save $1619.25 per month.  29. (a) The Ostedts' monthly payments are $2726.10.  (b) They will pay $517,830 in interest.  (c) After 5 years, their equity is $117,508.93.

31. The monthly payments are $1,049.00. The equity after 10 years is $21,576 plus any down payment.  33. Mr. Graff should pay $119,432 for the mine.  35. Mr. Doody needs $40,557.64 to meet his goals.  37. Mr. Jones will have saved $8966.18.  39. The monthly payments are $141.22.

41. The effective rate of interest is 9.38%.  43. John will receive $1156.60 every 6 months for 15 years.  45. After 30 months there will be $2087.09 in the fund.

Mathematical Questions from Professional Exams (p. 353)

1. b  2. c  3. 6  4. b  5. d  6. a  7. c
CHAPTER 7  

Probability

Exercise 7.1 (p. 363)


21.  $\emptyset$  23.  $\{a, b, d, c, f, g\}$  25.  (a)  $\{0, 1, 2, 3, 5, 7, 8\}$  (b)  $\{5\}$  (c)  $\{5\}$  (d)  $\{0, 1, 2, 3, 4, 6, 7, 8, 9\}$  (e)  $\{4, 6, 9\}$

(f)  $\{0, 1, 5, 7\} = \mathbb{A}$  (g)  $\mathbb{D}$  (h)  $\{5\}$

27.  (a)  $\{b, c, d, e, f, g\}$  (b)  $\{c\}$  (c)  $\{a, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

(d)  $\{a, b, d, c, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

29.

31.

33.

35.

37.

39.

$A \cap B = (A \cap B) \cup (A \cap C)$

$A \cap B = \overline{A} \cap \overline{B}$

Exercise 7.2 (p. 369)

1.  False  2.  $\emptyset$  3.  $\{a\}$  4.  $\{a, b\}$  5.  $n(A) = 6$  6.  $n(A \cap B) = 3$  7.  $n[(A \cap B) \cup A] = 6$  8.  $n(A \cup B) = 5$  9.  $n(A) = 2$  10.  $n(A) = 10$

15.  452 cars were manufactured.  17.  $n(A) = 24$  19.  $n(A \cup B) = 34$  21.  $n(A \cap B) = 15$  23.  $n(A \cup B \cup C) = 54$  25.  $n(A \cap B \cap C) = 3$

27.  (a)  4,050,000 households are maintained by a mother in the South.  (b)  274,000 households in the Midwest are maintained by a divorced parent.  (c)  600,000 households are maintained by a father who was never married or widowed.  29.  (a)  13,414,000 U.S. civilians 20 years or older were unemployed.  (b)  85,764,000 U.S. civilians 20 years or older were unemployed or not in the labor force.  (c)  184,887,000 U.S. civilians 20 years or older were female or employed.  (d)  27,403,000 U.S. civilians 20 years or older were male and not in the labor force.  31.  (a)  256 layoffs were executive managerial, faculty, or professional nonfaculty.  (b)  168 layoffs were female.  (c)  93 layoffs were black female or faculty.  (d)  125 layoffs were white male or executive managerial.
### AN–36  Answers to Odd-Numbered Problems

#### 33. (a)

![Exercise Bike and Stair Stepper Diagram]

- 222 members surveyed use an exercise bike or stair stepper regularly.
- 103 members surveyed regularly use an exercise bike but not a stair stepper.
- 52 members surveyed regularly use a stair stepper but not an exercise bike.
- 28 members surveyed use neither an exercise bike nor a stair stepper regularly.

#### 35. (a)

![User Name Diagram]

- 91 clients own only GE.
- 78 clients own only F.
- 68 clients own only MMM.
- 179 clients own GE or MMM, but not F.
- 20 clients own GE and MMM, but not F.
- 661 clients own none of the three stocks.

#### 39. (a) 40 cars had satellite radio and heated seats.
- 35 cars had GPS and heated seats.
- 40 cars had neither satellite radio nor GPS.
- 205 cars were sold in July.
- 155 cars were sold with GPS or heated seats or both.

#### 43. 46 use only one of the three brands.

#### 45. \( \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c,d\} \). There are 16 subsets of \( \{a,b,c,d\} \).

#### Exercise 7.3 (p. 376)

1. There are 15 shirt and tie combinations.
2. 9000 four-digit numbers are possible.
3. There are 604,800 telephone numbers without repeated digits.
4. There are 200 different one-topping pizzas.
5. 960 lunches are possible.
6. The people can sit 720 ways.
7. Display 24 cars.
8. There are 36 ways to enter through a window and exit through a door.
9. 2610 = 4,569,760,000 user names are possible.
10. The people can sit 720 ways.
11. 2610 = 4,569,760,000 user names are possible.
12. There are 2625 = 9,261,562,500 user names with no adjacent repeats.
13. There are 224 possible match-ups.
14. Adam can get 486 different types of coverage.
15. There are 8 different blood types.
16. There are 84 possible lineups.
17. The audits can be scheduled 3,628,800 ways.
18. (a) There are 604,800 telephone numbers without repeated digits.
19. There are 344,320 telephone numbers without repeated digits that do not begin with zero.
20. (a) There are 604,800 telephone numbers without restrictions.
21. There are 107 telephone numbers without restrictions.
22. (a) There are 5040 ways to arrange P, R, O, B, L, E, M.
23. There are 720 ways to arrange R, O, B, L, E.
24. (a) 6,760,000 different license plates are possible.
25. 3,407,040 license plates without repeated digits can be made.
26. There are 16 distinguishable car types produced.
27. 60 types of homes can be built.
28. 256 different numbers can be formed.
29. 125,000 different lock combinations are possible.
30. There are 8 different paths through the maze.

#### Exercise 7.4 (p. 385)

1. False
2. 0.6
3. Sample space
4. True
5. Event
6. False
7. A sample space is \( \{RA, RB, RC, GA, GB, GC\} \) where \( R = \) Red and \( G = \) Green.
8. A sample space is \( \{AA, AB, AC, BA, BR, BC, CA, CB\} \).
9. A sample space is \( \{AA1, AB1, AC1, BA1, BB1, BC1, CA1, CB1, CC1, AA2, AB2, AC2, RA2, RB2, RC2, CA2, CB2, CC2, AA3, AB3, AC3, BA3, BB3, BC3, CA3, CB3, CC3, AA4, AB4, AC4, BA4, BB4, BC4, CA4, CB4, CC4\} \).
10. A sample space is \( \{RA1, RB1, RC1, GA1, GB1, GC1, RA2, RB2, RC2, GA2, GB2, GC2, RA3, RB3, RC3, GA3, GB3, GC3, RA4, RB4, RC4, GA4, GB4, GC4\} \) where \( R = \) Red and \( G = \) Green.
11. 16
12. 216
13. 2652
14. 2167
15. 23
16. 33
17. 25
18. 11
19. 23
20. 23
21. 23
22. 23
23. 23
24. 23
25. Assignment B should be used.
26. 27. If \( W \) is the event a white ball is picked, then \( P(W) = \frac{3}{23} \).
29. If \( G \) is the event a green ball is picked, then \( P(G) = \frac{7}{23} \).
31. If \( E \) is the event a white ball or a red ball is picked, then \( P(E) = \frac{8}{23} \).
33. If \( H \) is the event a white ball or a blue ball is picked, then \( P(H) = \frac{11}{23} \).
35. If \( E \) is the event the ace of hearts is drawn, \( P(E) = \frac{1}{52} \).
37. If \( S \) is the event a spade is drawn, then \( P(S) = \frac{1}{4} \).
39. If $F$ is the event a picture card is drawn, then $P(F) = \frac{3}{13}$.  

41. If $E$ is the event a card with a number less than 6 is drawn, then $P(E) = \frac{5}{13}$.  

43. If $A$ is the event the card is not an ace, then $P(A) = \frac{12}{13}$.  

45. $S = \{ MD, MR, MO, FD, FR, FO \}$, where $M$ (male), $F$ (female), $D$ (dry), $R$ (regular), and $O$ (oily). There are 6 outcomes in the sample space.

**Exercise 7.6**

47. (a) 

<table>
<thead>
<tr>
<th>$S$</th>
<th>GGGG</th>
<th>GGGB</th>
<th>GGBG</th>
<th>GGBB</th>
<th>GBGG</th>
<th>GBGB</th>
<th>BBGG</th>
<th>BBGB</th>
<th>BGBB</th>
<th>BBGB</th>
<th>BBBB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(e_i)$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{16}$</td>
</tr>
</tbody>
</table>

(b) (i) $P(1\text{st two children are girls}) = \frac{1}{4}$  

(ii) $P(\text{all children are boys}) = \frac{1}{16}$  

(iii) $P(\text{at least one girl}) = \frac{15}{16}$  

(iv) $P(1\text{st and last children are girls}) = \frac{1}{4}$

49. 

<table>
<thead>
<tr>
<th>Income Level</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;$25,000</td>
<td>$0.247$</td>
</tr>
<tr>
<td>$25,000–$49,999</td>
<td>$0.249$</td>
</tr>
<tr>
<td>$50,000–$74,999</td>
<td>$0.179$</td>
</tr>
<tr>
<td>$75,000–$99,999</td>
<td>$0.119$</td>
</tr>
<tr>
<td>=$100,000</td>
<td>$0.205$</td>
</tr>
</tbody>
</table>

51. $P($50,000–$90,999) = 0.298$  

53. $P($<$50,000) = 0.496$  

55. $P($MasterCard) = 0.352$  

57. $P($American Express or Discover) = 0.179$  

59. $P($Price > $50,000) = \frac{7}{15}$

61. $P($40.00 ≤ Price ≤ $50.00) = \frac{1}{15}$

63. 0.132

65. 0.212

67. 0.058

69. 0.013

71. (a) $P($American has health insurance) = 0.846  

(b) $P($American has no health insurance) = 0.154  

75. $P(R) = 0.3$, $P(W) = 0.7$, answers will vary but the results should be fairly close to the actual probabilities.

77. $P(A) = 0.22$, $P(B) = 0.60$, $P(C) = 0.18$, answers will vary but the results should be fairly close to the actual probabilities.

**Exercise 7.5 (p. 400)**

1. event  

2. True  

3. False  

4. False  

5. $P(E \cup F) = 0.7$  

7. $P(E \cap F) = 0.4$  

9. $P(F) = 0.3$  

11. $P(\overline{E}) = 0.6$  

13. (a) $P(A \cup B) = 0.7$  

(b) $P(A \cap \overline{B}) = 0.3$  

(c) $P(B \cap \overline{A}) = 0.2$  

(d) $P(A \cup \overline{B}) = 0.3$  

15. (a) $P(A \cap B) = 0$  

(b) $P(A \cup B) = 0.8$  

(c) $P(\overline{A} \cup \overline{B}) = 0.2$

17. $P(E) = \frac{3}{4}$  

19. $P(E) = \frac{5}{12}$  

21. $P(E) = \frac{1}{2}$  

23. The odds for $E$: 3 to 2; the odds against $E$: 2 to 3

25. The odds for $F$: 3 to 1; The odds against $F$: 1 to 3

27. Define $E$: The Bears win; $F$: The Bears tie. $P(\overline{E} \cup F) = 0.30$  

29. Define $M$: Anne passes mathematics; $E$: Anne passes English. $P(M \cap E) = 0.2$  

31. Define $T$: Car needs a tune-up; $R$: Car needs a brake job. (a) $P(T \cup R) = 0.68$  

(b) $P(T \cap \overline{R}) = 0.58$  

(c) $P(T \cup \overline{R}) = 0.32$  

33. (a) $P(1 \text{ or } 2) = 0.57$  

(b) $P(1 \text{ or more}) = 0.95$  

(c) $P(0, 1, 2 \text{ or } 3) = 0.83$

35. Define $E$: A person selected at random has Rh-positive blood. $P(E) = 0.82$  

37. Define $E$: A person selected at random has blood that contains the A antigen. $P(E) = 0.41$

39. Define $E$: A person selected at random is type O; $F$: a person selected at random is Rh-positive. $P(E \cup F) = 0.91$  

41. If $E$ is the event a woman aged 15–44 who gave birth in 2006 had her first child, then $P(\overline{E}) = 0.543$.  

43. Odds for a randomly selected bridge being deficient or obsolete are 150 to 453.

45. The probability a smart phone owner chosen at random will not have an iPhone is 0.856.

47. If $D$ is the event the patent was for design and $B$ the event it was for botanical plants, then $P(D \cup B) = 0.129$.  

49. If $E$ is the event a fund outperforms the market in year one and $F$ the event it outperforms the market in year two, then $P(E \cup F) = 0.19$.  

51. $P(A \cup B) = \frac{11}{15}$

53. Probability of a repeated digit is 0.940.  

55. Probability of no repeated letters is 0.0055.  

57. The probability at least two students choose the same number is almost 1.

59. The probability at least 2 of 3 people are born in the same month is 0.236.  

61. The probability at least 2 senators have the same birthday is almost 1.

**Exercise 7.6 (p. 409)**

1. True  

2. False  

3. $E = 1.2$  

5. $E = 50,800 \text{ fans}$  

7. Mary should pay 80 cents per game.  

9. Dave should pay $2 to play.  

11. The price exceeds the expected value by $0.75.

13. (a) $E = 0.75$  

(b) No, the game is not fair.  

(c) To make the game fair, a player should lose $2.00 if 1 tail is thrown.

15. It is not fair to you; your expected loss is $0.43.  

17. The expected loss is 1.2 cents, so Sarah should not play.
AN–38 Answers to Odd-Numbered Problems

19. The expected receipts of the chosen corporation are $216,194. 21. The expected sale price of a new house is $244,384. 23. (a) The truck rental agency expects 9 customers per day. (b) To maximize expected profit the agency should have 10 trucks on hand each day. (c) The expected profit if 10 trucks are available is $601. 25. (a) The insurance company can expect a profit of $100. (b) The company should set the premium at $600.

27. (a) Management should choose the second location to maximize expected profit. (b) Management should choose the first location to maximize expected profit. 29. (a) The investor's expected return is 12.50%. (b) To realize a return of 14%, the investor's portfolio should be split with 20% Wal-Mart stock and 80% Viacom stock.

Review Exercises (p. 413)

1. $\subset, \subset$ 3. none of these 5. none of these 7. $\subset, \subset$ 9. $\subset, \subset$ 11. $\subset, \subset$ 13. $\subset, \subset$ 15. $\subset, \subset$

29. $n(A \cap B) = 3$ 31. $n(A \cup B) = 20$ 33. $n(A \cap B) = 1; n(A \cup B) = 5$ 35. (a) 165 cars were sold in June.

(b) 40 cars had only a GPS.

37. $\{1, [1, 2], [1, 3], [1, 2, 3]\}$

39. $S = \{0, 1, 2, 3, 4, 5\}$

41. The outcomes for each child are boy (B) and girl (G). The sample space is \{BB, BG, GB, GG\}.

43. \(P\) (penny) = \(\frac{4}{15}\) 45. \(P(1) = \frac{1}{8}, P(2) = \frac{1}{4}, P(3) = \frac{1}{8}, P(4) = \frac{1}{8}, P(5) = \frac{1}{4}, P(6) = \frac{1}{8}\)

47. (a) Numbers of girls

<table>
<thead>
<tr>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0  1  2  3  4</td>
</tr>
<tr>
<td>1/16  1/4  3/8  1/4  1/16</td>
</tr>
</tbody>
</table>

49. (a) Probability both are blue is \(\frac{10}{91}\) (b) Probability exactly 1 is blue is \(\frac{45}{91}\) (c) Probability at least 1 is blue is \(\frac{55}{91}\)

51. (a) \(P(A \cup B) = 0.6\) (b) \(P(\overline{A}) = 0.7\) (c) \(P(\overline{A} \cup \overline{B}) = 0.4\) (d) \(P(\overline{A} \cap \overline{B}) = 0.8\) 53. (a) \(P(H) = \frac{243}{300} = 0.81\) (b) \(P(T) = \frac{57}{300} = 0.19\)

55. (a) \(P(F) = 0.50\) (b) \(P(\overline{E} \cup F) = 0.78\) (c) \(E\) and \(F\) are not mutually exclusive because \(P(E \cap F) \neq 0\).

57. (a) \(P(E \cup F) = 0.7\) (b) \(P(\overline{E} \cup F) = 0.8\) (c) \(P(\overline{E} \cap \overline{F}) = 0.9\) 59. (a) \(P(T) = 0.75\) (b) \(P(\overline{T}) = 0.7\) (c) \(P(E \cap F) = 0\) (d) \(P(\overline{E} \cap \overline{F}) = 1\) (e) \(P(\overline{E} \cap F) = 0.45\) (f) \(P(\overline{E} \cup F) = 1\)

61. (a) No. (b) Outcome 0,0,0 has the highest probability. (c) \(P(F) = \frac{9}{64}\)

63. There are 72 different styles. 65. There are 1024 different ways to answer. 67. Probability all are born on different days is 0.017.

69. The odds of having 4 boys in a 4-child family are 1 to 15. 71. The probability the Giants win Sunday's game is \(\frac{5}{8}\). The probability the Giants lose Sunday's game is \(\frac{3}{8}\). 73. Frank has paid \(\frac{1}{3}\) cents too much for the game. 75. The expected value of the game is \(\frac{28}{3}\) cents. The game is not fair.

77. The game is not fair to the player; he expects to lose almost 3 cents on a play.

Mathematical Questions From Professional Exams

1. a 2. a 3. b 4. b 5. b or c 6. d
Exercise 8.1 (p. 429)

3. False 4. \( P(E|F) = \frac{P(E \cap F)}{P(F)} \) 5. \( P(E) = 0.5 \) 7. \( P(E|F) = 0.429 \) 9. \( P(E \cap F) = 0.3 \) 11. \( P(E) = 0.5 \) 13. \( P(E|F) = 0.25, P(F|E) = 0.5 \)

15. \( P(F) = 0.5 \) 17. \( P(E \cap F) = \frac{4}{13} \) 19. (a) \( P(E) = \frac{1}{2} \) (b) \( P(F) = \frac{2}{3} \) 21. \( P(C) = 0.69 \) 23. \( P(C|A) = 0.9 \) 25. \( P(C|B) = 0.2 \)

27. \( P(E \cap F) = 0.1 \) 29. \( P(F|E) = 0.2 \) 31. \( P(E|F) = 0.667 \) 33. (a) Probability exactly 2 girls, given 1st child is a girl is \( \frac{1}{2} \). (b) Probability exactly 1 girl, given 1st child is a boy is \( \frac{1}{2} \).

35. (a) Probability of drawing a heart and then red is \( \frac{25}{204} \). (b) Probability of drawing red and then a heart is \( \frac{25}{204} \).

37. Probability of drawing 1 white and 1 yellow ball is \( \frac{1}{5} \). 39. (a) Probability of drawing a red ace is \( \frac{1}{26} \). (b) Probability of drawing a red ace given an ace was drawn is \( \frac{1}{2} \). (c) Probability of drawing a red ace given a red card was drawn is \( \frac{1}{13} \).

41. \( P(E) = 0.40 \) 43. \( P(H) = 0.24 \) 45. \( P(E \cap H) = 0.10 \)

47. \( P(G \cap H) = 0.08 \) 49. \( P(E|H) = \frac{10}{24} \approx 0.417 \) 51. \( P(G|H) = \frac{1}{3} \approx 0.333 \) 53. \( P(E|G) = \frac{9}{13} \approx 0.692 \); the probability the customer likes the deodorant given he/she is in group 1 is \( \frac{9}{13} \). 55. \( P(H|E) = \frac{22}{69} \approx 0.319 \); the probability the customer is from group II given he/she likes the deodorant is approximately 0.319.

57. \( P(F|G) = \frac{3}{13} \approx 0.231 \); the probability the customer does not like the deodorant given he/she is in group I is approximately 0.231. 59. \( P(H|F) = \frac{17}{42} \approx 0.405 \); the probability the customer is from group II given he/she does not like the deodorant is approximately 0.405.

61. \( P(F|E) = \frac{5}{11} \approx 0.455 \); the probability the resident is male given he/she is a Democrat is approximately 0.455. 65. \( P(M|R \cup I) = \frac{70}{125} = 0.56 \)

The probability the resident is male given he/she is either a Republican or Independent is 0.56. 67. (a) \( P(M) = \frac{724}{1009} \) (b) \( P(A) = \frac{333}{1009} \)

(c) \( P(F \cap B) = \frac{72}{1063} \) (d) \( P(F|E) = \frac{51}{724} \) (e) \( P(A|M) = \frac{171}{724} \) (f) \( P(F|A \cup E) = \frac{213}{724} \) (g) \( P(M \cap B) = \frac{383}{1009} \) (h) \( P(F|E) = \frac{117}{373} \)

69. \( P(B|Rh^+) = \frac{9}{82} \approx 0.110 \) 71. \( P(Rh^+|O) = \frac{39}{48} \approx 0.813 \) 73. \( P(\text{receives mailing and responds}) = 0.014 \)

75. (a) 45% of the purchases were made by persons under 30 years of age. (b) The probability a purchase of at least $500 was made by someone older than 50 is 0.067.

(c) There is 0.65265 probability that a purchase between $100 and $499.99 was made by a person between the ages of 30 and 50. 77. The probability the issue declined given it is high yield is 0.388. 79. The probability an issue is at a 52-week high given it is investment grade is 0.054.

81. (a) \( P(\text{no health insurance}|<18 \text{ years old}) = 0.099 \) (b) \( P(\text{individual <18 years old}|\text{no health insurance}) = 0.178 \)

(c) \( P(\text{individual <18 years old}|\text{has health insurance}) = 0.249 \) 83. \( P(W|E) = \frac{5}{11} \) 85. \( P(\text{a person completes training}) = 0.715 \)

87. \( P(\text{person smokes}|\text{over 16 years of age}) = 0.215 \)

Exercise 8.2 (p. 441)

1. False 2. \( P(E) \cdot P(F) \) 3. \( P(E \cap F) = 0.24 \) 5. \( P(F) = 0.125 \) 7. No, \( P(E \cap F) = \frac{2}{9} \neq \frac{1}{9} = P(E)P(F) \).

9. (a) \( P(E|F) = 0.2 \)

(b) \( P(F|E) = 0.4 \) (c) \( P(E \cap F) = 0.08 \) (d) \( P(E \cup F) = 0.52 \) 11. \( P(E \cap F \cap G) = \frac{4}{147} \) 13. \( P(E|F) = 0.5 \)

No, \( P(E \cap F) = 0.1 \neq 0.06 = P(E)P(F) \). 15. \( P(RRR) = \frac{1}{12}, P(RRL) = \frac{2}{12}, P(RLR) = \frac{1}{12}, P(LLR) = \frac{1}{12}, P(LLR) = \frac{2}{12}, P(LRL) = \frac{2}{12}, P(RRR) = \frac{1}{12} \)

17. (a) Probability both are red is \( \frac{9}{25} \). (b) Probability neither child is diseased is \( \frac{1}{16} \).

(c) Probability exactly 1 has disease is \( \frac{3}{8} \). 21. \( P(\text{seed produces a violet}) = 0.20 \) 23. (a) \( P(\text{both inspectors miss a defect}) = 0.04 \)

(b) Three inspectors should be hired to ensure the probability of failing to identify the defect is less than 0.01. 25. \( P(\text{both stocks increase}) = 0.36; P(\text{at least one stock does not increase}) = 0.64 \)

27. (a) \( P(<18 \text{ years old}) = 0.239; P(<18 \text{ years old}|\text{no health insurance}) = 0.178 \)

(b) The events are not independent since \( P(<18 \text{ years old}) \neq P(<18 \text{ years old}|\text{no health insurance}) \). 29. (a) The failures in the two pumps are not mutually exclusive; there is 0.25% chance they both fail. (b) \( P(\text{at least 1 pump fails}) = 0.0975 \)

(c) Failures are independent because \( P(\text{both fail}) = 0.0025 = P(\text{one fails}) \cdot P(\text{second fails}) \)
31. (a) \( P(\text{burglary} | \text{suburban}) = 0.205 \)  
(b) \( P(\text{rural} | \text{vehicle theft}) = 0.143 \)  
(c) The events “rural” and “vehicle theft” are not independent since 
\[ P(\text{rural}) \cdot P(\text{VT}) = 0.15 \neq P(\text{rural} \cap \text{VT}) = 0.008. \]  
(d) The events “urban” and “burglary” are independent since 
\[ P(U) \cdot P(B) = 0.088 = P(U \cap B). \]

33. (a) The probability both vote for the candidate is \( \frac{4}{9} \).  
(b) The probability neither votes for the candidate is \( 9 \cdot 9 = 9 \).  
(c) The probability one votes for the candidate is \( 4 \cdot 9 = 9 \).  
\[ P(U) \cdot P(B) = 0.088 = P(U \cap B). \]  
\[ P(R) \cdot P(\text{VT}) = 0.015 \]  
\[ P(R \cap \text{VT}) = 0.008. \]

35. (a) \( p = 0.8 \)

<table>
<thead>
<tr>
<th>Group Size</th>
<th>Expected Tests Saved per Component</th>
<th>Percent Saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( p^2 - \frac{1}{2} )</td>
<td>14.0</td>
</tr>
<tr>
<td>3</td>
<td>( p^3 - \frac{1}{3} )</td>
<td>17.9</td>
</tr>
<tr>
<td>4</td>
<td>( p^4 - \frac{1}{4} )</td>
<td>16.0</td>
</tr>
<tr>
<td>5</td>
<td>( p^5 - \frac{1}{5} )</td>
<td>12.8</td>
</tr>
<tr>
<td>6</td>
<td>( p^6 - \frac{1}{6} )</td>
<td>9.5</td>
</tr>
<tr>
<td>7</td>
<td>( p^7 - \frac{1}{7} )</td>
<td>6.7</td>
</tr>
<tr>
<td>8</td>
<td>( p^8 - \frac{1}{8} )</td>
<td>4.3</td>
</tr>
</tbody>
</table>

The optimal group size is 3. Its percent saving is 17.9%.

(c) \( p = 0.99 \)

<table>
<thead>
<tr>
<th>Group Size</th>
<th>Expected Tests Saved per Component</th>
<th>Percent Saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( p^2 - \frac{1}{2} )</td>
<td>48.01</td>
</tr>
<tr>
<td>3</td>
<td>( p^3 - \frac{1}{3} )</td>
<td>63.70</td>
</tr>
<tr>
<td>4</td>
<td>( p^4 - \frac{1}{4} )</td>
<td>71.06</td>
</tr>
<tr>
<td>5</td>
<td>( p^5 - \frac{1}{5} )</td>
<td>75.87</td>
</tr>
<tr>
<td>6</td>
<td>( p^6 - \frac{1}{6} )</td>
<td>77.48</td>
</tr>
<tr>
<td>7</td>
<td>( p^7 - \frac{1}{7} )</td>
<td>78.92</td>
</tr>
</tbody>
</table>

The optimal group size is 3. Its percent saving is 17.9%.

(b) \( p = 0.95 \)

<table>
<thead>
<tr>
<th>Group Size</th>
<th>Expected Tests Saved per Component</th>
<th>Percent Saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( p^2 - \frac{1}{2} )</td>
<td>40.3</td>
</tr>
<tr>
<td>3</td>
<td>( p^3 - \frac{1}{3} )</td>
<td>52.4</td>
</tr>
<tr>
<td>4</td>
<td>( p^4 - \frac{1}{4} )</td>
<td>56.5</td>
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<tr>
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<td>( p^5 - \frac{1}{5} )</td>
<td>57.4</td>
</tr>
<tr>
<td>6</td>
<td>( p^6 - \frac{1}{6} )</td>
<td>56.8</td>
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<tr>
<td>7</td>
<td>( p^7 - \frac{1}{7} )</td>
<td>55.5</td>
</tr>
<tr>
<td>8</td>
<td>( p^8 - \frac{1}{8} )</td>
<td>53.8</td>
</tr>
</tbody>
</table>

The optimal group size is 5. Its percent saving is 57.4%.

37. (a) The probability pooled test is positive is \( 1 - (1 - p)^{20} \).

(b) The expected number of tests needed is \( 21 - 20(1 - p)^{20} \).

(c) The pooled method saves \( 20(1 - p)^{20} - 1 \) tests per individual.
Exercise 8.3 (p. 453)

2. Partition 3. True 4. False 5. \(P(E|A) = 0.4\) 7. \(P(E|B) = 0.2\) 9. \(P(E|C) = 0.7\) 11. \(P(E) = 0.31\) 13. \(P(A|E) = \frac{12}{31}\) 15. \(P(C|E) = \frac{7}{31}\)

17. \(P(B|E) = \frac{12}{31}\) 19. \(P(E) = 0.024\) 21. \(P(E) = 0.016\) 23. \(P(A|E) = 0.5; P(A_2|E) = 0.5\) 25. \(P(A_1|E) = 0.375; P(A_2|E) = 0.375; P(A_3|E) = 0.25\)

27. (a) \(P(\text{Red}) = \frac{5}{16}\) (b) \(P(\text{White}) = \frac{5}{16}\) (c) \(P(\text{Blue}) = \frac{3}{8}\) (d) \(P(\text{Jar I}|\text{Red}) = \frac{1}{3}\) (e) \(P(\text{Jar II}|\text{Blue}) = \frac{1}{2}\) (f) \(P(\text{Jar III}|\text{White}) = \frac{1}{3}\)

29. \(P(\text{male}|\text{color blind}) = 0.953\) 31. (a) The probability is 0.34 that the customer had never cruised on Castaway. (b) The probability is 0.38 that the person was male given the caller had never cruised on Castaway. 33. Since the plane is late, there is a 0.659 probability she is on Southwest.

35. (a) There is a 0.0935 probability that the survey has negative feedback. (b) If the survey contains negative feedback, there is a 0.342 probability it is from the Riverside restaurant. (c) If the survey contains negative feedback, there is a 0.321 probability it is from the Springfield restaurant. 37. If the tax return is audited, the probability is 0.025 that thefiler made more than $1,000,000. 39. (a) 46.7% of employed persons in the United States are women. (b) \(P(\text{management, professional, and related|woman}) = 0.395\) (c) \(P(\text{service|woman}) = 0.206\) (d) \(P(\text{sales and office|woman}) = 0.332\)

41. (a) \(P(\text{Democrat|voted}) = 0.385;\) (b) \(P(\text{Republican|voted}) = 0.39\) (c) \(P(\text{Independent|voted}) = 0.225\) 43. (a) \(P(\text{Rock|positive}) = 0.385;\) (b) \(P(\text{Clay|positive}) = 0.209;\) (c) \(P(\text{Sand|positive}) = 0.405\) 45. (a) \(P(\text{Republican}) = 0.466;\) (b) \(P(\text{Northeast|Republican}) = 0.343\)

47. The probability the nurse forgot is \(\frac{9}{11}\). 49. (a) The probability the accident involved a young driver is 0.28. (b) The probability the driver was 25 or older is 0.72. 51. (a) Given a student tests positive, the probability of having HIV is 0.5. (b) In a high risk population, if the test is positive the probability of having HIV is 0.963. (c) If Jack tests positive twice, then he has a 0.996 probability of having the HIV virus. 53. If \(P(F) \neq 0\) and \(F\) is a subset of \(E\), then \(E \cap F = F\) and \(P(E \cap F) = P(F) \neq 0\). So \(P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1\).

Exercise 8.4 (p. 463)

3. 1; 6 4. \(\frac{m!}{(n-r)!}\) 5. False 6. False 7. 60 9. 90 11. 9 13. 28 15. 42 17. 40,320 19. 1 21. 56 23. 1

25. The ordered arrangements of length 3 formed from the letters \(a, b, c, d, \) and \(e\) are:

\[
\text{abc, abd, abe, acb, acd, ace, adb, ade, aeb, ace, acd, ade, bca, bcd, bde, bca, bcd, bde, cade, cde, ceb, cea, ced, eab, eac, ebc, ecd, eca, ecd, eda, edb, eca, ecb, eda, edb, eca, ecb, use letters.}
\]

\(P(5, 3) = 60\)

47. \(P(21, 8) = 8,204,716,800\) different itineraries are possible. 49. There are 132,860 ways 2 people can have different birthdays. 51. (a) There are 720 arrangements of letters S, U, N, D, A, Y (b) There are 120 arrangements if S comes first. (c) There are 24 arrangements if S must come first and Y last.

53. There are 19,958,400 ways the books can be distributed. 55. There are 3,368,253,000 ways to win. 57. There are 32,760 ways officers can be chosen.

Exercise 8.5 (p. 476)

1. combination 2. 10 3. False 4. True 5. 15 7. 21 9. 5 11. 28 13. \(abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde, C(5, 3) = 10\)

15. 123, 124, 134, 234; C(4, 3) = 4 17. 35 ways 19. 2380 ways 21. 1140 ways 23. 56 8-bit strings have exactly three 1s. 25. 90,720 different 9 letter words. 27. 27,720 ways 29. 336 different committees 31. 27,720 ways 33. There are 1,467,962 different Powerball tickets.

35. There can be 4,680,720,000 different negotiating teams. 37. The auditor can choose 1,916,797,311 different samples.

39. There are \(\frac{22! \cdot 13! \cdot 10! \cdot 5! \cdot 16! \cdot 17!}{1000!} = 1.157 \times 10^{76}\) ways \(41. (a) \frac{1}{15}\) (b) \(\frac{7}{15}\) (c) \(\frac{8}{15}\)

43. (a) AT&T, Verizon, Kraft, Pfizer, Home Depot

\(P(\text{no account with errors is selected}) = 0.620\) (b) \(P(\text{one account with errors is selected}) = 0.332\)

(c) \(P(\text{at least two accounts with errors are selected}) = 0.048\) 47. (a) There are 1,757,600,000 different phone codes. (b) \(P(\text{code ends in AA}) = 0.00148\)

(c) \(P(\text{code begins and ends in A}) = 0.00148\) (d) \(P(\text{code has no repeats}) = 0.268\) (e) \(P(\text{code has no repeated letters, but all the same number}) = 0.0000888\)

49. \(P(\text{all 5 refrigerators are defective}) = 2.83 \times 10^{-6}; P(\text{at least two refrigerators are defective}) = 0.103\)

51. (a) \(P(\text{favorable extension}) = \frac{1}{100}\) (b) \(P(\text{favorable extension if first digit cannot be 0}) = \frac{9}{1000}\) 53. (a) \(P(\text{committee is all Democrat}) = 0.0213\)

(b) \(P(\text{committee is all Republican}) = 0.0014\) (c) \(P(\text{committee has 4 Democrats and 3 Republicans}) = 0.3031\) 55. \(P(\text{shipment rejected}) = 0.098\)

57. (a) \(P(\text{all hearts}) = 0.0005\) (b) \(P(\text{exactly 4 spades}) = 0.0107\) (c) \(P(\text{no errors}) = 0.274\) (d) \(P(\text{4 of a kind}) = 0.401 \times 10^{-4}\) (e) \(P(\text{full house}) = 0.0014\) (f) \(P(\text{flush}) = 0.0020\)

(f) \(P(\text{straight}) = 0.0039\)
Exercise 8.6 (p. 491)

1. False  2. True  3. \( \binom{n}{k} p^k (1-p)^{n-k} \)  4. \( b(7, 3; 0.3) \)  5. True  6. \( np \)  7. \( b(7, 4; 0.20) = 0.0287 \)  9. \( b(15, 8; 0.80) = 0.0138 \)

11. \( b(15, 10; \frac{1}{2}) = \frac{3003}{32,768} = 0.0916 \)  13. 0.2969  15. \( b(3, 2; \frac{1}{3}) = \frac{2}{9} \)  17. \( b(3, 0; \frac{1}{6}) = \frac{125}{216} \approx 0.5787 \)  19. \( b(5, 3; \frac{2}{3}) = \frac{80}{243} \approx 0.3292 \)

21. \( b(10, 6; 0.3) = 0.0368 \)  23. \( b(12, 9; 0.8) = 0.2362 \)  25. \( P(\text{at least 5 successes}) = 0.0580 \)  27. \( b(8, 1; \frac{1}{2}) = \frac{1}{32} \)

29. \( P(\text{at least 5 tails}) = \frac{93}{256} = 0.3633 \)  31. \( P(\text{2 H at least 1 H occurs}) = \frac{28}{255} = 0.1098 \)

33. \( b(5, 2; \frac{1}{6}) = \frac{625}{3888} = 0.1608 \)  35. (a)

(b) \( b(6, 5; 0.5) + b(6, 6; 0.5) = 0.1094 \)  (c) \( b(6, 2; 0.5) + b(6, 3; 0.5) = \frac{35}{64} = 0.5469 \)

41. (a) Probability of getting all answers correct is \( b(20, 20; \frac{1}{2}) = \left( \frac{1}{2} \right)^{20} = 9.537 \times 10^{-7} \).  (b) Probability of passing or the probability of getting at least 12 correct is 0.2517.  (c) The odds in favor of passing are about 1 to 3.

43. (a) Probability that at least 5 of those surveyed think it is acceptable to cheat is 0.0265.  (b) Probability fewer than 3 people surveyed think it is acceptable to cheat is 0.7346.  (c) Probability that no one thinks it is acceptable to cheat is 0.0004.  (d) Probability that no one think it is acceptable to cheat is 0.1470.

45. (a) Probability two of the returns were filed on April 15 is \( b(6, 2; 0.10) = 0.1488 \).  (b) Probability five of the returns were filed on April 15 is \( b(8, 5; 0.10) = 0.0004 \).  (c) Probability that none of the returns were filed on April 15 is \( b(8, 0; 0.10) = 0.4305 \).

47. (a) Probability that all like their jobs is \( b(10, 10; 0.8) = 0.107 \).  (b) Probability that 9 persons like their jobs is \( b(10, 9; 0.8) = 0.2684 \).  (c) Probability that no more than 7 persons like their jobs is 0.3222.

49. (a) Probability that exactly 14 of those selected use coupons is \( b(18, 14; 0.77) = 0.2205 \).  (b) Probability that at least 14 of those selected use coupons is 0.5988.  (c) Probability that at least 16 of those selected use coupons is 0.1813.  (d) Probability of getting all answers correct is 0.0004.  (e) Probability that at least one of the 12 claims involves fraud is 0.1422.

53. (a) Probability two passengers of the 44 do not show is 0.0635.  (b) Probability all 44 passengers show up is 0.0036.

(c) Probability at least two passengers of the 44 do not show is 0.9748.  55. (a) Probability the player has at least two hits in four times at bat is 0.2617.

(b) Probability the player has at least one hit in four times at bat is 0.6336.  57. \( b(8, 8; 0.40) = (0.40)^8 = 0.0007 \)  59. \( b(10, 4; 0.23) = 0.1225 \)

61. (a) \( P(\text{At least 5 correct}) = \frac{7}{64} \approx 0.1094 \)  (b) \( P(\text{Fewer than 5 correct}) = 0.3446 \)  (c) \( P(\text{At most 5 are over 65}) = 0.9995 \)

65. \( E = \frac{2000}{6} = 333 \frac{1}{3} \)  67. \( E = 10 \) lightbulbs.  69. The expected number of adults surveyed who would say that health care costs are paramount is 0.0001.

71. \( E = 1 \) person would be expected to have an unfavorable reaction.

73. The probability the message is correctly received is 0.9647.  75. The probability the code is correctly received is 0.9349.

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Review Exercises (p. 498)
1. \( P(E|A) = 0.82 \)  3. \( P(E|B) = 0.10 \)  5. \( P(B \cap E) = 0.738 \)  7. \( P(B \cap E) = 0.01 \)  9. \( P(A|E) = 0.9866 \)  11. \( P(B|E) = 0.0134 \)
13. \( P(E|A) = 0.5 \)  15. \( P(E|B) = 0.4 \)  17. \( P(E|C) = 0.3 \)  19. \( P(A \cap E) = 0.2 \)  21. \( P(B \cap E) = 0.2 \)  23. \( P(C \cap E) = 0.03 \)
25. \( P(E) = 0.43 \)  27. \( P(A|E) = 0.4651 \)  29. \( P(B|E) = 0.4651 \)  31. \( P(C|E) = 0.0698 \)  33. \( 1 \)  35. 210  37. 12  39. There are 10 ways to form the committee.  41. There are 6 ways to place the books.  43. (a) There are 120 words.  (b) There are 20 words if order is not important.
45. (a) There are 525 different committees.  (b) There are 1715 different committees.  47. There are 12,441,600 ways.  49. There are 20,790 different committees.  51. (a) There are 4845 samples that will contain only good plums.  (b) There are 5700 samples that will contain 3 good plums and 1 rotten plum.  (c) There are 7805 samples that will contain one or more rotten plums.  53. 360 words can be made.  55. There are 302,400 ways she can arrange the books.
57. \( E = \{ (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6) \} \)  59. Define \( E \): A person has blue eyes, \( F \): A person has brown eyes, \( G \): A person is left handed.  (a) \( P(E \cap G) = 0.025 \)  (b) \( P(G) = 0.0625 \)  (c) \( P(E|G) = 0.4 \)  61. Define \( E \): scored over 80% and \( F \): took form A (a) \( P(F|E) = \frac{2}{5} \)  (b) \( P(E|F) = \frac{1}{5} \)  (c) Yes, because \( P(E \cap F) = 0.08 = P(E)P(F) \).  (d) Yes, because \( P(E \cap \overline{F}) = 0.12 = P(E)P(\overline{F}) \).
63. \( P(E|F) = 0.5 \)  65. (a) Probability (misses the 1st and gets the next 3) = 0.1029.  (b) Probability (makes 10 in a row) = 0.00282.  67. Probability at least one matched is \( \frac{2}{3} \).  69. (a) Probability all are underweight is 0.0002.  (b) Probability 2 are underweight is 0.083.  (c) Probability at most 1 is underweight is 0.910.
71. (a) The probability is 0.3438 that the defective part came from machine \( A_1 \).  (b) The probability is 0.375 that the defective part came from machine \( A_2 \).  (c) The probability is 0.2813 that the defective part came from machine \( A_3 \).  73. (a) \( b(5, 0, 0.20) = 0.3277 \)  (b) \( b(5, 3; 0.20) = 0.0512 \)  75. One would expect \( E = np = \frac{7}{2} = 3.5 \) heads when tossing a coin 7 times.

Mathematical Questions from Professional Exams (p. 502)
**AN-44**  Answers to Odd-Numbered Problems

**CHAPTER 9  Statistics**

**Exercise 9.1 (p. 508)**
1. Variable  
2. simple random sample  
3. False  
4. biased  
5. Number of heads; discrete  
7. Average gas mileage; continuous  
9. Time a person waits in line; continuous  
11. Number of airplane flights; discrete  
13. Number of people; discrete  
15. Life of a phone battery; continuous  
17, 19, 21. Answers will vary. All answers should include a method to choose a sample in which each member of the population has an equal chance of being selected.  
23. Answers will vary.  
25. Answers will vary.

**Exercise 9.2 (p. 514)**
1. bar graphs; pie charts  
2. False  
3. False  
4. True

5. (a) ![Bar graph showing median family income by region](image)
   (b) West has the highest median income.  
   (c) South has the lowest median income.  
   (d) Answers will vary. All discussions should conclude that pie charts are inappropriate.

7. (a) ![Bar graph showing families in thousands by region](image)
   (b) West 22.5%  
   (c) Northeast 21%  
   (d) South 20.9%  
   (e) Midwest 38.7%  
   (d) The South has the most families.  
   (e) The Midwest has the fewest families.

(b) Married-couple families have the highest median income.  
(c) Female householder—no spouse families have the lowest median income.
11. (a) Pie chart is appropriate since all categories are represented.
(b) Heart disease was the leading cause of death in 2006.

Exercise 9.3 (p. 528)
2. True
3. Class Intervals
4. Ogive
5. (a) | Score | Frequency | Score | Frequency |
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(c) | Class    | Frequency | Class    | Frequency | Class    | Frequency | Class    | Frequency |
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13. (a) Hawaiian Air had the highest percentage of on-time flights. (b) Comair had the lowest percentage of on-time flights. (c) About 85% of United Airlines’ flights were on time.
15. (a) Housing, fuel, and utilities are the largest component of the CPI, making up 42% of the CPI. (b) Other goods and services form the smallest component of the CPI. This sector comprises 3% of the CPI.
7. (a) | Class | Frequency | Class | Frequency | Class | Frequency | Class | Frequency |
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7. (e) Skewed right

9. Skewed right
   11. (a) There are 13 class intervals. (b) The lower class limit of the 1st class interval is 20 years; the upper class limit is 24 years. (c) The class width is 5 years. (d) There are about 1,500,000 drivers between the ages of 70 and 84. (e) The interval 45–49 years has the most drivers. (f) The interval 80–84 years has the fewest drivers. (g) The distribution is skewed right.
13. (a) There are 13 class intervals.  
(b) The lower class limit of the 1st class interval is 20 years; the upper class limit is 24 years.  
(c) The class width is 5 years.  
(d) The frequency distribution is as follows:

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<tr>
<td>80–84</td>
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</tbody>
</table>

(e) Class interval 45–49 has the most licensed drivers.  
(g) Class interval 80–84 has the fewest licensed drivers.

15. (a) There are 19 class intervals.  
(b) The lower class limit for the first interval is 0; the upper limit is $1999.  
(c) The class width is $2000.  
(d) The frequency distribution is as follows:

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<td>0</td>
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<tr>
<td>$38,000–39,999</td>
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</tr>
</tbody>
</table>

(f) Tuition is most frequently in the $4000–$5999 range.
Exercise 9.6 (p. 562)

1. (a) mean, median, mode  
   (b) True  
   (c) True  
   (d) Bi-modal  
   (e) False  
   (f) 6. (a) mean: 31.25  
       (b) median: 30.5  
       (c) no mode  
   (d) 9. (a) mean: 70.4  
       (b) median: 70  
       (c) mode: 55  
   (e) 10. (a) mean: 78.6  
         (b) median: 82  
         (c) mode: 82  
   (f) 13. (a) mean: 73.33  
       (b) median: 77.5  
       (c) mode: 80  
   (g) Exercise 9.6 (p. 562)  
   (h) 15. (a) mean: 28.25 years of age  
        (b) median: 27 years of age  
        (c) mode: 24 and 25 years of age  
   (i) The mean age of a new mother in 2007 was approximately 27.85 years.  
   (j) The median age of a new mother in 2007 was approximately 27.6 years.  
   (k) The mean age of a licensed driver was approximately 47.49 years.  
   (l) The mean tuition in 2006–07 was approximately $14,389.94.  
   (m) About 95% of the clothing can be expected to last between 2 and 3 years.  
   (n) About 77% of the figures are between 20 and 30.  
   (o) About 5% of people have IQs either below 70 or above 130.  
   (p) Approximately 70% of the figures differ from the mean by at least 1500 pounds.  
   (q) Answers will vary.  
   (r) We would expect 70% of the students to weigh between 124.61 and 135.39 pounds.  
   (s) Exact; the data are not grouped.  
   (t) Kathleen had the highest relative standing.
39. (a) The distribution is skewed right. 
(b) The distribution is skewed right. 
(c) mean 4.5, standard deviation 1.775
41. (a) The approximate probability of having at least 80 but no more than 90 hits is 0.25. (b) The approximate probability of having 85 or more hits is 0.10.
43. The approximate probability of selecting at least 10 unsealed packages is 0.93.
45. The graph assumes its maximum at \( x = 0 \).

47. \( P[X \geq 10] \approx 0.93; \)

Review Exercises (p. 566)

1. circumference, continuous
3. number of people, discrete
5. number of defective products, discrete
7. Answers will vary. All answers should include a method to choose a sample of 100 students from the population in which each student has an equal chance of being chosen.
9. Answers will vary. All answers should give examples of possible bias.
11. (a) American Indians made up the smallest percentage of 4-year college enrollment in 2007. (b) Asian Americans were overrepresented in 4-year colleges in 2007. 
13. (a) 
15. (a) Approximately 116,000 Hispanic students were enrolled in 4-year colleges in 2007. (b) Approximately 916,000 Hispanic students were enrolled in 4-year colleges in 2007. 
17. (a) High school graduate represents the highest level of educational attainment of most Americans in 2009. (b) Approximately 58,000,000 Americans have at least a bachelor’s degree. (c) Approximately 28,000,000 Americans do not have a high school diploma. (d) Approximately 50,000,000 Americans have gone to college but do not have a bachelor’s degree.
19. (a) | Score | Frequency | Score | Frequency | Score | Frequency | Score | Frequency |
<table>
<thead>
<tr>
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<td>62</td>
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<td>74</td>
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<td>75</td>
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<tr>
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<tr>
<td>44</td>
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<td>1</td>
<td>80</td>
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<td>92</td>
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</tr>
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<td>72</td>
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<td>85</td>
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<td>73</td>
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</tr>
</tbody>
</table>

The range is 79.

(b) 

(c) 

(d) 

(e) The distribution is skewed left.

(f) | Score | Cumulative Frequency | Score | Cumulative Frequency |
<table>
<thead>
<tr>
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<tr>
<td>72</td>
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(g)
21. (a) Time Frequency Time Frequency Time Frequency Time Frequency Time Frequency

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<th>Frequency</th>
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<tr>
<td>4'14&quot;</td>
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<td>4'50&quot;</td>
<td>1</td>
<td>4'52&quot;</td>
<td>1</td>
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<tr>
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<td>5'04&quot;</td>
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<td>5'11&quot;</td>
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<tr>
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<td>5'31&quot;</td>
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<td>5'49&quot;</td>
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<td>5'54&quot;</td>
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The range is 3 minutes, 3 seconds.

(b) Frequency

(c) Class Interval | Frequency

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<tr>
<th>Class Interval</th>
<th>Frequency</th>
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<tbody>
<tr>
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<tr>
<td>5'00&quot;–5'29&quot;</td>
<td>11</td>
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<tr>
<td>5'30&quot;–5'59&quot;</td>
<td>9</td>
</tr>
<tr>
<td>6'00&quot;–6'29&quot;</td>
<td>4</td>
</tr>
<tr>
<td>6'30&quot;–6'59&quot;</td>
<td>3</td>
</tr>
<tr>
<td>7'00&quot;–7'29&quot;</td>
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</table>

(d) Frequency

(e) Frequency

(f) The distribution is skewed right.

(g) Class Interval | Cummulative Frequency

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>Cummulative Frequency</th>
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</thead>
<tbody>
<tr>
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<tr>
<td>4'30&quot;–4'59&quot;</td>
<td>14</td>
</tr>
<tr>
<td>5'00&quot;–5'29&quot;</td>
<td>25</td>
</tr>
<tr>
<td>5'30&quot;–5'59&quot;</td>
<td>33</td>
</tr>
<tr>
<td>6'00&quot;–6'29&quot;</td>
<td>37</td>
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<tr>
<td>6'30&quot;–6'59&quot;</td>
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<tr>
<td>7'00&quot;–7'29&quot;</td>
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(h) Cumulative frequency
23. (a) | Age | Frequency | Age | Frequency |
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<tr>
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(c) There are 5 class intervals.

<table>
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<tr>
<td>30.0–34.9</td>
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<td>35.0–39.9</td>
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</tr>
<tr>
<td>40.0–44.9</td>
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</tbody>
</table>

(e) Skewed right

(g) | Class       | Frequency | Cumulative Frequency |
<table>
<thead>
<tr>
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</table>
AN–54 Answers to Odd-Numbered Problems

25. (a) (b)

27. (a) Mean: 5.7273 (b) Median: 5 (c) Mode: 0, 4, 8 and 10 (d) Range: 12 (e) Standard deviation: 4.149

29. (a) Mean: 16.2 (b) Median: 7 (c) Mode: 7 (d) Range: 11 (e) Standard deviation: 3.6515

31. (a) Mean: 7 (b) Median: 7 (c) Mode: 7 (d) Range: 11 (e) Standard deviation: 3.6515

33. (a) Answers may vary. We assume they are a sample of Joe’s scores calculating parts (b) and (c). (b) Joe’s mean score is 75.57. (c) The approximate standard deviation of Joe’s scores is 2.99.

35. (a) The approximate mean age of a female in 2009 was 38.2 years. (b) The approximate median age of a female in 2009 was 38.1 years. (c) The approximate standard deviation of the ages of females in 2009 was 23.3 years.

37. (a) 441; 759 (b) Approximately 95% of the lightbulbs will last between 494 and 706 hours. (c) About 81.5% of the lightbulbs will last between 547 and 706 hours.

39. The company expects to replace 0.15% of the lightbulbs.

41. The probability a bag weighs less than 9.5 or more than 10.5 pounds is less than or equal to 0.25.

43. Z = −0.667 45. Z = 1.4 47. Z = 1.667

49. A = 0.0855 51. A = 0.7555 53. (a) 68.26% of the scores are between 20 and 30. (b) 2.28% of the scores are above 35.

55. 0.17% of the dogs will die before reaching the age of 10 years, 4 months.

57. Bob scored equally well on both exams.

59. There is a probability of 0.95 that this week’s production will be between 30 and 50.

61. The probability of obtaining more than 160 positive results is approximately 0.0008.

63. The probability is approximately 0.74 that the gate will fail to open between 20 and 30 times in the next 500 attempts.

Mathematical Questions from Professional Exams (p. 574)

1. e 2. b 3. c
Exercise 10.1 (p. 584)

5. False  6. True  7. \( X \times m \)  8. \( v^{(k-1)}, v^{(0)} \)
9. (a) The entry represents the probability that an object in state 2 will move to state 1.
   (b) \( v^{(1)} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}, v^{(2)} = \begin{bmatrix} 5 & 13 \\ 18 & 18 \end{bmatrix} \)
   (c) \( v^{(1)} = \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix}, v^{(2)} = \begin{bmatrix} 13 & 35 \\ 48 & 48 \end{bmatrix} \)

15. \( a = 0.4, b = 0.1, c = 1 \)  17. \( v^{(5)} = \begin{bmatrix} 0.504 & 0.496 \end{bmatrix} \)

After 5 years, 50.4% of the residents live in the city and 49.6% live in the suburbs.

19. (a) This is a Markov chain because it represents a sequence of experiments each of which results in one of two states and the probability of being in a particular state depends only on the previous state.
   (b) \( P = \begin{bmatrix} 0.90 & 0.10 \\ 0.05 & 0.95 \end{bmatrix} \)
   (c) \( P^2 = \begin{bmatrix} 0.815 & 0.185 \\ 0.0925 & 0.9075 \end{bmatrix}, P^3 = \begin{bmatrix} 0.74275 & 0.25725 \\ 0.128625 & 0.871375 \end{bmatrix} \)

21. (a) This is a Markov chain because it represents a sequence of experiments each of which results in one of nine states and the probability of being in a particular state depends only on the previous state.
   (b) \( P = \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \)

23. 57% of wine drinkers will be drinking Pinot Grigio after 2 months.
   25. \( v^{(10)} = \begin{bmatrix} 0.3200 & 0.2892 & 0.1631 & 0.2277 \end{bmatrix} \)

27. (a) \( P = \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \)
   (b) \( v^{(0)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \)
   (c) \( v^{(10)} = \begin{bmatrix} 0 & 0.001 & 0 & 0 & 0.001 & 0.998 \end{bmatrix} \)
   (d) The mouse is most likely in room 6.

29. \( uA = \begin{bmatrix} u_1a_{11} + u_2a_{12} + u_1a_{12} \\ u_1a_{21} + u_2a_{22} \end{bmatrix} \)

Exercise 10.2 (p. 597)

3. False  4. True  5. No, when you multiply row 2 by column 1 you will always get a 0. So \( p_{21} = 0 \) for every power of \( P \).
   7. No, when you multiply row 1 by column 2 you will always get a 0. So \( p_{12} = 0 \) for every power of \( P \).

9. Yes, because \( P^2 = \begin{bmatrix} 7 & 13 & 4 \\ 36 & 36 & 9 \\ 1/8 & 3/8 & 1/2 \\ 5/4 & 11 \\ 24 & 3/4 \end{bmatrix} \) has all positive entries.
   11. No, when you multiply row 3 by column 1 you will always get a 0. So \( p_{31} = 0 \) for every power of \( P \).

13. Yes, because \( P^2 = \begin{bmatrix} 1/3 & 3/16 & 48 & 24 \\ 1/3 & 3/16 & 5 & 24 \\ 1/3 & 3/16 & 18 & 24 \\ 1/3 & 3/16 & 6 & 24 \\ 1/3 & 3/16 & 5 & 24 \end{bmatrix} \) has all positive entries.

15. \( t = \begin{bmatrix} 3 & 4 \\ 7 & 7 \end{bmatrix} \)
   17. \( t = \begin{bmatrix} 2 & 3 \\ 5 & 5 \end{bmatrix} \)
   19. \( t = \begin{bmatrix} 15 & 12 \\ 43 & 43 \end{bmatrix} \)
AN–56  Answers to Odd-Numbered Problems

21. (a) \[
\begin{pmatrix} A & B & C \\ 0.7 & 0.15 & 0.15 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}
\] (b) \[
\begin{pmatrix} 4 & 6 & 3 \\ 13 & 13 & 13 \end{pmatrix}
\]
In the long run, the grocers’ stock is 30.8% brand A, 46.2% brand B, and 23.1% brand C.

23. (a) The probability the grandson of a Labourite votes Socialist is 0.09.  (b) In the long run 55.3% will vote Conservative.

25. (a) The salesperson’s movement among the universities can be thought of as a sequence of experiments each of which ends in one of a finite number of states.

\[
P = \begin{pmatrix} U_1 & U_2 & U_3 \\ 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \end{pmatrix}
\]

(c) In the long run, she sells at \(U_1\) 42.9% of the time, at \(U_2\) 45.7% of the time, and at \(U_3\) 11.4% of the time.

Exercise 10.3 (p. 608)


19. 21.

Exercise 10.4 (p. 614)

2. True  3. Katy’s game matrix is

\[
\begin{pmatrix} \text{one} & \text{two} \\ \text{finger} & \text{fingers} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}
\]

where the entries are numbers of dimes

5. Katy’s game matrix is

\[
\begin{pmatrix} \text{1 finger} & \text{2 fingers} \\ \text{one} & \text{two} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 7 \\ -2 & 5 & -8 \\ 1 & -2 & 5 \end{pmatrix}
\]

where the entries are numbers of dimes

7. The game is strictly determined; the value is \(-2\).  9. The game is strictly determined; the value is 3.  11. The game is not strictly determined.

13. The game is strictly determined; the value is 2.  15. The game is not strictly determined.  17. The game is strictly determined if \(0 \leq a \leq 3\).

19. \(a \cdot b = 0\) for the game to be strictly determined.
**Exercise 10.5 (p. 618)**

3. $E = \$1.42$  
5. $P = 9 / 4 = 2.25$  
7. $E = 19 / 8 = 2.375$  
9. $E = 17 / 9$  
11. $E = 1 / 3$  
13. $H \begin{bmatrix} 1 & T \\ 1 & -1 \end{bmatrix} \ E = 0$

**Exercise 10.6 (p. 625)**

2. True

3. The optimal strategy for player I is $P = \begin{bmatrix} 3 / 4 \\ 1 / 4 \end{bmatrix}$. The optimal strategy for player II is $Q = \begin{bmatrix} 5 / 6 \\ 1 / 6 \end{bmatrix}$. The expected payoff is $E = 7 / 4 = 1.75$.

5. The optimal strategy for player I is $P = \begin{bmatrix} 1 / 6 \\ 5 / 6 \end{bmatrix}$. The optimal strategy for player II is $Q = \begin{bmatrix} 2 / 3 \\ 1 / 3 \end{bmatrix}$. The expected payoff is $E = 1 / 3$.

7. The optimal strategy for player I is $P = \begin{bmatrix} 5 / 8 \\ 3 / 8 \end{bmatrix}$. The optimal strategy for player II is $Q = \begin{bmatrix} 8 / 3 \\ 1 / 3 \end{bmatrix}$. The expected payoff is $E = 7 / 8$.

9. (a) The optimal strategy for the Democrat is to spend 37.5% of the time on domestic issues and 62.5% on foreign issues. The optimal strategy for the Republican is to spend 50% of the time on domestic issues and 50% of the time on foreign issues. (b) The expected payoff is $E = 1.5$, so the Democrat gains 1.5 units by employing the optimal strategy.  
11. The optimal strategy for the spy is to try the deserted exit 8.5% of the time and try the heavily used exit 91.5% of the time. The optimal strategy for the spy's opponent is to wait at the deserted exit 22.5% of the time and wait at the heavily used exit 77.5% of the time. The expected payoff is favoring the spy.

**Review Exercises (p. 628)**

1. Not regular.  
3. Not regular.  
5. Yes, because $P^3 \approx \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix}$  
7. $t = \begin{bmatrix} 4 \\ 7 \\ 7 \end{bmatrix}$  
9. $\begin{bmatrix} 3 & 1 \\ 8 & 4 \\ 3 & 8 \end{bmatrix}$

11. $t = \begin{bmatrix} 48 \\ 10 \\ 21 \end{bmatrix} \approx \begin{bmatrix} 0.608 \\ 0.127 \\ 0.266 \end{bmatrix}$

13. (a) The situation forms a Markov chain because the shifts in market shares can be thought of as a sequence of experiments each of which results in one of a finite number of states.  
(b) After one year, A will have 46.7% of the market, B will have 28.3% of the market, and C will have 25% of the market.  
(c) After two years, A will have 47.2% of the market, B will have 26.9% of the market, and C will have 25.9% of the market.  
(d) In the long run, A will have 47.3% of the market, B will have 26.6% of the market, and C will have 26.0% of the market.

15. (a) The Markov chain is not absorbing.  
17. (a) The Markov chain is absorbing.  
(b) The absorbing state is state 2.

(c) $P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$

19. (a) The Markov chain is absorbing.  
(b) The absorbing state is state 1.

(c) $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 3 & 3 \end{bmatrix}$

(d) In the long run, A will have 47.3% of the market, B will have 26.6% of the market, and C will have 26.0% of the market.
AN–58 Answers to Odd-Numbered Problems

19. (a) The Markov chain is not absorbing.  
21. (a) The Markov chain is absorbing.  
(b) The absorbing states are states 1 and 4.  
(c)  
\[
\begin{array}{c|cccc}
\text{State} & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 \\
3 & 0.3 & 0.1 & 0.5 & 0.1 \\
4 & 0.25 & 0.2 & 0.35 & 0.2 \\
\end{array}
\]

23. (a)  
\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0.55 & 0.45 & 0 & 0 & 0 \\
2 & 0 & 0.55 & 0.45 & 0 & 0 \\
3 & 0 & 0 & 0.55 & 0 & 0.45 \\
4 & 0 & 0 & 0 & 0.55 & 0.45 \\
5 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
(b) The optimal strategy for player I is \( \tilde{P} = \frac{1}{2} \) and the optimal strategy for player II is \( \tilde{Q} = \frac{1}{2} \).  
(c) Given that the man started with $2.00, on the average the process will be in state one 1.298 times; in state two 2.361 times; in state three 1.412 times and in state four 0.635 time.  
(d) The expected length of the game is 5.71 bets.  
(d) The probability the man loses all his money is 0.714. The probability he wins $5 is 0.286.

25. The game is not strictly determined.  
27. The game is strictly determined; the value of the game is 9.  
29. The game is strictly determined.  

The value of the game is 12.  
31. \( E = \frac{1}{3} \)  
33. \( E = 1.75 \)  
35. \( E = \frac{4}{3} \)  
37. The optimal strategy for player I is \( P = \frac{1}{2} \) and the optimal strategy for player II is \( Q = \frac{1}{2} \).  
39. (a) The investor should invest in A 41.9% of the time and invest in B 58.1% of the time.  
(b) The percentage gain is 8.37%.

41. The builder should use 60% of the land for the shopping center and 40% of the land for the houses.
### Exercise 11.1 (p. 639)

1. Proposition
2. $p \land q$
3. False
4. True
5. Proposition
6. Not a proposition
7. Proposition
8. $p \land q$
9. Proposition
10. Proposition

11. A fox is not an animal.
12. I am not buying stocks.
13. Someone wants to buy my house.
14. Everybody has a car.
15. John is an economics major or John is a sociology minor (or both).
16. John is an economics major and a sociology minor.
17. John is not an economics major or John is not a sociology minor (or both).
18. John is not an economics major or John is a sociology minor (or both).

### Exercise 11.2 (p. 647)

3. True
4. True
5. False
6. False

7. | $p$ | $q$ | $\sim q$ | $p \lor \sim q$ |
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9. | $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \land \sim q$ |
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10. | $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim (p \land q)$ |
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12. | $p$ | $q$ | $\sim p$ | $p \lor q$ | $p \land q$ | $\sim p \land \sim q$ |
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13. | $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \land \sim q$ | $\sim p \lor \sim q$ | $\sim (p \lor \sim q)$ |
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14. | $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim (p \lor q)$ | $\sim q \land \sim p$ |
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23. | $p$ | $q$ | $\sim p$ | $q \land \sim p$ | $p \land (q \land \sim p)$ |
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AN–60  Answers to Odd-Numbered Problems

25. | p  | q  | \sim p | \sim q | p \land q | \sim p \land \sim q | (p \land q) \lor (\sim p \lor \sim q) | \sim q \lor r | p \lor (q \lor r) | \sim p \lor (\sim q \lor r) | \sim p \lor (q \lor r) |
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29. | p  | q  | r  | p \land q | (p \land q) \land r | q \land r | p \land (q \land r) | (p \lor q) \lor r | q \lor r | p \lor (q \lor r) |
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31. | p  | q  | p \land q | p \lor q | p \lor (p \land q) | p \land (p \lor q) |
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33. | p  | q  | \sim q | \sim q \lor q | p \land (\sim q \lor q) |
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35. | p  | \sim p | \sim (\sim p) |
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39. 

\( (p \lor q) \land r = r \land (p \lor q) \) (commutative property)
\( = (r \land p) \lor (r \land q) \) (distributive property)
\( = (p \land r) \lor (q \land r) \) (commutative property)

37. The compound proposition, “Smith is an ex-convict and Smith is rehabilitated,” is equivalent to the compound proposition, “Smith is rehabilitated and Smith is an ex-convict.” The compound proposition, “Smith is an ex-convict or Smith is rehabilitated” is equivalent to the compound proposition, “Smith is rehabilitated or Smith is an ex-convict.”

41. Mike cannot hit the ball well or he cannot pitch strikes.

43. The baby is not crying and the baby is not talking all the time.

Exercise 11.3 (p. 654)

1. Hypothesis; conclusion
2. Tautology
3. False
4. True
5. Converse: \( q \Rightarrow \sim p \); contrapositive: \( \sim q \Rightarrow p \); inverse: \( p \Rightarrow \sim q \)
7. Converse: \( \sim p \Rightarrow \sim q \); contrapositive: \( p \Rightarrow q \); inverse: \( q \Rightarrow p \)
9. Converse: If the grass is wet, then it is raining. Contrapositive: If the grass is not wet, then it is not raining. Inverse: If it is not raining, then the grass is not wet.
11. Converse: If it is not cloudy, then it is not raining. Contrapositive: If it is cloudy,
then it is raining. Inverse: If it is raining, then it is cloudy.  
13. Converse: If it is cloudy, then it is raining. Contrapositive: If it is not cloudy, then it is not raining. Inverse: If it is not raining, then it is not cloudy.

15. \[
\begin{array}{ccc}
p & q & \sim p & p \land q & \sim p \lor (p \land q) \\
T & T & F & T & T \\
T & F & F & F & F \\
F & T & T & F & T \\
F & F & T & F & T \\
\end{array}
\]

17. \[
\begin{array}{ccc}
p & q & \sim p & \sim p \land q & p \lor (\sim p \land q) \\
T & T & F & F & T \\
T & F & F & F & T \\
F & T & T & F & T \\
F & F & T & F & T \\
\end{array}
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19. \[
\begin{array}{ccc}
p & q & \sim p & \sim p \implies q \\
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T & F & F & T \\
F & T & T & T \\
F & F & T & F \\
\end{array}
\]

21. \[
\begin{array}{ccc}
p & \sim p & \sim p \lor p \\
T & F & T \\
F & T & T \\
\end{array}
\]

23. \[
\begin{array}{ccc}
p & q & p \implies q & p \land (p \implies q) \\
T & T & T & T \\
T & F & F & F \\
F & T & T & F \\
F & F & F & T \\
\end{array}
\]

25. \[
\begin{array}{cccccccc}
p & q & r & p \land q & q \land r & (p \land q) \land r & p \land (q \land r) & p \land (q \land r) \equiv (p \land q) \land r \\
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\end{array}
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27. \[
\begin{array}{cccccc}
p & q & p \lor q & p \land (p \lor q) & p \land (p \lor q) \iff p \\
T & T & T & T & T \\
T & F & T & T & T \\
F & T & T & F & T \\
F & F & F & F & T \\
\end{array}
\]

29. \(p \implies q\)  
31. \(\sim q \iff \sim p\)  
33. \(q \implies p\)

35. \(p \implies q = \sim p \lor q\) (Hint)  
\(p \implies q = q \lor \sim p\) (Commutative property)  
\(p \implies q = \sim q \implies \sim p\) (Hint)
### AN–62  Answers to Odd-Numbered Problems

#### 37. (a)

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<td>$q$</td>
<td>$r$</td>
<td>$\sim q$</td>
<td>$\sim r$</td>
<td>$p \land q$</td>
<td>$p \land \sim r$</td>
<td>$(p \land q) \Rightarrow r$</td>
<td>$(p \land \sim r) \Rightarrow \sim q$</td>
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#### (b)  

$(p \land q) \Rightarrow r = \sim (p \land q) \lor r$

$(p \land q) \Rightarrow r = \sim p \lor \sim q \lor r$  (De Morgan’s property)

$(p \land q) \Rightarrow r = \sim p \land (\sim q \lor r)$  (Associative property)

$(p \land q) \Rightarrow r = \sim p \land (r \lor \sim q)$  (Commutative property)

$(p \land q) \Rightarrow r = (\sim p \land r) \lor \sim q$  (Associative property)

$(p \land q) \Rightarrow r = (p \land \sim r) \lor \sim q$  (De Morgan’s property)

$(p \land q) \Rightarrow r = (p \land \sim r) \Rightarrow \sim q$

### Exercise 11.4 (p. 660)

1. Direct; Indirect or Proof by Contradiction  
2. True
3. Let $p$ and $q$ be the statements:
   $p$: It rains.  
   $q$: John goes to school.

Prove: $\sim p$ is true.

**Direct Proof:**

<table>
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<tr>
<th>True Statements</th>
<th>Justification</th>
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<tbody>
<tr>
<td>$p \Rightarrow \sim q$</td>
<td>Assumption</td>
</tr>
<tr>
<td>$q$</td>
<td>Assumption</td>
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<tr>
<td>$q \Rightarrow \sim p$</td>
<td>Contrapositive</td>
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<tr>
<td>$\sim p$</td>
<td>Law of Detachment</td>
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</table>

We conclude it is not raining.
5. Let \( p, q, \) and \( r \) be the statements:
\( p: \) Smith is elected president; \( q: \) Kuntz is elected secretary;
\( r: \) Brown is elected treasurer.
Prove: \( \sim r \) is true.

(a) Direct Proof:

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<th>Justification</th>
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<tbody>
<tr>
<td>( p \Rightarrow q )</td>
<td>Assumption</td>
</tr>
<tr>
<td>( q \Rightarrow \sim r )</td>
<td>Assumption</td>
</tr>
<tr>
<td>( p \Rightarrow \sim r )</td>
<td>Law of Syllogism</td>
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<tr>
<td>( p )</td>
<td>Assumption</td>
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<tr>
<td>( \sim r )</td>
<td>Law of Detachment</td>
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</table>

Since \( \sim r \) is true, Brown is not elected treasurer.

(b) Indirect Proof:
Assume \( \sim r \) is false.

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<th>True Statements</th>
<th>Justification</th>
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<tr>
<td>( r )</td>
<td>Law of Contradiction</td>
</tr>
<tr>
<td>( p \Rightarrow q )</td>
<td>Assumption</td>
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<tr>
<td>( p \Rightarrow q )</td>
<td>Assumption</td>
</tr>
<tr>
<td>( q \Rightarrow \sim r )</td>
<td>Assumption</td>
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<tr>
<td>( \sim r )</td>
<td>Law of Detachment</td>
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\* The argument resulted in a contradiction. So the assumption \( \sim r \) is false is incorrect so \( \sim r \) is true and we have shown that Brown was not elected treasurer.

7. Invalid. If the hypotheses \( p \Rightarrow q \) and \( \sim p \) are both true, then \( p \) is false. In an implication, if \( p \) is false, \( q \) can be either true or false so the conclusion, \( \sim q \), could be either true or false.

9. Valid. Let \( p: \) Tami studies, \( q: \) Tami fails, and \( r: \) Tami plays with dolls too often. We are given \( p \Rightarrow \sim q, \sim r \Rightarrow p, \) and we must show \( r \) is true. Using the Law of the Contrapositive we have \( q \Rightarrow \sim p \) and \( \sim p \Rightarrow r \) are true. Then by the Law of Syllogism, \( q \Rightarrow r. \) Finally, we are given \( q. \) By the Law of Detachment, we conclude \( r \) is true.

Exercise 11.5 (p. 665)

1. The output is 1 when (a) both \( p = 1 \) and \( q = 1 \) or (b) both \( p = 0 \) and \( q = 0 \), or (c) both \( p = 0 \) and \( r = 1 \).

3. The output is 1 when \( p \) and \( q \) are both 1.

5. \( p \quad q \)

7. \( p \quad q \)

9. (For Problem 1):

(For Problem 3):

(For Problem 5):

(For Problem 7):

11. The truth table and two possible diagrams for this circuit is either

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or

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13. \( p \) AND

Review Exercises (p. 666)

11. If I go to the math learning center, then I complete my math homework. 13. If I don’t go to the math learning center, then I don’t complete my math homework. 15. I don’t go to the math learning center and I don’t complete my math homework.

25. 29. \( q \Rightarrow p \) 31. \( p \Leftrightarrow q \) 33. Define the statements \( p \): The temperature outside is below 30\(^o\)C; \( q \): I wear gloves. (a) \( p \Rightarrow q \) (b) \( q \Rightarrow p \): If I wear gloves, then the temperature outside is below 30\(^o\)C. (c) \( \sim q \Rightarrow \sim p \): If I do not wear gloves, then the temperature outside is not below 30\(^o\)C. (d) \( \sim p \Rightarrow \sim q \): If the temperature is not below 30\(^o\)C, then I do not wear gloves.

35. Define the statements \( p \): Stu works on the project; \( q \): Julie helps. (a) \( q \Rightarrow p \) (b) \( p \Rightarrow q \): If Stu works on the project, then Julie helps. (c) \( \sim p \Rightarrow \sim q \): If Stu does not work on the project, then Julie does not help. (d) \( \sim q \Rightarrow \sim p \): If Julie does not help, then Stu does not work on the project.

37. Define the statements \( p \): Kurt goes to the club; \( q \): Jessica comes to town. (a) \( q \Rightarrow p \) (b) \( p \Rightarrow q \): If Kurt goes to the club, then Jessica comes to town. (c) \( \sim p \Rightarrow \sim q \): If Kurt does not go to the club, then Jessica does not come to town. (d) \( \sim q \Rightarrow \sim p \): If Jessica does not come to town, then Kurt does not go to the club.

39. Define the statements \( p \): Brian comes to the gym; \( q \): Mike works out. (a) \( q \Rightarrow p \) (b) \( p \Rightarrow q \): If Brian comes to the gym, then Mike works out. (c) \( \sim p \Rightarrow \sim q \): If Brian does not come to the gym, then Mike does not work out. (d) \( \sim q \Rightarrow \sim p \): If Mike does not work out, then Brian does not come to the gym.

41. Define the statements \( p \): Patrick goes to practice; \( q \): Patrick starts the game. Show that \( p \Rightarrow q \) is equivalent to \( \sim p \lor q \).

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<th>( p \Rightarrow q )</th>
<th>( \sim p \lor q )</th>
<th>( p \Rightarrow q \equiv (\sim p \lor q) )</th>
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<td>F</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

43. See the table for Problem 41.

45. Define the statements \( p \): I paint the house; \( q \): I go bowling. Assume the premises \( \sim p \Rightarrow q \) and \( \sim q \) are true. Show \( p \) is true.

Prove: \( p \) is true.

(a) Indirect Proof:

Assume \( p \) is false.

(b) Direct Proof:

\[ \begin{array}{ccc}
\text{True Statements} & \text{Justification} \\
\sim p \Rightarrow q & \text{Assumption} \\
\sim q \Rightarrow p & \text{Contrapositive} \\
\sim q & \text{Assumption} \\
p & \text{Law of Detachment} \\
\end{array} \]

So \( p \) is true; I paint the house.

* The proof resulted in a contradiction, so the assumption that \( p \) is false is incorrect. We conclude \( p \) is true, and I paint the house.
47. Define the statements: p: John is in town, q: Mark gets tickets, and r: We go to the game. Assume the premises p \implies q, \neg r \implies \neg q, and p are true. Prove: r is true.

(a) Direct Proof:

<table>
<thead>
<tr>
<th>True Statements</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>p \implies q</td>
<td>Assumption</td>
</tr>
<tr>
<td>\neg r \implies \neg q</td>
<td>Assumption</td>
</tr>
<tr>
<td>q \implies r</td>
<td>Law of Contrapositive</td>
</tr>
<tr>
<td>p \implies r</td>
<td>Law of Syllogism</td>
</tr>
<tr>
<td>p</td>
<td>Assumption</td>
</tr>
<tr>
<td>r</td>
<td>Law of Detachment</td>
</tr>
</tbody>
</table>

So r is true, and we go to the game.

49. Define the statements: p: I pay a finance charge, q: My payment is late, r: Colleen sends the mail. Assume the premises q \implies p, r \lor q, and \neg r are true. Prove p is true.

(a) Direct Proof:

<table>
<thead>
<tr>
<th>True Statements</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>r \lor q</td>
<td>Assumption</td>
</tr>
<tr>
<td>\neg r</td>
<td>Assumption</td>
</tr>
<tr>
<td>q</td>
<td>A true disjunction has at least one true proposition</td>
</tr>
<tr>
<td>q \implies p</td>
<td>Assumption</td>
</tr>
<tr>
<td>p</td>
<td>Law of Detachment</td>
</tr>
</tbody>
</table>

So p is true, and I pay a finance charge.

51. Define the statements: p: Rob is a bad boy, q: Danny is crying, r: Laura is a good girl. Assume the premises p \lor q, r \implies ~p, and ~q are true. We want to prove ~r.

<table>
<thead>
<tr>
<th>True Statements</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>p \lor q</td>
<td>Assumption</td>
</tr>
<tr>
<td>\neg r</td>
<td>Assumption</td>
</tr>
<tr>
<td>q</td>
<td>A true disjunction has at least one true proposition</td>
</tr>
<tr>
<td>q \implies p</td>
<td>Assumption</td>
</tr>
<tr>
<td>p \implies \neg r</td>
<td>Contrapositive</td>
</tr>
<tr>
<td>\neg r</td>
<td>Law of Detachment</td>
</tr>
</tbody>
</table>

We have shown that ~r is true. So we conclude: Laura is not a good girl.
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Chapter 8

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Chapter 10

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Standard Normal Curve Table

An entry in the table is the area under the curve between $Z = 0$ and a positive value of $Z$. Areas for negative values of $Z$ are obtained by symmetry.

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### Lines

**Slope**
The slope \( m \) of the line containing the points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) is

\[
m = \frac{y_2 - y_1}{x_2 - x_1} \text{ if } x_1 \neq x_2
\]

\( m \) is undefined if \( x_1 = x_2 \)

**Point-Slope Equation**
The equation of the line with slope \( m \) containing the point \((x_1, y_1)\) is

\[
y - y_1 = m(x - x_1)
\]

**Slope-Intercept Equation**
The equation of the line with slope \( m \) and \( y \)-intercept \((0, b)\) is

\[
y = mx + b
\]

**General Equation**
Any line can be expressed by an equation of the form

\[
Ax + By = C
\]

where \( A, B, \) and \( C \) are real numbers and either \( A \neq 0 \) or \( B \neq 0 \).

### Pairs of Lines

- **Two Lines are Either Coincident (identical)**
  - All points on each line are the same.
  - Coincident lines that are vertical have undefined slope and the same \( x \)-intercept.
  - Coincident lines that are non-vertical have the same slope and the same \( x \)-intercepts.

- **Parallel**
  - The two lines have no points in common.
  - Parallel lines that are vertical have undefined slope and different \( x \)-intercepts.
  - Parallel lines that are non-vertical have the same slope and different \( x \)-intercepts.

- **Intersecting**
  - The two lines have one point in common.
  - Intercepting lines have different slopes.

### Finance

**Simple Interest Formula**
If a principal \( P \) is borrowed at a simple interest rate of \( R\% \) per annum (where \( R\% \) is expressed as the decimal \( r = \frac{R}{100} \)) for a period of \( t \) years, the interest \( I \) is

\[
I = Prt
\]

**Discounted Loans**
If \( r \) is the per annum rate of interest, \( t \) is the time in years, and \( L \) is the amount of the loan, then the proceeds \( R \), the amount received, is

\[
R = L - Lrt
\]

where \( Lrt \) is the discount, the interest deducted from the amount of the loan.

**Compound Interest Formulas**
The amount \( A \) after \( t \) years due to a principal \( P \) invested at an annual interest rate \( r \) compounded \( n \) times per year is

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt}
\]

The amount \( A \) after \( t \) years due to a principal \( P \) invested at an annual interest rate \( r \) compounded continuously is

\[
A = Pe^{rt}
\]

**Amount of an Annuity**
Suppose \( P \) is the deposit made at each payment period for an annuity paying \( i \) percent interest per payment period. The amount \( A \) of the annuity after \( n \) deposits is

\[
A = P \left( 1 + \frac{i}{n} \right)^n - 1 \frac{i}{i}
\]

**Amortization**
The payment \( P \) required to pay off a loan of \( V \) dollars borrowed for \( n \) payment periods at a rate of interest \( i \) per payment period is

\[
P = \frac{V}{1 - (1 + i)^{-n}}
\]

### Counting

**Counting Formula**
For two sets \( A \) and \( B \),

\[
n(A \cup B) = n(A) + n(B) - n(A \cap B)
\]

**Factorials**
0! = 1, 1! = 1, \( n! = n(n - 1)(n - 2) \ldots (3)(2)(1) \)

An ordered arrangement of \( r \) objects chosen from \( n \) objects.

The number of ordered arrangements of \( r \) objects chosen from \( n \) objects, in which the \( n \) objects are distinct and repetition is allowed, is \( n \).

The number of ordered arrangements of \( r \) objects chosen from \( n \) objects, where \( r \leq n \), in which the \( n \) objects are distinct and repetition is not allowed, is

\[
P(n, r) = \frac{n!}{(n - r)!}
\]
The number of permutations of \( n \) objects, of which \( n_1 \) are of one kind, \( n_2 \) are of a second kind, . . . , and \( n_k \) are of a \( k \)th kind, is
\[
\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}
\]
where \( n_1 + n_2 + \cdots + n_k = n \).

**Combinations**

An arrangement, without regard to order and without repetition, of \( r \) objects selected from \( n \) distinct objects, where \( r \leq n \). The number of such arrangements is
\[
C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

**Binomial Theorem**

For \( n > 1 \) a positive integer, \((x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\)

**Probability**

**Equally Likely Events**

If the sample space \( S \) of an experiment has \( n \) equally likely outcomes and the event \( E \) in \( S \) contains \( m \) outcomes, then
\[
P(E) = \frac{m}{n}
\]

**Additive Rule**

For any two events \( E \) and \( F \) of a sample space \( S \),
\[
P(E \cup F) = P(E) + P(F) - P(E \cap F)
\]

**Complement**

For any event \( E \) in a sample space \( S \),
\[
P(E) = 1 - P(E')
\]

**Conditional Probability**

If \( E \) and \( F \) are two events in a sample space \( S \) and if \( P(F) \neq 0 \),
\[
P(E|F) = \frac{P(E \cap F)}{P(F)}
\]

**Product Rule**

If \( E \) and \( F \) are two events in a sample space \( S \),
\[
P(E \cap F) = P(E) P(F)
\]

**Independent Events**

Two events \( E \) and \( F \) of a sample space \( S \) are independent if and only if
\[
P(E \cap F) = P(E) P(F)
\]

**Bayes’ Formula**

Let \( S \) be a sample space partitioned into \( n \) events, \( A_1, A_2, \ldots, A_n \). If \( E \) is an event in \( S \) for which \( P(E) > 0 \), then
\[
P(A_j|E) = \frac{P(A_j) P(E|A_j)}{P(E)} = \frac{P(A_j) P(E|A_j)}{P(A_1) P(E|A_1) + P(A_2) P(E|A_2) + \cdots + P(A_n) P(E|A_n)}
\]
for \( j = 1, 2, \ldots, n \).

**Binomial Probabilities**

In a Bernoulli trial, the probability of exactly \( k \) successes in \( n \) trials is
\[
b(n, k, p) = \binom{n}{k} p^k q^{n-k}
\]
where \( p \) is the probability of success and \( q = 1 - p \) is the probability of failure.

**Statistics**

**Sample Data**

The data used are a sample \( x_1, x_2, \ldots, x_n \), taken from the population of \( N \) items, \( n < N \).

**Sample Mean**

\[
\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}
\]

**Sample Standard Deviation**

\[
s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1}}
\]

**Population Data**

The data used are from the entire population \( x_1, x_2, \ldots, x_N \) of \( N \) items.

**Population Mean**

\[
\mu = \frac{x_1 + x_2 + \cdots + x_N}{N}
\]

**Population Standard Deviation**

\[
\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}}
\]

**Z – score**

\[
Z = \frac{x - \mu}{\sigma}
\]